PRACTICAL ASTRONOMY

BY

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SECOND EDITION.
FIRST THOUSAND.

NEW YORK:
JOHN WILEY & SONS,
53 EAST TENTH STREET.
1893.

Robert Drummond,
Electrotyper,
444 & 446 Pearl Street,
New York.

Ferris Bros.,
Printers,
326 Pearl Street,
New York.

PREFACE.

THIS volume, both in respect to matter and arrangement, is designed especially for the use of the cadets of the U. S. Military Academy, as a supplement to the course in General Astronomy at present taught them from the text-book of Professor C. A. Young. It is therefore limited to that branch of Practical Astronomy which relates to Field Work, and more particularly to those subjects which are not discussed at sufficient length for practical work in Professor Young's volume. It is believed, however, that it will find a useful application in the hands of officers of the Army, who may be called upon to conduct such explorations and surveys for military purposes as the War Department may from time to time direct.

The more usual methods of determining Time, Latitude, and Longitude, on Land, are explained, and the requisite reduction formulas are deduced and explained. In addition, there is given a short explanation of the principles relating to the Construction of Ephemerides, to the Figure of the Earth, the determination of Azimuths, and the projection of Solar Eclipses.

The instruments described are those used by the cadets in the Field and Permanent Observatories of the Military Academy during the summer encampment.

The principal sources of information from which the matter in this volume has been derived are the published Reports of the United States Lake, Coast, and Northern Boundary Surveys; the publications of the Hydrographic Office, U. S. Navy, and the works of Brünnow and Chauvenet.

U. S. MILITARY ACADEMY,
WEST POINT, N. Y., October, 1892.

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OTHER METHODS OF DETERMINING LONGITUDE.

PRACTICAL ASTRONOMY.

EPHEMERIS.

Ephemeris.—The numerical values of the coördinates of the principal celestial bodies, together with the elements of position of the circles of reference, are recorded for given equidistant instants of time in an Astronomical Ephemeris.

The "American Ephemeris and Nautical Almanac" is published by the United States Government, generally three years in advance of the year of its title, and comprises three parts, viz.:

Part I.—**Ephemeris for the Meridian of Greenwich,** which gives the heliocentric and geocentric positions of the major planets, the ephemeris of the sun, and other fundamental astronomical data for equidistant intervals of mean Greenwich time.

Part II.—**Ephemeris for the Meridian of Washington,** which gives the ephemerides of certain fixed stars, sun, moon, and major planets, for transit over the meridian of Washington, and also the mean places of the fixed stars, with the data for their reduction.

Part III.—**Phenomena,** which contains prediction of phenomena to be observed, with data for their computation.

EPHEMERIS OF THE SUN.

To construct the ephemeris of the sun it is necessary to compute its tables: these are

1. The table of Epochs.
2. The table of Longitudes of Perigee.
3. The table of Equations of the Center, and its corrections.
4. The table of the Equations of the Equinoxes in Longitude.

In Mechanics* it was shown that the Earth's undisturbed orbit is an ellipse, having one of its foci at the sun's center, and that the earth's angular velocity is

$$\frac{d\theta'}{dt} = \frac{h}{r^2}; \quad \ldots \ldots \ldots \quad (551)$$

its radius vector,

$$r = \frac{a\,(1 - e^2)}{1 + e \cos \theta}; \quad \ldots \ldots \ldots \quad (610)$$

its constant double sectoral area,

$$h = \sqrt{\mu' a\,(1 - e^2)}; \quad \ldots \ldots \ldots \quad (615)$$

and its periodic time,

$$\tau = 2\,\pi \sqrt{\frac{a^3}{\mu'}} = \frac{2\,\pi}{n}. \quad \ldots \ldots \ldots \quad (616)$$

In these expressions θ' is the angle made by the earth's radius vector with any assumed right line drawn through the sun's center, θ that included between the radius vector and the line of apsides estimated from perihelion, and n is the mean motion of the earth in its orbit.

From (551), (615) and (616), we have

$$\frac{d\theta'}{dt} = \frac{d\theta}{dt} = \frac{\sqrt{\mu' a\,(1 - e^2)}\;(1 + e \cos \theta)^2}{a^2\,(1 - e^2)^2}$$

$$= \sqrt{\frac{\mu'}{a^3}}\,\frac{(1 + e \cos \theta)^2}{(1 - e^2)^{\frac{3}{2}}} = n\,\frac{(1 + e \cos \theta)^2}{(1 - e^2)^{\frac{3}{2}}}: \qquad (1)$$

and therefore

$$n\,d\,t = (1 - e^2)^{\frac{3}{2}}\,(1 + e \cos \theta)^{-2}\,d\,\theta'. \qquad (2)$$

Since e varies but little from 0.01678 (see Art. 185, Young †), we may omit all terms containing the third and higher powers of e in the development of the second member of the preceding equation.

* Michie's Mechanics, 4th Edition.

† Young's General Astronomy.

$(x+y)^m = x^m + m x^{m-1} y + \frac{m(m-1)}{2} x^{m-2} y^2 + \&c.$

$n\,dt = (1 - \frac{3}{2} e^2 + \&c)(1 - 2e\cos\theta + 3e^2\cos^2\theta + \&c)\,d\theta'$

$n\,dt = (1 - 2e\cos\theta + \frac{3e^2}{2}(1 + \cos 2\theta) - \frac{3}{2}e^2 + \&c)\,d\theta'$

$= d\theta' - 2e\cos\theta\,d\theta' + \frac{3}{4}\cos 2\theta\, d(2\theta)$ $\quad d(2\theta) = 2\,d\theta$

Then after substituting $\frac{1 + \cos 2\theta}{2}$ for $\cos^2\theta$, we have

$$n\,d\,t = d\,\theta' - 2\,e\cos\theta\,d\,\theta + \tfrac{3}{4}e^2\cos 2\,\theta\,d\,(2\,\theta) + \text{etc.} \tag{3}$$

Integrating we have

$$n\,t + C = \theta' - 2\,e\sin\theta + \tfrac{3}{4}e^2\sin 2\,\theta + \text{etc.} \tag{4}$$

The earth's orbit is, however, not entirely undisturbed. Due to the perturbating action of other bodies of the solar system the earth is never exactly in the place which it would occupy in an undisturbed orbit. Moreover the line of apsides has a direct motion, i.e., in the direction in which longitudes are measured, of about 11''.7 per annum, and the vernal equinox an irregular retrograde motion whose mean value is about 50''.2 per annum.

Therefore (Fig. 1), let the line from which θ' is estimated be

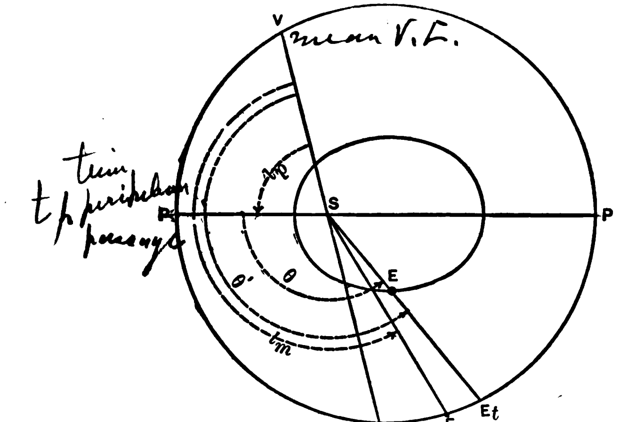


FIG. 1.

that drawn through the sun and the position of the mean vernal equinox V at some fixed instant, called the epoch. Then when θ is zero, θ' will be the longitude of perihelion, *estimated from this point.* Let this be denoted by l_p, and the time of perihelion passage by t_p; then from (4) we have,

$$n\,t_p + C = l_p. \tag{5}$$

Subtracting from (4) we have

$$n\,(t - t_p) = \theta' - l_p - 2\,e\sin\theta + \tfrac{5}{4}\,e^2 \sin 2\,\theta, \tag{6}$$

which since

$$\theta' - l_p = \theta \tag{7}$$

reduces to

$$n\,(t - t_p) = (\theta' - l_p) - 2\,e\sin(\theta' - l_p) + \tfrac{5}{4}\,e^2 \sin 2\,(\theta' - l_p). \tag{8}$$

Transposing l_p, we have

$$n\,(t - t_p) + l_p = l_m = \theta' - 2\,e\sin(\theta' - l_p) + \tfrac{5}{4}\,e^2 \sin 2\,(\theta' - l_p), \tag{9}$$

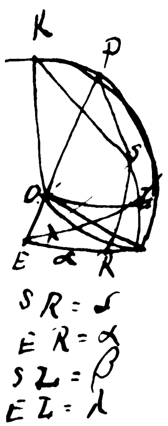
in which l_m is the longitude of the mean place of the earth at the time t, referred to the same origin.

Let L be the longitude of the earth's mean place at the epoch, also referred to the same origin, and T any *interval* of time before or after this epoch. Then will

$$l_m = L + n\,T, \tag{10}$$

and we have

$$L + n\,T = \theta' - 2\,e\sin(\theta' - l_p) + \tfrac{5}{4}\,e^2 \sin 2\,(\theta' - l_p). \tag{11}$$



To find the values of the four unknown quantities, L, n, e, and l_p, take four observations of R. A. and declination at different times, and having reduced the declination to its geocentric value by correcting for refraction and parallax, find the corresponding longitudes (Art. 180, Young).

Each longitude is necessarily referred to the true equinox of its own date. Reduce each to the mean equinox of the epoch by correcting for aberration, nutation, precession, and perturbations, add 180°, and the results will be the longitudes of the true place of the earth referred to a common point—the mean equinox of the epoch.

They will therefore be the values of θ' corresponding to the values of T in the following equations, the solution of which will give L, n, e, and l_p.

$$\left.\begin{aligned} L + n\,T_1 &= \theta_1' - 2\,e\sin(\theta_1' - l_p) \\ L + n\,T_2 &= \theta_2' - 2\,e\sin(\theta_2' - l_p) \\ L + n\,T_3 &= \theta_3' - 2\,e\sin(\theta_3' - l_p) \\ L + n\,T_4 &= \theta_4' - 2\,e\sin(\theta_4' - l_p) \end{aligned}\right\}. \tag{12}$$

The value of n derived from these equations is evidently the earth's mean motion from a fixed point.

Its mean motion from the moving mean vernal equinox (or mean motion in longitude) is evidently given by

$$n' = n \frac{360^\circ}{360^\circ - 50.''2}.$$

These observations repeated at different times will determine the changes that take place in n, e, and l_p; from the last two the variations in the eccentricity and the rate of motion of perihelion can be found.

Having in this manner found the elements of the earth's place and motion, the corresponding mean longitude of the sun at any instant can be obtained by adding to that of the earth 180°. $L + n' T + 180^\circ$ will then give for any instant the mean longitude of the sun's mean place. The difference between the longitudes of the sun's true and mean places at any instant is the Equation of the Center for that instant.

From the preceding elements let it be required to construct the Ephemeris of the Sun.

1. The Table of Epochs.—Take mean midnight, December 31—January 1, 1890, as the epoch. To the mean longitude of the sun's mean place at that epoch, add the product of the sun's mean motion n', by the number of mean solar days after the epoch, subtracting 360° when this sum is greater than 360°. These longitudes with their corresponding times being tabulated, form the table of epochs, from which the mean longitude of the mean place of the sun can be found by inspection for any day, hour, minute or second.

2. The Table of Longitudes of Perigee.—The longitude of perihelion increased by 180° is the corresponding longitude of perigee. Hence the former being found, and its rate of change determined, the addition of 180° to each longitude of perihelion will give the longitude of perigee, and these values being tabulated form the table of longitudes of perigee.

3. The Table of Equations of the Center.—The difference between the true and mean anomalies at any instant, given by the first of Eqs. (650), Mechanics,

$$\theta - nt = 2e \sin nt + \tfrac{5}{4} e^2 \sin 2nt + \text{etc.}, \tag{13}$$

arc a : arc a' :: r : r'

arc a' = $\frac{\text{arc } a \times r'}{r}$ a' = 2πr'

is called the Equation of the Center, and is known when n **and** e are known; t being the time since perihelion passage.

Assuming e to be constant and causing $n\,t$ to vary from 0° to 360°, the resulting values of the second member of the equation will form a table of the equations of the center. The errors in these values arise from the small variations in the values of e; these errors can be found by substituting in the second member of the above equation the actual values of e at the time, and the differences being tabulated will give a table by which the equations of the center may be corrected from time to time.

4. The Perturbations in Longitude of the earth arising from the attractions of the planets (especially Venus and Jupiter), are the same for the sun; these are computed by the methods indicated in Physical Astronomy, (see Art. 174, Mechanics,) and then tabulated.

5. The Sun's Aberration is taken to be constant, amounting to $-20''.25$ and is included in the table of epochs.

Ephemeris of the Sun.—The above tables having been computed, we proceed as follows:

1. From the table of epochs take out the mean longitude of the sun's mean place corresponding to the exact instant considered.

2. From the table of longitudes of perigee take the mean longitude of perigee; the difference between this and the mean longitude of the sun's mean place is the mean anomaly.

3. With the mean anomaly as an argument find the corresponding value of the equation of the center from its table, and add it with its proper sign to the mean longitude of the sun's mean place; the result will be the mean longitude of the sun's true place; hence the

Sun's true longitude = Mean longitude of sun's mean place ± Equation of center ± Perturbations in longitude ± Corrections to pass from the mean equinox of date to true equinox of date. These latter corrections are due to Nutation and constitute the Equation of the Equinoxes in Longitude.

4. Having the true longitude of the sun and the obliquity of the ecliptic, the corresponding Right Ascension and Declination of the sun can be computed for the same instant by the method explained in Art. 180, Astronomy.

6. Earth's Radius Vector.—Substituting the values of e and $n\,t$, in the second of Eqs. (650), Mechanics, will give the values of the distance of the sun from the earth in terms of the mean distance a: thus

$$r = a\left(1 - e\cos nt + \frac{e^2}{2}(1 - \cos 2\,nt) - \frac{3e^3}{8}(\cos 3\,nt - \cos nt) + \text{etc.}\right). \quad (14)$$

7. The Sun's Horizontal Parallax.—From astronomical observations the value of a (and hence of r) is found in terms of the earth's equatorial radius, ρ_e. (Young, Chapters XIII and XVI.)

The sun's equatorial horizontal parallax, P, at any time is then given by

$$P = \frac{\omega}{r}, \quad (15)$$

ω being the number of seconds in a radian $= 206264''.8$, and r being expressed as just stated.

At any place where the earth's radius in terms of the equatorial radius is ρ, we shall have for the horizontal parallax $\frac{\rho\omega}{r} = \rho P$.

8. The Sun's Apparent Semi-Diameter.—Knowing P, the sun's linear semi-diameter s', is known in terms of ρ_e. Its angular apparent semi-diameter s, for any day, is then given by

$$s = Ps'. \quad (16)$$

9. Equation of the Equinoxes in Longitude.—Due to physical causes, the pole of the equator completes a revolution about the pole of the ecliptic in about 26,000 years. The plane of the equator conforming to this motion of the pole, its intersection with the plane of the ecliptic, called the line of the equinoxes, turns with a retrograde motion of about $50''.2$ per annum about the sun as a fixed point.

This motion is not however, perfectly uniform. The true pole describes once in 19 years around the moving mean place above referred to, a small ellipse, whose transverse axis directed toward the pole of the ecliptic is $18''.5$ in angular measure, and whose conjugate axis is $13''.74$. The corresponding irregularity in the motion of the line of the equinoxes causes a slight oscillation of the true on either side of the moving mean equinox. Both are on the ecliptic; and their distance apart at any time is called the *Equation of the Equinoxes in Longitude,* its projection on the equator the *Equation of the Equinoxes in Right Ascension,* and the intersection of the declination circle which projects the mean equinox with the equator, the *Reduced Place of the Mean Equinox.* The maximum value of the Equation of the Equinoxes in Longitude is

$$\frac{13''.74}{2} \div \sin 23^\circ 28' = 17''.25.$$

To illustrate, P, in Fig. 2, is the pole of the equator, VE the ecliptic, VM the equator, V the true, V' the mean, and V'' the re-

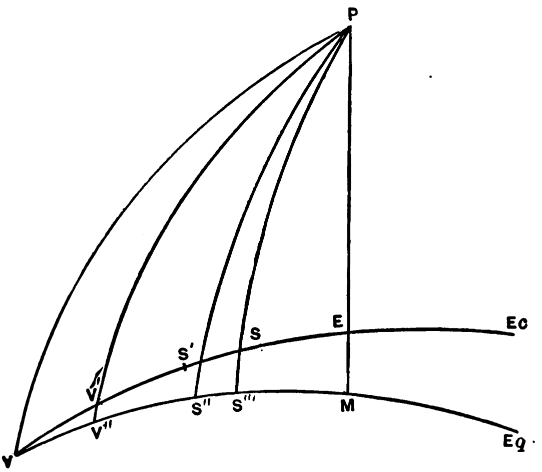


FIG. 2.

duced place of the mean vernal equinox. VV' is the equation of the equinoxes in longitude, and VV'' in Right Ascension.

The equation of the equinoxes in longitude is a function of the longitude of the moon's node, the longitude of the sun, and the obliquity of the ecliptic. Separate tables are constructed for this correction, in which the arguments for entering them are the *obliquity* and *longitude of the moon's node,* and the *obliquity* and *the longitude of the sun;* the sum of the two corrections is the value of the equation of the equinoxes in longitude at the corresponding times.

10. Equation of Time.—If, at the instant when the true sun's mean place coincides with the mean equinox, an imaginary point should leave the reduced place of the mean equinox and travel with uniform motion on the celestial equator, returning to its starting-point at the instant the true sun's mean place next again coincides with the mean equinox, such a point is called a *Mean Sun.* Time measured by the hour angles of this point is called *Mean Solar Time.* The angle included between the declination circles passing through the centre of the true sun and this point at any instant is called the *Equation of Time* for that instant; its value, at any instant, added algebraically to mean or apparent solar time will give the other. As the apparent time can be found by direct observation the equation of time is usually employed as a correction to pass from apparent to mean solar time. Thus in Fig. 2, PM is the meridian, S the true sun, S' its mean place, S'' the mean sun, VS''' the true R. A. of the true sun, $V''S''$ the mean R. A. of the mean sun $= V'S' =$ sun's mean longitude, angle MPS''' or arc MS''' apparent solar time, MS'' mean solar time, and $S''S'''$ the Equation of Time $= VS''' - (V''S'' + VV'')$.

Hence we have for the Equation of Time,

$$\epsilon = \text{True sun's true Right Ascension} - (\text{sun's mean longitude} + \text{equation of equinoxes in R. A.}). \quad (17)$$

The mean sun (S'') moving in the equator and used in connection with time, must not be confused with the mean sun (S') before referred to, moving in the ecliptic.

11. Referring to the American Ephemeris, we see that Page I of each month contains the Sun's Apparent R. A., Declination, Semi-diameter, Sidereal time of semi-diameter passing the meridian, at Greenwich apparent noon, together with the values for their respective hourly changes; the latter being computed from the values of their differential co-efficients. From these we can find the corresponding data for any other meridian. Page II contains similar data for the epoch of Greenwich mean moon, and in addition the sidereal time or R. A. of the mean sun. Page III contains the sun's true longitude and latitude, the logarithm of the earth's radius vector and the mean time of sidereal noon. The obliquity, precession, and sun's mean horizontal parallax for the year, are found on page 278 of the Ephemeris. All these constitute an Ephemeris of the Sun.

From the hourly changes the elements for any meridian can be readily computed.

THE EPHEMERIS OF THE MOON.

The Ephemeris of the Moon consists of tables giving the Moon's Right Ascension and Declination for every hour of Greenwich mean time, with the changes for each minute; the Apparent Semi-diameter, Horizontal Parallax, Time of upper transit on the Greenwich Meridian, and Moon's Age. In order to compute these, it is first necessary to find the True Longitude of the Moon, its True Latitude, the Longitude of the Moon's Node, the Inclination of the Moon's Orbit to the Ecliptic, and the Longitude of Perigee.

1. The Elements of the Lunar Orbit.—Let DC be the intersection of the celestial sphere by the plane of the lunar orbit; VB the

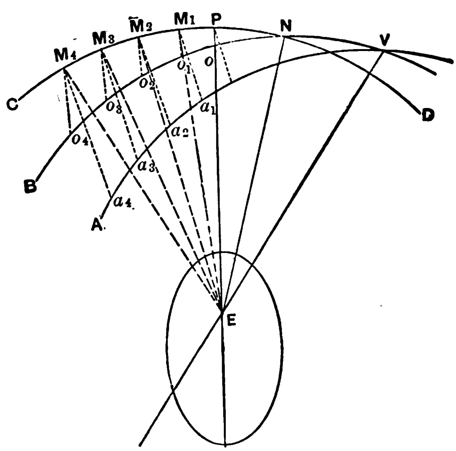


FIG. 3.

ecliptic, and VA the equinoctial; V the mean vernal equinox, N the ascending node, P the Perigee, all relating to some assumed epoch. Also let M_1, M_2, M_3, M_4, be the geocentric places of the moon's center at the four times, t_1, t_2, t_3, t_4. These places are obtained as in case of the sun by observed Right Ascensions and Declinations, corrected for refraction, semi-diameter, parallax, and perturbations, then converted into the corresponding latitudes and longitudes, and finally referred to the mean equinox of the epoch, by correcting for aberration, nutation, and precession.

Referring to the figure, assume the following notation:

$\nu = VN$, the longitude of the node;
$i = CNB$, the inclination of the orbit;
$l_1 = VO_1$, the longitude of M_1;
$l_2 = VO_2$, the longitude of M_2;
$\lambda_1 = M_1 O_1$, the latitude of M_1;
$\lambda_2 = M_2 O_2$, the latitude of M_2;
$v_1 = VEN + NEM_1$, the orbit longitude of M_1;
$p = VEN + NEP$, the orbit longitude of perigee;
$\phi = PEM_1 = v_1 - p$, the true anomaly of M_1;
e = eccentricity of orbit;
m = mean motion of moon in its orbit;
t_1 = time since epoch for M_1;
L = mean orbit longitude at epoch.

To find ν and i, we have from the right-angled spherical triangles $M_1 N O_1$ and $M_2 N O_2$,

$$\left.\begin{aligned}\sin (l_1 - \nu) &= \cot i \tan \lambda_1 \\ \sin (l_2 - \nu) &= \cot i \tan \lambda_2\end{aligned}\right\} \qquad (18)$$

and by division,

$$\frac{\sin (l_1 - \nu)}{\sin (l_2 - \nu)} = \frac{\tan \lambda_1}{\tan \lambda_2}. \qquad (19)$$

Adding unity to both members, reducing, then subtracting each member from unity, again reducing, and finally dividing one result by the other, we obtain

$$\frac{\sin (l_2 - \nu) + \sin (l_1 - \nu)}{\sin (l_2 - \nu) - \sin (l_1 - \nu)} = \frac{\tan \lambda_2 + \tan \lambda_1}{\tan \lambda_2 - \tan \lambda_1}, \qquad (20)$$

or by reduction formulas, page 4 (Book of Formulas),

$$\tan \left[\frac{l_2 + l_1}{2} - \nu\right] = \tan \tfrac{1}{2} (l_2 - l_1) \frac{\sin (\lambda_2 + \lambda_1)}{\sin (\lambda_2 - \lambda_1)}. \qquad (21)$$

from which ν can be found; i is found from either of equations (18), when ν is known.

To find L, m, e, and p, we proceed as in the determination of the table of epochs in the case of the sun, using a similar equation, thus:

$$\left.\begin{aligned} L + m\,T_1 &= v_1 - 2\,e \sin\,(v_1 - p), \\ L + m\,T_2 &= v_2 - 2\,e \sin\,(v_2 - p), \\ L + m\,T_3 &= v_3 - 2\,e \sin\,(v_3 - p), \\ L + m\,T_4 &= v_4 - 2\,e \sin\,(v_4 - p), \end{aligned}\right\} \tag{22}$$

in which

$$v_1 = \nu + \tan^{-1} \frac{\tan\,(l_1 - \nu)}{\cos i}; \tag{23}$$

and similar values for v_2, v_3, and v_4.

To find the *ecliptic longitude of perigee* $V\,O$, represented by p_1, we have from the right-angled triangle $N\,P\,O$,

$$\tan N\,O = \tan\,(p - \nu)\,.\,\cos i, \tag{24}$$

from which

$$p_1 = \nu + \tan^{-1}\,(\tan\,(p - \nu)\,.\,\cos i). \tag{25}$$

Similarly the *mean ecliptic longitude of the moon*, L_1, at the epoch is

$$L_1 = \nu + \tan^{-1}\,(\tan\,(L - \nu)\,.\,\cos i). \tag{26}$$

To find the *sidereal period*, s, we have

$$s = \frac{360°}{m} \tag{27}$$

in which s is the length of the sidereal period in mean solar days.

2. The Ephemeris of the Moon.—The motion of the moon is much more irregular and complicated than the apparent motion of the sun, owing mainly to the disturbing action of this latter body. But this and other perturbations have been computed and tabulated, and from these tables, including those of the node and inclination, the places of the moon in her orbit are found in the same way as those of the sun in the ecliptic. The mean orbit longitude of the moon and of her perigee are first found and corrected; their difference gives her mean anomaly, opposite to which in the appropriate table is found the equation of the center, and this being applied with its proper sign to the mean orbit longitude gives the true orbit longitude, after reduction to true equinox of date.

The Right Ascension and Declination of the Moon can now be computed for any instant of time, thus: subtract the longitude of the node from the orbit longitude of the moon, and we have the moon's angular distance from her node, represented in the figure by NM_1. This, with the inclination i, will give us the moon's latitude and the angular distance NO_1; the latter added to the longitude of the node will give the moon's longitude VO_1. The latitude, longitude, and obliquity of the ecliptic suffice to compute the right ascension and declination. The radius vector, equatorial horizontal parallax, apparent diameter, etc., are computed as in the case of the sun.

THE EPHEMERIS OF A PLANET.

From the tables of a planet its true orbit longitude as seen from the sun is found, as in the case of the moon as seen from the earth. The heliocentric longitude and latitude, and the radius vector are found from the heliocentric orbit longitude, heliocentric longitude of the node, and inclination, in the same way as the geocentric elements of the moon are found from similar data in the lunar orbit.

To pass from heliocentric to geocentric coördinates, let P, Fig. 4, be the planet's center, E that of the earth, S that of the sun, and

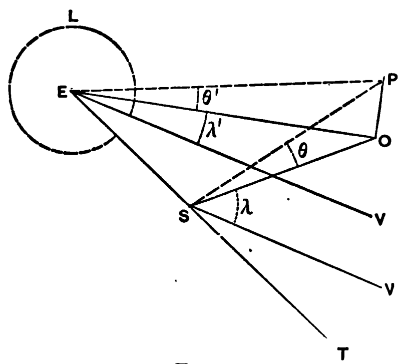


FIG. 4.

O the projection of P on the plane of the ecliptic. SV and EV are drawn to the vernal equinox; then let

$r\ = E\,S$, be the earth's radius vector;
$r' = S\,P$, be the planet's radius vector;
$\lambda\ = V\,S\,O$, be the heliocentric longitude of planet;
$\lambda' = V\,E\,O$, be the geocentric longitude of planet;
$\theta\ = P\,S\,O$, be the heliocentric latitude of planet;
$\theta' = P\,E\,O$, be the geocentric latitude of planet;
$S\ = O\,S\,E$, be the commutation;
$O\ = S\,O\,E$, be the heliocentric parallax;
$E\ = S\,E\,O$, be the elongation;
$L\ = V\,E\,S$, be the longitude of the sun;
$r'' = E\,P$, be the distance of planet from the earth.

To find the *geocentric longitude,*

$$S\,O = r' \cos \theta, \tag{28}$$

$$V\,S\,T = V\,E\,S = 360^\circ - L, \tag{29}$$

$$S = 180^\circ - (360^\circ - L) - \lambda = L - 180^\circ - \lambda, \tag{30}$$

from which S is known.

In the plane triangle $O\,E\,S$, we have

$$r' \cos \theta + r : r' \cos \theta - r :: \tan \tfrac{1}{2}(E + O) : \tan \tfrac{1}{2}(E - O). \tag{31}$$

$$S + O + E = 180^\circ, \tag{32}$$

$$\tfrac{1}{2}(E + O) = 90^\circ - \frac{S}{2}, \tag{33}$$

hence

$$\tan \tfrac{1}{2}(E - O) = \cot \tfrac{1}{2} S \, \frac{r' \cos \theta - r}{r' \cos \theta + r}, \tag{34}$$

nd placing

$$\tan p = \frac{r' \cos \theta}{r}, \tag{35}$$

we have

$$\tan \tfrac{1}{2}(E - O) = \cot \tfrac{1}{2} S \, \frac{\tan p - 1}{\tan p + 1} = \cot \tfrac{1}{2} S \tan (p - 45^\circ) \tag{36}$$

therefore E and O are known: and we have

$$\lambda' = E - (360^\circ - L) = E + L - 360^\circ. \tag{37}$$

To find the *geocentric latitude,* we have

$$P\,O = E\,O \tan\theta' = S\,O \tan\theta \tag{38}$$

$$\frac{\tan\theta'}{\tan\theta} = \frac{S\,O}{E\,O} = \frac{\sin E}{\sin S}; \tag{39}$$

whence

$$\tan\theta' = \tan\theta\,\frac{\sin E}{\sin S}. \tag{40}$$

To find r'', we have

$$E\,O = r''\cos\theta',$$

$$S\,O = r'\cos\theta.$$

In the triangle $E\,S\,O$, we have

$$r''\cos\theta' : r'\cos\theta :: \sin S : \sin E,$$

whence

$$r'' = r'\,\frac{\cos\theta}{\cos\theta'}\,\frac{\sin S}{\sin E}. \tag{41}$$

With these data we can readily find the right ascension, declination, horizontal parallax, and apparent diameter as in the case of the sun and moon.

INTERPOLATION.

Interpolation.—Whenever the differences of the quantities recorded in the Ephemeris tables are directly proportional to the differences of the corresponding times, simple interpolation will enable us to find the numerical value of the quantity in question. When this is not the case, the value is determined by the "method of interpolation by differences." Bessel's form of this formula, usually employed, is

$$F_n = F + n\,d_1 + \frac{n\,(n-1)}{2}\,d_2 + \frac{n\,(n-1)\,(n-\frac{1}{2})}{2.3}\,d_3 + \frac{(n+1)\,n\,(n-1)\,(n-2)}{2.3.4}\,d_4 + \text{etc.} \qquad (42)$$

In this formula, F_n is the value of the function to be determined; F, the ephemeris value from which we set out; d_1, d_2, d_3, etc., are the terms of the successive orders of differences, determined as explained below; n is the fractional value of the time interval, in terms of the constant interval taken as unity corresponding to which the values of the function F are computed and recorded in the tables. To use this formula, draw a horizontal line below the value of F from which we set out, and one above the next consecutive value taken from the ephemeris. These lines are to enclose the values of the odd differences d_1, d_3, d_5, etc. The values of the even differences d_2, d_4, d_6, etc., being each the mean of two numbers, one above and one below in their respective columns, are then inserted in their proper places. The following example is given to illustrate the application of Bessel's formula.

Find the distance of the moon's center from Regulus at 9 P.M. West Point mean time March 24th, 1891.

The longitude of West Point is 4.93 hrs. west of Greenwich; hence the Greenwich time corresponding to 9 P.M. West Point mean time is 13.93 hrs. Referring to pages 54 and 55 American Ephemeris we take out the following data, namely:

A March 24.	F	d_1	d_2	d_3	d_4
6^h	27° 01′ 24″				
		1° 28′ 9″			
9^h	28° 29′ 33″		+11″		
		1° 28′ 20″		+1″	
12^h	29° 57′ 53″		+12″		−2″
$13^h.93$	30° 54′ 47″.31	1° 28′ 32″	(+11″.5)	−1″	−1″
15^h	31° 26′ 25″		+11″		0
		1° 28′ 43″		−1″	
18^h	32° 55′ 8″		+10″		
		1° 28′ 53″			
21^h	34° 24′ 1″				

Whence, substituting in the formula, we have

$$F = 29^\circ\ 57'\ 53'' + 0.643\ (1^\circ\ 28'\ 32'') + 0.643\ \left(-\frac{0.357}{2}\right)\ (11''.5)$$
$$+ (0.643)\ \tfrac{1}{2}\ (-0.357)\ \tfrac{1}{3}\ (0.143)\ (-1'').$$
$$= 29^\circ\ 57'\ 53'' + 56'\ 55''.616 - 1''.32 + 0''.01,$$
$$\doteq 30^\circ\ 54'\ 47''.31 \text{ the required distance.}$$

Instruments.—The principal instruments used in field astronomical work are the Transit, Sextant, Zenith Telescope, and Altazimuth or Astronomical Theodolite. A short description of each instrument will be given in connection with the first problem involving its use. But since much relating to the transit is applicable also to the zenith telescope and altazimuth, that instrument will be explained first.

THE TRANSIT.

The Transit is an instrument usually mounted in the meridian, and employed in connection with a chronometer for observing the meridian passage of a celestial body. Since the R. A. of a body is equal to the sidereal time at the instant of its meridian passage, or is equal to the *chronometer* time plus its error ($\alpha = T + E$), it is seen that by noting T, E will be given when α is known, and conversely α will be given when E is known. *The very accurate determination of E is the chief use of the transit in field work.*

The instrument consists essentially of a telescope mounted upon and at right angles to an axis of such shape as to prevent easy flexure. The ends of this axis called the pivots, are usually of hard bell metal or polished steel, and should be portions of the *same* right cylinder with a circular base. They rest upon Y's, which in turn are supported by the metal frame or stand. At one end of the axis there is a screw by which its Y may be slightly raised or lowered in order that the axis may be made horizontal. At the other end of the axis is another screw by which its Y may be moved backward or forward, in order that the telescope may be placed in the meridian. The telescope is provided with an achromatic object glass, at the principal focus of which is a wire frame carrying an odd number of parallel vertical wires as symmetrically disposed as possible with reference to the middle; also two horizontal wires near to each other, between which the image of the point

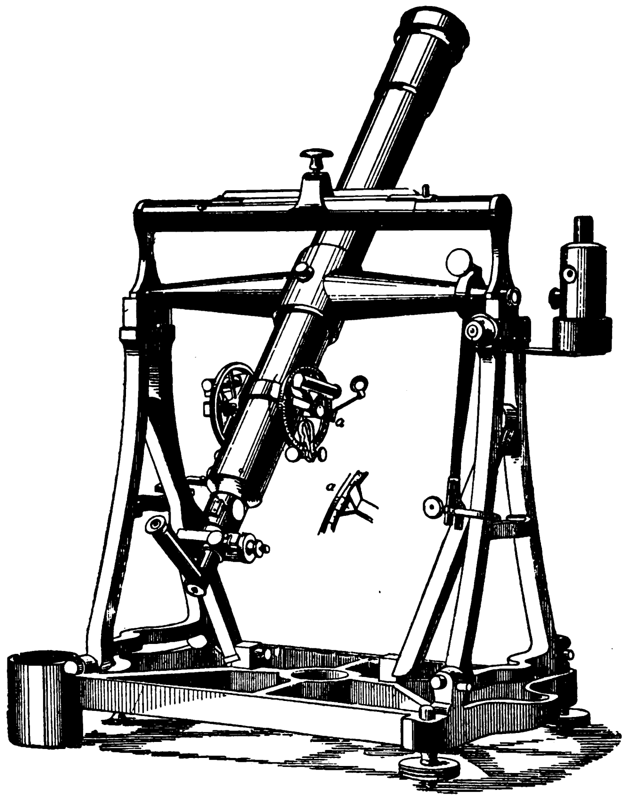


FIG. 5.—THE TRANSIT.

observed should always be placed. This system of wires is viewed by a positive or Ramsden's eye-piece, which can be moved bodily in a horizontal direction to a position directly opposite any wire, thus practically enlarging the field of *direct* view. The wires are rendered visible in the daytime by the diffuse light of day, but at night artificial illumination is required. This is effected by passing light from a small lamp along the length of the perforated axis,

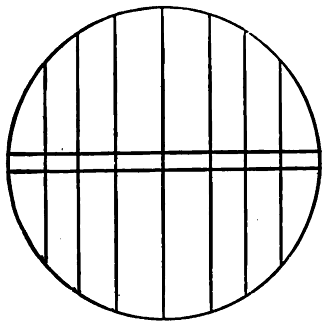


FIG. 6.

whence it is thrown toward the eye by a small reflector placed at the junction of the axis and the telescope tube, thus producing the effect of "a bright field and dark wires."

The right line passing through the optical center of the object glass intersecting and at right angles to the axis of rotation of the instrument, is called the "line of collimation."

The wire frame should be so placed that this line will pass midway between the two horizontal wires, and intersect the middle vertical wire; which latter should also be at right angles to the axis of rotation of the instrument.

These conditions being fulfilled, it is manifest that if the axis be placed in a true east and west line, and be made exactly level, the line joining any point of the middle wire and the optical center of the objective will, as the instrument is turned on its pivots, trace on the celestial sphere the true meridian; and the sidereal time when any body appears on the *middle wire*, will, if correctly estimated, be the value of T required in the equation,

$$\alpha = T + E.$$

The improbability of estimating T with precision leads to the use of more than one wire, although the advantage of increasing the number beyond five is, according to Bessel, very slight. If the wires are grouped in perfect symmetry with reference to the middle, evidently the mean of the times when a star, as it passes across the field of view, is bisected by each wire will give a more trustworthy time of meridian passage than if a single wire be used. Even if they are not grouped in perfect symmetry, the same will be true, after applying a correction deduced from the "Equatorial Intervals" to be explained hereafter. Every transit instrument is provided with a level, a diagonal eye-piece, one or more setting circles, and usually with a R. A. micrometer. In the case of field transits a striding level is generally used. Its feet are provided with Y's which are placed on the pivots of the instrument. Before using, it should be put in adjustment according to the principles explained in connection with surveying instruments.

The diagonal eye-piece facilitates the observation of stars near the zenith by reflecting the rays at right angles after they pass the wires.

The setting circles are firmly attached to the telescope tube and are read by an index arm carrying a vernier, to which is also attached a small level. They may be arranged to point out the position of a star either by its declination or its meridian altitude. In the latter case, the altitude is computed by the formula

$$\text{Mer. Alt.} = \text{Dec.} + \text{Co-Latitude},$$

for stars south of the zenith, and by

$$\text{Mer. Alt.} = \text{Latitude} \pm \text{Polar Distance},$$

for stars north of the zenith, the upper sign being used for stars above the pole. In any case having determined the "setting," place the index arm to mark it, and turn the instrument on its pivots until the bubble plays. The star will appear to pass through the field from west to east, except in case of sub-polars, which move from east to west. An equatorial star passes through the field with considerable velocity, only 40 to 60 seconds being required for its passage, the apparent path being a right line. For other stars the time required is greater, and the path becomes more curved, until as we approach the pole several minutes are required, and the curvature becomes very apparent.

These facts are of importance in determining when and where to look for the star.

The curvature of path must be considered in determining the "Equatorial Intervals." The eye-piece should be moved horizontally, keeping pace with the star, presenting the latter always in the *middle* of the field of view.

The uses of the R. A. micrometer will be explained hereafter.

ADJUSTMENTS OF THE TRANSIT.

From the above it is manifest that, assuming the objective to be properly adjusted, there are five adjustments to be made before the instrument is ready for use.

1. To Place the Wires in the Principal Focus of the Objective.—Push in or draw out the eye-piece till the wires are seen with perfect distinctness, using an eye-piece of high power. Direct the telescope to a small well-defined terrestrial object, not nearer than two or three miles. Now if the wires are not in the focus of the objective, the object will appear to move with reference to the wire as the eye is moved from side to side.

The wire frame must then be carried slightly toward or from the objective until this parallax is corrected.

After the instrument has been placed in the meridian, and the horizontal wire made truly horizontal, as explained in the following adjustments, let an equatorial star run along the wire, and if it does not remain accurately bisected while the eye is moved up and down, the wires are not exactly in the *principal* focus. Other stars must then be used until the parallax is removed. The wires are then at the common focus of the objective and eye-piece.

2. To Level the Axis.—The striding level is usually graduated from the center toward each end.

The pivots are assumed to be equal.

If when its Y's are applied to the transit pivots, the axis of the tube is parallel to the axis of the pivots * (*i.e.*, if the level be in

* The axis of the tube is of course a circular arc of long radius. Strictly speaking, it is the chord of this arc which, when the level is perfectly adjusted, will be parallel to the axis of the pivots.

perfect adjustment), and if w and e denote the readings of the west and east ends of the bubble respectively, then

$$\frac{w-e}{2}$$

will denote the reading of the middle of the bubble, and will therefore measure the inclination of the axis of the pivots *in level divisions.* But the accurate adjustment of the level is never to be assumed. If the axis of the level be inclined to the axis of the pivots by such an amount as to increase the west reading and therefore diminish the east reading by x divisions, then w and e still denoting the actual readings, we shall have for the true inclination of the axis of the pivots,

$$\frac{w-e}{2}-\frac{2x}{2}.$$

Upon reversing the level, the west and east readings will be as much too small and too large respectively as they were too large and too small before reversal; therefore w' and e' denoting the actual readings, we shall have for the true inclination this second value,

$$\frac{w'-e'}{2}+\frac{2x}{2}.$$

The mean of these two values,

$$\tfrac{1}{2}\,\frac{(w-e)+(w'-e')}{2}, \text{ or } \frac{(w+w')-(e+e')}{4}$$

is expressed only in actual level readings and is free from x, the unknown effect of maladjustment of level.

Hence to level the axis—Take direct and reverse readings with the level, altering the inclination of the axis till the sum of the west equals the sum of the east readings.

If the level be graduated from end to end, a similar discussion will show the level error to be

$$\frac{(w+e)-(w'+e')}{4}.$$

3. To Place the Wires at Right Angles to the Rotation Axis.—Bisect a very distant small terrestrial object by the middle wire, and the axis being level, note whether the bisection remains perfect from end to end of the wire as the telescope is alternately elevated and depressed. If not, rotate the box carrying the wire frame, until the above condition is fulfilled.

The side wires are parallel and the horizontal wires perpendicular, to the middle wire.

After the instrument has been finally placed in the meridian, this adjustment must be verified by noting whether an equatorial star will remain accurately bisected by the horizontal wire during its passage through the field.

4. To Place the Middle Wire in the Line of Collimation.—Bisect the same distant object as before. Lift the telescope carefully from the Y's and replace it with the axis reversed. If the object is still perfectly bisected the collimation adjustment is complete. If not, move the wire frame laterally by the proper screws over an estimated half of the distance required to reproduce bisection. If the half distance has been correctly estimated, the middle wire is now in the line of collimation. Repeat the operation from the beginning until the condition is fulfilled.

If a proper terrestrial point can not be obtained, the cross-wires in an ordinary surveyor's transit or theodolite adjusted to stellar focus, will answer quite as well. If two theodolites are placed, one north and the other south of our transit, pointing toward and accurately adjusted on each other, the reversal of the axis above referred to may be avoided.

In all these cases, the R. A. micrometer is of great convenience for measuring the distance whose half is to be taken.

The parts of the instrument are now in adjustment among themselves. It remains to adjust the instrument as a whole with reference to the celestial sphere; *i.e.*, to so place the instrument that when turned on its pivots, the line of collimation shall trace the true meridian.

5. To Place the Line of Collimation in the Meridian.—This is most easily effected by the aid of a sidereal chronometer whose error is known. The instrument is first placed as nearly in the proper position as can be estimated, and its supporting frame turned in azimuth until the telescope can be pointed at a slow moving star at about the time of its meridian passage.

Now level the axis carefully, set the telescope to the meridian altitude of a circum-polar star whose place is given in the Ephemeris, and bring the middle vertical wire upon this star a short time before its meridian passage. Hold the wire upon the moving star by turning the screw which moves one of the Y's in azimuth, until the chronometer corrected for its error indicates a time equal to the star's R. A. for the date. The transit is now very approximately in the meridian, although the adjustment should be tested by other stars.

Since the observations to be made with the transit will be for the purpose of an accurate determination of the chronometer error, this latter will usually be known only approximately. It may however be found with sufficient accuracy for making the adjustment by noting that since all vertical circles intersect at the zenith, the time of a zenith star's passage over the middle wire will be its time of passage over the meridian even though the transit be not in the meridian. The difference between the chronometer time of this event and the star's R. A. will therefore be the clock error.

In the absence of a zenith star, two circum-zenith stars, at opposite and nearly equal zenith distances, will give values of the clock error differing about equally and in opposite directions from its true value.

Alternating observations on circum-polar and circum-zenith stars will now give the required adjustment with two or three trials.

As a final test, the values of the chronometer error determined from stars which cross the meridian at widely separated points should be practically identical.

INSTRUMENTAL CONSTANTS.

These must be determined before the instrument can be used, and are five in number. The transit is supposed to be in good adjustment.

1. The Value in Time of One Division of the R. A. Micrometer Head.—The micrometer head, which is usually divided into 100 equal parts, carries a movable wire which is always parallel to the fixed vertical wires of the transit, and as nearly as possible in their plane. As it moves across the field of view it apparently coincides with each of them in succession.

If s denote the angular distance, measured from the optical center of the objective, between two positions of the micrometer wire, one of which coincides with the middle wire or the meridian of the instrument, then $\frac{s}{15} = i$ will be the interval of *time* required for a star exactly on the celestial equator to pass from one position to the other; since it is *only* such stars whose diurnal path is a great circle, and since also intervals of time are measured by arcs of a great circle—the equator.

With a star exactly on the equator, the process of finding the value of one division of the R. A. micrometer head would therefore consist in noting the time required for the star to pass from one position of the wire to the other; the quotient of which by the number of turns or divisions through which the head has been moved would give the value of one turn or division.

In the absence of such a star we must select one whose declination, δ, is not zero. The interval of time required for such a star to pass from one position to the other will be given by the equation

$$\sin I = \sin i \sec \delta. \qquad (43)$$

To prove this, let AB in Figure 7, which represents the sphere projected on the plane of the horizon, be the meridian, P the pole,

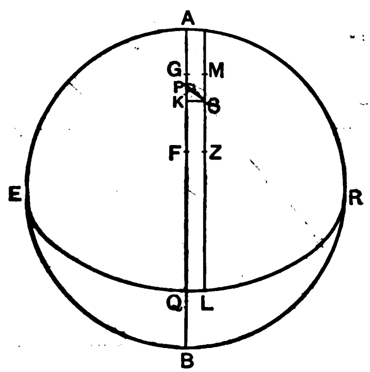


FIG. 7.

EQR the equator, S the place of the star, MZ the first position of the wire, and PF, coinciding with the meridian, the second.

Through S pass an arc of a great circle, KS, perpendicular to AB. This arc will be equal to QL, and will therefore, from what precedes, be denoted by s.

Hence, in the right-angled triangle SPK, we have

$$\sin P = \frac{\sin s}{\cos \delta}. \tag{44}$$

But P is the hour angle of the star at S, and s is the hour angle of an equatorial star at an equal angular distance from the meridian, *i.e.*, at L.

Hence denoting the time equivalent of the former by I, and of the latter by i as before, we have

$$\sin I = \sin i \sec \delta,$$

and therefore

$$\sin i = \sin I \cos \delta. \tag{a}$$

From this equation we may compute i, δ being taken from the Ephemeris, and I, which is directly observed, being the sidereal time required for the star to pass from S to the meridian.

After which, if R denote the value of a revolution or division of the micrometer head, and N the number of revolutions or divisions corresponding to I, we have for the value in time

$$R = \frac{i}{N}. \tag{b}$$

If the star be not within 10° of the pole we may write

$$i = I \cos \delta, \tag{c}$$

and

$$R = \frac{i}{N}, \tag{d}$$

thus avoiding the "Correction for Curvature" involved in the trigonometric functions.

By examining the equations

$$\sin i = \sin I \cos \delta, \text{ and } i = I \cos \delta, \tag{45}$$

it is seen that for the accurate determination of i, it is better to use stars near the pole, since errors in the observed values of I will then be multiplied by the cosine of an angle near 90°.

Therefore, to determine this constant, proceed as follows:

Shortly before the time of culmination of some slow-moving (circum-polar) star set the instrument so that the star will pass through the field. Set the micrometer head at some exact division, with the wire on the side of the field where the star is about to enter. Note the reading of the micrometer head, and record the time of passage of the star over the wire, using a sidereal chronometer whose rate is well determined. Set the wire again a short distance ahead of the star, note the reading, and record the time of passage. In this manner "step" the screw throughout its entire length. Then, remembering that I is the sidereal interval (corrected for rate if appreciable) between any given passage and that obtained when the wire was nearest to the meridian or the center of the field of view, apply to each pair of observations equations (a) and (b), or (c) and (d), according to the value of δ.

Where δ is *considerably* less than 90° and equations (c) and (d) are used, the correction for curvature of path becomes very small, and the same necessity does not exist for comparing each observation with the one made at the center of the field.

No correction for difference of refractions between any two positions of the star is required, since at its meridian passage the star is moving almost wholly in azimuth.

In any case the adopted value of the constant should rest on many such determinations.

Very convenient stars to use are α, δ, β, Ursæ Minoris. Their declinations are accurately given in the Ephemeris, the first two for every day, and the last one for every ten days.

The first two require equations (a) and (b).

The last one not necessarily so.

2. The Equatorial Intervals.—By the "Equatorial Interval" of a given wire is meant the interval of sidereal time required for a star on the celestial equator to pass from this wire to the middle wire, or *vice versa.*

The method of determinating this constant for each wire is manifestly identical in principle with the process just described, omitting the application of equations (b) or (d), and remembering that I is the *observed* interval with a star whose declination is δ, and i is the required Equatorial Interval.

Another method, which may either be used independently or as a verification, is to measure the intervals between the wires (in time) by the R. A. micrometer. The adopted constants should rest upon many determinations.

3. The Reduction to the Middle Wire.—The mean of the times of transit of a celestial body over the several wires of a transit instrument is called the time of transit over the mean of the wires or *the mean wire.* The mean does not usually coincide with the middle wire, due to the improbability of grouping the wires in perfect symmetry with reference to the middle.

Since it is the *middle* wire which has been placed in the meridian, it becomes necessary to determine the distance, in time, of the mean from the middle wire. Then, the mean of the times of transit being corrected by this constant, we will have a very accurate determination of the time of transit over the meridian. Suppose the instrument to have seven wires, and to be in good adjustment. A star at its upper culmination will apparently move over these wires from west to east; therefore (with the instrument in a given position, say with "illumination east") let the wires be successively numbered from the west towards the east.

Let a star whose declination is δ pass through the field, and let $t_1, t_2, t_3, t_4, t_5, t_6, t_7$, be the accurate instants of passing the corresponding wires; let $i_1, i_2, i_3, 0, i_5, i_6, i_7$, be the equatorial intervals from the middle wire. Then the time of passing the *mean* wire is

$$\frac{t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7}{7}. \qquad (46)$$

The time of passing the *middle* wire is either

$$t_1 + i_1 \sec\delta,\ t_2 + i_2 \sec\delta,\ t_3 + i_3 \sec\delta,\ t_4,\ t_5 - i_5 \sec\delta,\ t_6 - i_6 \sec\delta,$$
$$\text{or } t_7 - i_7 \sec\delta$$

(note the minus sign in the last three). Hence the most probable time of passing the middle wire is

$$\frac{\Sigma\,(t + i\sec\delta)}{7} = \frac{\Sigma\, t}{7} + \frac{\Sigma\, i}{7}\sec\delta. \qquad (47)$$

The difference between this and the time of passing the mean wire is evidently the second term, or

$$\frac{\Sigma i}{7} \sec \delta = \frac{(i_1 + i_2 + i_3) - (i_5 + i_6 + i_7)}{7} \sec \delta. \qquad (48)$$

The equatorial value of this reduction (the desired constant) will then be

$$\frac{\Sigma i}{7} = \Delta i,$$

and for any given star the actual reduction will be this value multiplied by sec δ. The adopted value of Δi should rest upon many determinations. Its sign is evidently changed by reversing the axis of the instrument.

Hence, to find the time of a star's passage over the middle wire, we have the rule: *To the mean of the times add Δi sec δ, noting the signs of both factors.*

The Equatorial Intervals are also used for finding the time of passage over the middle wire when actual observation on some of the wires has been prevented by clouds or other cause. Thus suppose observations have only been made on the second, third, and seventh wires. The most probable time of passing the middle wire is

$$\frac{(t_2 + i_2 \sec \delta) + (t_3 + i_3 \sec \delta) + (t_7 - i_7 \sec \delta)}{3} = \frac{\Sigma t}{3} + \frac{\Sigma i}{3} \sec \delta,$$

t and i referring only to the wires used.

4. Value of One Division of the Level.—In practical astronomy the level is used not merely for testing and regulating the horizontality of a given line, but also for *measuring* either in arc or time those small residual inclinations to the horizontal which no process of mechanical adjustment can either eliminate or maintain at a constant value.

Hence we must determine the value of one division of the striding level of the transit; *i.e.*, the increment or decrement of inclination which will throw the bubble one division of the graduation.

The best method of determining this quantity in case of a detached level is by use of the "Level-trier," which consists simply of a metal bar resting at one end on two firm supports, and at the other on a vertical screw. Then if d be the distance from the screw to the middle of the line joining the two fixed supports, and h the distance between two threads of the screw (obtained by counting the number of threads to the inch), the inclination of the bar to the horizon would be changed by $\frac{h}{d \sin 1''}$ due to one revolution of the screw. The level is then placed on the bar and the number n of divisions passed over by the bubble due to one turn (or division) of the screw is noted. The value of one division of the level in angle is then $\frac{h}{n\, d \sin 1''}$. The mean of several observations, using both ends of the bubble, should be adopted. The value in time is $\frac{h}{15\, n\, d \sin 1''}$. If no level-trier is available, the level should be placed on the body of the telescope connected with a vertical circle *reading to seconds:* as for example the meridian circle of a fixed observatory. Move the instrument slowly by the tangent screw and note the number of level divisions corresponding to a change of 1″ in the reading of the circle, taking the means as before. By either method the level may be tested throughout its entire length.

We have seen that the inclination of a line *in level divisions* is $\frac{(w + w') - (e + e')}{4}$; hence if D denote the constant just found, the inclination of the line *in arc* will be

$$B = \frac{(w + w') - (e + e')}{4} \cdot \frac{h}{n\, d \sin 1''} = \frac{(w + w') - (e + e')}{4} D, \quad (49)$$

the west end being higher if $(w + w') > (e + e')$, or when this expression is positive.

5. Inequality of the Pivots.—The construction of the pivots being one of the most delicate operations in the manufacture of the whole instrument, their equality must never be assumed.

In transit observations it is manifestly the axis of rotation (the axis of the pivots) which should be made horizontal, or whose inclination should be measured. If the pivots are unequal they may be regarded as portions of the same right cone; in which case it is evident that the striding level applied to the upper element might indicate horizontality when the axis was really inclined, and *vice* *versa.* We must therefore correct our level indications by the effect of this "Inequality of Pivots."

To determine this, let $w\,x\,y\,z$ in Figure 8 represent the cone of

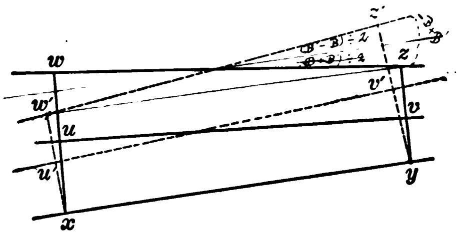


FIG. 8.

the pivots, $u\,v$ being the axis. Let the inclination of the upper element $w\,z$ be measured with the level, giving

$$B = \frac{(w + w') - (e + e')}{4} D.$$

Lift the axis from the Y's and turn it end for end. In this position $w'\,x\,y\,z'$ will represent the cone of the pivots.

Measure as before the inclination of $w'\,z'$, and denote it by B'. Then by inspection of the figure it is seen that $B' - B$ is the angle between the two positions of the upper element, $\frac{B' - B}{2}$ is the angle between the upper and lower elements of the cone, and $\frac{B' - B}{4} = p$ is consequently the angle between the upper element and the axis $u\,v$.*

* B and B' are manifestly the inclinations, in the two positions, which the upper element would have if the pivots were equal, minus twice the effect of the inequality:—this effect being the angle subtended by the difference of the radii, $r - r'$. Of course if the pivots are unequal, the inclination obtained by applying the level Y's to the pivots is not strictly that of the upper element; but if the angles of the transit and level Y's are equal (as is usually the case), it will evidently be, as before, the inclination which the upper element would have if the pivots were equal, minus twice the effect of the inequality:—the effect in this case being (Fig. 8*a*, which represents a cross-section of the pivots and level Y) the angle subtended by $\frac{r - r'}{\sin \frac{1}{2} a}$. Hence the algebraic difference, $B' - B$, will be four times the effect of the inequality, as before.

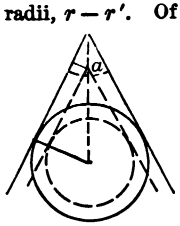


FIG. 8*a*.

This quantity, $\frac{B' - B}{4} = p$, is therefore the desired constant, and as the figure indicates, it is a correction to be added algebraically to the level determination of the unreversed instrument, or to be subtracted from that of the reversed instrument.

Its value should rest upon many determinations.

The inclination of the axis of a transit will hereafter be denoted by b, which is therefore either $B + p$, or $B' - p$, according as the instrument is direct or reversed.

✠ The cross-sections of the pivots should be perfect circles. Any departure from this form may be discovered and corrected as follows:

With instrument direct, determine the value of B with the telescope placed successively at every 10° of altitude. Call the mean B_0.

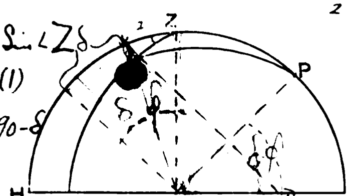
Then $B_0 - B$ is the correction for *irregularity* of pivots *for the reading corresponding to B* with instrument direct. Do the same with instrument reversed. Then $B_0' - B$ will be the correction for irregularity with instrument reversed. $\frac{B_0' - B_0}{4}$ will be the correction for *inequality*. Both corrections must be applied to obtain the true value of b.

EQUATION OF THE TRANSIT INSTRUMENT IN THE MERIDIAN.

The transit, having been adjusted and the instrumental constants determined, is ready for use. Hitherto it has been assumed that an adjustment was perfect:—that the middle wire had been placed *exactly* in the line of collimation, that the axis of rotation had been made exactly level, and that the line of collimation would trace with mathematical accuracy the true meridian. Manifestly, however, this theoretical accuracy cannot be attained by mechanical means. It will therefore be proper, having performed each adjustment as accurately as possible, not to regard the outstanding small errors as zero, but to introduce them into a given problem as additional unknown quantities having an ascertainable effect on the result, and then to make independent determinations of their value, or leave these values to be revealed by the observations themselves.

Any departure from perfect adjustment is *positive* when its effect is to make stars south of the zenith cross the middle wire *earlier* than they otherwise would.

1. To Ascertain the Effect of an Error in Azimuth on the Time of Passage of the Middle Wire.—Let a denote the horizontal angular deviation of the axis of rotation from a true east and west line, positive when the west pivot is south of the east pivot. (This should never exceed 15″, and will usually be even less.) The line of collimation will then, as the instrument is moved in altitude, describe a great circle of the celestial sphere intersecting the meridian in the zenith, and making with it the angle a (HZA in Figure 9).



Then from the ZPS triangle we have (S being the position of a star when on the middle wire),

$$\sin P : \sin a :: \sin z : \cos \delta,$$

or

$$\sin P = \frac{\sin a \sin z}{\cos \delta}.$$

If the star were exactly on the meridian, z would be equal to $\phi - \delta$. Being less than 15″ therefrom, the change required in z to give $\phi - \delta$ is entirely negligible. Again P and a are exceedingly small angles. Hence we may write with great precision, expressing a and P in time,

$$P = a \frac{\sin (\phi - \delta)}{\cos \delta}. \qquad (50)$$

That is, if the instrument have an azimuth error in time, of a seconds, a star when passing the middle wire is distant from the true meridian $a \dfrac{\sin(\phi - \delta)}{\cos \delta}$ seconds of time, and the *recorded* time of transit must be corrected accordingly.

2. To Ascertain the Effect of an Inclination of the Axis on the Time of Passage of the Middle Wire.—Let b denote the angular deviation of the axis of rotation from the horizontal, positive when the west pivot is higher than the east. The line of collimation will then, as the instrument is moved in altitude, describe a great circle of the celestial sphere intersecting the meridian at the north and south points of the horizon, and making with it the angle b (ZHS, in Figure 10).

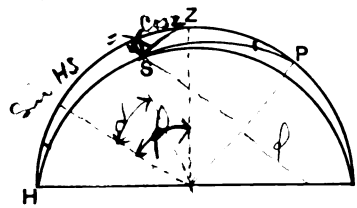


FIG. 10.

Then from the triangle PHS (S being the position of a star when on the middle wire)

$$\sin P : \sin b :: \cos z : \cos \delta.$$

Or, as before, expressing b in time,

$$P = b \frac{\cos(\phi - \delta)}{\cos \delta}. \qquad (51)$$

This is interpreted as in the preceding case.

3. To Ascertain the Effect of an Error in Collimation on the Time of Passage of the Middle Wire.—Let c denote the angular distance of the middle wire from the line of collimation, positive when the wire is *west* of its proper position. The line of *sight* will then, as the instrument is moved in altitude, describe a small circle of the celestial sphere, east of the meridian and parallel to it. Through S, the place of the star, Fig. 11, pass the arc of a great circle, SM, perpendicular to the meridian. This arc will be the measure of c. Then in the right-angled triangle $P\,S\,M$ we have

$$\sin P = \frac{\sin c}{\cos \delta}.$$

$\frac{\sin P}{\sin M} = \frac{\sin c}{\sin SP}$ $\angle M = 90°$, $SP = 90 - \delta$, $\sin M = 1$, $\sin SP = \cos \delta$

$\therefore \sin P = \frac{\sin c}{\cos \delta}$

Or, as before, expressing c in time,

$$P = \frac{c}{\cos \delta} = c \sec \delta. \tag{52}$$

Hence when all these errors, a, b, and c, exist together, called respectively the azimuth, level, and collimation error, we have for the Equation of the Transit Instrument in the Meridian,

$$\alpha = T + E + a\frac{\sin(\phi - \delta)}{\cos \delta} + b\frac{\cos(\phi - \delta)}{\cos \delta} + c \sec \delta. \tag{53}$$

In this equation α is the apparent R. A. of the star for the date, T is the clock time of transit over the middle wire, obtained from the time of transit over the mean wire by applying the "Reduction

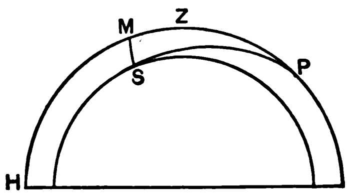


FIG. 11.

to Middle Wire," E is the chronometer error, positive when slow, negative when fast, ϕ the latitude, δ the star's apparent declination for the date, and a, b, and c are expressed in time.

When great precision is desired, for example in longitude work, the equation must be modified by the introduction of a small correction for Diurnal Aberration, additive to α. The value of the correction is $0^s.021 \cos \phi \sec \delta$.

Hence the complete form of the above equation is

$$\alpha = T + E + a\frac{\sin(\phi - \delta)}{\cos \delta} + b\frac{\cos(\phi - \delta)}{\cos \delta} + (c - 0^s.021 \cos \phi) \sec \delta.$$

Each correction is deduced under the supposition that it alone exists, we thus find an expression for each and then in the final result they are all taken into account. & are the final equation express the true state of affairs when all errors exist simultaneously? See Chauvenet

Or, placing

$$c' = c - 0.021 \cos \phi,$$

$$\alpha = T + E + a \frac{\sin (\phi - \delta)}{\cos \delta} + b \frac{\cos (\phi - \delta)}{\cos \delta} + c' \sec \delta. \qquad (54)$$

After an observation has been made we shall have in this equation four unknown quantities, E, a, b, c', since ϕ is supposed to be known, and α and δ are found in the Ephemeris. We may either determine a, b, and c independently, as will next be explained (in which case an observation on a single star will then give E), or leave all four to be determined by observation of at least four stars.

The sign of c is changed by reversing the axis, since the middle wire is thus placed on the other side of the line of collimation.

✠ This value, $0^s.021 \cos \phi \sec \delta$, which we will denote by R, may be deduced in an elementary manner as follows: Due to the earth's rotation on its axis, all celestial bodies are apparently displaced toward the east point of the horizon. If the body be on the meridian, this displacement is wholly in R. A. Hence the R. A. of the object *as seen* will not be α, but $\alpha + R$.

The direction of a ray of light received from a body on the meridian is at right angles to the direction of the observer's diurnal motion. Under this condition, the absolute amount of apparent displacement in seconds of a great circle may be written (Young, pa. 142),

$$R = \frac{u}{V \tan 1''},$$

where u is the observer's velocity, and V that of light. If the observer be at the equator, we shall have

$$u = \frac{20926062 \times 2\pi}{5280 \times 24 \times 60 \times 60} \text{ miles per second,}$$

where 20926062 is the number of feet in the earth's equatorial radius (Clarke).

According to Newcomb and Michelson,

$$V = 186330 \text{ miles per second.}$$

Hence

$$R = \frac{20926062 \times 2\pi}{5280 \times 24 \times 3600 \times 186330 \times \tan 1''} = 0''.319.$$

This angular displacement in a great circle perpendicular to the meridian corresponds to $0^s.021$ if the star be on the equator, or to $0^s.021 \sec \delta$ if the star's declination be δ, since, as we have seen before, equal angular distances from the meridian correspond to hour angles varying with $\sec \delta$.

If the observer be not on the equator, but at latitude ϕ, his velocity will be diminished in the ratio of the radius of his circle of latitude to that of the equator: or regarding the earth as a sphere, in the ratio $\cos \phi : 1$.

Hence, for an observer in any latitude, with a star at any declination,

$$R = 0^s.021 \cos \phi \sec \delta.$$

DETERMINATION OF INSTRUMENTAL ERRORS.

1. To Determine the Level Error *b*.—This is found from the formula already deduced, viz.:

$$b = B + p = \frac{(w + w') - (e + e')}{4} D + p \qquad (55)$$

or

$$b' = B' - p = \frac{(w + w') - (e + e')}{4} D - p. \qquad (56)$$

according as the instrument is direct or reversed. D and p must be expressed in time, by dividing their values in arc by 15, thus giving b in time.

2. To Determine the Collimation Error *c*.—Turn the instrument to the horizon, select some well-defined distant point whose image is near the middle wire, measure the distance between them with the micrometer, making the distance positive when the middle wire is *west* of the image of the point. Reverse the axis, and measure the new distance, with same rule as to sign. Subtract the second from the first, and one half the difference gives the collimation error *in micrometer divisions* for instrument direct.

This multiplied by the value of one division in time, gives c in time.

The rule will be evident from an inspection of Fig. 12 (which is a horizontal projection), where w is the west, and e the east end of the axis, TE the horizontal line of collimation, P the image of the point in the field of view, a the direct and b the reversed position of the middle wire. Ea is equal to Eb, and c is positive.

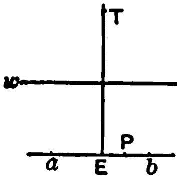


FIG. 12.

Instead of a terrestrial point we may use the intersection of the cross hairs in the focus of a surveyor's transit adjusted to stellar focus, the two instruments facing each other. The intersection referred to will then be optically at an infinite distance, and its image will be found at the principal focus of our transit.

It is sometimes necessary to determine c by independent stellar observations, in which case the following method is always employed: Point the telescope to a circumpolar star and note the times of its passage over as many wires as possible on one side of the middle wire. Reverse the axis. As the star moves out of the field of view, it will cross the same wires in reverse order, the times of passage being noted as before.

By means of the Equatorial Intervals reduce each time to the middle wire, and let T and T' denote the mean of those before and after reversal, respectively.

T and T' are therefore the times of passage of the same star over two different positions of the middle wire—one as much to the east as the other was to the west of the true line of collimation. From their difference therefore we have double the collimation error, thus:

For instrument direct,

$$\alpha = T + E + a\frac{\sin(\phi - \delta)}{\cos\delta} + b\frac{\cos(\phi - \delta)}{\cos\delta} + \frac{c}{\cos\delta} - \frac{0^s.021\cos\phi}{\cos\delta}.$$

For instrument reversed,

$$\alpha = T' + E + a\frac{\sin(\phi - \delta)}{\cos\delta} + b'\frac{\cos(\phi - \delta)}{\cos\delta} - \frac{c}{\cos\delta} - \frac{0^s.021\cos\phi}{\cos\delta},$$

$$0 = T' - T + (b'-b)\frac{\cos(\phi-\delta)}{\cos\delta} - \frac{2c}{\cos\delta}, \text{ or } c = \tfrac{1}{2}(T'-T)\cos\delta + \tfrac{1}{2}(b'-b)\cos(\phi-\delta) \quad (57)$$

4/-4'

allowance being made for a change in level error due to a possible inequality of pivots, and c changing its sign by reversal of the instrument.

By subtraction and solution we have

$$c = \tfrac{1}{2}(T' - T)\cos\delta + \tfrac{1}{2}(b' - b)\cos(\phi - \delta). \qquad (57)$$

If the pivots are equal and the instrument be undisturbed in level, the last term disappears and we have

$$c = \tfrac{1}{2}(T' - T)\cos\delta. \qquad (58)$$

A slow-moving star must be used in order to give time for careful reversal.

There are various other methods of finding both b and c, based principally upon observation of the wires and their images as seen by reflection from mercury.

3. To Determine the Azimuth Error, *a*.—Observe in the usual manner the time of transit, T, of a star of known declination. Then, b and c having been measured, let the corresponding corrections, $b\,\dfrac{\cos(\phi - \delta)}{\cos\delta}$ and $c'\sec\delta$, be added to T, giving t. This is called correcting the time for level and collimation. The equation of the instrument as applied to this star will now read

$$\alpha = t + E + a\,\frac{\sin(\phi - \delta)}{\cos\delta}. \qquad (m)$$

Similarly for another star,

$$\alpha' = t' + E + a\,\frac{\sin(\phi - \delta')}{\cos\delta'}. \qquad (n)$$

From which

$$a\left(\frac{\sin(\phi - \delta')}{\cos\delta'} - \frac{\sin(\phi - \delta)}{\cos\delta}\right) = (\alpha' - \alpha) - (t' - t)$$

Sin (φ - δ') = Sin φ cos δ' - cos φ Sin δ'

$$a(\sin\phi - \cos\phi\tan\delta' - \sin\phi + \cos\phi\tan\delta) = (\alpha' - \alpha) - (t' - t).$$

$$a = \frac{(\alpha' - \alpha) - (t' - t)}{\cos\phi(\tan\delta - \tan\delta')}. \qquad (59)$$

The value of the clock error does not enter. If however it be not constant, its rate, r, must be known, positive when losing, negative when gaining. Then if E_0 be the unknown error at some assumed instant T_0, the errors at the two instants of observation will be $E_0 + (T - T_0)\,r$, and $E_0 + (T' - T_0)\,r$. These should be substituted for E in equations (m) and (n), and the known terms, $(T - T_0)\,r$ and $(T' - T_0)\,r$, be united to T and T' in forming t and t' as are the corrections for level and collimation. The time is then said to be corrected for rate. By subtraction to obtain (59), E_0 will disappear. Hence while the rate must be known, the error need not be.

Examining the value of a, we see that the following conditions must be fulfilled in order to obtain an accurate determination.

First, α and α' must be known exactly; therefore only Ephemeris stars should be used.

Again, if the rate of the clock be not well determined, the interval between the observations must be as small as possible in order that the correction for rate may affect a but slightly. Therefore if both stars are at upper culmination, they should be nearly equal in R. A. Or, if one be above and the other below the pole, they should differ in R. A. by as nearly 12 hours as possible.

Again the larger numerically the factor ($\tan \delta - \tan \delta'$), the less the effect of errors in $t' - t$. Hence, if both stars are at upper culmination, one should be as near and the other as far from the pole as possible. Or, if one be at upper and one at lower culmination, they should both be as near the pole as possible; the declination of the lower star being then taken to be $90° +$ Polar Distance.
Declination from Z P S Δ.

REFRACTION TABLES.

A ray of light passing from a celestial body to a point on the earth's surface, may be supposed to pass through successive spherical strata of the atmosphere, the densities of which continually increase toward the center. Under these circumstances, as has been previously shown, the ray will be bent toward the normal, resulting in an apparent displacement of the body toward the zenith.

It has also been previously shown that the actual amount of such displacement increases with the zenith distance, and with the density of the air, which latter depends on its pressure and temperature. In order to facilitate the calculation of this displacement or refraction in any particular case, tables have been constructed con-containing certain functions of the zenith distance, temperature, and pressure, from which, with observed data as arguments, the refraction may be computed.

Such tables are called Refraction Tables. Those of Bessel are the best and most usually employed. In these tables the adopted value of the refraction function is given by

$$r = \alpha\,\beta^{A}\gamma^{\lambda}\tan z,$$

in which r is the refraction; A, λ, and α are quantities varying slowly with the zenith distance; β is a factor depending on the pressure, and γ upon the temperature of the air; z is the apparent zenith distance; β therefore depends upon the reading of the barometer, and γ upon the reading of the thermometer. But since the actual height indicated by a barometer depends not only upon the pressure of the air, but upon the temperature of the mercury, β is really composed of two factors B and T, the first of which depends upon the actual reading of the barometer, and T involves the correction due to the temperature of the mercury.

Nearly all the collections of astronomical tables contain "Tables of Refraction," from which may be found the various quantities in the equation

$$r = \alpha\,(B\,T)^{A}\gamma^{\lambda}\tan z.$$

The first portion of the table consists of three columns giving the values of A, λ, and log α, with the apparent zenith distance z as the argument.

The second part contains B, with the height of the barometer as the argument. The third part gives the value of T with the reading of the *attached* thermometer as the argument, and the fourth part gives γ with the reading of the *external* thermometer as the argument; z is the observed zenith distance. A substitution of these quantities gives the refraction, which must then be added to z to give the *true* zenith distance.

The attached thermometer gives the temperature of the mercury of the barometer. The external thermometer should be screened from the direct and reflected heat of the sun, but be so fully exposed as to give accurately the temperature of the external air.

A similar table is sometimes given for passing from true to apparent zenith distances. The mode of using is exactly the same, subtracting the resulting refraction from the true zenith distance to obtain z. It is of use in "setting" instruments for observation.

A "Table of Mean Refractions" is also given in nearly every collection, and contains the refractions for a temperature of 50° F., and 30 in. height of barometer, with apparent zenith distances or altitudes, as the argument, which may be used when a very precise result is not required.

The above relates only to refraction in altitude. But a change in a star's place due to refraction will in the general case cause a change in its *observed* R. A. and Dec. In order to ascertain these two coördinates as affected by refraction at a given sidereal time T, we first compute the body's hour angle from $P = T -$ R. A., and then its true zenith distance (z) and parallactic angle (ψ) from the astronomical triangle, knowing P, φ, and δ. Then if r denote the refraction in altitude, found as just explained, the refraction in declination will be

$$\Delta \delta = r \cos \psi,$$

and the refraction in R. A.,

$$\Delta \alpha = \frac{r \sin \psi}{\cos \delta}$$

TIME.

The perfect uniformity with which the earth rotates on its axis makes its motion a standard regulator for all time-pieces. No clock or chronometer can run with perfect uniformity, and therefore the time indicated by them must ever be in error. To find these errors at any instant is the object of the time problems in Practical Astronomy.

Time is measured by the hour angle of some point or celestial body. If the point be the true Vernal Equinox its hour angle is *true sidereal time.*

If the point be the mean Equinox, it is *mean sidereal time;* but since the greatest difference between true and mean sidereal time can never exceed 1.15 seconds in 19 years, astronomical clocks are run on true sidereal time. To pass from true to mean sidereal time, apply the correction known as the Equation of Equinoxes in Right Ascension.

If the point be the Mean Sun its hour angle is *mean solar time;* all solar time pieces are run on mean solar time.

If the point be the center of the True Sun, its hour angle is *true or apparent solar time;* to pass from true to mean solar time apply the correction known as the Equation of Time.

Before proceeding to the time problems, it is necessary to determine the relation existing between sidereal and mean solar *intervals*, and especially the relation existing between the sidereal and mean solar *time* at any instant.

Relation between Sidereal and Mean Solar Intervals.—The interval of time between two consecutive returns of the sun to the mean vernal equinox, called the mean tropical year, is according to Bessel 365.2422 mean solar days. Since, while the earth is rotating on its axis from west to east, the mean sun is moving uniformly in the same direction, the interval between two consecutive passages of the meridian over the mean sun will be $1 + \frac{1}{365.2422}$ times the interval between two passages over the mean vernal equinox: for in one mean solar day the mean sun must advance $\frac{1}{365.2422}$ of the whole circuit from equinox to equinox, and each mean solar day must correspond to $1 + \frac{1}{365.2422}$ passages of the mean vernal equinox. Hence 365.2422 mean solar days correspond to 366.2422 sidereal days.

Hence we have the relations,

$$\textit{One mean solar day} = \frac{366.2422}{365.2422} = 1.00273791 \textit{ sidereal days,}$$

$$= 24^h\ 3^m\ 56^s.555 \textit{ sidereal time.}$$

$$One\ sidereal\ day = \frac{365.2422}{366.2422} = 0.99726957\ mean\ solar\ days,$$
$$= 23^h\ 56^m\ 4^s.091\ mean\ solar\ time.$$

The same relation manifestly exists between the corresponding hours, minutes, and seconds. Now since the sidereal unit is shorter than the mean solar in the ratio of 1 : 1.00273791, it follows that the number of these units in a given interval of time is to the number of mean solar units as 1.00273791 to 1.

Hence the relations,

$$\text{Sidereal Interval} = \text{Mean Solar Interval} \times 1.00273791.$$
$$\text{Mean Solar Interval} = \text{Sidereal Interval} \times 0.99726957.$$

Or, denoting these intervals respectively by I' and I,

$$I' = I + 0.00273791\ I$$
$$I = I' - 0.00273043\ I',$$

Tables II and III, Appendix to the Ephemeris, give the values of the corrections 0.00273791 I and 0.00273043 I', for each second in the 24 hours.

Again, since 24 sidereal hours equals $23^h\ 56^m\ 4^s.091$ mean solar time, it follows that a mean solar clock loses $3^m\ 55^s.909$ on a sidereal clock in one sidereal day, or $9^s.8296$ in one sidereal hour.

Also, since 24 mean solar hours equals $24^h\ 3^m\ 56^s.555$ sidereal time, it follows that a sidereal clock gains $3^m\ 56^s.555$ on a mean solar clock in one mean solar day, or $9^s.8565$ in one mean solar hour.

These two facts may be thus expressed:

(1) *The hourly rate of a mean solar clock on sidereal time is* $+\ 9^s.8296$.

(2) *The hourly rate of a sidereal clock on mean solar time is* $-\ 9^s.8565$.

From (2) it is seen that the R. A. of the mean sun increases $9^s.8565$ per hour, or in other words, the sidereal time of mean noon occurs $9^s.8565$ later for each hour of west longitude.

These deductions are of importance in what follows.

Relation between Sidereal and Mean Solar Time. That is, having given either the sidereal or mean solar time at a certain instant, to find the other.

Suppose first the sidereal time to be given and let the circle in Figure 13 represent the celestial equator, M being the point where it is intersected by the meridian, V the vernal equinox and S the place of the mean sun.

Then $MV =$ the sidereal time at the instant, supposed to be known, $MS =$ the mean solar time required, and $VS =$ right ascension of mean sun.

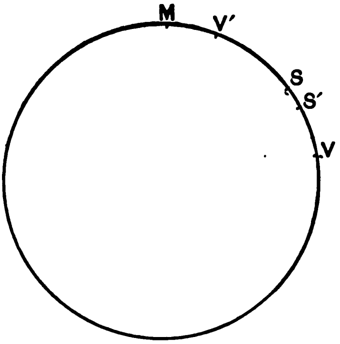


Fig. 13.

At the preceding mean noon the mean sun's R. A. was less than at the moment considered by an amount which may be represented by SS'.

At that time, therefore, the mean sun was at M, and the Vernal Equinox at a position V' such that $V'M = VS'$. Hence at the instant considered, the sidereal time elapsed since the preceding mean noon is $MV - MV'$. The time since mean noon having thus been found in sidereal units, the mean solar equivalent of this interval will necessarily be the mean solar time at the instant considered. Hence the rule:

From the given sidereal time subtract the R. A. *of the mean sun at the preceding mean noon. Convert the result into a mean solar interval by the Ephemeris Tables or the formula* $I = I' - 0.00273043I'$. *The result is the required mean solar time.*

To find the sidereal from the given mean time, this operation must obviously be performed in the inverse order, viz.:

Convert the given mean solar time into a sidereal interval by the Ephemeris Tables or by the formula $I' = I + 0.00273791I$. *To the result add the* R. A. *of the mean sun at the preceding mean noon. The result is the required sidereal time.*

On Page II, Monthly Calendar of the Ephemeris, will be found the R. A. of the mean sun at the preceding *Greenwich* mean moon. To find this element for the *local* mean noon, multiply the hourly change $9^s.8565$ (heretofore deduced) by the longitude in hours, and add the result to the Ephemeris value.

The above rules are not only of great use in astronomical calculations, but they enable us to determine the error of either a sidereal or mean time clock, knowing that of the other, by "the method of coincident beats." Suppose both clocks to beat seconds. Then from the relative rate heretofore deduced it is seen that their beats will be coincident once in about 6 minutes. Note the seconds given by each clock when this occurs, and then supply the hours and minutes. Apply the known error to the mean solar for example; and the result will be the correct m. s. time. Find the corresponding sidereal time by the rule just given. The difference between this and the time given by the sidereal clock will be its error.

EXAMPLE.

At West Point, N. Y., Nov. 27, 1891, Longitude $4^h.93$ west, the mean solar and sidereal clocks were compared at the instant of coincident beats, with the following result:

Mean Solar, $0^h - 46^m - 29^s.00$.
Sidereal, $17^h - 15^m - 55^s.00$.

The error of the mean solar was $0^s.17$ slow on Standard Time, which is itself $4^m\ 9^s.45$ slow on local time.

It is required to find the error of the sidereal clock.

Indicated m. s. time..........................	0^h	46^m	$29^s.00$
Error on standard time.......................			0.17
Reduction to local time......................		4	9.45
Corrected m. s. time.........................	0	50	38.62
Reduction to sidereal interval................			8.32
Sidereal interval since mean noon.............	0	50	46.94
R. A. of mean sun at Greenwich mean noon.....	16	24	27.25
Correction = $9^s.8565 \times 4.93$..................			48.59
True sidereal time............................	17	16	2.78
Clock indication..............................	17	15	55.00
Error of sidereal clock.			+ 7.78

Hence the sidereal clock was $7^s.78$ slow.

TO FIND THE TIME BY ASTRONOMICAL OBSERVATIONS.

This general problem usually presents itself as a question of determining the error of a time-piece at a given instant. The different methods of obtaining this error may, as far as considered here, be grouped under three heads.

I. Time by Meridian Transits.

II. Time by Single Altitudes.

III. Time by Equal Altitudes.

The first is the method of precision when properly carried out with the transit instrument. The second and third, being usually carried out with the sextant, can only be relied upon as giving an approximate result more or less exact.

I. TIME BY MERIDIAN TRANSITS.

1. To Find the Error of a Sidereal Time-piece by the Meridian Transit of a Star. (See Form 1.)—The general statement of the problem is briefly this: since the time-piece, if correct, ought to indicate the R. A. of the star at the instant of culmination, the difference in time is the error required. The transit instrument being supposed to be approximately in the meridian, *i.e.*, to have been carefully adjusted, for the practical solution it is necessary to find by observation and computation the quantities in the following equation (heretofore deduced) and solve it.

$$\alpha = T + E + aA + bB + c'C, \qquad (60)$$

in which A, B, and C have for brevity been substituted for $\frac{\sin(\phi - \delta)}{\cos \delta}$, $\frac{\cos(\phi - \delta)}{\cos \delta}$, and sec δ, respectively. Then having measured a, b, and c; computed A, B, and C; observed T; and taken α from the Ephemeris ($\alpha =$ the star's apparent R. A. for the date), the value of E follows from the solution of the equation.

In finding T, record to quarter seconds (or if possible to tenths of a second), the time of passage of each wire. Take the mean and apply the "Reduction to middle wire." T, corrected by $aA + bB + c'C$ is evidently the *chronometer* time of the star's transit over the meridian.

Form 1 indicates the proper method of recording the observations, it being arranged for five stars. Under the head of "Transit," record its number and the maker. The "Illumination" should be recorded as east or west, this showing whether the instrument is direct or reversed.

The adopted value of E should be the mean of the results from several stars. Stars within the polar circle, or those whose declination exceeds about 67°, are not used for time determinations, since the exact instant when a slow-moving star is bisected by a wire cannot be judged with the greatest precision, and since also slight errors in measuring a, b, and c will then be greatly magnified by A, B, and C, all of which become ∞ for $\delta = 90°$. But by including in the observing list two circum-polar stars upon one of which the instrument is reversed after half the wires are passed, both a and c may be found by Equations (57) and (59). b is found from level readings by Equation (55) or (56).

If only a single star is available, it should be one given in the Ephemeris, and which passes near the zenith ($\delta = \phi$), since at the zenith Aa disappears, and this is the only one of the three corrections which *requires* star observations for its determination.

For very accurate work, such as is required in connection with the telegraphic determination of longitude, it is usual to employ at least ten stars for each determination of time, half the stars being observed with the instrument reversed; and of each half, two should be circum-polar and three equatorial stars. In this case, b is ordinarily the only instrumental error actually measured (by level readings); each star then gives an equation of the form (60), and E together with a and c are found from a solution of the equations by Least Squares.

These matters will be explained more fully hereafter.

The clock rate is found from errors determined at different times.

To find the error at a given instant, as for example at the middle of the time consumed in a series of observations extending over several hours, this rate should be applied as explained when treating of the azimuth error.

✠ It may sometimes be desirable to find the error of a sidereal clock from a meridian transit of the sun, although in field work this would be exceptional. In such a case it may be assumed, with an error entirely negligible, that during the short time consumed in the observation the sun's motion is uniform, that the time required for the sun to pass from the mean to the middle wire, and from the middle wire to the meridian is the same as that for a star of the same declination.

For example, the reduction to middle wire not exceeding $0^s.5$, the error committed by the second assumption could not exceed $\frac{0^s.00274}{2} \times \sec(23° 28') = 0^s.0015$. Hence that reduction may be computed as usual.

Therefore, note the time of transit of each limb of the sun over each wire, and take the mean. Reduce to the middle wire as usual, and apply the correction $aA + bB + c'C$. The result is the clock time of culmination of the sun's *center*. The *true* sidereal time of this event, or the R. A. of the sun at apparent noon, is found on page 1, Monthly Calendar, by interpolation. The difference gives E.

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2. To Find the Error of a Mean Solar Time-piece by a Meridian Transit of the Sun. (See Form 2.)—Apparent noon at any place is the instant of culmination of the sun's center at that place. This epoch may be expressed in three different times, viz.:

In apparent time, or 12 o'clock apparent time.

In mean time, or 12 o'clock plus the equation of time.

In clock time, or that indicated by a mean solar time-piece.

At apparent noon a mean solar time-piece should therefore indicate 12 o'clock plus the equation of time at the instant.

Therefore the general equation of the Transit Instrument becomes for this case

$$12^h + \epsilon = T + E + aA + bB + c'C, \qquad (61)$$

ϵ denoting the equation of time.

Note the order and directions that follow:

1. The mean of all the observed times is the chronometer time of transit of sun's center over the mean of the wires.
2. The reduction to middle wire, as well as the three corrections, are found as in Form 1. By adding them to the above-men-

tioned mean, we have the chronometer time of apparent noon. The declination of the sun, used in computing these corrections, is to be taken from the Ephemeris, allowance being made for the observer's longitude. Use page 1, Monthly Calendar.

3. The mean time of apparent noon is 12 hours $\pm \epsilon$. In computing ϵ use page 1, Monthly Calendar, and make allowance for observer's longitude. The Ephemeris gives the sign of ϵ.
4. Subtract the chronometer time of apparent noon from the mean time of apparent noon, and the remainder is the error of the chronometer:—plus if slow, minus if fast.
5. Time-pieces at West Point are run on 75th Meridian mean time, *i.e.* $4^m\ 9^s.45$ slower than local mean time. Hence in finding the error at West Point subtract $4^m\ 9^s.45$ from $12^h \pm \epsilon$, before proceeding with step No. 4.

✠ Should the necessity arise for finding the error of a mean solar time-piece by a meridian transit of a star, it may be done by the same methods, the reduction to the middle wire and corrections for instrumental errors being computed as usual, since the equatorial value of the first, as before, being taken as not exceeding $0^s.5$, the greatest error thus produced cannot exceed $\frac{0^s.00273}{2}$ sec 67° $= 0^s.0035$. Stars within the polar circle, or whose declination exceeds about 67° are not used for time determinations.

Therefore having *observed* the *clock* time of transit (corrected by $aA + bB + cC$), and having *computed*, as heretofore explained, the *correct* mean solar time of transit from the star's R. A., the difference gives the clock error.

THE SEXTANT.

As problems under the second and third heads arising in field work are usually solved by aid of the sextant, a short description of that instrument and the manner of using it becomes necessary.

The sextant is a hand reflecting instrument designed for the measurement of the angular distance between two objects. In its construction it embodies the following principle of Optics, viz.:

When a ray of light is reflected successively by two plane mirrors, the angle between the first and last direction of the ray is twice the angle between the mirrors, provided the ray and its two reflections are all in the same plane perpendicular to both mirrors. For astronomical work the sextant is mainly used for measuring vertical angles, *i.e.*, the altitude of some celestial body. In the measurement of Lunar Distances, however, the angle will usually be inclined.

The instrument consists essentially of a graduated circular arc, usually somewhat over 90° in extent, connected with its center by several radii and braced by cross pieces, forming what is known as the *frame*. Attached to the center of the arc is a movable *index-arm* provided with clamp and tangent screw, carrying at its outer end a vernier and microscope for reading the sextant arc. Attached to the index-arm at its center of motion, and therefore rotating with it, is a small mirror known as the *index-glass*, whose plane is perpendicular to that of the frame. Perpendicular to the frame, attached thereto and therefore immovable, is a second small mirror, known as the *horizon-glass*. These two mirrors are so placed with reference to each other that when the index-arm vernier points to the zero of the arc, they shall be exactly parallel and facing each other. In this position a ray reflected by both mirrors will have its original direction unchanged. The horizon-glass is divided into two parts by a line parallel to the frame. The first part next the frame is a mirror, and is the horizon-glass proper. The outer part, consisting of unsilvered glass, is not a mirror. A small telescope screwing into a fixed ring, is held by the latter with its axis parallel to the frame and pointing to the horizon-glass. The distance to the axis from the frame is so regulated that the objective will receive rays passing through the unsilvered, as well as rays reflected from the silvered, part of the horizon-glass. Since each portion of an objective forms as perfect an image as does the whole, the difference being only in degree of brightness of the image, it is manifest that by pointing the telescope at one object and placing another so that its reflected rays will be received by the objective, an image of each object may be seen in the field of view, each perfect in detail, but less bright than if formed with the whole aperture of the objective. The relative brightness of the two images may be varied at will by simply moving the telescope bodily to or

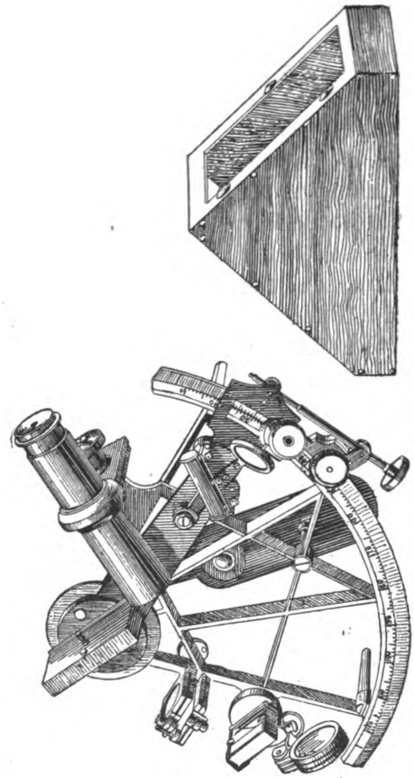


Fig. 14.—The Sextant and Artificial Horizon.

from the frame, thus presenting more or less of the objective to the silvered part of the horizon-glass. For observation, they should be equally bright.

Excessive brightness, as in case of the sun, is reduced by two sets of colored shades of different degrees of opacity, one set for the reflected, and one for the direct rays. These are supposed to be of plane glass, but to eliminate any errors due to a possible prismatic form, they admit of easy reversal. A disk containing a set of colored glasses is arranged to screw over the eye end of the telescope. This should be used when practicable, since any prismatic form in *these* glasses will affect both direct and reflected rays equally.

Two parallel wires are placed in the focus of the objective, the middle point between which marks the center of the field of view. The line joining this point and the optical center of the objective is the axis of the telescope. It is this line which should be parallel with the frame of the instrument.

Suppose now with the index-arm set at zero (in which case the mirrors are parallel), the telescope is accurately directed to some very distant point. Rays will pass through the unsilvered part of the horizon glass and form an image at the center of the field of view. Rays sensibly parallel to these will fall upon the index-glass, be reflected to the horizon-glass, and thence into parallelism with the original direction, since the angle between the mirrors is zero.

These reflected rays being parallel to the direct rays, will be brought to the same focus, and there will be presented at the middle of the field of view, apparently one, in reality two, images of the point, accurately coinciding.

Retaining the direct image at the middle of the field, let the index-arm be moved forward, say 25°. According to the principle of Optics cited, there will be superimposed on the first image that of another point, separated from the first point by an angular distance of 50°. Accordingly in order to give the real value of an angle, the sextant graduations are marked double their true value.

Also according to the same principle of Optics, it follows that if the reading is 50° when the distance is 50°, the ray from the second point and all its reflections must determine a plane perpendicular to both mirrors and hence parallel to the frame. If the instrument and index-arm be so moved as to produce coincidence of images *on* *either side* of the field, evidently the last direction of the ray is not parallel to the frame, the fundamental principle of the sextant is violated, and the position assumed by the index-arm to give this coincidence gives an incorrect value of the angle. The frame of the instrument must therefore always be held in the plane of the two points, which condition is fulfilled when coincidence of their images can be produced at the centre of the field.

Hence, to measure an angle with a sextant:—Direct the telescope to the fainter of the two objects and bring its image to the middle of the field. *Retaining it in this position,* rotate the instrument about the line of sight and move the index-arm slowly back and forth until accurate coincidence of the two images is produced at the middle of the field. Perfection of coincidence is produced by use of the tangent screw.

In measuring altitudes (*e.g.* of the sun) at sea, it is sufficient to bring the reflected image tangent to the sea horizon, and correct the resulting altitude for dip. On land the natural horizon cannot be used for obvious reasons. Recourse is therefore had to an "artificial horizon" consisting of a small vessel of mercury with its surface protected against wind, etc., by a glass roof. An observer placing himself in the plane of the "object" and the perpendicular to the artificial horizon, will by placing the eye at the proper angle see an image of the object reflected from the mercury. Since the angles of incidence and reflection are equal, this image may be regarded as another body at the distance of the object and at the same angular distance *below* the horizon as the real object is *above* it. The measurement of the angle between the two will therefore give the double altitude of the object. This measurement is accomplished by regarding the image seen in the mercury as the "fainter of the two objects" mentioned in the foregoing rule, and then proceeding as there indicated.

If the body have a sensible diameter, as the sun, the altitude of the *center* is the quantity sought, since all data in the Ephemeris relating to the sun is given for its center. Nevertheless since it is easier to judge of the exact tangency of the two images than of their exact coincidence, it is the altitude of a limb which is always measured. This, corrected for refraction, semi-diameter, and parallax, will give the true geocentric altitude of the *center*.

The sextant being usually held in the hand and therefore somewhat unstable, being also of small dimensions and graduated on the arc only to 10′, a single measurement of an angle never suffices for any astronomical purpose. Altitudes are therefore always taken in "sets" and the corresponding times noted. There are two methods of taking these sets according as the body is moving rapidly or slowly in altitude. The first case evidently applies to extra-meridian observations, and the second to circum-meridian and circum-polar observations.

To explain the first case, suppose it were required to take a set of forenoon altitudes of the sun's upper limb. First it is to be noted that the image in the horizon as viewed by the telescope, is erect; it having been inverted once by the reflection and again by the telescope. The image reflected from the mirrors is however inverted, and its lowest point corresponds to the upper limb of the sun. Now point the telescope to the mercury (having applied the proper shades), and place the upper limb of this image at the center of the field. By the rotation and movement of the index-arm before described bring the image from the mirrors into the field *above* the other. Since the sun is rising, this image (inverted) will appear to be slowly falling in the field of view toward the other. Set the vernier at the nearest outward exact division of the limb, and note the instant when the two images are just tangent. Set the vernier at the next exact outward division of the limb (which operation separates the images), and note again the time when they come to tangency, which will be only a few seconds later. So proceed until the set is complete. The altitudes are thus equidistant, involve no reading of the vernier, and while waiting for contact the instrument can be held steady by both hands.

To take altitudes of the lower limb, allow the falling image to pass over the other and note the instants of separation.

In the afternoon, the image here described as falling, is rising.

In the second case, when the body is about to pass the meridian or is near the pole, it is moving so slowly in altitude that we cannot set the index-arm ahead by successive equal steps and wait for the body to reach that altitude. Moreover upon passing the meridian the motion in altitude is reversed. In this case we must therefore measure the altitudes of the selected limb in as quick succession as possible according to the ordinary method.

The same principles apply in case of a star.

The glass forming the roof of the horizon may be somewhat prismatic. The effect of this may be eliminated by taking another set with the roof reversed.

ADJUSTMENTS OF THE SEXTANT.

Hitherto it has been assumed that both mirrors were accurately perpendicular to the frame, that when they were parallel to each other the index-arm vernier reads zero, that the center of motion of the arm was the center of the graduated limb, and that the telescope axis was parallel to the frame. The mirrors and telescope are however not rigid in their connections, but each is susceptible of a slight motion to perfect the adjustment. Well-known optical principles together with the preceding remarks render any explanation of these adjustments unnecessary.

1st. Adjustment:—To make the index-glass perpendicular to the frame.

Set the index near the middle of the arc; remove the telescope and place the eye near the index-glass nearly in the plane of the frame. Observe at the right-hand edge of the glass whether the arc as seen directly and its reflected image form one continuous arc, which can only be the case when the glass is perpendicular. If not, tip the glass slightly by the proper screws until the above test is fulfilled.

2d. Adjustment:—To make the horizon-glass perpendicular to the frame.

The first adjustment having been perfected, the second is tested by noting whether the two mirrors are parallel for some one position of the index-glass. If so, the horizon-glass must also be perpendicular to the frame.

Point the telescope to a 3d or 4th magnitude star, or to a distant terrestrial point, the plane of the frame being vertical. Move the index-arm slowly back and forth over the zero. This will cause the reflected image to pass through the field; if it passes exactly over the direct image the two mirrors must be perpendicular to the frame. If it passes to one side, tip the horizon-glass by the proper screws until the test is fulfilled.

3d. Adjustment:—To make the axis of the telescope parallel to the frame.

Turn the telescope until the wires before referred to are parallel to the frame. (An adjusting telescope in which the wires are well separated is to be preferred.) Select two objects which are at a considerable distance apart, as the sun and moon when distant 100° or more from each other. Point the telescope to the moon and bring the image of the sun tangent to it on one of the wires. Move the instrument till the images appear on the other wire. If the tangency still exists, the telescope is adjusted. Otherwise tip the ring holding it, by means of the proper screws, till the test is fulfilled.

4th. Adjustment:—To make the mirrors parallel when the reading of the arc is zero.

Set the index exactly at zero and point to the distant object described in the second adjustment. If the two images are exactly coincident, the adjustment is perfect. Otherwise turn the horizon-glass around an axis perpendicular to the frame, by the proper screws, until coincidence is secured. The mirrors are now parallel.

Errors of the Sextant.

It should be remembered that to whatever division of the arc the index may point *when the mirrors are parallel*, this division is the temporary zero, and from it all angle readings must be reckoned. The fourth adjustment will not remain perfect; it is therefore easier to determine the temporary zero from time to time, note its distance and direction from the zero of the graduation, and apply the correction to all readings. The distance in arc of the temporary from the fixed zero is called the "Index Error," positive if the temporary zero lie beyond the graduated arc, negative if on. To facilitate its measurement when positive, the graduations are carried 4 to 5 degrees to the right of the zero, constituting what is called the "extra arc."

To measure the index-error, bring the mirrors to parallelism by producing a perfect coincidence of the direct and reflected images of a star or distant point; read the vernier, giving the result the proper sign.

Another method specially applicable at sea is as follows:

Measure the *horizontal* diameter of the sun (so that the two limbs may not be affected by unequal refraction), first on the arc and then on the extra arc. Evidently one reading will exceed, and the other be less than the diameter, by the index-error. One half the difference will then be the index error, positive if the larger reading be on the extra arc.

As a verification, one fourth the sum should be the sun's semi-diameter as given for the date in the Ephemeris.

Another error which must be attended to with equal care is the "Eccentricity." This arises when the center of motion of the index-arm is not coincident with the center of the graduated arc. The effect of such maladjustment is seen from Figure 15, *a* being the center of motion, and *b* that of the arc. When the arm is in the position *ae* in prolongation of the line joining the two centers, there is manifestly no error in the reading. When at *ad* perpendicular to that line, there is an error *cd*. Between these two positions the error will be intermediate in value.

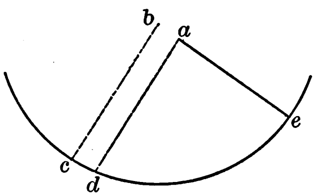


FIG. 15.

To determine this error, measure with a theodolite the angular distance between two distant points. Then take the mean of several measurements of the same angle with the sextant. The difference will be the effect of eccentricity *for that reading of the sextant.* This operation should be repeated at short angular intervals for the whole arc, and the results tabulated.

Other methods may be adopted when the appliances of a fixed observatory are at hand.

Nomenclature of the Astronomical Triangle.

A = azimuth angle = angle at the zenith.
P = hour angle = angle at the pole.
ψ = parallactic angle = angle at the body.
$90° - \phi$ = side from zenith to pole.
$90° - \delta = d$ = side from pole to body = polar distance.
$90° - a = z$ = side from zenith to body = zenith distance.

In which

ϕ = latitude of place.
δ = declination of body.
a = altitude of body.

II. TIME BY SINGLE ALTITUDES.

1. To Find the Error of a Sidereal Time-piece by a Single Altitude of a Star. (See Form 3.)—The solution of this problem consists in finding the value of the hour angle $Z P S$ in the astronomical triangle (see Fig. 16), having given the three sides of the triangle, viz.: $Z P$, the complement of the latitude, $P S$ the polar distance of the star, and $Z S$ its zenith distance. The latitude ϕ is supposed to be known, the polar distance d is taken from the

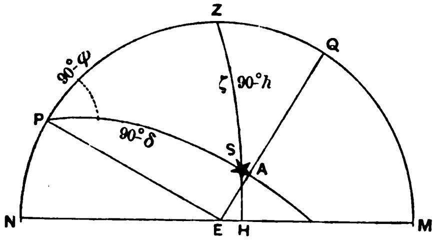


FIG. 16.

Ephemeris for the date, and the altitude a, the complement of the zenith distance, is measured by the sextant and artificial horizon. The measured altitude having been corrected for errors of the sextant and refraction, the above data substituted in the formula

$$\sin \tfrac{1}{2} P = \pm \sqrt{\frac{\cos m \sin (m - a)}{\cos \phi \sin d}}, \qquad (62)$$

$$\left[m = \frac{\phi + d + a}{2} \right],$$

will give the value of P, the star's hour angle, which divided by 15 will give the hour angle in time. (The negative sign is to be used if the star be east of the meridian.)

This plus the star's R. A. for the date will give the sidereal time, which by comparison with the chronometer time noted at the instant of taking the altitude, will give the chronometer error.

As heretofore stated, reliance is not to be placed upon a single measurement by so defective an instrument as the sextant. A set of observations, from 5 to 10, is therefore made by recording the times corresponding to successive changes of 10′ in the star's double altitude. These altitudes will thus be equidistant and involve no measurement of seconds of arc.

In the computations it is usual to assume that the mean of the times corresponds to the mean of the altitudes, as shown on Form 3, which implies that the star's motion in altitude is uniform. This in general is not true. We must therefore, to be as accurate as possible, either apply a correction to the mean of the times to obtain the time when the star was at the mean of the altitudes, or a correction to the mean of the altitudes to give the altitude at the mean of the times. Whether corrected or not, the means are used as a single observation. Also, since the refraction varies *uniformly* with the altitude, the refraction corresponding to the mean of the altitudes requires, in strictness, a slight correction; although of much less importance than the first. These corrections may as a rule be omitted. Their deduction is given in the following paragraph.

✠ To determine the correction to be applied to the mean of the altitudes or the mean of the times, the following deduction is appended essentially as given by Chauvenet.

To find the change in *altitude* of a star in a given *interval of time*, having regard to second differences, let

$$a = f(P).$$

Then $$a + \Delta a = f(P + \Delta P).$$

Expanding by Taylor's Theorem,

$$a + \Delta a = f(P) + \frac{df(P)}{dP}\Delta P + \frac{d^2 f(P)}{dP^2}\frac{(\Delta P)^2}{2} + \ldots$$

$$\Delta a = \frac{da}{dP}\Delta P + \frac{d^2 a}{dP^2}\frac{(\Delta P)^2}{2}.$$

From the astronomical triangle,

$$\sin a = \cos d \sin \phi + \sin d \cos \phi \cos P.$$
$$\cos a \, d a = - \sin d \cos \phi \sin P \, d P.$$

$$\frac{da}{dP} = -\frac{\sin d \cos \phi \sin P}{\cos a} = -\cos \phi \sin A, \tag{63}$$

A being the azimuth.

$$\frac{d^2 a}{dP^2} = -\cos \phi \cos A \frac{dA}{dP}. \tag{64}$$

Also from the astronomical triangle in a similar manner,

$$\frac{dA}{dP} = -\frac{\cos\psi \sin A}{\sin P}, \tag{65}$$

ψ being the parallactic angle.

Whence

$$\Delta a = -\cos\phi \sin A \Delta P + \frac{\cos\phi \sin A \cos A \cos\psi}{\sin P} \frac{(\Delta P)^2}{2}. \tag{66}$$

Expressing Δa and ΔP in seconds of arc and time respectively, we have, after reduction,

$$\Delta a = -\cos\phi \sin A\,(15\,\Delta P) + \frac{\cos\phi \sin A \cos A \cos\psi}{\sin P} \frac{(15\,\Delta P)^2}{2} \sin 1'', \tag{67}$$

which gives the variation in altitude due to a lapse of ΔP seconds of time.

The last term may be written

$$\frac{(15\,\Delta P)^2}{2} \sin 1'' = \frac{2 \sin^2 \frac{1}{2} \Delta P}{\sin 1''} = m.$$

Values of m are given in tables under the head of Reduction to the Meridian.

Placing also, for brevity,

$$g = \cos\phi \sin A, \quad k = \frac{\cos A \cos\psi}{\sin P},$$

we have,

$$\Delta a = -15\, g \Delta P + g\, k\, m,$$

a more convenient expression of the same relation.

Now let H, H', H'', etc., denote the altitudes (corrected for sextant errors), T, T', T'', etc., the corresponding times, a_0 the mean of the altitudes, t_0 the mean of the times, and a_0' the altitude corresponding to t_0, since this cannot be a_0. It is now required to determine the relation between a_0' and a_0 in order that the whole set of observations may be resolved into *one*—a single altitude taken at the mean of the times.

The change $H - a_0'$ required the time $T - t_0$.
The change $H' - a_0'$ required the time $T' - t_0$,
etc.

Therefore from the relation $\triangle a = -15\, g \triangle P + g\, k\, m$ we have, denoting the different m's by m_1, m_2, etc.,

$$\left.\begin{array}{l} H - a_0' = -15 g\,(T - t_0) + g\,k\,m_1, \\ H' - a_0' = -15 g\,(T' - t_0) + g\,k\,m_2, \\ \quad\text{etc.} \qquad\qquad\qquad \text{etc.} \end{array}\right\} \tag{68}$$

If there were n observations, the mean gives.

$$a_0 - a_0' = g\,k\,\frac{m_1 + m_2 + m_3 + \text{etc.}}{n} = g\,k\,m_0. \tag{69}$$

Or

$$a_0' = a_0 - g\,k\,m_0. \tag{70}$$

The last term is therefore the desired correction to the mean of the altitudes in order that it may correspond to the mean of the times.

It will however be more convenient to find such a correction as applied to the mean of the times will cause it to correspond to the mean of the altitudes.

Let t_0' denote the time corresponding to the mean of the altitudes.

The change $a_0 - a_0'$ required the time $t_0' - t_0$. Hence from the preceding, we have, since $t_0' - t_0$ is very small,

$$15\,g\,(t_0 - t_0) = -(a_0 - a_0') = -g\,k\,m_0:$$

$$t'_0 = t_0 - \tfrac{1}{15}\,k\,m_0.$$

Expressing $k = \dfrac{\cos A \cos \psi}{\sin P}$ in known quantities,

$$k = \frac{\sin P \cos \phi \sin d \sin a}{\cos^2 a} - \cot P. \tag{71}$$

$$t_0' = t_0 + \tfrac{1}{15}\left[\cot P - \frac{\sin P \cos \phi \sin d}{\cos a_0 \cot a_0}\right]\frac{1}{n}\Sigma\frac{2\sin^2\frac{1}{2}(T-t_0)}{\sin 1''}. \quad (72)$$

The refraction, r, belonging to the mean of the altitudes is corrected, if desired, by the quantity

$$2\,\frac{\sin r}{\sin^2 a_0}\frac{1}{n}\Sigma\frac{2\sin^2\frac{1}{2}(a_0 - H)}{\sin 1''}.$$

H denoting the different altitudes.

It is important to ascertain what stars are suited to the solution of this problem.

Differentiating the equation derived from the astronomical triangle (regarding a and P as variable),

$$\sin a = \sin \phi \sin \delta + \cos \phi \cos \delta \cos P, \quad (73^1)$$

and reducing by

$$\frac{\cos a}{\cos \delta} = \frac{\sin P}{\sin A},$$

we have

$$d\,P = -\frac{1}{\cos \phi \sin A}\,d\,a. \quad (74)$$

From this it is seen that any error ($d\,a$) in the measured altitude will have the least effect on the computed hour angle when $\phi = 0°$, and $A = 90°$. That is, the method is less exposed to error in low latitudes; but whatever the latitude, the star should be near the prime vertical. The worst position of the star is when on the meridian.

Differentiating the same equation regarding ϕ and P as variable, reducing by

$$\cos a \cos A = \sin \delta \cos \phi - \cos \delta \sin \phi \cos P. \quad (75)$$

and the same equation as before, we have

$$d\,P = \frac{1}{\cos \phi \tan A}\,d\,\phi. \quad (76)$$

From this it is seen that any uncertainty as to the exact latitude will also have least effect when the star is near the prime vertical and the observer near the equator.

Differentiating with reference to δ and P, we have

$$d\,P = \frac{1}{\cos \delta \tan \psi} d\,\delta, \tag{77}$$

and it thus appears that an erroneous value of δ will also produce the least effect when the star is on the prime vertical, since from the equation

$$\sin \psi = \frac{\cos \phi}{\cos \delta} \sin A$$

$\sin \psi$ and therefore $\tan \psi$ will be a maximum when $\sin A$ is also a maximum.

From the three foregoing differential equations it is also seen that the effect of constant errors either in the measured altitude, the assumed latitude, or assumed declination, may be eliminated by combining the results from two stars, one east and one west of the meridian, and in as nearly corresponding positions as possible; since then the corresponding values of $\sin A$, $\tan A$, and $\tan \psi$ will be numerically nearly equal and of opposite signs.

Hence the following general rule should be observed: *In order to reduce to a minimum the effect of errors either in the observations or the assumed data, select a star which will cross the prime vertical at some distance from the zenith* ($\delta < \phi$), *and make the observations near that circle. As the latitude increases, greater accuracy in the observations and data is required in order to give a constant degree of precision in the results. Stars very near the horizon should be avoided on account of excessive and irregular refraction. The adopted value of the clock error should be the mean of the results from an east and a west star.*

In the computation, if great accuracy be not essential, mean refractions may be employed; their values are given in tables.

2. To Find the Error of a Mean Solar Time-piece by a Single Altitude of the Sun's Limb. (See Form 4.)—This problem does not differ in principle from the preceding. The observations are made on the sun's limb, and therefore in addition to refraction the correction for semi-diameter at the time of observation must be applied. Also, since the sun has an appreciable parallax, and since also the Ephemeris data supposes the observer to be at the earth's center, the altitude must be further corrected for "parallax in altitude." Parallax in altitude = Equatorial Horizontal Parallax $\times \rho \times$ cos altitude, ρ being the ratio of the earth's radius at the equator to that at the place of observation. At West Point log $\rho = 9.999368 - 10$. The Equatorial Parallax is given in the Ephemeris, page 278.

The sun's declination (or polar distance) which is given in the Ephemeris for certain instants of Greenwich time, varies quite rapidly; and in order to determine this element at the instant of observation we must know our longitude and the error of the chronometer, to obtain which is the object of the problem. In practice, however, the error will usually be known with sufficient accuracy to find approximately the time elapsed since Greenwich mean noon. With this assumed difference we find by interpolation in page II, Monthly Calendar, the declination for the instant. The same remarks apply to the determination of the semi-diameter referred to above, and the Equation of Time below.

With the data thus found, compute P (in time) as in the preceding problem.

Then if it be a morning observation,

$$\text{Apparent time} = 12^h - P.$$

If an afternoon observation,

$$\text{Apparent time} = P.$$

Apparent time $\pm$ Equation of Time = Mean Time. This compared with the mean of the recorded times gives the chronometer error, and if this is found to differ very materially from the assumed error, the declination and possibly also the Semi-diameter and Equation of Time, must be redetermined, and the computation repeated. The sun should be observed as near the prime vertical as is consistent with avoiding irregular refraction.

In all cases where time is to be determined by altitudes of the sun, it is better to make a set of observations on each limb, and reduce each set separately. If a difference of personal error in estimating contact of images as compared with their separation exists, it will thus be discovered and in a great measure eliminated.

III. TIME BY EQUAL ALTITUDES.

1. To Find the Error of a Sidereal Time-piece by Equal Altitudes of a Star. (See Form 5.)—If the times when a star reaches equal altitudes on opposite sides of the meridian be noted, the "middle chronometer time" will be the time of transit, provided the chronometer has run uniformly. Hence we would have

$$E = \alpha - \frac{T_e + T_w}{2}. \tag{78}$$

But if the refraction is different at the times of the two observations, the *true* altitudes will be unequal when the observed are equal; which latter will consequently not correspond to equal hour angles. Manifestly therefore one of the chronometer times (*e.g.*, the last), requires a correction equal to the hour angle corresponding to the change in *true* altitude,—this change being the difference between the E. and W. refractions,—and the middle chronometer time will require one half this correction.

Hence we have in full (see note at end of next problem),

$$E = \alpha - \left[\frac{T_e + T_w}{2} + \frac{1}{2}\frac{(r_e - r_w)\cos a}{15\cos\phi\cos\delta\sin t}\right], \tag{79}$$

E being the chronometer error at time of meridian passage, α the star's apparent R. A., T_e and T_w the chronometer times of observation, r_e and r_w the east and west refractions, and t one half the elapsed time between the observations. The above equation evidently applies even when the times have been noted by a mean solar chronometer, provided α be replaced by the computed mean time of meridian passage.

Use an Ephemeris star and make the first set of observations as prescribed under "Time by Single Altitudes." Then with the same sextant use the same altitudes in the second set, of course in the reverse order.

From the preceding Equation it is seen that the actual altitudes are not required. Therefore unless the correction for refraction is to be applied, no record need be made of the sextant readings or errors. Also, under the same condition, the method is independent of errors in the assumed latitude or the star's declination.

As before, the observations should be made as near the prime vertical as is consistent with avoiding irregular refraction. By selecting a star whose declination is but a little less than ϕ, it will be on the prime vertical near the zenith, and we can probably avoid the correction for refraction since the elapsed time will be small. The sextant and chronometer also will be but little liable to changes.

If the eastern observations have been prevented by clouds or other cause, we may still take the western observations, and the eastern at the next prime vertical transit of the star; thus giving the chronometer error at time of star's *lower* meridian passage.

2. To Find the Error of a Mean Solar Time-piece by Equal Altitudes of the Sun's Limb. (See Form 6.)—The general principles involved and the methods of observation are the same as in the preceding problem. But since the sun changes in declination between the times of the E. and W. observations, equal altitudes do not correspond to equal hour angles. For example, when the sun is moving north, the morning will be less than the afternoon hour angle at the same altitude. Manifestly therefore the afternoon hour angle requires to be diminished by the change due to the change of declination, and the middle chronometer time by *half* this amount, which is accomplished in practice by adding the correction with its sign changed. This correction is called the "Equation of Equal Altitudes."

The middle chronometer time thus corrected gives the chronometer time of apparent noon. $12^h \pm$ the Equation of time at Apparent Noon gives the mean time of apparent noon, and the difference is the chronometer error on mean time at apparent noon.

Hence in full

$$E = 12^h \pm \epsilon - \left[\left(\frac{T_e + T_w}{2} + 6^h\right) + \frac{1}{2}\frac{(r_e - r_w)\cos a}{15\cos\phi\cos\delta\sin t} + (AK\tan\phi + BK\tan\delta)\right]. \quad (80)$$

A –, B +

The last term in the bracket is the Equation of Equal Altitudes. For its deduction, see note at end of problem.

A and B are taken from tables. K is the sun's hourly *increase* in declination at apparent noon, taken from the Ephemeris by interpolation; δ is the sun's declination at same time.

If a sidereal chronometer had been used, the above equation would evidently still apply, substituting for $12^h \pm \epsilon$ the sun's R. A. at apparent noon, and omitting 6^h in the parenthesis.

For the application of this method to midnight, and effect of errors in data, see Note.

Correction for Refraction.—To deduce the correction for refraction employed in the two preceding problems, resume the differential equation of the last note,

$$d\,P = \frac{\cos a}{\cos \phi \cos \delta \sin P}\, d\,a \text{ (numerically)},$$

which gives the change in hour angle (in arc) for a change in altitude of da.

If the west refraction be less than the east, the sun will, in falling, reach the altitude a too soon, and the west hour angle must be increased. Hence in this case the correction must be positive and additive, and in any case the correction with its proper sign in time will be obtained from the expression

$$\frac{(r_e - r_w) \cos a}{15 \cos \phi \cos \delta \sin t},$$

since $r_e - r_w$ is the change in altitude da, and t, or one half the elapsed time, is practically P.

For the middle chronometer time, we therefore have

$$\text{Cor. for Ref.} = \tfrac{1}{2}\, \frac{(r_e - r_w) \cos a}{15 \cos \phi \cos \delta \sin t}. \qquad (81)$$

The equation reduced as in the preceding note, gives

$$d\,P = \frac{r_e - r_w}{30 \cos \phi \sin A}. \qquad (82)$$

Since $r_e - r_w$ may denote an error in altitude from any cause whatever, it follows that the observations should be made near the prime vertical.

Equation of Equal Altitudes.—In order to deduce the Equation of Equal Altitudes, resume the equation

$$\sin a = \sin \phi \sin \delta + \cos \phi \cos \delta \cos P.$$

Differentiate, regarding δ and P as variable, and solving, we have

$$d\,P = \frac{\sin \phi \cos \delta - \cos \phi \cos P \sin \delta}{\cos \phi \cos \delta \sin P}\, d\,\delta,$$

$$= \left(\frac{\tan \phi}{\sin P} - \frac{\tan \delta}{\tan P}\right) d\,\delta, \qquad (83)$$

which gives the change in hour angle due to a change $d\,\delta$ in declination.

Now if t denote half the elapsed time in hours, and K the hourly *increase* in the sun's declination at the middle instant (assumed to be apparent noon), we will have

$$d\,\delta = 2\,t\,K.$$

Again assuming P to be the mean hour angle $= t$, and δ to be the declination at the middle instant (assumed to be apparent noon), we shall have for the change in hour angle in time due to the increase in declination

$$d\,P = \tfrac{1}{15}\left(\frac{\tan \phi}{\sin t} - \frac{\tan \delta}{\tan t}\right) 2\,K t. \qquad (84)$$

Since K denotes an increase in declination, the afternoon hour angle will be too large by the above quantity, and the middle chronometer time too large by half the same quantity. Hence *in any case*, the quantity to be *added* to the middle chronometer time to reduce it to chronometer time of apparent noon is

$$-\frac{K\,t \tan \phi}{15 \sin t} + \frac{K\,t \tan \delta}{15 \tan t}.$$

Making $A = -\frac{t}{15 \sin t}$, $B = \frac{t}{15 \tan t}$, we have

$$\text{Eq. of Equal Altitudes} = A\,K \tan \phi + B\,K \tan \delta. \qquad (85)$$

As in the preceding case, observations may be made in the afternoon and the following morning to obtain the chronometer error at midnight. Such a set may be regarded as A. M. and P. M. observations respectively made by a person at the other extremity of the earth's diameter, and therefore in latitude $-\phi$.

Hence for midnight the Eq. would be

$$\frac{Kt \tan \phi}{15 \sin t} + \frac{Kt \tan \delta}{15 \tan t}. \qquad (86)$$

Since t is always less than 12^h, its sine is always positive. Also tan t will be positive when t is less than 6^h, and negative when more. From which it is seen that we may use a single equation for both noon and midnight, viz.:

$$A\,K \tan \phi + B\,K \tan \delta,$$

by noting the following rule as to signs.

For noon, A is always negative, for midnight positive. For noon or midnight B is positive when the elapsed time is less than 12^h and negative when more.

The effect of errors in ϕ and δ is readily seen by a differentiation of the Equation.

Time of Sunrise or Sunset.—This problem is precisely similar to that of single altitudes, except that the altitude of the sun is known and therefore no observation is required. The zenith distance of the sun's center at the instant when its upper limb is on the horizon is assumed to be 90° 50′, which is made up of 90°, plus 16′ (the mean semi-diameter of the sun), plus 34′ (the mean refraction at the horizon). The resulting hour angle replaces P in Form 4.

Duration of Twilight.—The zenith distance in this case is 108°, as twilight is assumed to begin in the morning or end in the evening when the sun's center is 18° below the horizon. (See Art. 130, Young.)

From the solution of the ZPS triangle it can readily be shown that the time required for the sun to pass from the horizon to a zenith distance z is

$$t = \frac{2}{15} \sin^{-1} \sqrt{\frac{1 - \sin z \cos (\psi - \psi')}{2 \cos^2 \phi}}, \tag{87}$$

in which ψ and ψ' (called the sun's parallactic angles) are the angles included between the declination and vertical circles through the sun's center for any zenith distance z, and for the horizon respectively, and ϕ is the observer's latitude. Making z equal to 108° this becomes

$$t = \frac{2}{15} \sin^{-1} \sqrt{\frac{1 - \cos 18° \cos (\psi - \psi')}{2 \cos^2 \phi}}, \tag{88}$$

from which the duration of twilight for any latitude and any season of the year can be found; the values of ψ and ψ' are given by

$$\cos \psi = \frac{\sin \phi - \sin \delta \cos z}{\cos \delta \sin z}, \tag{89}$$

and

$$\cos \psi' = \frac{\sin \phi}{\cos \delta}. \tag{90}$$

When ψ is equal to ψ' then t is a minimum, and we have, after replacing $1 - \cos 18°$ by $2 \sin^2 9°$,

$$t = \tfrac{2}{15} \sin^{-1} (\sin 9° \sec \phi), \tag{91}$$

from which the duration of the shortest twilight is found. Under the same condition we have from Eqs. (89) and (90),

$$\sin \delta = -\tan 9° \sin \phi; \tag{92}$$

from which the sun's declination at the time of shortest twilight at any latitude can be found.

LATITUDE.

The latitude of a place on the earth's surface is the declination of its zenith. The *apparent zenith* is the point in which the plumb-line, if produced, at the point of observation would pierce the celestial sphere. The *central zenith* is the point in which the radius of the earth, if produced, would pierce the celestial sphere. The latitude measured from the central zenith is called the *geocentric latitude*, and that from the apparent zenith is called the *astronomical latitude* or simply the *latitude*. The difference between the latitude and the geocentric latitude is called the *reduction of latitude.*

The direction of the plumb-line is affected by the local attraction of mountain masses on the plumb-bob, or on account of the unequal variations of density of the crust of the earth, at or near the locality of the station. The Astronomical latitude is determined from the actual direction of the plumb-line, and therefore includes all abnormal deviations. The Geographical or Geodetic latitude is that which would result from considering the earth a perfect spheroid of revolution, without the abnormal deviations above referred to.

Form and Dimensions of the Earth.—Before proceeding to the latitude problems it is important to derive some necessary formulas from the form and dimensions of the earth. For this purpose, let us assume that the earth is an oblate spheroid about the polar axis.

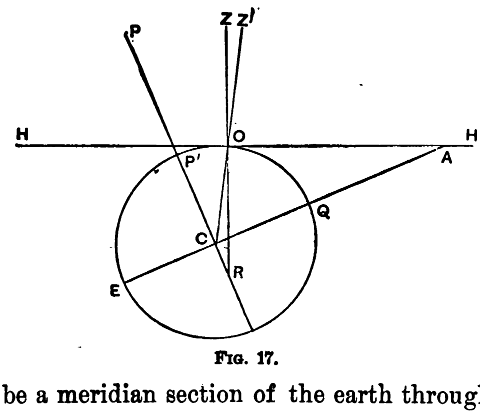


FIG. 17.

Let $E P' O$ be a meridian section of the earth through the observer's place O; $C P'$ the earth's axis; $E Q$ the earth's equator and

$H\,H'$ the observer's horizon. Let P be the pole of the heavens; Z the apparent and Z' the central zenith; ϕ the latitude and ϕ' the geocentric latitude. The equation of the observer's meridian referred to its center and axes is

$$a^2y^2 + b^2x^2 = a^2b^2, \tag{93}$$

in which a and b are the equatorial and polar radius of the earth. The coördinates of O being x' and y', we have the following analytical conditions.

For the *tangent* at O, coincident with the horizon, from

$$a^2yy' + b^2xx' = a^2b^2; \tag{94}$$

and the *normal* at O, through the apparent zenith Z, from

$$a^2y'\,(x - x') - b^2x'\,(y - y') = 0. \tag{95}$$

From Eq. (94), we have

$$\tan O\,A\,C = \tan\,(90° - \phi) = \frac{b^2\,x'}{a^2\,y'}; \tag{96}$$

whence

$$b^2\,x' \tan\phi = a^2\,y'. \tag{97}$$

Substituting in

$$a^2\,y'^2 + b^2\,x'^2 = a^2b^2 \tag{98}$$

and eliminating b by

$$e^2 = \frac{a^2 - b^2}{a^2}, \tag{99}$$

we have

$$\left.\begin{aligned} x' &= \frac{a\cos\phi}{\sqrt{1 - e^2\sin^2\phi}}, \\ y' &= \frac{a\,(1 - e^2)\sin\phi}{\sqrt{1 - e^2\sin^2\phi}}. \end{aligned}\right\} \tag{100}$$

Let s be the length of any portion of the meridian; then for the elementary arc, its projection on the major axis x, is

$$d\,s\cos O\,A\,C = d\,s\sin\phi = -\,d\,x', \tag{101}$$

since x' is a decreasing function of the latitude. Differentiating the first of Eqs. (100), we have

$$d x' = -a \frac{(1 - e^2) \sin \phi \, d \phi}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}. \qquad (102)$$

Equating (101) and (102), we have

$$d s = a \frac{(1 - e^2) \, d \phi}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}, \qquad (103)$$

and for any other latitude $\phi_{,}$,

$$d s_{,} = a \frac{(1 - e^2) \, d \phi}{(1 - e^2 \sin^2 \phi_{,})^{\frac{3}{2}}}. \qquad (104)$$

Let $d \phi = 1°$, then dividing (103) by (104), we have

$$\frac{d s}{d s_{,}} = \frac{(1 - e^2 \sin^2 \phi_{,})^{\frac{3}{2}}}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}} = \frac{1 - \frac{3}{2} e^2 \sin^2 \phi_{,}}{1 - \frac{3}{2} e^2 \sin^2 \phi} \text{ nearly}, \qquad (105)$$

which, after solving with reference to e^2, reduces to

$$e^2 = \frac{2}{3} \frac{d s - d s_{,}}{d s \sin^2 \phi - d s_{,} \sin^2 \phi_{,}}, \qquad (106)$$

from which the value of *the eccentricity of the meridian* can be found when the measured lengths ds and $ds_{,}$ of any two portions of the meridian line, each 1° in latitude, and the latitudes ϕ and $\phi_{,}$ of their middle points are known; for the earth, this has been found to be about 0.0816967.

To find the *equatorial* and *polar radii*, we have from Eq. (103) after making $d\phi = 1°$,

$$a = \frac{ds}{1 - e^2} (1 - e^2 \sin^2 \phi)^{\frac{3}{2}}, \qquad (107)$$

and from the property of the ellipse,

$$b = a \sqrt{1 - e^2}. \qquad (108)$$

To find the *radius of curvature R* at any point of the meridian. After substituting the values of dx, dy, and d^2y, taken from Eqs. (100), in the general formula for radius of curvature,

$$R = \pm \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\, d^2y}, \tag{109}$$

we have

$$R = a \frac{1 - e^2}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}; \tag{110}$$

and hence the *length of one degree of latitude* at any latitude ϕ is,

$$\beta = \frac{2\,\pi\, R}{360} = \frac{2\,\pi\, a}{360} \frac{1 - e^2}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}. \tag{111}$$

To find the *length of a degree on a section perpendicular to the meridian* at any latitude ϕ we proceed as follows: The radius ρ of the earth at the observer's place, is the minor axis, and the equatorial radius a is the major axis of the elliptical section, cut out of the earth by a plane perpendicular to the meridian plane, passed through the center and the observer's place.

Squaring and adding Eqs. (100) and extracting the square root, we have the radius of the earth at the observer's place; or

$$\rho = a\sqrt{1 - \frac{e^2(1 - e^2)\sin^2\phi}{1 - e^2 \sin^2 \phi}} = a\sqrt{\frac{1 - 2\,e^2 \sin^2 \phi + e^4 \sin^2 \phi}{1 - e^2 \sin^2 \phi}}. \tag{112}$$

The square of the *eccentricity of the section* is

$$e'^2 = \frac{a^2 - \rho^2}{a^2} = \frac{e^2\,(1 - e^2)\sin^2 \phi}{1 - e^2 \sin^2 \phi};$$

which being substituted for e^2 in Eq. (111) after making $\phi = 90°$, gives

$$\beta' = \frac{2\,\pi}{360} a \sqrt{\frac{1 - e^2 \sin^2 \phi}{1 - e^2\,(2 - e^2)\sin^2 \phi}}. \tag{113}$$

To find the *length of a degree of longitude* at any latitude ϕ, we know, Eqs. (100), that the radius of the parallel is x'; therefore we have

$$\alpha = \frac{2\,\pi}{360}\,x' = \frac{2\,\pi}{360}\,a\,\frac{\cos\phi}{\sqrt{1 - e^2 \sin^2\phi}}. \tag{114}$$

The value of the *radius of the earth*, at any latitude ϕ, is derived from Eq. (112) or,

$$\rho = a\sqrt{\frac{1 - 2\,e^2 \sin^2\phi + e^4 \sin^2\phi}{1 - e^2 \sin^2\phi}},$$

which, for logarithmic reduction, when a is made unity may be placed under the form

$$\log\rho = 9.9992747 + 0.0007271 \cos 2\,\phi - 0.0000018 \cos 4\,\phi. \tag{115}$$

From the figure and Eqs. (100), we have

$$x' = \rho\cos\phi' = \frac{a\cos\phi}{\sqrt{1 - e^2 \sin^2\phi}}, \tag{116}$$

$$y' = \rho\sin\phi' = \frac{a\,(1 - e^2)\sin\phi}{\sqrt{1 - e^2 \sin^2\phi}}. \tag{117}$$

Multiplying these equations by $\cos\phi$ and $\sin\phi$ respectively, adding and reducing we have

$$\cos(\phi - \phi') = \frac{a}{\rho}\sqrt{1 - e^2 \sin^2\phi}, \tag{118}$$

and from (116),

$$\cos\phi' = \frac{a}{\rho}\,\frac{\cos\phi}{\sqrt{1 - e^2 \sin^2\phi}}. \tag{119}$$

Whence by combination we have

$$\cos\phi' \cos(\phi - \phi') = \frac{a^2}{\rho^2}\cos\phi; \tag{120}$$

and solving with reference to ρ we have

$$\rho = a\sqrt{\frac{\cos\phi}{\cos\phi'\cos(\phi-\phi')}}, \tag{121}$$

which is capable of logarithmic computation.

To find the *reduction of latitude* $\phi - \phi'$. Since ϕ is the angle made by the normal with the axis of x we have

$$\tan\phi = -\frac{dx}{dy}, \tag{122}$$

and from the figure we have

$$\tan\phi' = \frac{y}{x}. \tag{123}$$

Differentiating the equation of the meridian section we have

$$\frac{y}{x} = -\frac{b^2}{a^2}\frac{dx}{dy}. \tag{124}$$

Whence

$$\tan\phi' = \frac{b^2}{a^2}\tan\phi = (1-e^2)\tan\phi. \tag{125}$$

Developing into a series, we have

$$\phi - \phi' = \frac{e^2}{2-e^2}\sin 2\phi - \left(\frac{e^2}{2-e^2}\right)^2\sin 4\phi + \text{etc.} \tag{126}$$

But since $e = 0.0816967$ this reduces to

$$\phi - \phi' = 690''.65 \sin 2\phi - 1''.16 \sin 4\phi \text{ very nearly.} \tag{127}$$

Latitude Problems.—The general problem of latitude consists in finding the side ZP in the ZPS triangle, any other three parts being given.

Differentiating (73′), regarding first a and ϕ and next P and ϕ as variable, and reducing by (75) we obtain

$$d\phi = \sec A\, d a,$$

and

$$d\phi = \tan A \cos\phi\, d P.$$

$\sin a = \sin\phi\sin\delta + \cos\phi\cos\delta\cos P$

$\cos a\cos A = \sin\delta\cos\phi - \cos\delta\sin\phi\cos P$

Whence observations for latitude should as a rule be made upon a body at or near the time of its culmination.

The following are the methods usually employed.

1. **By Circumpolars.**—This depends on the fact that the altitude of the pole is equal to the astronomical latitude of the place. Let a and a' be the altitudes of a circumpolar star at upper and lower culmination respectively, corrected for refraction and instrumental errors; d and d' the corresponding polar distances, and ϕ the latitude; then we have

$$\phi = a - d, \quad \phi = a' + d', \quad \phi = \tfrac{1}{2}(a + a') + \tfrac{1}{2}(d' - d).$$

The change from d to d' is ordinarily so small in the interval (12 hours) between the observations as to be negligible; it is due solely to precession and nutation. This method is free from declination errors, but subject to changes and errors in the refraction. It is therefore an independent method, and is the one used in fixed observatories where the observations can be made with great accuracy even during daylight by the transit circle. With the sextant the method is applicable only in high latitudes during the winter so that both culminations occur during the night time. A star with a small polar distance is to be preferred, to avoid irregular refraction at the lower culmination.

The sextant, however, is not well adapted to this method, since the least count of its vernier is usually 10″, and at culmination only a single altitude can be measured, even if the instant of culmination be accurately noted by a chronometer. But if Polaris be the star chosen, a series of observations may be made during the five minutes immediately preceding and following culmination, and at no time during these ten minutes will the star's altitude differ from its meridian altitude by more than 1″.1. Errors within this limit would not be detected by even the best sextant observations, and the mean of the measured altitudes will therefore be the meridian altitude with the usual precision.

Even if a be regarded as too small when found in this manner, a' will be too large by practically the same amount, and $\frac{1}{2}(a + a')$ will be correct.

2. **By Meridian Altitudes or Zenith Distances.**—This method depends on the fact that the astronomical latitude of a place is equal

to the declination of its zenith. If the star culminate between the pole and the zenith, then

$$\phi = \delta - z_1,$$

where Z_1 is the meridian zenith distance of the star. If between the zenith and equator, then

$$\phi = \delta + z_1.$$

We have therefore only to measure z_1, take δ from the Ephemeris, and substitute in one of these equations.

This method is a very exact one when the observations are made with an instrument, such as the transit circle, accurately adjusted to the meridian, and whose least count is small. It is subject to errors of both declination and refraction; although the latter as well as any constant errors in the measured altitudes may be nearly eliminated, as is seen from the preceding equations, by combining the result with that from another star which culminates at about the same time at a nearly equal altitude on the opposite side of the zenith.

For reasons stated above, the sextant is not well adapted to this method except at sea, where the highest accuracy is not requisite.

3. **By Circum-meridian Altitudes.**—If the altitude of a celestial body be measured within a few minutes of culmination, we may by noting the corresponding time very readily compute the difference between the measured altitude and the altitude which the body will have when it reaches the meridian. This difference is called the "Reduction to the Meridian," and by addition to the observed will give the meridian altitude. If several altitudes be measured and each be reduced to the meridian, we may evidently, by taking the mean of the results, obviate the inaccuracies incident to the use of the sextant in the last problem.

These are called "Circum-meridian Altitudes," and their reduction to the meridian is rendered very simple by the special formula

$$a_{\prime} = a + \frac{\cos\phi\cos\delta}{\cos a_{\prime}} \cdot \frac{2\sin^2\frac{1}{2}P}{\sin 1''} - \left(\frac{\cos\phi\cos\delta}{\cos a_{\prime}}\right)^2 \tan a_{\prime} \frac{2\sin^4\frac{1}{2}P}{\sin 1''} + \ldots, \quad (128)$$

the deduction of which will be given hereafter.

In this formula a is the true altitude, δ the declination, and P the hour angle, all relating to the instant of observation; $a_{\prime}$ is the desired meridian altitude. Values of $\frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}$ and $\frac{2 \sin^4 \frac{1}{2} P}{\sin 1''}$ are given in tables with P as the argument. For small values of P the series will converge rapidly provided $a_{\prime}$ is not too large.

Having the meridian altitude, the latitude follows as in the last method.

For stars below the pole, we have simply to reckon P from the time of lower transit, and δ as exceeding 90° by the star's polar distance. Cos δ will thus change its sign and both terms of the reduction become negative.

From (128) it is seen that for computing $a_{\prime}$ we require both $a_{\prime}$ and ϕ; but as will appear later, approximate values will suffice. If an approximate value of ϕ be known, that of $a_{\prime}$ follows from

$$a_{\prime} = \delta + 90° - \phi. \tag{129}$$

If not, one may be found as follows: In this method, double altitudes are taken in as quick succession as possible from a few minutes before until a few minutes after meridian passage. The greatest altitude measured will therefore be very near the meridian altitude, and its substitution in (129) will give a value of ϕ sufficiently accurate for the purpose.

The mode of making observations and reductions in case of a star with sidereal chronometer, will be at once apparent from an explanation in case of the sun with a mean time chronometer. It must be borne in mind that the declination of the sun is constantly changing.

The observations are made as just explained on a limb of the sun, viz.: Several double altitudes are taken as near together as possible, as many before, as after meridian passage, and the corresponding chronometer times noted. (Note the difference between this, and sextant observations for time.)

Now if we suppose each observation to have been reduced to the meridian, after correcting for refraction, parallax and semidiameter, we would have several equations of the form

$$a_{\prime} = a + Am - Bn,$$

in which m and n are the tabular values of $\frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}$, and $\frac{2 \sin^4 \frac{1}{2} P}{\sin 1''}$, and A and B the remaining factors of the corresponding terms in Equation (128). Any one of the equations will give for the latitude,

$$\phi = \delta + 90° - (a + A m - B n). \tag{130}$$

In this equation, δ is the declination at the time of observation. For, since the reduction to the meridian has been made with this value of δ in obtaining A and B, $a + A m - B n$ is manifestly the meridian altitude of a body whose declination is *constantly* δ. In fact, the reduction to the meridian by the formula given, can be computed only on the hypothesis of a constant declination. We are thus dealing with a fictitious sun, whose declination on the meridian differs from that of the true sun. But since declination and meridian altitude always preserve a constant difference (the colatitude), we see that Equation (130) will give the correct value of ϕ, due to *perfect* balance in the errors of δ and $(a + A m - B n)$.

The mean of all the equations due to the several observations will be

$$\phi = \delta_0 + 90° - (a_0 + A_0 m_0 - B_0 n_0). \tag{131}$$

Reasoning in the same manner with reference to the *mean* fictitious sun (with its declination δ_0 and meridian altitude $[a_0 + A_0 m_0 - B_0 n_0]$), we see that the result will be perfectly rigorous in *theory* if we employ the mean, δ_0, of the actual declinations, not only for the first term in the value of ϕ, but for the *single* computation of A_0 and B_0. We thus avoid a separate computation for each observation.

The result will moreover be perfectly rigorous in *practice* if we use for δ_0 the declination corresponding to the mean of the times; since in the 30 minutes covered by the observations, the departure of the sun's declination from a *uniform* increase or decrease, is negligible.

We have still to determine the value of P from the chronometer time of each observation, and in this determination it must be borne in mind that P (in arc) is the angular distance of the *true* sun from the meridian at the instant of observation.

There are two reasons why this distance (in time) cannot be given directly by a mean time chronometer. First, the chronometer will usually be gaining or losing, *i.e.*, it will have a "*rate.*" Secondly, a mean time chronometer, even when running without rate, indicates the angular motion of the *mean* sun, which may be quite different from that of the *true* sun, as shown by the continual change in the Equation of Time.

We therefore proceed as follows: From Page I, Monthly Calendar of the Ephemeris (knowing the longitude), take out the Equation of Time. Add this algebraically to 12 hours, apply the error of the chronometer, and the result will be the *chronometer time* of apparent noon. The difference between this and the chronometer time of each observation, gives the several values of P in time, each subject to the two corrections mentioned. To find the correction for rate, let r represent the number of seconds gained or lost in 24 hours (a losing rate being positive for the same reason that an error slow is positive). Then if P' be the corrected hour angle, we will have

$$P' : P :: 86400 : 86400 - r. \qquad [86400 = 60 \times 60 \times 24].$$

Or

$$P' = P \frac{86400}{86400 - r}.$$

Or

$$\frac{2 \sin^2 \frac{1}{2} P'}{\sin 1''} = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''} \left(\frac{86400}{86400 - r} \right)^2 = k \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Hence we will also have

$$A\,m \text{ (corrected for rate)} = k \frac{\cos \phi \cos \delta}{\cos a_{\prime}} \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Hence if we compute A by the formula

$$A = k \frac{\cos \phi \cos \delta}{\cos a_{\prime}},$$

we may employ the actual chronometer intervals and pay no further attention to the question of rate.

From $k = \left(\frac{86400}{86400 - r}\right)^2$, values of k are tabulated with the *rate* as the argument.

The second correction depends, as just stated, on the difference between the motions of the true and mean sun, while the former is passing from the point of observation to the meridian. In other words it depends on the change in the Equation of Time in the same interval, or, which is the same thing, *upon the rate of an accurate mean solar chronometer on apparent time.*

If therefore we let e represent the change in the Equation of Time for 24 hours (positive when the Equation of Time is increasing algebraically), it is evident that $r - e$ will be the rate of the given chronometer on *apparent* time, and that the correction for this *total* rate may be computed as just explained for r, or taken from the same table, using $r - e$ as the argument instead of r alone.

The operation of reducing the observations is then, in brief, as follows.

By Circum-Meridian Altitudes of the Sun's Limb.—Form 7.

Correct the mean of the double altitudes for eccentricity and index error. Correct the resulting mean *single* altitude for refraction, semi-diameter, and parallax in altitude. Denote the result by a_0.

From the Equation of Time (Page I, Monthly Calendar), longitude and chronometer error, find the chronometer time of apparent noon.

Take the difference between this and each chronometer time of observation, denote the difference by P, and their mean by P_0.

With each value of P, take from tables the corresponding values of m and n. Denote their respective means by m_0 and n_0.

From Page II, Monthly Calendar, take the sun's declination corresponding to the local apparent time P_0, and denote it by δ_0.

If ϕ can be assumed with considerable accuracy, determine the corresponding $a_{\prime}$ by $a_{\prime} = \delta_0 + 90° - \phi$.

If not, take the greatest measured altitude, correct it for refraction, etc., call it $a_{\prime}$, and deduce ϕ from the above equation.

From the rate of the chronometer and change in Equation of Time, (both for 24 hours,) take k from the table.

With these values of k, ϕ, $a_{,}$, and δ_0, compute

$$A_0 = \frac{\cos\phi\cos\delta_0}{\cos a_{,}}\,k, \quad \text{and} \quad B_0 = A_0^2 \tan a_{,}.$$

The latitude then follows from

$$\phi = \delta_0 + 90^\circ - (a_0 + A_0 m_0 - B_0 n_0). \qquad (132)$$

By Circum-Meridian altitudes of a Star.—Form 8.

With a star observed with a *sidereal* chronometer, the observations are the same, and the reduction is only modified by the fact that parallax, semi-diameter, equation of time and longitude do not enter, while the declination is constant.

If the star lie between the zenith and pole, the formula becomes

$$\phi = (a_0 + A_0 m_0 - B_0 n_0) - 90^\circ + \delta_0. \qquad (133)$$

If below the pole,

$$\phi = (a_0 - A_0 m_0 - B_0 n_0) + 90^\circ - \delta_0. \qquad (134)$$

1. An Ephemeris star is to be preferred to the sun, since the reduction is more simple, its declination is better known and constant, it presents itself as a *point,* which is of advantage in sextant observations, and we have a greater choice both in time and the place of the object to be observed.

2. By comparing Eqs. (132) and (133) we see that constant errors in the measured altitudes, and in refraction, will be nearly eliminated by combining the results of two stars, one as much north as the other is south, of the zenith.

Also from the principal term of the Reduction to the Meridian, $\frac{\cos\phi\cos\delta}{\cos a_{,}} \cdot \frac{2\sin^2\frac{1}{2}P}{\sin 1''}$, it is seen that the effect of an imperfect knowledge of the chronometer error, giving an incorrect value of P may be eliminated by taking another observation at about an equal altitude on the other side of the meridian; since, P being very small, $\sin^2\frac{1}{2}P$ will be as much too large in one case as it will be too small in the other.

The double altitudes should therefore be taken at as nearly equal intervals of time and be as symmetrically arranged with reference to the meridian, as practicable.

3. By rewriting the *assumed* formula for the reduction, expressing the first term as a function of ϕ and δ only, and including the third term which has heretofore been omitted, we have (Formula 2, P. 4, Book of Formulas),

$$a_, - a = \frac{1}{\tan\phi - \tan\delta} m - \frac{\tan a_,}{(\tan\phi - \tan\delta)^2} n + \frac{\frac{2}{3}(1 + 3\tan^2 a_,)}{(\tan\phi - \tan\delta)^3} s,$$

and

$$a_, - a = \frac{1}{\tan\delta - \tan\phi} m - \frac{\tan a_,}{(\tan\delta - \tan\phi)^2} n + \frac{\frac{2}{3}(1 + 3\tan^2 a_,)}{(\tan\delta - \tan\phi)^3} s,$$

for south and north stars respectively.

$$\left[m = \frac{2\sin^2 \frac{1}{2}P}{\sin 1''}, \quad n = \frac{2\sin^4 \frac{1}{2}P}{\sin 1''}, \quad s = \frac{2\sin^6 \frac{1}{2}P}{\sin 1''} \right].$$

From these equations it is seen that if a star be selected which culminates at a considerable distance from the zenith, *either north or south*, each term of this development is much smaller than in case of a star culminating near the zenith, either north or south.

Since the third term has been entirely neglected in the previous discussion, it becomes desirable to select our star in such a manner that the omitted term (and hence all following it) shall be small; and this, as just seen, will occur when there is considerable difference between the latitude and the star's declination in either direction.

From the above expression for the third term, knowing the approximate latitude, we may readily find the hour angle of any given star, within which if the observations be confined, the value of the term will not exceed any desired limit—say 0''.01 or 1''. Similarly for the second term. We thus ascertain how long before culmination the observations may safely be begun when it is proposed to omit one or both terms in the reduction.

For example in latitude 40° N., if we observe a star at declination 0°, the observation may be made at 20^m from meridian passage and yet the third term amount only to .01″, which would affect the resulting latitude by one linear foot. Or it may be made at 27^m from culmination, and the third term amount only to .1″, affecting the resulting latitude by ten feet.

A star at an equal altitude north of the zenith, declination 80° (for combination with the preceding as recommended), may be observed at 48 and 62 minutes from culmination, with no larger errors.

With other latitudes the figures will vary, but the principle remains the same.

From an inspection of the third term

$$\frac{\frac{2}{3}(1+3\tan^2 a_{\prime})}{(\tan\phi-\tan\delta)^2}\cdot\frac{2\sin^6\frac{1}{2}P}{\sin 1''},$$

it is seen that an unfavorable position of the star, causing the first factor to be excessive, may be counterbalanced by diminishing the hour angle, P.

Hence the general rule: Select a star whose declination differs considerably from the latitude. This will give ample time for taking a series of altitudes. As the declination of the selected star approaches the latitude, restrict the observations to a shorter time, greater care in this respect being necessary for south stars. Arrange the observations as symmetrically with reference to the meridian as practicable, and use at least two stars—on opposite sides of the zenith.

4. Finally, if a mean solar chronometer be used with a star, the corrected m. s. intervals defined by the equation $P' = P\frac{86400}{86400-r}$ must evidently be reduced to sidereal intervals by multiplying by 1.00272791 heretofore deduced. That is

$$P' = P\frac{86400}{86400-r}1.00273791,$$

and the factor for rate will be $k\,(1.00273791)^2$ instead of k.

Similarly if a sidereal chronometer be used with the sun, the factor for rate will be $k\,(0.99726957)^2$ instead of k.

5. **To Determine the Reduction to the Meridian.**—The difference between a *circum-meridian* and the *meridian* altitude of a body is called the "Reduction to the Meridian."

Its nature will be understood from Figure 18. S is the place of the star; S' the point where it crosses the meridian, ($PS = PS'$); and SS'' the arc of a small circle of which Z, the zenith, is the pole, ($ZS'' = ZS$). ZS' will therefore be the *meridian* zenith distance $= z_{\prime} = 90° - a_{\prime}$; ZS or ZS'' will be the *circum-meridian* zenith distance $= z = 90° - a$; and if x denote the Reduction to the Meridian $= S'S''$, we shall have

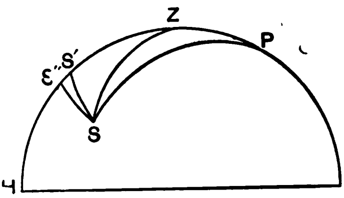


FIG. 18.

$$a_{\prime} = a + x. \qquad (a)$$

The several terms of Equation (128) after a therefore represent x; and it is required to deduce this value of x arranged, as is seen, in a series according to the ascending powers of $\sin^2 \frac{1}{2} P$,

The equation heretofore deduced, viz.:

$$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta - 2 \cos \phi \cos \delta \sin^2 \tfrac{1}{2} P,$$

gives by reduction (since $\sin \phi \sin \delta + \cos \phi \cos \delta = \cos (\phi - \delta) = \cos z_{\prime}$),

$$\cos z_{\prime} - \cos z - 2 \cos \phi \cos \delta \sin^2 \tfrac{1}{2} P = 0. \qquad (b)$$

Putting for convenience $2 \cos \phi \cos \delta = m$, and $\sin^2 \frac{1}{2} P = y$, we have

$$\cos z_{\prime} - \cos z - m y = 0. \qquad (b')$$

We also have

$$x = a_{\prime} - a, \quad \text{or} \quad z = x + z_{\prime}, \qquad (c)$$

$$\cos z = \cos x \cos z_{\prime} - \sin x \sin z_{\prime}.$$

Hence from (b')

$$\cos z_{\prime} - \cos x \cos z_{\prime} + \sin x \sin z_{\prime} - m y = 0. \qquad (d)$$

Now let

$$x = A y + B y^2 + C y^3 + \text{etc.}, \qquad (e)$$

be the undetermined development desired. From the relation expressed by (d), we are to determine such constant values of A, B, and C, as will make the series, when convergent, true for all values of y. Therefore let the values of $\cos x$ and $\sin x$ derived from (e) be substituted in (d). The resulting equation will, from the condition imposed on (e), be an identical equation.

To find $\cos x$ and $\sin x$ for this substitution, we have from calculus,

$$\cos x = 1 - \frac{x^2}{2} + \text{etc.},$$

$$\sin x = x - \frac{x^3}{6} + \text{etc.},$$

and from (e),

$$\cos x = 1 - \tfrac{1}{2}\,(A^2y^2 + 2\,A\,B\,y^3 + \text{etc.}),$$

$$\sin x = A\,y + B\,y^2 + C\,y^3 - \tfrac{1}{6}\,A^3y^3 - \text{etc.}$$

Substituting in (d),

$$\cos z_, - \cos z_, + \tfrac{1}{2}\,A^2 \cos z_,\,y^2 + A\,B \cos z_,\,y^3 + \sin z_,\,A\,y$$
$$+ \sin z_,\,B\,y^2 + \sin z_,\,C\,y^3 - \tfrac{1}{6} \sin z_,\,A^3y^3 - m\,y = 0.$$

Collecting the terms,

$$\left.\begin{matrix} \sin z_,\,A \\ -m \end{matrix}\right\}y + \left\{\begin{matrix} \tfrac{1}{2}\,A^2 \cos z_, \\ + B \sin z_, \end{matrix}\right\}y^2 + \left\{\begin{matrix} A\,B \cos z_, \\ + \sin z_,\,C \\ - \tfrac{1}{6} \sin z_,\,A^3 \end{matrix}\right\}y^3 = 0.$$

From the principles of identical equations

$$\sin z_,\,A - m = 0. \quad A = \frac{m}{\sin z_,}.$$

$$\frac{1}{2}\frac{m^2 \cos z_,}{\sin^2 z_,} + B \sin z_, = 0. \quad B = -\frac{1}{2}\,\frac{m^2 \cot z_,}{\sin^2 z_,}.$$

$$\frac{1}{2}\,\frac{m^3 \cot^2 z_,}{\sin^2 z_,} + \frac{1}{6}\,\frac{m^3}{\sin^3 z_,} - \sin z_,\,C = 0. \quad C = \frac{1}{6}\,\frac{m^3}{\sin^3 z_,}(1 + 3 \cot^2 z_,).$$

Therefore

$$x = \frac{\cos\phi\cos\delta}{\cos a_{,}} 2\sin^2\frac{1}{2}P - \left(\frac{\cos\phi\cos\delta}{\cos a_{,}}\right)^2 \tan a_{,}\, 2\sin^4\frac{1}{2}P$$

$$+ \frac{2}{3}\left(\frac{\cos\phi\cos\delta}{\cos a_{,}}\right)^3 (1 + 3\tan^2 a_{,})\, 2\sin^6 \tfrac{1}{2} P.$$

Reducing the terms of the series from radians to seconds of arc, we have for the value of $a_{,}$,

$$a_{,} = a + \frac{\cos\phi\cos\delta}{\cos a_{,}} \frac{2\sin^2\frac{1}{2}P}{\sin 1''} - \left(\frac{\cos\phi\cos\delta}{\cos a_{,}}\right)^2 \tan a_{,} \frac{2\sin^4\frac{1}{2}P}{\sin 1''}$$

$$+ \frac{2}{3}\left(\frac{\cos\phi\cos\delta}{\cos a_{,}}\right)^3 (1 + 3\tan^2 a_{,})\, 2\frac{\sin^6\frac{1}{2}P}{\sin 1''} - \text{etc.}$$

THE ZENITH TELESCOPE.

This instrument, being employed in the next latitude problem, will now be briefly described, and the manner of determining its constants explained. Its use, as will be seen, is limited to field work, and it therefore forms no essential part of the equipment of a permanent observatory.

The instrument consists of a telescope like that of the transit, mounted at *one end* of a horizontal axis, counterpoised by a weight at the other. The telescope turns freely in altitude about this axis, which is in turn supported by a conical vertical column rising from the centre of a horizontal graduated circle, the circle resting on a small frame consisting of three legs whose feet are levelling screws.

The horizontal axis with the telescope attached turns freely in azimuth about the vertical column. the amount of such motion being indicated by a vernier sweeping over the horizontal circle. By this motion the instrument is placed in the meridian.

The setting circle is similar to the one described in connection with the transit. It is rigidly attached to the body of the telescope, and reads to single minutes of zenith distance. The *attached level*, connected with the movable vernier arm of the setting circle, being intended to *measure* as well as to *indicate* differences of inclina-

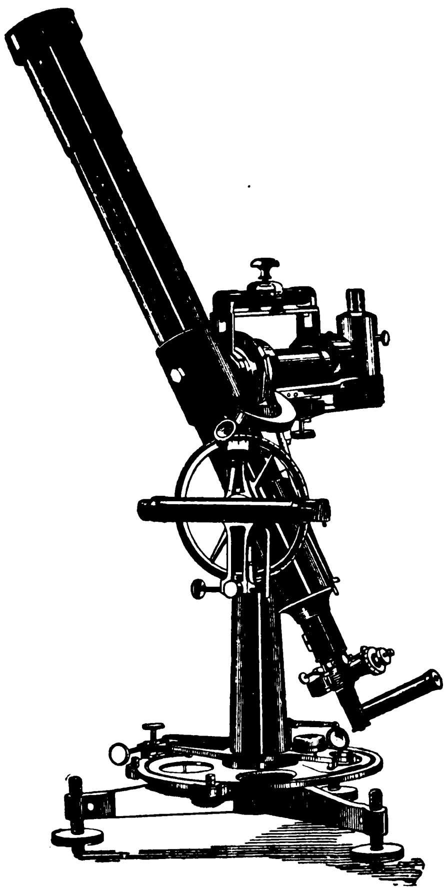


FIG. 19.—THE ZENITH TELESCOPE.

tion, is of considerable delicacy. The instrument is provided with clamp and tangent screws for both motions, also the usual adjusting screws.

The field of view presents the appearance shown in Figure 20; sometimes however the number of vertical wires is increased so that the instrument may if necessary be used as a transit. The wires are all fixed except *i k*, which can be moved up or down parallel to itself, and is called the declination micrometer wire. The comb-scale *f g* is so cut that one turn of the micrometer head carries the wire *i k* exactly from one tooth to the next, thus recording the number of whole revolutions between two positions of the wire. Hundredths of a revolution are shown on the micrometer head by a fixed index. These are called divisions. (Arrangements for illuminating the wires are the same as with the transit.)

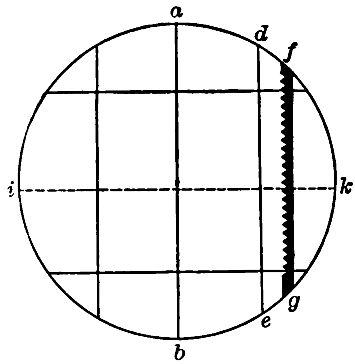


FIG. 20.

Therefore it is seen that if, when the instrument is adjusted to the meridian, two stars cross the middle wire at different times and in different places, but yet within the same field of view, we may find the difference of their meridian zenith distances by bisecting each in succession by the *movable* at the instant of its passing the *middle* wire, noting the difference of micrometer readings, and multiplying the result by the value in arc of one division of the micrometer head: or if the attached level shows the telescope to have altered its angle of elevation between the observations, thus apparently displacing the second star in the field of view, we may still correct the micrometer reading provided we know the value in arc of one division of the level.

It is therefore necessary to determine for each instrument these two constants.

The Attached Level and Declination Micrometer of the Zenith Telescope.—Since the level is neither detached, nor attached to a circle reading to seconds, neither of the modes of finding the level constant given in connection with the transit, is available. The same is true of the micrometer constant, since the micrometer wire is now parallel, not perpendicular, to the apparent path of a star at its meridian passage. With the zenith telescope the usual method of finding these two constants is to find the value of a division of the level *in terms of a revolution of the micrometer head.* Then after finding the latter (which involves the former) we may find the actual value of a division of the level in seconds of arc, as will now be explained. The formulas come at once from the astronomical triangle, remembering that at the time of a star's elongation the triangle is right-angled: ($\psi = 90°$).

Direct the telescope to a small, well-defined, distant, terrestrial object, and set the level so that the two ends of the bubble will give different readings. Bisect the object with the micrometer wire, note the reading, also that of each end of the bubble. Move the telescope and level together by the tangent screw until the bubble plays near the other end of the tube. Again bisect the mark by the micrometer wire and note all three readings as before. The mean of the number of divisions passed over by the two ends of the bubble is then the number of divisions passed over by the bubble. The difference of the micrometer readings is the run of the micrometer. Dividing the second by the first, we have the value of a division of the level in terms of a revolution of the micrometer. Take a mean of several determinations and denote it by d.

We can now find the value of one division of the micrometer. For reasons stated when treating of the R. A. micrometer, we use a circumpolar star, and at the instant that its path is perpendicular to the wire in question. *This requires us to take the star at its elongation.* Manifestly the same principles apply to the two cases, since the principal difference is that the star and wire have each been apparently shifted 90°; the motion of the star with reference to the wire not having changed. Some changes in detail are however necessary. In the first place, since the motion of the star is almost wholly in altitude, we cannot as before neglect differences in refraction between two transits. Again, since the pressure of the hand in working the micrometer head is in a direction to cause a possible disturbance of the instrument even though firmly clamped, we must read the level at every transit, and if any change has occurred, correct the micrometer readings accordingly.

As a preliminary, we must determine the time of elongation (in order to know when to begin our observations), and the setting of the instrument, *i.e.*, the azimuth and zenith distance of the star at the time of elongation. The hour angle is found from

$$\cos P_0 = \cot \delta \tan \phi,$$

from which the sidereal time of elongation is given by

$$T_0 = \alpha \pm P_0 - E, \tag{135}$$

in which α is the star's apparent R. A. for the instant, and E is the error of the chronometer. The plus sign is used for western and the minus for eastern elongations.

The azimuth is given by

$$\sin A = \frac{\cos \delta}{\cos \phi}, \tag{136}$$

and the zenith distance by

$$\cos z_0 = \frac{\sin \phi}{\sin \delta}. \tag{137}$$

Set the instrument in accordance with these coördinates 20 or 30 minutes before the time of elongation, and as soon as the star enters the field, shift the telescope if necessary so that it will pass nearly through the center.

The observations are now conducted in exactly the same manner as for the R. A. micrometer, with the addition that each end of the level bubble is read in connection with each transit.

Then, as before, each observation is compared with the one made nearest the time of elongation, T_0; the interval of time being computed from either

$$\sin i = \sin I \cos \delta, \tag{137½}$$

or

$$i = I \cos \delta,$$

according to the declination of the star. After which we have in arc (neglecting for the present differences of refraction and level),

$$R' = \frac{15\, i}{M}.$$

M being the number of micrometer revolutions or divisions between the two positions of the star, and R' the value of *one* revolution or division.

But if the reading of the level is different at the two observations, manifestly M must be corrected accordingly.

For instance, if the level shows that between the two observations the telescope had moved *with* the star in its diurnal path, then evidently the micrometer will indicate only a *part* of the angular distance between the two positions of the star, and the level correction must be *added* to the micrometer interval. Conversely, if the telescope has moved *against* the motion of the star. This level correction is found as follows : if d is the value of one division of the level in terms of a revolution of the micrometer, and L the number of divisions which the level has shifted, then Ld will be the value (in micrometer revolutions) of the correction to be applied to M. The method of finding d has already been explained.

Hence the value of R' becomes,

$$R' = \frac{15\,i}{M \pm Ld}.$$

Since, however, refraction affects the two positions of the star unequally, it is seen that $M \pm L\,d$ is only the difference of *apparent* zenith distances (*i.e.*, the instrumental difference), while 15 i being derived directly from the time interval, is the difference of *true* zenith distances. If therefore 15 i be corrected by the difference of refraction, the numerator will denote the difference of apparent zenith distance in arc, and the denominator this same difference in micrometer revolutions.

Denote by $\Delta\, r$ the difference of refraction in seconds for 1′ of zenith distance at z_0; then for $15\, i''$ it may be taken as $\frac{1}{60}\, 15\, i\, \Delta\, r$, which is the desired correction. The above formula therefore becomes, denoting the true value of a revolution by R,

$$R = \frac{15\,i - \frac{1}{60}\,15\,i\Delta r}{M \pm L\,d} = R' - \frac{R'\Delta\, r}{60}; \qquad (138)$$

$\Delta\, r$ is taken from refraction tables.

The adopted value of R should be a mean of the results from all the observations.

Having now found R, the value in arc of one division of the level is evidently

$$D = R\,d, \tag{139}$$

since d is the value in micrometer revolutions. Both constants are therefore determined.

One of the most convenient and accurate modes of employing formula (138) in practice, is as follows: Suppose the star to be approaching eastern elongation, and the micrometer readings to increase as the zenith distance decreases. Let Z_0, M_0, and L_0 be the zenith distance, micrometer, and level readings at elongation (all unknown), and Z', M', and L' the corresponding quantities at the time of any one of the recorded transits. Then remembering that in (138), $15\,i$ is the true difference of zenith distance $= Z' - Z_0$, $M = M_0 - M'$, $L = L_0 - L'$, and reserving the correction for refraction to be applied finally, we have

$$Z' - Z_0 = (M_0 - M')\,R + (L_0 - L')\,R\,d.$$

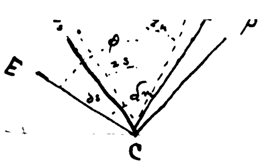
Similarly for another transit,

$$Z'' - Z_0 = (M_0 - M'')\,R + (L_0 - L'')\,R\,d.$$

Subtracting and solving,

$$R = \frac{(Z' - Z_0) - (Z'' - Z_0)}{(M'' - M') + (L'' - L')d}. \tag{140}$$

Then $Z - Z_0$ having been computed for each transit by (137½), these differences may be taken by pairs for substitution in (140), in any manner desired. For example, if forty transits have been recorded, it is usual to pair the first difference with the twenty-first, the second with the twenty-second, etc., when if the successive micrometer readings have been equidistant, the divisors will be equal, save for the slight level correction. We thus obtain twenty determinations of R, the mean of which should be corrected for refraction as shown in (138), viz.: by subtracting $\frac{R\,\Delta r}{60}$.



The preceding method of finding these two constants of the zenith telescope is regarded as the best; but provision is made in the construction of the instrument for turning the box containing the wire frame through an angle of 90°. When this is done, the declination micrometer becomes virtually a R. A. micrometer, and the value of a revolution may be found as described for that micrometer, and then the box revolved back to its proper place and clamped. In this case however the result must be in *arc*. The level constant must be found as just described.

4. Latitude by Opposite and nearly equal Meridian Zenith Distances. Talcott's Method. See Form 9.

This method depends upon the principle that the astronomical latitude of a place is equal to the declination of the zenith.

Let z_n and z_s represent the observed *meridian* zenith distances of two stars, the first north and the second south of the zenith; r_n and r_s the corresponding refractions; and δ_n and δ_s their apparent declinations. Then, ϕ denoting the latitude,

$$\phi = \delta_s + z_s + r_s, \tag{141}$$

$$\phi = \delta_n - z_n - r_n. \tag{142}$$

From which

$$\phi = \frac{\delta_s + \delta_n}{2} + \frac{z_s - z_n}{2} + \frac{r_s - r_n}{2}. \tag{143}$$

Since refraction is a direct function of the zenith distance, this equation shows that any constant error in the adopted refraction will be nearly or wholly eliminated if we select two stars which culminate at very nearly the same zenith distance, and provided also that the time between their meridian transits is so short that the refractive power of the atmosphere cannot be changed appreciably in the mean time.

Again, since absolute zenith distances are not required, but only their *difference*, if the stars are so nearly equal in altitude that a telescope directed at one, will, upon being turned around a *vertical axis* 180° in azimuth, present the other in its field of view, then manifestly the *difference* of their zenith distances may be measured directly by the *declination micrometer*, and the use of a graduated circle (with its errors of graduation, eccentricity, etc.) be entirely dispensed with, except for the purpose of a rough finder. The instrument used in this connection is called a "*Zenith Telescope.*" Its construction, and application to the end in view, are best learned from an examination of the instrument itself.

Again, since errors in the declinations will affect the resulting latitude directly, we should be very careful to employ only the *apparent declinations for the date.*

The following conditions should therefore be fulfilled in selecting the stars of a pair:

1st. They should culminate not more than 20°, or at most 25°, from the zenith.

2d. They should not differ in zenith distance by more than 15′, and for very accurate work, by not more than 10′. The field of view of the telescope is about 30′. The limit assigned prevents observations too near the edge of the field, and lessens the effect of an error in the adopted value of a turn of the micrometer head. This limit also requires a very approximate knowledge of the latitude, which may be found with the sextant, or by measuring the meridian zenith distance of a star by the zenith telescope itself.

3d. They should differ in R. A. by not less than one minute of time, to allow for reading the level and micrometer, and by not more than fifteen or twenty minutes, to avoid changes in either the instrument or the atmosphere.

Since the Ephemeris stars, whose apparent declinations are given with great accuracy for every ten days, are comparatively few in number, it becomes necessary, in order to fulfil the above conditions, to resort to the more extended star catalogues.

But since in these works only the stars' *mean* places are given, and those for the epoch of the catalogue (which fact involves reduction to apparent places for the date), and moreover since these mean places have often been inexactly determined, it becomes desirable to rest our determination of latitude on the observation of more than one pair. For example, on the "Wheeler Survey," west of the 100th meridian, the latitude of a primary station was required to be determined by not less than 35 separate and distinct pairs of stars, these observations being distributed over five nights.

Preliminary Computations.—We should therefore form a list of all stars not less than 7th magnitude which culminate not more than 25° from the zenith and within the limits of time over which we propose to extend our observations, arrange them in the order of their R. A., and from this list select our pairs in accordance with the above conditions, taking care that the time between the pairs is sufficient to permit the reading of the level and micrometer, and setting the instrument for the next pair; say *at least* two minutes.

A "Programme" must then be prepared for use at the instrument, containing the stars arranged in pairs, with the designation and magnitude of each for recognition when more than one star is in the field; their R. A., to know when to make ready for the observation; their declinations, from which are computed their approximate zenith distances; a statement whether the star is to be found north or south of the zenith, and finally the "setting" of the instrument for the pair, which is always the mean of the two zenith distances.

The declinations here used, being simply for the purpose of so pointing the instrument that the star shall appear in the field, may be mean declinations for the beginning of the year, which are found with facility as hereafter indicated. Similarly for the R. A. For this Programme, see Form 9.

Adjustment of Instrument.—The Instrument must next be prepared for use. The column is made vertical by the levelling screws, and the adjustment tested by noting whether the striding level placed on the horizontal axis will preserve its reading during a revolution of the instrument 360° in azimuth. The horizontality of the latter axis is secured by its own adjusting screws, and tested by the level in the usual way. The focus and verticality of the wires are adjusted as explained for the transit. The collimation error should, as far as is mechanically possible, be reduced to zero. This may be accomplished *approximately* by the ordinary reversals upon a terrestrial point distant not less than 5 or 6 miles (to reduce the parallax caused by the distance of the telescope from the vertical column); or very perfectly by two collimating telescopes, as explained for the transit. The instrument is adjusted to the meridian as explained for the transit. When this is perfected, one of the movable stops on the horizontal circle is moved up against one side of the clamp which controls the motion in azimuth, and there fixed by its own clamp-screw. The telescope is then turned 180° around the vertical column and again adjusted to the meridian by a circum-polar star; the other stop is then placed against the other side of the clamp, and fixed. The instrument can now be turned exactly 180° in azimuth, bringing up against the stops when in the meridian.

Observations.—The circle being set to the mean of the zenith distances of the two stars of a pair, the bubble of the attached level is brought as nearly as possible to the middle of its tube, and when the first star of the pair arrives on the middle transit wire (the instrument being in the meridian) it is bisected by the declination micrometer wire, the sidereal time noted, and the micrometer and level read. The telescope is then turned 180° in azimuth, the clamp bringing up against its stop. The same observations and records are now made for the second star. The instrument is then reset for the next pair, and so on. The time record is not necessary unless it be found that the instrument has departed from the meridian, or unless observation on the middle wire has been prevented by clouds, and it becomes desirable to observe on a side wire rather than lose the star. In these cases the hour angle is necessary to obtain the "reduction to the meridian."

The observations are recorded on Form 9 *a*. In the column of remarks should be noted any failure to observe on middle wire, weather, and any circumstance which might affect the reliability of the observations.

Reduction of Observations.—By referring to Eq. (143) the general nature of the reduction will be evident. The principal term in the value of ϕ is $\delta_n + \delta_s$, which, as before stated, must be found for *the date*. Since $z_s - z_n$ has been measured entirely by the micrometer and level, this term involves two corrections to $\delta_n + \delta_s$; $r_s - r_n$ involves another, and the very exceptional case of observation on a side wire involves another.

1st. The reduction from mean declination of the epoch of the catalogue to apparent declination of the date. Let us take the case of the B. A. C. (British Association Catalogue).

The star's *mean* place is first brought up to the beginning of the current year by the formula

$$d''' = d'' + \left(p' + \mu' + \frac{s'}{100}\frac{y}{2}\right)y,$$

in which d'' = mean north polar distance as given in catalogue, p' = annual precession in N. P. distance, s' = secular variation in same, μ' = annual proper motion in N. P. distance (all given in catalogue for each star), y = number of years from epoch of catalogue to beginning of current year, and d''' = the *mean* N. P. distance at the latter instant. To this, the corrections for precession, proper motion, nutation, and aberration, since the beginning of the year, are applied by the formula

$$d = d''' + \tau\mu' + Ac' + Bd' + Ca' + Db',$$

in which τ = fractional part of year already elapsed at date, given on pp. 285–292, Ephemeris; A, B, C, D, are the Besselian Star Numbers, given on pp. 281–284 Ephemeris for each day; a', b', c', d', are star constants, whose logarithms are given in the catalogue; and d = star's apparent N. P. distance at date. Then $\delta = 90° - d$.

The quantities a', b', c', d', are not strictly constant; indeed many of their values have changed perceptibly since 1850, the epoch of B. A. C. If it be desired to obviate this slight error, it may be done by recomputing them by formulas derived from Physical Astronomy, or, in part, by using a later catalogue. In this connection a work prepared under the "Wheeler Survey," entitled "Catalogue of Mean Declinations of 2018 Stars, Jan. 1, 1875," will be found most convenient, embracing stars between 10° and 70° N. Dec., and therefore applicable to the whole area of the U. S. exclusive of Alaska.

With this catalogue, the reductions are made directly in declination, not N. P. distance, and by the formulas,

$$\delta' = \delta'' + (p' + \mu')\,y$$

$$\delta = \delta' + \tau\,\mu' + Aa' + Bb' + Cc' + Dd',$$

in which everything relates to *declination*.

Exactly analogous formulas hold for reduction in R. A.

2d. The micrometer and level corrections to $\frac{\delta_n + \delta_s}{2}$, viz.:

$$\frac{z_s - z_n}{2}.$$

Let us suppose that, with the telescope set at a given inclination, the micrometer readings are greater as the body viewed is nearer the zenith; and in the first instance, that the inclination as shown by the attached level is not changed when the instrument is turned 180° in azimuth.

Then $\frac{z_s - z_n}{2}$ will be given *wholly* by the micrometer, and be either $\frac{m_s - m_n}{2}R$, or $\frac{m_n - m_s}{2}R$, in which m_s and m_n are the micrometer readings on the south and north stars respectively, and R the value in arc of a division of the micrometer head. Since the readings increase as the zenith distance decreases, it is manifest that $\frac{m_n - m_s}{2}R$ is *the one* of the two expressions which will represent $\frac{z_s - z_n}{2}$ with its proper sign.

But as a rule the upright column will not be truly vertical, and therefore the inclination of the optical axis of the telescope will change slightly due to the necessary revolution between the observations of the stars of a pair,—the fact being indicated by a different reading of the level. In this case, the difference of micrometer readings will not be strictly the difference of zenith distance as before, but will be that difference ± the amount the telescope has moved. The micrometer readings therefore require correction before they can give $\frac{z_s - z_n}{2}$. Since it is immaterial which star of the pair is observed first, let us suppose it to be the southern, and let l_n and l_s be the readings of the ends of the bubble. Then $\frac{l_n - l_s}{2}$ will be the reading of the level, it being graduated from the center toward each end. Now if, on turning to the north, the level shows that the angle of elevation of the telescope has increased, the micrometer reading on the northern star will be too small, by just the amount corresponding to the motion of the telescope in altitude; and this whether the star be higher or lower than the southern star. Consequently m_n must be *increased* to compensate. If l'_n and l'_s be the reading of the present north and south ends of the bubble, then the bubble reading will be $\frac{l'_n - l'_s}{2}$; the change of level, in level divisions, will be

$$\frac{l_n - l_s}{2} + \frac{l'_n - l'_s}{2}, \text{ and in arc } \frac{(l_n + l'_n) - (l_s + l'_s)}{2} D.$$

Since, upon turning to the north, the angle of elevation of the telescope was supposed to increase, this quantity is *positive;* and being the angular change of elevation, it is the correction to be applied to m_n.

If the telescope diminished its elevation on being turned to the north, it would be necessary to *diminish* m_n by the same amount. But in this case the above correction is obviously negative, and the result will be obtained by still adding it algebraically.

The correction to $\frac{m_n}{2}$ will be half the above amount; hence in *all* cases we have the rule. Subtract the sum of the south readings from the sum of the north. One-fourth the difference multiplied by the value of one division of the level, will be the level correction. The true difference of observed zenith distances of the two stars, is therefore

$$R\frac{m_n - m_s}{2} + \frac{(l_n + l'_n) - (l_s + l'_s)}{4} D.$$

3d. *The correction for refraction,* or $\frac{r_s - r_n}{2}$. Since the stars are at so small and so nearly equal zenith distances, differences of actual refractions will be practically equal to differences of mean refractions (Bar. 30 in., F. 50°), which latter may therefore be substituted for $r_s - r_n$. If $\frac{dr}{dz}$ denote the change in mean refraction for a difference of 1′ in zenith distance, then for $z_s - z_n$ (expressed in seconds) it will be $\frac{z_s - z_n}{60}\frac{dr}{dz}$. Hence we may write

$$\frac{r_s - r_n}{2} = \tfrac{1}{2}\frac{z_s - z_n}{60}\frac{dr}{dz}.$$

To determine $\frac{ar}{dz}$, we have for the equation of mean refraction Young, p. 64),

$$r = \alpha \tan z.$$

Differentiating,

$$\frac{dr}{dz} \text{ (for } 1') = \frac{\alpha \sin 1'}{\cos^2 z},$$

α being taken from refraction tables, and z representing the mean of the zenith distances of the pair. The following table of values of $\frac{dr}{dz}$ is given, in which we may interpolate at pleasure.

z	$\frac{dr}{dz}$
0°	0.0168″
5°	0.0169″
10°	0.0173″
15°	0.0180″
20°	0.0190″
25°	0.0205″

The principal term in $\frac{z_s - z_n}{2}$ is $\frac{m_n - m_s}{2} R$. Hence we may write

$$\frac{r_s - r_n}{2} = \tfrac{1}{2} \frac{m_n - m_s}{60} R \frac{dr}{dz},$$

and the correction for refraction will have the same sign as the micrometer correction.

Hence the rule: Multiply the micrometer correction in minutes by the tabular value of $\frac{dr}{dz}$, and add the result algebraically to the other corrections.

4th. The correction to the zenith distance when the observation has not been made in the meridian; *i.e.*, when not made on the middle vertical wire.

This will be an exceptional correction, but one which must occasionally be made.

If a plane be passed through the middle horizontal wire and the optical centre of the objective, it will cut from the celestial sphere a great circle; and the zenith distance of a star anywhere on this circle will, as measured by this fixed position of the instrument, be the inclination of the plane to the vertical.

Therefore, *if the zenith distance of a star between the zenith and equinoctial be measured by an instrument which moves only in the meridian, it will have its greatest value when on the meridian. For a star which crosses any other part of the meridian, the ordinary rule as to relative magnitude applies.*

But whatever the position of the star, the numerical value of *this* "reduction to the meridian," due to an observation on a side wire, is different from that heretofore discussed, where the instrument was in the vertical plane of the star; being in this case $\frac{1}{4}(15\,P)^2 \sin 1'' \sin 2\,\delta$; P being the hour angle. For the deduction of this expression, see ✠ following. For a star below the equinoctial or below the pole $\sin 2\,\delta$ would be negative; hence from the as to relative magnitudes above given, it is seen that if in using the zenith telescope, a star south of the zenith be observed on a side wire, the above correction must be added algebraically to the *observed* to obtain the *meridian* zenith distance; and north of the zenith it must be subtracted algebraically.

By inspecting the term $\frac{z_s - z_n}{2}$, we see that in any case one half this reduction, or

$$\tfrac{1}{8}(15\,P)^2 \sin 1'' \sin 2\,\delta = [6.1347]\,P^2 \sin 2\,\delta,$$

is to be added to the deduced latitude, or to the sum of the other corrections in order to obtain the latitude. The hour angle P in seconds of time is known from $P = t + E - \alpha$, t being the chronometer time of observation, E the error, and α the star's R. A. We therefore have the following complete formula for the latitude

$$\phi = \frac{\delta_s + \delta_n}{2} + R\,\frac{m_n - m_s}{2} + D\,\frac{(l_n + l'_n) - (l_s + l'_s)}{4}$$
$$+ \tfrac{1}{2}\,\frac{m_n - m_s}{60}\,R\,\frac{dr}{dz} + [6.1347]\,P^2 \sin 2\,\delta_s + [6.1347]\,P'^2 \sin 2\,\delta_n. \qquad (144)$$

For the reduction see Form 9*b*. The results of all the pairs may be discussed by Least Squares.

This method, although extremely simple in theory, involves considerable labor. It has however been employed almost exclusively on the Coast and other important Government surveys, with results which compare favorably with those obtained by the first-class instruments of a fixed observatory.

To Determine the Reduction to the Meridian for an Instrument in the Meridian.—Let S Fig. 21 be the place of the star when on a side wire. Then $C S S''$ will be the projection of the great circle cut from the celestial sphere by the plane of the middle horizontal wire and the optical center of the objective, $Z S''$ will be the recorded zenith distance $= z'$. Let $S S'$ be an arc of the star's diurnal path, preserving always the same distance from the equator.

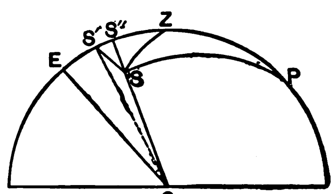


FIG. 21.

Then $Z S'$ will be the true meridian zenith distance $= z_{,}$, and $E S' = \delta$. Represent $E S''$ by δ'.

The Reduction to the Meridian, $S'\ S''$, being denoted by x, we have

$$z_{,} = z' + x, \quad \text{and} \quad \delta = \delta' - x. \qquad (a)$$

Let it now be required to develop x into a series arranged according to the ascending powers of $\sin^2 \frac{1}{2} P$, as before.

The triangle $P S S''$, right angled at S'', gives

$$\tan \delta = \cos P \tan \delta' = \tan \delta' - 2 \tan \delta' \sin^2 \tfrac{1}{2} P. \qquad (b)$$

Replacing for brevity $\sin^2 \frac{1}{2} P$ by y,

$$\tan \delta = \tan \delta' - 2\, y \tan \delta', \qquad (c)$$

$$\tan \delta = \tan (\delta' - x) = \frac{\tan \delta' - \tan x}{1 + \tan \delta' \tan x}$$

$$= \tan \delta' - 2\, y \tan \delta',$$

$$\tan \delta' - \tan x = \tan \delta' - 2\, y \tan \delta' + \tan^2 \delta' \tan x$$

$$- 2\, y \tan^2 \delta' \tan x. \qquad (d)$$

Let

$$x = A\,y + B\,y^2 + \text{etc.} \qquad (e)$$

be the undetermined development desired. If the value of tan x derived from this equation be substituted in (d), the resulting equation will be identical.

From Trigonometry,

$$\tan x = x + \frac{x^3}{3} + \text{etc.},$$

and from this and (e),

$$\tan x = A\,y + B\,y^2 + \text{etc.}$$

Substituting in (d), and transposing,

$$Ay + By^2 - 2\,y \tan \delta' + \tan^2 \delta' \,(Ay + By^2) - 2 \tan^2 \delta' (Ay + By^2) y = 0.$$

From the principles of identical equations,

$$A - 2 \tan \delta' + A \tan^2 \delta' = 0.$$

$$A = \frac{2 \tan \delta'}{1 + \tan^2 \delta'} = 2 \frac{\sin \delta'}{\cos \delta'} \cos^2 \delta' = \sin 2\,\delta'.$$

$$B + B \tan^2 \delta' - 2\,A \tan^2 \delta' = 0. \qquad B = 2 \sin^2 \delta' \sin 2\,\delta'.$$

Therefore, expressing x in seconds of arc, from (e),

$$x = \frac{\sin^2 \frac{1}{2} P \sin 2\,\delta'}{\sin 1''} + \frac{2 \sin^4 \frac{1}{2} P \sin 2\,\delta' \sin^2 \delta'}{\sin 1''}.$$

Omitting the last term as insensible, expressing P in seconds of time, and remembering that since P is very small,

$$\sin^2 \frac{15\,P}{2} = \left(\frac{15\,P}{2}\right)^2 \sin^2 1'', \text{ we have } x = \tfrac{1}{4}\,(15\,P)^2 \sin 1'' \sin 2\,\delta'.$$

In computing this term, δ may be substituted for δ'.

✠ To Determine the Probable Error of the Final Result.

From equation (143) it is seen that the probable error of a latitude deduced from a single pair of stars will be composed of two parts: 1st, the probable error of the half sum of the declinations derived from the catalogue used; 2d, the probable error of the half difference of the measured zenith distances, which may be called the error of observation.

Consider first a single pair of stars observed once. Let R_2 denote the probable error of the deduced latitude, R' that of the half sum of the declinations, and R'' that of observation, all unknown as yet. Then, Johnson,* Art. 89,

$$R_2 = \pm \sqrt{R'^2 + R''^2}, \qquad (a)$$

and for this pair observed n times, *i.e.*, on n nights,

$$R_1 = \pm \sqrt{R'^2 + \frac{R''^2}{n}}. \qquad (b)$$

If now we employ m different pairs,

$$R = \pm \sqrt{\frac{R'^2}{m} + \frac{R''^2}{n'}}, \qquad (c)$$

in which n' denotes, as before, the total number of observations.

It may be observed at this point, that as shown by (*c*), if a skilled observer be provided with a catalogue not of the first order of excellence, (R' large, R'' small), it is better to employ many pairs, rather than repeat observations on a few pairs; thus augmenting both m and n, instead of n alone.

To determine R'', form the differences between the mean of all the latitudes resulting from the first pair and the separate latitudes from that pair.

The residuals denoted by v_1', v_1'', v_1''', etc., will manifestly be free from any effect of error in the half sum of the declinations employed. Do the same with the results from each of the other pairs, giving v_2', v_2'' .... · v_3', v_3'' .... etc.,

Then, Johnson, Art. 138,

$$R'' = \pm\, 0.6745 \sqrt{\frac{\Sigma v^2}{n' - m}}. \qquad (d)$$

* Johnson's "Theory of Errors and Method of Least Squares," 1890.

The value of R'' should not exceed about 0″.8, and cannot be expected to fall below 0″.3. On the Coast Survey, its value has usually been slightly less than 0″.5.

To determine R', we have from (b)

$$R'^2 = R_,^2 - \frac{R''^2}{n}, \qquad (e)$$

in which it must be remembered that $R_,$ is the probable error of the latitude as deduced from a *single* pair of stars observed n times.

Select several (m') pairs, which are observed on an equal number of nights in order that the results from each pair may be of equal weight. Then, as before, form the differences between the mean of the n results for each pair and the mean of these m' means.

Then the mean value of $R_,$ will be, Johnsou, Art. 72,

$$R_, = 0.6745 \sqrt{\frac{\Sigma v^2}{m' - 1}}. \qquad (f)$$

Substituting this value of $R_,$ together with that of n in (e), we have R', and the probable error of the final result is given by (c), as before seen.

If R' be determined from a great number of stars taken from a single catalogue, it may be considered as constant for that catalogue. With the one employed on the Lake Survey, R' usually fell between 0″.53 and 0″.60.

If it be desired to combine the mean results from each pair according to their weights in order to obtain the weighted mean latitude, we have from (b), (since the weight of an observation is proportional to the reciprocal of the square of the probable error,)

$$p = \frac{n}{n R'^2 + R''^2},$$

p denoting the weight of the mean result from a pair observed n times.

The weighted mean latitude will be, Johnson, Art. 66,

$$\frac{\Sigma (p \phi)}{\Sigma (p)}$$

with a probable error, Johnson, Art. 72,

$$R = 0.6745 \sqrt{\frac{\Sigma\,(p\,v^2)}{(n-1)\,\Sigma p}}.$$

The errors which give rise to R' are those pertaining to the catalogue or catalogues used.

Those giving rise to R'' are due to various causes, viz.: imperfect bisection of one or both stars due to personal bias or unsteadiness of the stars, anomalous refraction, errors in determining the value of a division of the micrometer and level, changes in temperature affecting the instrument between the two observations of a pair, etc.

If any of the residuals (v) are unusually large, they should be examined by Peirce's Criterion before rejection.

Finally it must be remembered that in this, as in all other methods here given, the final result (supposed free from error) is the *astronomical* latitude, and will differ from the *geodetic* or *geographical* latitude by any abnormal deflection of the plumb-line which may exist at the station.

5. Latitude by Polaris off the Meridian. See Form 10.—This method depends upon the fact that the astronomical latitude of a place is equal to the altitude of the elevated pole.

This latter is obtained by measuring the altitude of Polaris at a given instant, and from the data thus obtained, together with the star's polar distance, passing to the altitude of the pole.

To explain this transformation:

Let P = star's hour angle, measured from the upper meridian.
a = altitude of star at instant P, corrected for refraction.
d = polar distance of star at instant P.
ϕ = latitude of place.

Then from the ZPS triangle we have

$$\sin a = \sin\phi \cos d + \cos\phi \sin d \cos P. \qquad (145)$$

This equation which applies to any star may be solved directly; but with a circum-polar star it is much simpler to take advantage of its small polar distance, and obtain a development of ϕ in terms of the ascending powers of d, in which we may neglect those terms which can be shown to be unimportant.

Now if we let $x =$ the difference in altitude between Polaris at the time of observation and the pole, we shall have

$$\phi = (a - x), \quad \sin\phi = \sin(a - x), \quad \cos\phi = \cos(a - x),$$

and from (145),

$$\begin{aligned} 1 &= \cos x\,(\cos d + \sin d \cot a \cos P) \\ &\quad - \sin x\,(\cos d \cot a - \sin d \cos P). \end{aligned} \tag{146}$$

Moreover, it is evident that if we can obtain the development of x in terms of the ascending powers of d, we will have the development of ϕ in the same terms, from $\phi = a - x$.

This is the end to be attained. Therefore let

$$x = A\,d + B\,d^2 + C\,d^3 + \text{etc.}, \tag{147}$$

be the undetermined development desired, in which A, B, C, etc., are to have such constant values, that the series, when it is convergent, shall give the true value of x, whatever may be the value of d.

It is manifest, then, that if this assumed value of x be substituted in (146), the resulting equation must be satisfied by every value of d which renders (147) convergent; that is, the resulting equation must be identical; otherwise (147) could not be true.

With a view, therefore, to this substitution, let it be noted that by the Calculus we have

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \text{etc.}, \tag{m}$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \text{etc.}, \tag{n}$$

and hence from (147),

$$\cos x = 1 - \frac{A^2\,d^2}{2} - A\,B\,d^3 + \text{etc.}, \tag{148}$$

$$\sin x = A\,d + B\,d^2 + \left(C - \frac{A^3}{6}\right) d^3 + \text{etc.} \tag{149}$$

Also,

$$\cos d = 1 - \frac{d^2}{2} + \frac{d^4}{24} - \text{etc.,} \tag{150}$$

and

$$\sin d = d - \frac{d^3}{6} + \frac{d^5}{120} - \text{etc.} \tag{151}$$

Now d is a very small angle; at present about 1° 16′, or 0.0221 radians; x can never be greater than d, and in the general case will be less. *Under these circumstances* the above series becomes very convergent, and the sum of a few terms will represent with great accuracy the sum of the series. It is for this reason that the problem under discussion is applicable only to close circum-polar stars, and therefore we take advantage of the small polar-distance of Polaris.

Substituting (148), (149), (150), and (151), in (146), we have, rejecting terms involving the 4th and higher powers of d,

$$\left.\begin{array}{l} \dfrac{A \cot a}{2} \\ + B \cos P \\ - \dfrac{\cos P \cot a}{6} \\ - \dfrac{A^2 \cos P \cot a}{2} \\ - A B \\ - \left(C - \dfrac{A^2}{6}\right) \cot a \end{array}\right\} d^3 + \left\{\begin{array}{l} A \cos P \\ - B \cot a \\ - \dfrac{A^2}{2} \\ - \dfrac{1}{2} \end{array}\right\} d^2 + \left\{\begin{array}{l} \cos P \cot a \\ - A \cot a \end{array}\right\} d = 0.$$

This equation being identical, the algebraic sum of the coefficients of each power of d must be separately equal to zero.

Hence we have by solution,

$$A = \cos P.$$

$$B = -\frac{\sin^2 P}{2} \tan a.$$

$$C = \frac{\cos P \sin^2 P}{3}.$$

Therefore, from (147),

$$x = d\cos P - \tfrac{1}{2}d^2\sin^2 P\tan a + \tfrac{1}{3}d^3\cos P\sin^2 P.$$

From (m) and (n), (150) and (151), it is seen that x and d are expressed in radians. Expressing them in seconds of arc,

$$x = d\cos P - \tfrac{1}{2}d^2\sin^2 P\tan a\sin 1'' + \tfrac{1}{3}d^3\cos P\sin^2 P\sin^2 1'', \text{ etc.},$$

which is the required development.

Therefore,

$$\phi = a - d\cos P + \tfrac{1}{2}d^2\sin 1''\sin^2 P\tan a$$
$$- \tfrac{1}{3}d^3\sin^2 1''\cos P\sin^2 P + \text{etc.} \qquad (152)$$

The last three terms are in seconds.

Hence we have the general rule:

Take a series of altitudes of Polaris at any convenient time. Note the corresponding instants by a chronometer, preferably sidereal, whose error is well determined. Correct each observed altitude for instrumental errors and refraction. Determine each hour angle by P = sidereal time − R. A.

Take from the Ephemeris the star's polar distance at the time, being careful to use pp. 302–313, where also the R. A. required above will be found.

Substitute *each set of values* in Equation (152), and reduce each set *separately*. The mean of the resulting values of ϕ is the one adopted. See Form 10.

As before stated, the method is applicable only to close circumpolar stars. Polaris is selected since it is the nearest *bright* star to the pole, a fact which is of importance in sextant observations.

On the last page of the Ephemeris are given tabular values of the correction x. They are however only approximate; and the complete solution, as given above, consumes but very little more time.

This is a very convenient method of determining latitude; our only restriction being that, with a sextant, the observations must be made at night. With the "Altazimuth" instrument, the observations may be made for some time before dark.

The last term in (152) is very small. In order to ascertain whether it is of any practical value, let us determine its maximum numerical value. Denoting the term by z, and its constant factors by c, we have

$$z = c \cos P \sin^2 P.$$

Replacing $\sin^2 P$ by $1 - \cos^2 P$, and differentiating twice, we have, after reduction,

$$\frac{dz}{dP} = 2c \sin P - 3c \sin^3 P.$$

$$\frac{d^2 z}{dP^2} = 2c \cos P - 9c \sin^2 P \cos P.$$

To obtain the maximum,

$$2c \sin P - 3c \sin^3 P = 0.$$

From the roots of this we have

$$\sin P = 0, \quad \sin P = +\sqrt{\tfrac{2}{3}}, \quad \sin P = -\sqrt{\tfrac{2}{3}},$$

the last two of which correspond to equal numerical maxima.* Hence the maximum value of the term is given when $\sin^2 P = \frac{2}{3}$, or when $z = \frac{1}{3} d^3 \sin^2 1'' \frac{2}{3} \sqrt{\frac{1}{3}}$. For $d = 1° 16'$, this gives $z = 0''.29$.

The *maximum* error committed by the omission of this term will therefore be about $0''.3$. Evidently its retention when the observations have been made with a sextant would be superfluous.

---

* With $\sin P = +\sqrt{\frac{2}{3}}$ we may have $\cos P = \pm\sqrt{\frac{1}{3}}$, and similarly for $\sin P = -\sqrt{\frac{2}{3}}$. By substituting in the second differential coefficient we see that $\pm\sqrt{\frac{2}{3}}$ with $+\sqrt{\frac{1}{3}}$ correspond to equal maxima, while $\pm\sqrt{\frac{2}{3}}$ with $-\sqrt{\frac{1}{3}}$ correspond to equal minima. With $\sin P = 0$, we may have $\cos P = \pm 1$, the former of which corresponds to a minimum and the latter to an equal maximum; viz., zero. Hence zero is a *lesser* and not the *greatest* maximum value of z; the latter, with which only we are concerned, being, from (152), $\frac{1}{3} d^3 \sin^2 1'' \frac{2}{3}\sqrt{\frac{1}{3}}$.

Fig. 22 gives the curve of values of z with P as the abscissæ, showing the inferior maximum at $P = 180°$, and the greatest maxima (numerical) at about 55°, 125°, 235°, and 305°.

The value of log sin 1″, not given in ordinary tables, is 4.6855575–10.

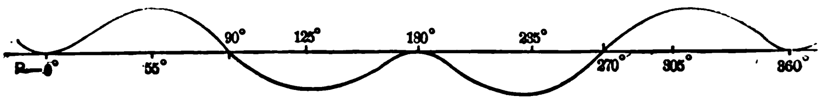


FIG. 22.

Any mistake as to the value of P will manifestly produce its greatest effect when the star is moving wholly in altitude. Hence if the chronometer error be not well determined, the times of elongation are the least advantageous for observation.

Since cos $(360° - P) = \cos P$, we may measure P from the upper meridian to 180° in *either direction.*

6. Latitude by Equal Altitudes of Two Stars. See Form 11.—By this method the latitude is found from the declinations and hour angles of two stars; the hour angles being subject to the condition that they shall correspond to equal altitudes of the stars.

Let θ and θ' = the correct sidereal times of the observations.
α and α' = the apparent right ascensions of the stars.
δ and δ' = the apparent declinations of the stars.
P and P' = the apparent hour angles of the stars.
a = the common altitude.
ϕ = the required latitude.

P and P' are given from

$$P = \theta - \alpha. \qquad P' = \theta' - \alpha'.$$

From the ZPS triangle we have

$$\sin a = \sin \phi \sin \delta + \cos \phi \cos \delta \cos P.$$

$$\sin a = \sin \phi \sin \delta' + \cos \phi \cos \delta' \cos P'.$$

Subtracting the first from the second and dividing by cos ϕ,

$$\tan \phi (\sin \delta' - \sin \delta) = \cos \delta \cos P - \cos \delta' \cos P'. \quad (153)$$

The value of tan ϕ might be derived at once from this equation, since it is the only unknown quantity entering it. The form is, however, unsuited to logarithmic computation. In order to obtain a more convenient form, observe that the second member may be written

$$\left(\frac{\cos\delta\cos P}{2} - \frac{\cos\delta'\cos P'}{2}\right) + \left(\frac{\cos\delta\cos P}{2} - \frac{\cos\delta'\cos P'}{2}\right).$$

Adding to the first parenthesis

$$\left(\frac{\cos\delta\cos P'}{2} - \frac{\cos\delta'\cos P}{2}\right),$$

and subtracting the same from the second, we have, after factoring,

$$\begin{aligned}\tan\phi\,(\sin\delta' - \sin\delta) = {} & \tfrac{1}{2}(\cos\delta - \cos\delta')(\cos P + \cos P') \\ & + \tfrac{1}{2}(\cos\delta + \cos\delta')(\cos P - \cos P').\end{aligned}$$

Solving with reference to tan ϕ, and reducing by Formulas 16, 17, and 18, Page 4, Book of Formulas,

$$\begin{aligned}\tan\phi = {} & \tan\tfrac{1}{2}(\delta' + \delta)\cos\tfrac{1}{2}(P' + P)\cos\tfrac{1}{2}(P' - P) \\ & + \cot\tfrac{1}{2}(\delta' - \delta)\sin\tfrac{1}{2}(P' + P)\sin\tfrac{1}{2}(P' - P).\end{aligned} \qquad (154)$$

The solution may be made even more simple by the use of two auxiliary quantities, m and M, such that

$$m\cos M = \cos\tfrac{1}{2}(P' - P)\tan\tfrac{1}{2}(\delta' + \delta). \qquad (155)$$

$$m\sin M = \sin\tfrac{1}{2}(P' - P)\cot\tfrac{1}{2}(\delta' - \delta) \qquad (156)$$

Then

$$\tan\phi = m\cos\left[\tfrac{1}{2}(P' + P) - M\right]. \qquad (157)$$

Equations (155) and (156) give m and M, and (157) gives ϕ, all in the simplest manner.

For example, to find M, divide (156) by (155), and we obtain

$$\tan M = \tan\tfrac{1}{2}(P' - P)\cot\tfrac{1}{2}(\delta' - \delta)\cot\tfrac{1}{2}(\delta' + \delta).$$

This admits of easy logarithmic solution.

The value of m follows from either (155) or (156), and that of ϕ from (157), both by logarithms.

The value of a does not enter; hence the resulting latitude will be entirely free from instrumental errors, those of graduation, eccentricity, and index error, and its accuracy will depend only upon the skill of the observer, and the accuracy of our assumed chronometer error and rate. Ephemeris stars should be chosen if possible, for the sake of accuracy in declinations, and their R. A. should permit the observations to be made with so short an interval that the refractive power of the atmosphere can not have changed materially in the mean time. The value of refraction is not required; it is only necessary that it remain practically constant.

Differentiating (153) with reference to ϕ, P, and P', solving, reducing by

$$\cos \delta' \sin P' = \cos a \sin A',$$

and

$$\sin \delta' = \sin \phi \sin a + \cos \phi \cos a \cos A',$$

we have, since a is the same for both stars,

$$d\phi = \cos \phi \frac{\sin A'}{\cos A' - \cos A} dP' - \cos \phi \frac{\sin A}{\cos A' - \cos A} dP,$$

from which it is seen that any error in the time or in the assumed chronometer correction will have least effect on the resulting latitude when the two stars reach the common altitude at about equal distances north and south of the prime-vertical, the nearer to the meridian the better.

When several observations with the sextant are taken in succession on each star, it is better to reduce separately the pair corresponding to each altitude.

LONGITUDE.

The difference of Astronomical Longitude between two places is the spherical angle at the celestial pole included between their respective meridians. By the principles of Spherical Geometry, the measure of this angle is the arc of the equinoctial intercepted by its sides; or it is the same portion of 360° that this arc is of the whole great circle.

But since the rotation of the earth upon its axis is perfectly uniform, the time occupied by a star on the equinoctial in passing from one meridian to another, is the same portion of the time required for a complete circuit that the angle between the meridians is of 360°, or, that the intercepted arc is of the whole great circle. Moreover, *all* stars whatever their position occupy equal times in passing from one meridian to another due to the fact that all points on a given meridian have a constant angular velocity.

The same facts apply also to the case of a body which, like the mean sun, has a proper motion, provided that motion be uniform and in the plane of, or parallel to, the equinoctial.

Hence it is that Longitude is usually expressed in *time;* and in stating the difference of longitude between two places in time, it is immaterial whether we employ sidereal or mean solar time: for the number of mean solar time units required for the mean sun to pass from one meridian to another, is exactly equal to the number of sidereal time units required for a star to pass between the meridians.

The astronomical problem of longitude consists, therefore, in determining the difference of local times, either sidereal or mean solar, which exist on two meridians at the same absolute instant.

Since there is no *natural* origin of longitudes or circle of reference as there is in case of latitude, one may be chosen arbitrarily, and which is then called the "first" or "prime meridian." Different nations have made different selections: but the one most commonly used throughout the world is the upper meridian of Greenwich, England, although in the United States frequent reference is made to the meridian of Washington.

The astronomical may differ slightly from the geodetic or geographical longitude, for reasons given under the head of latitude. In the following pages, only the former is referred to; it is usually found from the difference of time existing on the two meridians at the instant of occurrence of some event, either celestial or terrestrial. Up to about the year 1500 A.D., the only method available was the observation of Lunar Eclipses. But with the publication of Ephemerides and the introduction of improved astronomical instruments, other and better methods have superseded this one, of which the two most accurate and most generally used are the "Method by Portable Chronometers," and the "Method by Electric Telegraph." Longitude may also be found from "Lunar Culminations" and "Lunar Distances," in cases when other modes are not available.

1. By Portable Chronometers. Let A and B denote the two stations the difference of whose longitude is required. Let the chronometer error (E) be accurately determined for the chronometer time T, at one of the stations, say A ; also its daily rate (r).

Transport the chronometer to B, and let its error (E') on local time be there accurately determined for the chronometer time T'. Let i denote the interval in chronometer days between T and T'.

Then, *if r has remained constant during the journey*, the true local time at A corresponding to the chronometer time T' will be, $T' + E + ir$.

The true time at B *at the same instant* is, $T' + E'$.

Their difference = difference of Longitude is

$$\lambda = E + ir - E'. \tag{158}$$

Thus the difference of Longitude is expressed as the difference between the *simultaneous* errors of the same chronometer upon the local times of the two meridians, and the absolute indications of the chronometer do not enter except in so far as they may be required in determining i.

The rule as to signs of E and r, heretofore given, must be observed. If the result be positive, the second station is west of the first; if negative, east.

This method is used almost exclusively at sea, except in voyages of several weeks, the chronometer error on Greenwich time, and its rate, being well determined at a port whose longitude is known. Time observations are then made with a sextant whenever desired during the voyage, and the longitude found as above. The same plan may evidently be followed in expeditions on land, although extreme accuracy cannot be obtained since a chronometer's "traveling rate" is seldom exactly the same as when at rest.

In the above discussion, the rate was found only at the initial station. If the rate be determined again upon reaching the final station, and be found to have changed to r', then it will be better to employ in the above equation $\frac{r + r'}{2}$ instead of r. To redetermine the longitude of any intermediate station in accordance with this additional data, we have $x = \frac{r' - r}{i}$ = daily *change* in rate; and the accumulated error at any station, reached n days after leaving A, would be $E + \left(r + x\frac{n}{2}\right) n$, the quantity in parenthesis being the rate at the middle instant.

The above method is slightly inaccurate, since we have assumed that the chronometer rate as determined at one of the extreme stations (or *both*, if we apply the correction just explained), is its rate while en route. This is not as a rule strictly correct.

Therefore, when the difference of longitude between two places is required to be found with great precision, "Chronometric Expeditions" between the points are organized and conducted in such a manner as to determine this traveling rate.

As before,

let E	= chron.	error	on local	time	at A	at chron.	time	T.
" E'	= "	"	"	"	B	"	"	T'.
" E''	= "	"	"	"	B	"	"	T''.
" E'''	= "	"	"	"	A	"	"	T'''.

That is, the error on local time is determined at the first station for the time of departure, then at the second station for the time of arrival; again at the second station for the time of departure, and finally at the first station for the time of arrival.

Then the entire change of error is $E''' - E$. But of this $E'' - E'$ accumulated while the chronometer was at rest at the second station. The entire time consumed was $T''' - T$. But of this $T'' - T'$ was not spent in traveling. Therefore, the *traveling rate*, if it be assumed to be constant, will be

$$r = \frac{(E''' - E) - (E'' - E')}{(T''' - T) - (T'' - T')}. \tag{159}$$

This, then, is the rate to be employed in Eq. (158) instead of the stationary rate there used.

If the rate has not been constant, but, as is often the case, uniformly increasing or decreasing, the above value of r is the average rate for the whole traveling time of the two trips, whereas for use in Eq. (158), we require the average rate during the trip from A to B. This latter average will give a *perfectly correct* result provided the rate change *uniformly*. If the rate has been increasing, then r

in Eq. (159) will be too large numerically, by some quantity as x. Hence Eq. (158) becomes

$$\lambda = E + i\,(r - x) - E', \qquad (160)$$

in which r is found by (159). In order to eliminate x, let the chronometer be transported from B to A, and return; *i.e.*, take B instead of A as the initial point of a second journey. This is best accomplished by utilizing the return trip of the journey $A\ B\ A$, as the first trip of the journey $B\ A\ B$.

Then the new average rate r' having been found as before, it will, if the trips and the interval of rest have been practically equal to those of the first journey, exceed the value required, by the same quantity, x, due to the *uniformity* in the rate's change. Hence for this journey Eq. (158) becomes,

$$\lambda = E''' - [i\,(r' - x) + E'']. \qquad (161)$$

In the mean of (160) and (161), x disappears, giving,

$$\lambda = \frac{E''' + E}{2} + \frac{i\,(r - r')}{2} - \frac{E'' + E'}{2}. \qquad (162)$$

Hence, if our time observations are accurate, and the traveling rate constant, the difference of longitude between A and B may be determined by transporting the chronometer from A to B, and return. Or, if the rate be uniformly increasing or decreasing, the difference of longitude will be found by transporting the chronometer from A to B, and return, then back tc B; thus making three trips for the complete determination.

In a complete "Chronometric Expedition," however, many chronometers, sometimes 60 or 70, are used, to guard against accidental errors; and they are transported to and fro many times. As an example, in one determination of the longitude of Cambridge, Mass., with reference to Greenwich, 44 chronometers were employed and during the progress of the whole expedition, more than 400 exchanges of chronometers were made.

They are rated by comparison with the standard observatory clocks at each station, which are in turn regulated by very elaborately reduced observations on, as near as possible, the same stars.

Conducted as above described, "Chronometric Expeditions" give exceedingly accurate results, especially if corrections be made for changes in temperature during the journeys.

2. Longitude by the Electric Telegraph. See Form 12.—This method consists, in outline, in comparing the times which exist simultaneously on two meridians, by means of telegraphic signals. These signals are simply momentary "breaks" in the electric circuit connecting the stations, the instants of sending and receiving which are registered upon a chronograph at each station. Each chronograph is in circuit with a chronometer which, by breaking the circuit at regular intervals, gives a time scale upon the chronograph sheet, from which the instants of sending and receiving are read off with great precision.

Suppose a signal to be made at the eastern station (A) at the time T by the clock at A, which signal is registered at the western station (B) at the time T' by the clock at B.

Then if E and E' are the respective clock errors, each on its own local time; and if the signals were recorded *instantly* at B, then the difference of longitude would be $(T + E) - (T' + E')$. But it has been found in practice that there is always a loss of time in transmitting electric signals. Therefore in the above expression $(T' + E')$ does not correspond to the instant of sending the signal, but to a somewhat later instant. It is therefore too large, the entire expression is too small, and must be corrected by just the loss of time referred to. This is usually termed the "Retardation of Signals;" and if it be denoted by x, the *true* difference of longitude will be $(T + E) - (T' + E') + x = \lambda' + x = \lambda$. But x is unknown, and must therefore be eliminated.

In order to do this, let a signal be sent from the *western* station at the time T'' which is recorded at the eastern at the time T'''. Then if E'' and E''' are the new clock errors, the *true* difference of longitude will be

$$(T''' + E''') - (T'' + E'') - x = \lambda'' - x = \lambda.$$

By addition, x disappears, and if λ denote the longitude, we will have

$$\lambda = \frac{\lambda' + \lambda''}{2}.$$

Or, in full, assuming that the errors do not change in the interval between signals,

$$\lambda = \frac{[(T - T') + (E - E')] + [(T''' - T'') + (E - E')]}{2}. \quad (163)$$

T, T', T'', and T''' are given by the chronograph sheets; E and E' must be determined with extreme accuracy, since incorrect values will affect the resulting longitude *directly*.

Having established telegraphic communications between the two observatories (field or permanent), usually by a simple loop in an existing line, preliminaries as to number of signals, time of sending them, intervals, calls, precedence in sending, etc., are settled. At about nightfall messages are exchanged as to the suitability of the night for observations at the two stations. If suitable at *both*, each observer makes a series of star observations with the transit to find his chronometer error. The electric apparatus for this purpose, consisting of two or three galvanic cells, a break-circuit key, chronograph, and break-circuit chronometer, is arranged as shown in Fig. 23, the chronometer being placed in a separate

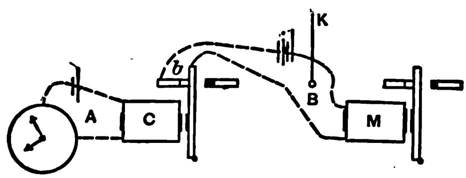


FIG. 23.

circuit with a single cell, connected with the principal circuit by a relay, to avoid the effects of too strong a current on its mechanism. The chronometer breaks the circuit A, releasing the armature of the chronometer relay, which therefore breaks circuit B at b. This releases the armature of the chronograph magnet to which is attached a pen, thus registering on the chronograph the beats of the chronometer. Circuit B may also be broken with the observing key, thus recording the transits of stars also on the chronograph. At least ten well-determined Ephemeris stars should be used—three equatorial and two circumpolar for each position of the transit.

Then as the time agreed upon for the exchange of signals approaches, the local circuit should be connected as shown in Fig. 24,

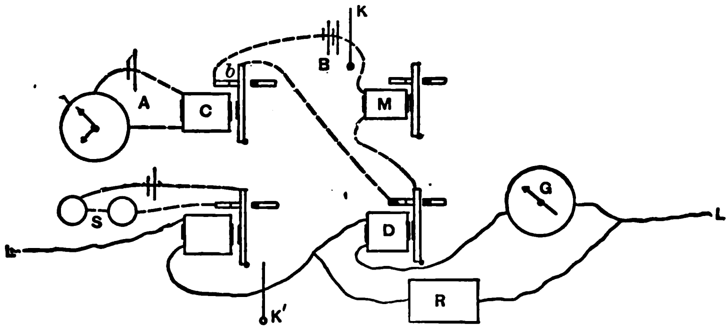


FIG. 24.

by a relay to the main line, which is worked by its own permanent batteries, and in which there is also a break-circuit key. The connections are the same at both stations. By this arrangement it is seen that *each chronograph will receive the time-record of its own chronometer; and also the record of any signals sent over the main line in either direction.*

Neither chronograph receives the record of the other's chronometer. Then at the time agreed upon, warning is sent by the station having precedence, and the signals follow according to any prearranged system. Notice being given of their completion, the second station signals in the same manner.

As an example of a system, let the break-circuit key in the main line be pressed for 2 or 3 seconds once in about ten seconds, but at irregular intervals: this being continued for five minutes will give 31 arbitrary signals from each station.

Each chronometer sheet when marked with the date, one or more references to actual chronometer time, and the error of chronometer, as soon as found, will, in connection with the sheet from the other station, afford the obvious means of finding all the quantities in Eq. (163) from which the longitude is computed. The sheets may be compared by telegraph, if desired.

The work of a single night is then completed by transit observations upon at least ten more stars under the same conditions as before, the entire series of twenty being so reduced as to give the chronometer error at the middle of the interval occupied in exchanging signals. The mode of making this reduction will be explained hereafter.

The preceding is called the method by "Arbitrary Signals," and is the one now usually employed. Sometimes however the method by "Chronometer Signals" is used, which will be readily understood by reference to Fig. 25, the connections being the same at both stations.

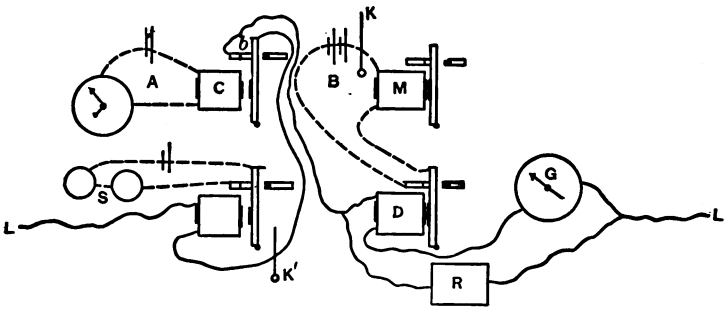


FIG. 25.

In this case it is seen that each chronometer, although in local circuit, graduates each chronograph, upon which we therefore have a direct comparison of the two time-pieces.

This method is subject to the inconvenience and possible inaccuracies in reading which may occur due to a close but not perfect coincidence in beats, unless special precautions are taken.

The arrangement of the galvanometer and rheostat, as shown in both figures (taken from the Coast Survey Report for 1880), insures the equality of the currents passing through the relays at the two stations, which point should be ascertained by exchange of telegraphic messages; therefore after the relays are properly adjusted they will be demagnetized by the signals with equal rapidity, and constant errors in this respect be avoided.

The final adopted value of the longitude should depend upon the results of at least five or six nights; outstanding errors in the electrical apparatus being nearly eliminated by an exchange between the two stations when the work is half completed.

"Longitude by the Electric Telegraph" had its origin in the

U. S. Coast Survey, and has since been employed considerably in Europe. As at first employed it consisted virtually in telegraphing to a western, the instant of a fixed star's culmination at an eastern station ; and afterwards, telegraphing to the eastern, at the instant of the same star's culmination at the western station.

In connection with Talcott's Method for Latitude, it has been used extensively in important Government Surveys, taking precedence, whenever available, over all other methods.

Reduction of the Time Observations. See Form 12*a*.—These observations, as just stated, are in two groups ; one before, and one after the exchange of signals or comparison of chronometers. From them is to be obtained the chronometer error at the epoch of exchange or comparison, which is assumed to be the middle of the interval consumed in the exchange; this latter being about 12 minutes.

Let us resume the equation of the Transit Instrument approximately in the meridian,

$$\alpha = T + E + aA + bB + C(c - .021 \cos\phi), \qquad (164)$$

and let T_0 denote the epoch, or the known chronometer time to which the observations are to be reduced. Let us suppose also, that of the three instrumental errors, a, b, and c, only b has been determined, this being found directly by reading the level for every star. The rate of the chronometer, r, is supposed to be known *approximately*, and it is to be borne in mind that E is the error at the time T. Then in the above equation E, a, and c are unknown.

Now if we denote the error *at the epoch* by E_0, we shall have

$$E = E_0 - (T_0 - T)r. \qquad (165)$$

And if E'_0 denote an *assumed approximate* value of E_0, and ϵ be the unknown error committed by this assumption, we shall have,

$$E = E'_0 + \epsilon - (T_0 - T)r. \qquad (166)$$

From which, Eq. (164) becomes

$$\epsilon + Aa + Cc + T - .021 \cos\phi\, C + E'_0 - (T_0 - T)r + Bb - \alpha = 0,$$

in which everything is known save ϵ (the correction to be applied to the assumed chronometer error at the epoch), a, and c.

Aa	is called the correction for azimuth.
Cc	" " " " collimation.
$-.021 \cos \phi\, C$	" " " " diurnal aberration.
$-(T_0 - T)r$	" " " " rate.
Bb	" " " " level.

Collecting the known terms, transposing them to the 2d member, and denoting the sum by n, we have

$$\epsilon + A\,a + Cc = n. \tag{167}$$

Each one of the twenty stars furnishes an Equation of Condition of this form, from which, by the principles of Least Squares, we form the three "Normal Equations,"

$$\Sigma\,(C)\,\epsilon + \Sigma\,(A\,C)\,a + \Sigma\,(C^2)\,c = \Sigma\,(Cn),$$
$$\Sigma\,(A)\,\epsilon + \Sigma\,(A^2)\,a + \Sigma\,(A\,C)\,c = \Sigma\,(A\,n),$$
$$\Sigma\,(1)\,\epsilon + \Sigma\,(A)\,a + \Sigma\,(C)\,c = \Sigma\,(n),$$

from a solution of which we find a, c, and the correction, ϵ, to be applied to the assumed chronometer error at the epoch.

If either c or a be known, say c, by methods given under "The Transit Instrument," then the correction for collimation for each star, $C\,c$, should be transferred to the 2d member and included in n. We then have only the two "Normal Equations,"

$$\Sigma\,(1)\,\epsilon + \Sigma\,(A\,a) = \Sigma\,(n),$$
$$\Sigma\,(A)\,\epsilon + \Sigma\,(A^2)\,a = \Sigma\,(A\,n),$$

from which to find ϵ and a.

It is to be remembered that the middle ten stars have been observed with the instrument reversed, and that such reversal changes the sign of c, and therefore of the term $C\,c$. Hence in forming the "Equations of Condition" for those stars, care should be taken to introduce this change by reversing the sign of C. The sign of c as found from the "Normal Equations" will then belong to the collimation error c of the unreversed instrument.

Also, since reversing the instrument almost invariably changes a, it is better to write a' for a in the *corresponding* "Equations of Condition," and treat a' as another unknown quantity. We will thus have four "Normal Equations" instead of three, and derive from them two values of the azimuth error, one for each position of the instrument.

Sometimes, and perhaps with even greater accuracy, the solution is modified as follows:

Independent determination of a and c are made, as explained heretofore, by the use of three stars.

Adopting these, each star gives a value of the chronometer error as per Form 1. The mean result compared with the similar mean of preceding and following nights, gives the rate. The principle of Least Squares is then applied (correcting also for rate) in the manner just detailed, to obtain the *corrections* to be applied to these values of a, c, and the mean chronometer error. With these corrected values of a and c, new values of the chronometer errors are found by direct solution (Form 1), the mean of which is adopted.

✠ **Personal Equation.**—From (163) it is seen that although errors in E and E' affect the deduced longitude directly, the effect will disappear if they are *equally* in error.

Practical observers acquire as a rule certain fixed habits of observation whereby the transits of stars are recorded habitually slightly too early or too late, thus affecting the deduced clock error correspondingly.

The difference between the result obtained by any observer and the true value is called his Absolute Personal Equation, and that between the results of two different observers their Relative Personal Equation. In Longitude work this latter should always be determined and applied to one of the clock errors, thus giving values of E and E' as though determined by a single observer, and causing them if in error at all, to be as nearly equally so as possible.

To determine this Relative Personal Equation, the two observers should, both before and after the longitude work, meet and compare as follows: one notes the transits of a star over half the wires of the instrument, and the other the transits over the remaining half. Each time of transit is then reduced to the middle wire by the

Equatorial Intervals, and the difference between their respective means will be a value of their relative personal equation. The adopted value should depend upon twenty or thirty stars, and the work be distributed over three or four nights.

Personal equation is not a constant quantity, and should be redetermined from time to time. On the Coast Survey it is largely eliminated by causing the observers to change places upon completion of half the observations for difference of longitude between the stations.

Application of Weights and Probable Error of Final Result.— The probable error of an observed star transit may be divided for practical purposes into two parts: the first, due to errors (apart from personal equation) in estimating the exact instants of the star's passage over the wires, unsteadiness of star, etc., is called the observational error; the second, called the culmination error, is due to abnormal atmospheric displacement of star, in exact determination of instrumental errors, anomalies and irregularities in the clock rate, etc. Evidently the first is the only part of the probable error which may be diminished by increasing the number of wires. It may be determined for each observer as follows:

Having made several (m) determinations of the Equatorial Intervals as before explained, let each be compared with its known value, giving for the probable error of a single determination (Johnson, Art. 72),

$$R_t = 0.6745 \sqrt{\frac{\Sigma v^2}{m-1}}. \qquad (a)$$

Since these intervals depend upon observed transits over two wires, we have for the probable error of an observed transit of an equatorial star over a single wire (Johnson, Art. 87),

$$R'' = 0.6745 \sqrt{\frac{\Sigma v^2}{2(m-1)}}. \qquad (b)$$

For any other star this will manifestly be

$$R'' \sec \delta,$$

and for N wires the probable error of the mean will be

$$\frac{R''}{\sqrt{N}}.$$

For the smaller instruments of the Coast Survey $R'' = 0^s.08$ about.

To determine the culmination error, R', for an equatorial star, let R denote the combined effect of both errors; then

$$R^2 = \frac{R''^2}{N} + R'^2. \qquad (c)$$

R may be found by comparing several (m) determinations of a star's R. A. (all reduced to the same equinox) with their mean, using the same formula as before. Multiplying the value thus found by cos δ, we have the probable error for an equatorial star. The mean result from many stars should be the adopted value of R.

For the smaller instruments of the Coast Survey $R = 0^s.06$ about.

Substituting in (c), making $N = 15$,

$$R' = 0^s.056.$$

For any other star this will evidently be R' sec δ. Hence for the probable error of the transit of an equatorial star over N, or the full number of wires,

$$R^2 = \frac{(0^s.08)^2}{N} + (0^s.056)^2; \qquad (d)$$

and for any less number of wires,

$$R_,^2 = \frac{(0^s.08)^2}{n} + (0^s.056)^2. \qquad (e)$$

Since the weights of observations are proportional to reciprocals of squares of probable errors, we have for the weight of an observation on n wires (that on the full number being taken as unity),

$$p = \frac{\frac{0.0064}{N} + 0.0032}{\frac{0.0064}{n} + 0.0032} = \frac{\frac{2}{N} + 1}{\frac{2}{n} + 1}. \qquad (f)$$

Again, from what precedes it is seen that the total probable error (R) of the transit of an equatorial star will become $R \sec \delta$ for any other. Hence different stars will have weights inversely as $\sec^2 \delta$. In practice, however, slightly different relations have been found to answer better. For the instruments above referred to, the formula

$$p' = \frac{1.6}{1.6 + \tan^2 \delta} \qquad (g)$$

has been adopted.

The report of the Chief of Engineers for 1873 gives

$$p' = \frac{1.3}{1 + 0.3 \sec^2 \delta}. \qquad (h)$$

Therefore if each Equation of Condition in the Reduction of the Time Observations be multiplied by the corresponding value of $\sqrt{p}$ (Johnson, Art. 126), it will be weighted for missed wires.

In the same way, if multiplied by $\sqrt{p'}$ it will be weighted for declination. It is, however, unusual to weight for declination when $\delta < 40°$.

The normal equations having been formed from the weighted equations of condition in the usual manner, their solution will give the chronometer error and its weight, p_e. (Johnson, Arts. 132, 133.)

The probable error of a single observation is then found by the formula, (Johnson Art. 138),

$$r = 0^s.6745 \sqrt{\frac{\Sigma v^2}{m - q}}, \qquad (i)$$

where the residuals, v, are formed from the m weighted equations of condition, and q is the number of normal equations.

The probable error of the chronometer correction as determined by a single night's work will then be

$$r_e = \frac{r}{\sqrt{p_e}}. \qquad (j)$$

Similarly we obtain p_e' for the weight of the chronometer correction at the other station, and the weight to be assigned to the resulting longitude, from the relation between weights and probable errors, will be

$$p_1 = \frac{p_e p_e'}{p_e + p_e'}. \qquad (k)$$

The weighted mean longitude as the result of m' nights' work will then be

$$\frac{\Sigma (p_1 \lambda)}{\Sigma (p_1)}, \qquad (l)$$

with a probable error

$$0^s.6745 \sqrt{\frac{\Sigma (p_1 v^2)}{(m' - 1)(\Sigma p_1)}}. \qquad (m)$$

Circumstances must, however, decide as to the relative weights to be assigned to the results of different nights. If the observations have been conducted on a uniform system, it will perhaps be better to give them all equal weight.

3. Longitude by Lunar Culminations.—The moon has a rapid motion in Right Ascension. If, therefore, we can find the local times existing on two meridians, when the moon had a certain R. A., their difference of longitude becomes known from this difference of times.

Determine the local sidereal time of transit or R. A. of the moon's bright limb, and denote it by α_l.

From pp. 385–392, Ephemeris, take out the R. A. of the *center* at the nearest Washington culmination. This ± the Sidereal Time of semi-diameter crossing the meridian, according as the east or west limb is bright, taken from same page, will give the R. A. of the bright limb, at its culmination at Washington. Denote this by α_w.

Now if an approximate longitude be *not* known, which will seldom be the case, one may be established as follows: Let $v =$ moon's change in R. A. for one hour of longitude, taken from same page of Ephemeris. Then upon the supposition that this is uniform, we will have

$$v : 1 :: \alpha_l - \alpha_w : L', \quad \text{or} \quad L' = \frac{\alpha_l - \alpha_w}{v},$$

L' being the approximate longitude from Washington, whose longitude from Greenwich is accurately known. With this value of L' take from the Ephemeris a new value of v corresponding to the mid-longitude $\frac{1}{2} L'$, and determine as before a closer approximate longitude, L''. If we are within two hours of Washington in longitude, L'' will be sufficiently close *for the purposes to which we are to apply it.* If farther away, make one or two more approximations, and call the final result L_{ap}.

L_{ap} will be true within a very few seconds of time even if the observing station be in Alaska, situated 6 hours from Washington, and even if the observations be made when the moon's irregularities in R. A. are most marked.

With the approximate longitude (and this is one of the uses to be made of this quantity, before referred to), we may now find the sidereal time required for moon's semi-diameter to cross the meridian of the place of observation by simple interpolation to 2d or 3d differences in the proper column of the same page of the Ephemeris. Denote this by T_l.

The *greatest* change in the time required for semi-diameter to cross the meridian, due to a change of one hour in longitude, is about 0.13^{sec}. Hence, even if we could possibly have made an error of 10 minutes in our determination of L_{ap}, the value of T_l can only involve an error of about $.02^{sec}$ *when at its maximum.* This would involve a maximum error of about 0.5^{sec} in the resulting longitude.

$\alpha_l \pm T_l = \alpha_c$ will then be the R. A. of the moon's *center at the instant of transit of the center.*

On Pages V to XII of the Monthly Calendar are found the

R. A. of the moon's *center* for each hour of Greenwich mean time. The problem now is to find at what instant (T_g) of Greenwich time the moon's center had the R. A. determined by our observation. This may be solved by an inverse interpolation; *i.e.,* instead of interpolating a R. A. corresponding to a given time not in the table, we are to interpolate a time to a given R. A. not in the table; and in this interpolation the use of second differences will be quite sufficient.

Therefore let T_0 and $T_0 + 1$ be the two Greenwich hours between which α_o occurs.

Let $\delta\,\alpha$ be the increase of moon's R. A. in one minute of mean time, at T_0. This is given on the same page.

Let $\delta'\,\alpha$ be the increase of $\delta\,\alpha$ in one hour. Found from same column by subtracting adjacent values of $\delta\,\alpha$.

Let α_0 be the R. A. given in the Ephemeris at T_0.

Then using second differences, we have

$$\alpha_o = \alpha_0 + \left(\delta\,\alpha + \frac{\delta'\,\alpha}{2} \cdot \frac{T_g - T_0}{3600}\right) \frac{T_g - T_0}{60}. \qquad (168)$$

In this equation $T_g - T_0$ is expressed in seconds; everything is known but *it,* and its value may be found by a solution of the quadratic. The result added to T_0 gives T_g, or the Greenwich *mean* time at which the moon's center had α_o for its R. A. Convert this into Greenwich *sidereal* time, call the result α_g, and our longitude is known from

$$\lambda = \alpha_g - \alpha_o. \qquad (169)$$

The preceding is the method to be followed where there is but a single station.

Imperfections in the Lunar Tables from which the Ephemeris is computed, render the tabular R. A. liable to slight errors. Therefore from Equation (168) our values of T_g and hence α_g may be incorrect from this cause, giving from Equation (169) an incorrect longitude.

Differences between two tabular values are, however, nearly correct.

Hence it is more accurate to have corresponding observations of the moon's transit on the same day taken at a station whose longitude is known.

Its longitude, *found as above,* will be

$$\lambda' = \alpha_g' - \alpha_o' \,;$$

and the difference of longitude between the two stations,

$$\lambda' - \lambda = (\alpha_g' - \alpha_g) - (\alpha_o' - \alpha^o),$$

inaccuracies of the Ephemeris being nearly eliminated in the difference $(\alpha_g' - \alpha_g)$.

No method of determining longitude by Lunar Culminations is sufficiently accurate for a fixed observatory. It may however be used in surveys and expeditions where telegraphic connection with a known meridian can not be secured. Even with the appliances of a fixed observatory, the mean of several determinations is sometimes subsequently found to be in error by from 4 to 6 seconds of time (Madras Observatory). Dependence should not therefore be placed upon a single observation, but the operation should be repeated upon each limb as many times as may seem desirable. The longitude derived from any determination may be employed as the *approximate* longitude required in any subsequent determination.

Before proceeding to any details as to the observations and reductions, it is well to note the effect of errors in either, upon our result. The main outline of the problem consists in determining the moon's *R. A.* at a certain instant, and then ascertaining from the Ephemeris the Greenwich time of the same instant. Both the moon's *R. A.* and the *instant* are denoted, at the place of observation, by $\alpha_o = \alpha_l \pm T_l$. α_l depends very largely upon accuracy of observation and reduction. T_l depends upon *interpolation* with an *approximate longitude.* As shown before, no error of *assumed longitude* that could ever occur in practice, would have any appreciable effect on T_l. If the *interpolation* be properly performed, T_l can involve only very slight errors. But whatever they may be, they enter with full effect in α_o, and when the final operation is performed to determine the corresponding Greenwich time, an inspection of the tables will show that any error in α_o is increased from

20 to 30 times in the resulting longitude. In this way, as before shown, an error of $.02^s$ in T_l is amplified into $.5^s$ in the result.

Errors in α_l affect α_o, and therefore the result, in the same manner; hence we see that considerable care is necessary in both observation and reduction. At the very best, the result is liable to be in error from 1 to 3 seconds. In latitude of West Point, 1 second of time = 1142 feet in longitude.

Observations and Reductions.—The transit instrument is supposed to be pretty accurately adjusted to the meridian, and the outstanding small errors a, b, and c, measured. The rate of the sidereal chronometer is also supposed to be known.

Note the chronometer time of transit of the moon's bright limb over each wire of the instrument. In this case, as with a star, the time of culmination is found by reducing the observations to the middle wire and then correcting for the three instrumental errors. See Form 1. But in case of the moon these reductions and corrections take a somewhat modified form due to the two facts that the moon has a proper motion in R. A., and also a very sensible parallax in R. A. when on a side wire. Hence (see note following) we have $\frac{\Sigma i}{n} F$ instead of $\frac{\Sigma i}{n} \sec \delta'$, for the reduction to the middle wire, and $(A a + B b + C c') F \cos \delta'$ instead of $A a + B b + C c$, for the instrumental correction; and the Equation of the Transit Instrument as applied to this case becomes,

$$\alpha_l = \frac{\Sigma T}{n} + \frac{\Sigma i}{n} F + E + (A a + B b + C c') F \cos \delta'. \quad (170)$$

In this equation ΣT is the sum of the observed times, n the number of wires used, Σi the sum of their equatorial intervals, δ' the moon's declination as seen, *i.e.*, as affected by parallax, and

$$F = [1 - \rho \sin \pi \cos (\phi' - \delta)] \sec \delta \, \frac{60.1643}{60.1643 - (\delta \alpha)},$$

ρ being the earth's radius at place of observation in terms of the equatorial radius, π the moon's equatorial horizontal parallax, ϕ' the geocentric latitude, $(\delta \alpha)$ as already stated, and δ the moon's geocentric declination. These quantities must be found before the reduction can be made. The mode of finding ρ and ϕ' has already been explained. To find π, δ, and $(\delta\,\alpha)$, note in addition to the transit of the moon's limb that of one or more stars at about the same altitude, and which culminate within a few minutes of the moon. The difference between the times of passing the middle wire applied to the star's R. A. will give an approximate value of α_l, from which an approximate longitude is determined as before explained. With this, π may be taken from page IV, and δ and $(\delta\,\alpha)$ from pp. V to XII, Monthly Calendar. F thus becomes known. Evidently

$$\delta' = \delta - \rho\,\pi \sin(\phi' - \delta)$$

with sufficient accuracy, and the computation of α_l can now be made.

One of the greatest inaccuracies to be apprehended is a failure to determine a very exact value of E *for the instant of transit.* This quantity may be eliminated, or very nearly so, as follows:

If two or more fundamental stars, those whose places have been established with the highest degree of accuracy, be selected so that the mean of the times of their transits shall be very closely the time of transit of the moon's limb, then the mean of their equations will be, corresponding to a mean star,

$$\alpha_s = \frac{\Sigma\,T_s}{n} + E_s + \frac{\Sigma\,i}{n} \sec\delta_s + (A\,a + B\,b + C\,c')_s. \qquad (171)$$

Subtracting from Eq. (170), since E and E_s denote errors at almost the same instant, we have

$$\alpha_l = \alpha_s + \frac{\Sigma\,T}{n} - \frac{\Sigma\,T_s}{n} + \frac{\Sigma\,i}{n}F - \frac{\Sigma\,i}{n}\sec\delta_s + \text{etc.}, \qquad (172)$$

in which E has disappeared.

If E and E_s differ, their difference will be simply the *change* of error in, for example, ten minutes, which can be accurately allowed for by the chronometer's well-established rate. Moreover, if the stars be selected so that their declinations differ but slightly from that of the moon, it is evident that the last terms of Eqs. (170) and (171) will be nearly the same, and that their difference in Eq. (172) will be a minimum. See expressions for A, B, and C, in connection with Form 1.

By this method, therefore, the R. A. of the moon's limb, α_l, is, from Eq. (172), made to depend very largely upon the R. A. of *fundamental stars;* instrumental and clock errors being reduced to a minimum of effect.

The stars should be selected from the Ephemeris in accordance with the above conditions, and observed in connection with the moon.

✠ *To deduce Equation* (170).

In the Equation of the Transit instrument, the quantities $\frac{\Sigma i}{n} \sec \delta$ (embraced in T), and $(Aa + Bb + Cc')$ denote respectively the times required for a star whose declination is δ to pass from the mean to the middle wire and from the middle wire to the meridian. In the case of the moon these intervals (or hour-angles) require modification, both on account of parallax and proper motion.

The Ephemeris values of R. A. and Declination are given for an observer at the earth's center; but on account of our proximity to the moon, an observer on the surface always sees that body displaced in a vertical circle, which results in a displacement or parallax both in declination and (unless the body be on the meridian) R. A. Hence it is that when the moon's limb appears tangent to a side wire as at M', Fig. 26, it is in reality at M. Therefore the

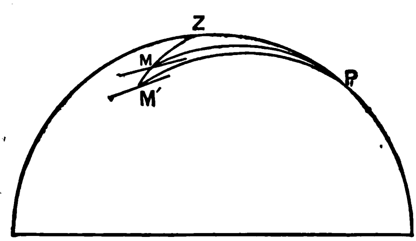


FIG. 26.

apparent hour-angle ZPM' requires a correction to reduce it to the true hour-angle ZPM, and the result is to be further modified due to the moon's own motion in R. A. The following is based on the method given by Chauvenet.

To deduce the relation between the true and apparent hour-angles, let them be represented respectively by P and P', the corresponding zenith distances by z and z', and the declinations by δ and δ', Z being the geocentric zenith.

Then

$$\sin P \quad : \sin A :: \sin z \quad : \cos \delta,$$

$$\sin P' \quad : \sin A :: \sin z' : \cos \delta',$$

$$\frac{\sin P}{\sin P'} : 1 :: \frac{\sin z}{\sin z'} : \frac{\cos \delta}{\cos \delta'},$$

$$\sin P = \sin P' \frac{\sin z}{\sin z'} \frac{\cos \delta'}{\cos \delta}.$$

Or, since P and P' are very small when the limb is on a side wire, we have, expressing them both in seconds,

$$P = P' \frac{\sin z}{\sin z'} \frac{\cos \delta'}{\cos \delta}.$$

P is the time which the limb with an hour-angle P' would require to reach the meridian if the moon had no proper motion. The *actual* interval is greater than P on account of the moon's continual motion eastward or increase in R. A., resulting in a retardation of its apparent diurnal motion.

To determine this, the Ephemeris gives at intervals of one hour the moon's motion in seconds of R. A. in one mean solar minute $= \delta \alpha$. One m. s. minute $= 60 \times 1.002738 = 60.1643$ sidereal seconds. Hence in one sidereal second the moon moves $\dfrac{\delta \alpha}{60.1643}$ seconds eastward, and therefore its apparent diurnal motion westward is only $1 - \dfrac{\delta \alpha}{60.1643}$ in the same interval. In other words, this is the apparent rate of the moon in diurnal motion at the instant considered. Denote it by R. Then the time required to traverse the true hour angle P (or the apparent, P'), will be

$$P' \frac{\sin z}{\sin z'} \frac{\cos \delta'}{\cos \delta} \frac{1}{R}.$$

When the limb is on the mean of the wires, the apparent hour angle, P', from the middle wire becomes $\frac{\Sigma i}{n}$ sec δ' (since δ', not δ, is the declination of the point as observed), and when on the middle wire P' becomes $[a \sin(\phi - \delta') + b \cos(\phi - \delta') + c'] \sec \delta'$.

Hence to pass from the mean of the wires to the meridian requires

$$\left[\frac{\Sigma i}{n} \sec \delta' + \left(a \sin(\phi - \delta') + b \cos(\phi - \delta') + c'\right) \sec \delta'\right]$$

$$\times \frac{\sin z}{\sin z'} \frac{\cos \delta'}{\cos \delta} \frac{1}{R} = \frac{\Sigma i}{n} \frac{\sin z}{\sin z'} \frac{1}{\cos \delta} \frac{1}{R} + (Aa + Bb + Cc')$$

$$\times \cos \delta' \frac{\sin z}{\sin z'} \frac{1}{\cos \delta} \frac{1}{R}.$$

Placing $\frac{\sin z}{\sin z'} \sec \delta \frac{1}{R} = F$, the Equation of the Transit instrument as applied to the moon, becomes, designating the R. A. of the limb by α_l,

$$\alpha_l = \frac{\Sigma T}{n} + E + \frac{\Sigma i}{n} F + (Aa + Bb + Cc') F \cos \delta'. \quad (170)$$

For purposes of computation the value of F may be simplified by expressing $\frac{\sin z}{\sin z'}$, in terms of quantities given in the Ephemeris. Let π = moon's equatorial horizontal parallax, p the parallax in altitude, and ρ as heretofore.

Then

$$\frac{\sin z}{\sin z'} = \frac{\sin (z' - p)}{\sin z'} = \frac{\sin z' \cos p - \cos z' \sin p}{\sin z'} =$$

$$\cos p - \cos z' \rho \sin \pi ;$$

$$\text{since} \quad \sin p = \rho \sin \pi \sin z'.$$

Expanding $\cos z' = \cos (z + p)$, placing $\sin^2 p = 0$ and $\cos z = \cos (\phi' - \delta)$, we have

$$\frac{\sin z}{\sin z'} = 1 - \rho \sin \pi \cos (\phi' - \delta)$$

and

$$F = [1 - \rho \sin \pi \cos (\phi' - \delta)] \sec \delta \frac{1}{R}.$$

Evidently we may also write,

$$\delta' = \delta - \rho \pi \sin (\phi' - \delta).$$

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4. Longitude by Lunar Distances.—On pp. XIII to XVIII of the Monthly Calendar in the Ephemeris are found the true or geocentric distances of the moon's center from certain fixed stars, planets, and the sun's center, at intervals of 3 hours Greenwich mean time. If then an observer on any other meridian determine by observation one of these distances, and note the local mean time at the instant, he can by interpolation determine the *Greenwich* mean time when the moon had this distance, and hence the longitude from the difference of times.

The planets employed are Venus, Mars, Jupiter, and Saturn, and the fixed stars, known as the 9 lunar-distance stars, are α Arietis (Hamal), α Tauri (Aldebaran), β Geminorum (Pollux), α Leonis (Regulus), α Virginis (Spica), α Scorpii (Antares), α Aquilæ (Altair), α Piscis Australis (Fomalhaut), and α Pegasi (Markab). From this list the object is so selected that the observed distance shall not be much less than 45°, although a less distance may be used if necessary.

The distance *observed* is that of the moon's bright limb from a star, from the estimated center of a planet, or from the *nearest* limb of the sun. If the sextant telescope be sufficiently powerful to give a well-defined disc, we may measure to the nearest limb of the planet, and treat the observation as in the case of the sun.

Thus in Fig. 27, letting Z represent the observer's zenith, and C'

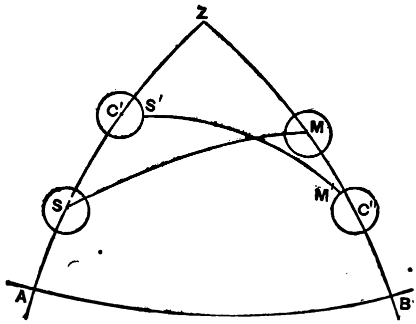


FIG. 27.

and C'' the observed places of the sun and moon respectively, the distance measured is $S' M'$, from limb to limb.

The effect of refraction is to make an object appear too high, and that of parallax, too low. In the case of the sun the former outweighs the latter. In the case of the moon the reverse is true. Hence the *true* or *geocentric* places of the two bodies would be represented relatively by S and M, and the distance $S M$, from center to center, is the one desired.

The outline of the method is as follows:

Having measured the distance $S' M'$, and corrected it for the two semi-diameters; and having also measured the altitudes of the two lower limbs and corrected them for the respective semi-diameters, we have in the triangle $Z C' C''$ the three sides given, from which we find the angle at Z. Then having corrected the observed altitudes for refraction, semi-diameter and parallax, we have in the triangle $Z S M$, two sides and the included angle Z, to compute the opposite side $S M$.

Before proceeding to the more definite solution, three points should be noticed.

1st. The semi-diameter of the moon as seen from the surface of the earth is greater than it would appear if measured from the center of the earth, due to its less distance. Hence $C'' M'$ is an "*augmented semi-diameter*," and must be treated accordingly. The augmentation in case of the sun is insignificant.

2d. Since refraction increases with the zenith distance, the refraction for the center of the sun or moon will be greater than that for the upper limb, and that of the lower limb will be greater than that of the center. The apparent distance of the limbs is therefore diminished, and the whole disc, instead of being circular, presents an oval figure, whose vertical diameter is the least, and horizontal diameter the greatest, as shown in Fig. 28. Therefore if $c\,d$ denote the direction of the measured distance, the assumed semi-diameter, $c\,f$, will be in excess by the amount $e\,f$, and must be corrected accordingly. This correction becomes of importance if the altitude of either sun or moon be less than 50° at the moment of observation.

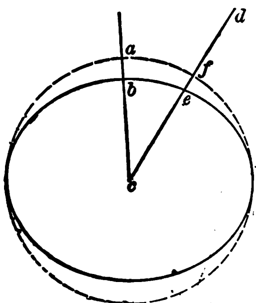


FIG. 28.

3d. Since the vertical line at the station does not in general pass through the earth's center, but intersects the axis at a point R. (see Fig. 17), it is most convenient to reduce our observations at first to the point R, regarding the earth as a sphere with $R\ O$ as a radius, and then to apply the small correction due to the distance $C\,R$, in order to pass to the true or geocentric quantities.

In the following explanation, the body whose distance from the moon is measured is taken to be the sun. The result will then apply equally to a planet if its limb be considered; if its center be considered, the expression for its semi-diameter becomes zero. If the body be a fixed star, the expressions for its semi-diameter and parallax become zero.

Let h'' = measured altitude of moon's lower limb, corrected for sextant errors.

H'' = measured altitude of sun's lower limb, also corrected.

d'' = measured distance between moon's bright limb and nearest limb of sun, also corrected.

T = local mean solar time at instant of measuring d''.

L' = an assumed approximate longitude.

ϕ = latitude.

Note the readings of the barometer and of the attached and external thermometers.

With T and L', take from the Ephemeris the following quantities:

s = geocentric semi-diameter of moon.
π = equatorial horizontal parallax of moon.
δ = geocentric declination.
S = semi-diameter of sun.
D = geocentric declination of sun.
P = equatorial horizontal parallax of sun.

The first two are obtained from page IV, monthly calendar, or pages 385 to 393 Ephemeris.

The third from pages V to XII, monthly calendar, or pages 385 to 393 Ephemeris.

The fourth and fifth from page I, monthly calendar, or from pages 377 to 385 Ephemeris.

The sixth from page 278 Ephemeris.

We must now correct d'' for both semi-diameters, augmented in case of the moon. Therefore with $h'' + s$ and s as arguments, enter the proper table and take out the amount of augmentation. In the absence of tables this may be computed by the formula,

$$\text{Augmentation} = k\,s^2 \sin(h'' + s) + \tfrac{1}{2}\,k^2\,s^3 + \tfrac{1}{2}\,k^2\,s^3 \sin^2(h'' + s);$$

in which $\log k = 5.25020 - 10$, and s is expressed in seconds. (For deduction of this series see Note 1.)

Add this correction to s and we have s' = moon's semi-diameter as seen from point of observation.

We now have (neglecting the distortion of discs), the following values of the observed quantities reduced to the centers of the observed bodies, viz.:

$$d' = d'' + s' + S. \qquad h' = h'' + s'. \qquad H' = H'' + S.$$

Using these quantities we may now find the correction due to distortion of discs (or refractive distortion), as follows: From tables of mean refraction take out the refractions corresponding to the altitude $(h' + s')$ of the upper limb, to that $(h' - s')$ of the lower, and that (h') of the center. The difference between the latter and each of the other two gives very nearly the contraction of the upper and lower semi-diameters of the moon. This may be repeated once if the refractions are very great due to a small altitude. The mean of the two is the contraction of the vertical semi-diameter due to refraction. Denote it by Δs, and the same quantity in case of the sun by ΔS.

These quantities are represented by $a\,b$ in Fig. 28, and from them we are to find $e\,f$, or the distortion in the direction of d''. This is found to vary very nearly as $\cos^2 q$, q being the angle which d'' makes with the vertical. (See Note 2.)

The values of q, or Q in case of the sun, will be found from the three sides of the triangle $Z\,C'\,C''$, Fig. 27. Their values, page 6, Book of Formulas, will be, if $m = \frac{1}{2}(d' + h' + H')$.

$$\sin^2 \tfrac{1}{2} q = \frac{\cos m \sin (m - H')}{\sin d' \cos h'}, \quad \sin^2 \tfrac{1}{2} Q = \frac{\cos m \sin (m - h')}{\sin d' \cos H'}.$$

And the refractive distortions will be, from the above,

$$\Delta s \cos^2 q, \text{ and } \Delta S \cos^2 Q.$$

Hence the fully corrected values of the measured quantities are

$$d' = d'' + (s' - \Delta s \cos^2 q) + (S - \Delta S \cos^2 Q),$$

$$h' = h'' + s' - \Delta s, \qquad H' = H'' + S - \Delta S.$$

We now have the distance (d'), between the centers and the altitudes of the centers (h' and H'), as these quantities would have been had we been able to measure them directly. We must now ascertain what they would have been had we measured them at the center of the earth; or, as a first step, had we measured them at the point R.

This is necessary, because the earth not being a perfect sphere, the transference of an observer to the center would not displace a body (apparently) toward the astronomical, but toward the geocentric zenith, and the angle at Z, Fig. 27, would no longer be common to the two triangles. But by regarding the earth as a sphere with radius $O\,R$, Fig. 17, the two zeniths will coincide, and the reduction therefore be easily made. Afterward a correction is to be applied due to a transference of the observer from R to C.

Therefore let $H_{,}$, $h_{,}$, and $d_{,}$, be the values of H', h', and d', when referred to R, and let $r_{,}$ and r be the actual refractions for

H' and h'. It will be shown in Note 3 that π (the angle subtended by the equatorial radius at the distance of the moon) is to $\pi_{,}$, the angle subtended by $O\,R$, as a, the equatorial radius, is to $\frac{a}{\rho}$. Therefore $\pi_{,}$ the parallax at R, equals $\frac{\pi}{\rho}$. On account of the greater distance of the sun, P will be practically the same for R as for C.

Therefore, Art. 83, Young,

$$h_{,} = h' - r + \pi_{,} \cos (h' - r)$$

$$H_{,} = H' - r_{,} + P \cos (H' - r_{,})$$

In order to find $d_{,}$, let S and M (Fig. 29) represent the places

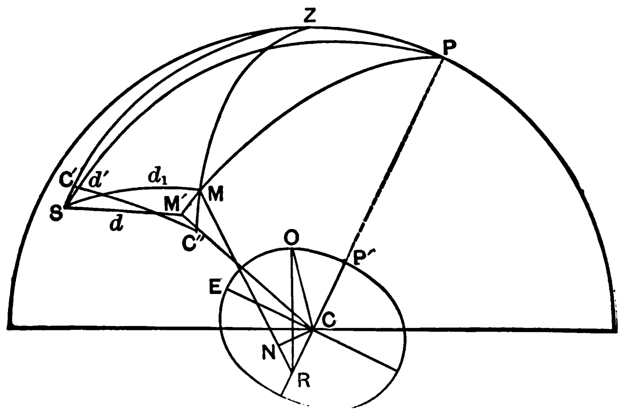


FIG. 29.

of the sun and moon as seen from the point R without refraction, given by $H_{,}$, $h_{,}$, and $d_{,}$; and C' and C'' the places as observed, (given by H', h' and d').

Then in triangle $Z\,C'\,C''$,

$$\cos Z = \frac{\cos d' - \sin h' \sin H'}{\cos h' \cos H'}.$$ Page 6, Book of Formulas.

From triangle $Z S M$,

$$\cos Z = \frac{\cos d_{,} - \sin h_{,} \sin H_{,}}{\cos h_{,} \cos H_{,}}.$$ Page 6, Book of Formulas.

Equating these two values of $\cos Z$, adding unity to both members and reducing,

$$\frac{\cos d' + \cos (h' + H')}{\cos h' \cos H'} = \frac{\cos d_{,} + \cos (h_{,} + H_{,})}{\cos h_{,} \cos H_{,}}.$$ Page 4.

Make $m = \frac{1}{2}(h' + H' + d')$, whence $\cos(h' + H') = \cos(2m - d')$. Substituting in the preceding equation, reducing the first member by formulas 4, page 4, 11, page 2, and 13, page 1, and the second member by formulas 9 and 10, page 2, we have

$$\frac{\cos m \cos (m - d')}{\cos h' \cos H'} = \frac{\cos^2 \frac{1}{2}(h_{,} + H_{,}) - \sin^2 \frac{1}{2} d_{,}}{\cos h_{,} \cos H_{,}}.$$

Whence,

$$\sin^2 \tfrac{1}{2} d_{,} = \cos^2 \tfrac{1}{2}(h_{,} + H_{,}) - \frac{\cos h_{,} \cos H_{,}}{\cos h' \cos H'} \cos m \cos (m - d').$$

This may be placed in a more convenient form by assuming

$$\frac{\cos h_{,} \cos H_{,}}{\cos h' \cos H'} \; \frac{\cos m \cos (m - d')}{\cos^2 \frac{1}{2}(h_{,} + H_{,})} = \sin^2 M.$$

Whence $$\sin \tfrac{1}{2} d_{,} = \cos \tfrac{1}{2}(h_{,} + H_{,}) \cos M.$$

We now have the distance between the centers as it would have been without refraction, if measured from the point R. This is represented by the line $S M$. (Fig. 29.)

The transference of the observer to the center will, since this motion lies wholly in the plane $P M R$, have the effect of apparently diminishing the declination of the moon, causing it to appear at M', while the position of S will not be sensibly changed.

It will be shown in Note 4 that the correction to be added to $d_{,}\,(S\,M)$ to give $d\,(S\,M')$ is

$$\frac{\pi\, e^2 \sin\phi}{\sqrt{1 - e^2 \sin^2\phi}}\left(\frac{\sin D}{\sin d_{,}} - \frac{\sin\delta}{\tan d_{,}}\right) = \pi\, i\left(\frac{\sin D}{\sin d_{,}} - \frac{\sin\delta}{\tan d_{,}}\right).$$

e being the eccentricity of the meridian = 0.0816967.

Hence we have finally, denoting the *geocentric* distance between centers by d,

$$d = d_{,} + \pi\, i\left(\frac{\sin D}{\sin d_{,}} - \frac{\sin\delta}{\tan d_{,}}\right).$$

This operation of finding d from the observed quantities is called "Clearing the Distance."

It is now necessary to find the Greenwich mean time when the moon and sun were separated by the distance d. For this purpose enter the Ephemeris at the pages before referred to, and find therein two distances between which d falls. Take out the nearer of these and the Greenwich hours at the head of the same column. Then if Δ denote the difference between the two distances, and Δ' the difference between the nearer one and d, both in seconds, we shall have, using only first differences, for the correction, t, to be applied to the tabular time taken out,

$$\Delta : 3^h :: \Delta' : t^h \mathrel{.\,\dot{}\,.} t^h = \frac{3^h}{\Delta}\,\Delta'.$$

Or $$\log t^h = \log\frac{3^h}{\Delta} + \log\Delta'.$$

Or in seconds, $$\log t^s = \log\frac{10800^s}{\Delta} + \log\Delta'.$$

The logarithms of $\dfrac{10800}{\Delta}$ are given in the columns headed "P. L. of Diff." (Proportional Logarithm of Difference.) Hence we have simply to add the common logarithm of Δ' in seconds to the proportional logarithm of the table to obtain the common logarithm of the correction in seconds of time.

To take account of second differences, take the difference between the preceding and following proportional logarithms. With this and t as arguments enter table 1, Appendix to Ephemeris, and take out the corresponding seconds, which are to be added to the time before found when the proportional logarithms are decreasing, and subtracted when they are increasing.

Denote the final result by T_g, and the difference of longitude by λ. Then

$$\lambda = T_g - T. \tag{173}$$

The mode given above for clearing the distance is quite exact, but somewhat laborious. There are, however, several approximative solutions, readily understood from the foregoing, which may be employed where an accurate result is not required, and which may be found in any work on Navigation.

The method by "Lunar Distances" is of great use in long voyages at sea or in expeditions by land, where no meridian instruments are available, and when the rate of the chronometers can no longer be relied upon.

It is important to note that if T in Eq. (173), denote the *chronometer* time of observation, instead of the *true* local time, $T_g - T$ will be the error of the chronometer on Greenwich time. In this way chronometers may be "checked." If, however, T denote the true local time, obtained by applying the error on local time to the chronometer time, then the same equation gives the longitude.

Observations.—It is necessary that h'', H'', and d'' should correspond to the same instant T. Hence observe the following order in making observations. Take an altitude of the sun's limb, then an altitude of the moon's limb, then the distance, carefully noting the time, then an altitude of the moon's limb, then an altitude of the sun's limb. A mean of the respective altitudes of the two limbs will give very nearly the altitudes at the instant of measuring the distance.

For greater accuracy, several measurements of the distance may be made, and the mean adopted. Also, when possible, at least two stars should be used on opposite sides of the moon, for the purpose of eliminating instrumental errors.

The accuracy of the result will depend upon the observer's skill with the sextant, and mode of reduction followed.

✠ 1. **To Find Augmentation of Moon's Semi-diameter.**—In determining the augmentation of the moon's semi-diameter due to its altitude, the ellipticity of the earth is practically insensible. Therefore (Young, p. 62), denoting the altitude of the center $(h'' + s)$ by h', the parallax in altitude by p, and the augmented semi-diameter by s',

$$s' = s \frac{\cos h'}{\cos (h' + p)}.$$

$$\text{Augmentation} = G = s' - s = s \left(\frac{\cos h' - \cos (h' + p)}{\cos (h' + p)} \right).$$

By page 4, Book of Formulas,

$$\cos h' - \cos (h' + p) = 2 \sin \tfrac{1}{2} (h' + h' + p) \sin \tfrac{1}{2} p.$$

$$G = \frac{2\,s}{\cos (h' + p)} \sin (h' + \tfrac{1}{2} p) \sin \tfrac{1}{2} p.$$

Expanding the sine and cosine of the sums, writing $\tfrac{1}{2} p$ for $\sin \tfrac{1}{2} p$, and unity for $\cos \tfrac{1}{2} p$, we have

$$G = \frac{s\,p\,(\sin h' + \tfrac{1}{2} p \cos h')}{\cos h' - p \sin h'};$$

$p = \pi \cos h'$ (Young, p. 61).

According to the Tables of the Moon, the relation between π and s is constant, such that

$$\pi = 3.6697\, s.$$

Hence $p = 3.6697\, s \cos h'$.

Designating this numeral by k',

$$G = \frac{k'\, s^2 (\sin h' + \tfrac{1}{2} k'\, s \cos^2 h')}{1 - k'\, s \sin h'}$$

By division,

$$G = k'\, s^2 \sin h' + \tfrac{1}{2} k'^2\, s^3 + \tfrac{1}{2} k'^2\, s^3 \sin^2 h' + \text{etc.}$$

Multiplying by sin 1″ to reduce G to seconds,

$$G = k\, s^2 \sin h' + \tfrac{1}{2} k^2 s^3 + \tfrac{1}{2} k^2 s^3 \sin^2 h' + \text{etc.}, \qquad (174)$$

in which $\log k = 5.2502 - 10$.

✠ 2. **To Deduce the Law of Refractive Distortion.**—In Fig. 28, let h' denote the altitude of the center, and h'' that of any point of the limb, as f. Then the difference of mean refraction for c and f will be (Young, p. 64),

$$F = a\,(\cot h' - \cot h''), \qquad (1)$$

in which a is the constant 60″.6.

Denoting the angle $a\,c\,f$ by q, and the semi-diameter by s,

$$h'' = h' + s \cos q. \qquad (2)$$

From Trigonometry,

$$\cot h'' = \frac{1 - \tan h' \tan (s \cos q)}{\tan h' + \tan (s \cos q)}.$$

Substituting in (1), and writing $s \cos q \tan 1''$ for $\tan (s \cos q)$, we have,

$$F = a \left(\frac{s \cos q \tan 1'' + s \cos q \tan 1'' \tan^2 h'}{\tan^2 h' + s \cos q \tan 1'' \tan h'}\right).$$

The last term in the denominator is insignificant compared with $\tan^2 h'$; hence

$$F = a \operatorname{cosec}^2 h'\, s \cos q \tan 1'', \qquad (3)$$

which by (1) and (2) will be the difference between that ordinate of the ellipse and the circle which passes through f.

Hence the line $e\,f$ will be,

$$\text{Refractive Distortion} = a \operatorname{cosec}^2 h'\, s \tan 1'' \cos^2 q.$$

If $q = o$, we have $a\,b$ = contraction of vertical semi-diameter = $\Delta s = a \operatorname{cosec}^2 h' \, s \tan 1''$. Hence finally,

$$\text{Refractive Distortion} = \Delta s \cos^2 q. \tag{175}$$

✠ 3. To Deduce the Parallax for the Point R.

By making $x = o$ in Equation (95), and reducing by (108), (99) and (100), we have for the distance $R\,C$ (Fig. 17 or 29),

$$\frac{a\,e^2 \sin\phi}{\sqrt{1 - e^2 \sin^2\phi}}.$$

Denoting the distance $R\,O$ by y, the triangle $R\,O\,C$ gives

$$\frac{a\,e^2 \sin\phi}{\sqrt{1 - e^2 \sin^2\phi}} : y :: \sin(\phi - \phi') : \cos\phi'.$$

$$y = \frac{a\,e^2 \sin\phi \cos\phi'}{\sin(\phi - \phi')\sqrt{1 - e^2 \sin^2\phi}}.$$

Developing $\sin(\phi - \phi')$, cancelling and applying (125),

$$y = \frac{a}{\sqrt{1 - e^2 \sin^2\phi}}.$$

Comparing with second part of (112) it is seen that the denominator is sensibly the value of ρ expressed in terms of a as unity. Hence

$$y = \frac{a}{\rho}.$$

The angles at the moon subtended by the two lines a and $\frac{a}{\rho}$ will be proportional to those lines. Therefore

$$\pi : \pi_{,} :: a : \frac{a}{\rho}.$$

$$\pi_{,} = \frac{\pi}{\rho}. \tag{176}$$

✠ 4. To Determine the Difference between $d_{,}$ and d, due to a Transference of the Observer from R to C.

By the previous note we have (Fig. 29),

$$R\,C = \frac{a\,e^2 \sin \phi}{\sqrt{1 - e^2 \sin^2 \phi}}.$$

The perpendicular distance, $C\,N$, from the center to the line $R\,M$, is, with an error entirely negligible,

$$C\,N = \frac{a\,e^2 \sin \phi \cos \delta}{\sqrt{1 - e^2 \sin^2 \phi}}.$$

As before, the angle at the moon subtended by this line will be

$$\frac{a\,e^2 \sin \phi \cos \delta}{\sqrt{1 - e^2 \sin^2 \phi}} \frac{\pi}{a} = \frac{\pi\, e^2 \sin \phi \cos \delta}{\sqrt{1 - e^2 \sin^2 \phi}},$$

which is therefore the angular apparent displacement of the moon, represented by the arc $M\,M'$ (Fig. 29).

Denote it by m. Then, in the triangles $P\,M'\,S$ and $M\,M'\,S$,

$$\cos M' = \frac{\cos d_{,} - \cos m \cos d}{\sin m \sin d} = \frac{\sin D - \sin \delta \cos d}{\cos \delta \sin d}.$$

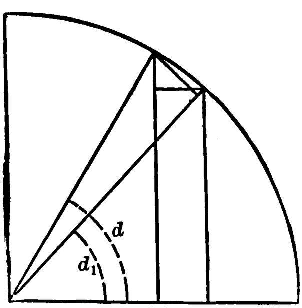


FIG. 30.

Reducing, replacing $\cos m$ by unity,

$$\cos d_{,} - \cos d = \sin m \left(\frac{\sin D}{\cos \delta} - \frac{\sin \delta \cos d}{\cos \delta}\right).$$

From Fig. 30, it is seen that when $d_{,}$ and d are nearly equal, as in the present case, we may replace $\cos d' - \cos d$ by $\sin (d - d_{,}) \sin d_{,}$.

Therefore

$$\sin (d - d_{,}) = \sin m \left(\frac{\sin D}{\cos \delta \sin d_{,}} - \frac{\sin \delta \cos d}{\cos \delta \sin d_{,}}\right).$$

Or

$$d - d_{,} = \frac{\pi\, e^2 \sin \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \left(\frac{\sin D}{\sin d_{,}} - \frac{\sin \delta}{\tan d_{,}}\right). \qquad (177)$$

OTHER METHODS OF DETERMINING LONGITUDE.

1st. If two stations are so near each other that a signal made at either, or at an intermediate point, can be observed at both, the time may be noted simultaneously by the chronometers at the two stations, and the difference of longitude thus deduced. An application of the same system, by means of a connected chain of signal stations, will give the difference of longitude between two remote stations. The signals are usually flashes of light—either reflected sunlight or the electric light, passed through a suitable lens.

2d. By noting the time of beginning or ending of a lunar or solar eclipse, or by occultations of stars by the moon. For these methods, see various Treatises on Astronomy.

3d. *By Jupiter's Satellites.* *a. From their eclipses.* The Washington mean-times of the disappearance of each satellite in the shadow of the planet, and reappearance of the same, are *accurately* given in the Ephemeris, pp.452–473, accompanied by diagrams of configuration for convenience of reference. A full explanation of the diagrams is given on p. 449. An observer who has noted one of these events, has only to take the difference between his own local time of observation and that given in the Ephemeris, to obtain his longitude. This method is defective, since a satellite has a sensible diameter and does not disappear or reappear instantaneously. The more powerful the telescope employed, the longer will it continue to show the satellite after the first perceptible loss of light. These facts give rise to discrepancies between the results of different observers, and even between those of the same observer with different instruments. Both the disappearance and reappearance should therefore be noted by the same person with the same instrument, and a mean of the results adopted. The first satellite is to be preferred, as its eclipses occur more frequently and more suddenly, although both disappearance and reappearance cannot be observed.

b. From their occultations by the body of the planet. The times of disappearance and reappearance to the *nearest minute only*, are given on same pages of the Ephemeris. Since the times are only approximate, they simply serve to enable two observers on different meridians to direct their attention to the phenomenon at the proper moment. A comparison of their times will then give their *relative longitude.*

c. From their transits over Jupiter's disc.

d. From the transits of their shadows over Jupiter's disc. The approximate times of ingress and egress, to be used as in case *b*, are given on same pages of the Ephemeris, for cases *c* and *d*.

Application to Explorations and Surveys.—On explorations, and reconnoissances for more exact surveys, the observer will usually be provided only with a chronometer, sextant, and artificial horizon, with probably the usual meteorological instruments.

The chronometer should be carefully rated and have its error on the local time of some comparison meridian (e.g., that of Washington) accurately determined for some given instant, so that, by applying the rate, its error on the same local time may be found whenever desired.

The sextant should have its eccentricity determined before starting, since this error often exceeds any ordinary index error, and cannot be eliminated by adjustment.

The observer should be able to recognize by name several of the principal Ephemeris stars. To determine the coördinates of his station when they are entirely unknown, he should first find the chronometer error on his own local time, using preferably the method by "equal altitudes of a star," since, as has been seen, he will then be independent of any knowledge of the star's coördinates, his own time, latitude, longitude, or instrumental errors.

Observations for latitude may be made at any convenient time by "circum-meridian altitudes" of a south and north star, or of a south star only, combined with "Polaris off the meridian," the reductions being made by aid of the chronometer error just referred to.

The method by "circumpolars" may also be used as a verification when applicable, the reduction being very simple.

The longitude is known as soon as the chronometer error on local time is known, by comparing this with its known error on the local time of the comparison meridian. However large the rate of a chronometer, it should be nearly constant; but after some time spent in traveling, with possible exposure to extremes of temperature, its indications of the comparison meridian time are rendered somewhat uncertain by the accumulation of unknown errors, thus introducing the same uncertainties into our longitudes. In such cases the method by "lunar distances" will afford an approximate reëstablishment of the chronometer error on the comparison meridian time, or a correction to an assumed approximate longitude.

If it be impracticable to find the local time by equal altitudes as recommended, on account of clouds or the length of time involved, it may be found by "single altitudes" of an east and a west star (or of a single star when necessary, either east or west), an approximate value of the latitude required in the computation being found from the best obtainable value of the meridian altitude of the star observed for latitude. With the error thus found the latitude is found as before, which, if it differs materially from the assumed approximate value, must be used in a recomputation of the time. From this the longitude follows as before.

If the latitude be known or approximately so, as at a fixed station or when tracing a parallel of latitude, time and longitude will be most expeditiously determined by "single altitudes."

In certain classes of work it is necessary to obtain approximate coördinates by day, in which case of course the sun must be used in accordance with the same general principles as far as applicable.

In all sextant work, except in methods by equal altitudes, its adjustments and errors must be carefully attended to.

In extensive surveys and geodetic work, where very precise results are required, the methods employed are "Time by Meridian Transits" with the reduction by Least Squares, Longitude by the Electric Telegraph, and Latitude by the Zenith Telescope. The observing-instruments should be mounted on small masonry piers or wooden posts set about four feet in the earth and isolated from the surrounding surface by a narrow circular trench one or two feet deep.

The exact location of an astronomical station is preserved, if desired (as when the station is one extremity of a base-line), by a cross on a copper bolt set in a block of stone embedded two or three feet below the surface, the exact location of which is recorded by suitable references to surrounding permanent objects.

Often it is required to determine the coördinates of a point where it is impracticable to locate an astronomical station, as for example a light-house or a central and prominent building of a city.

In such a case, having made the requisite observations at a suitable station in the vicinity, and having computed by (111) and (114) the length in feet of one second in latitude and longitude, measure the true bearing and distance of the point from the station, from which the coördinates of the former with respect to the latter are readily computed.

In locating points at intervals on a line which coincides with a parallel of latitude, sextant observations for latitude which can be quickly reduced will give, as just explained, the approximate distance of the observer from the desired parallel, to the immediate vicinity of which he is thus enabled to proceed. At this point a complete series of observations for latitude is made with the zenith telescope, and the resulting distance to the parallel carefully laid off due north or south.

In this manner points about twenty miles apart were located on the 49th parallel between the U. S. and the British Possessions.

TIME OF CONJUNCTION OR OPPOSITION.

Two celestial bodies are said to be in conjunction when either their longitudes or their right ascensions are equal; and in opposition when they differ by 180°. In the Ephemeris the conjunctions and oppositions of the moon or planets with respect to the sun refer to their longitudes. Conjunctions of the moon and planets or of the planets with each other refer to their right ascensions. In other cases, when used without qualification, the terms usually refer to longitudes.

The longitudes of the principal bodies of the solar system (or the data from which they may be computed) are given in the Ephemeris for (usually) each Greenwich mean moon. To find the time of conjunction, determine by inspection of the tables the two dates between which the longitudes of the bodies become equal, and denote the earlier date by T. Take from the tables four consecutive longitudes for each body—two next preceding and two next following the time of conjunction. Form for each the first and second differences, which give, from (42),

$$L_n = L + nd_, + \frac{n^2 - n}{2} d_2 \qquad (a)$$

and

$$L_n = L' + nd'_, + \frac{n^2 - n}{2} d_2' \qquad (b)$$

in which L_n is the unknown common longitude at opposition, and n in the second member is the required fractional portion of the interval between the consecutive epochs of the tables.

Subtracting and collecting the terms

$$\frac{d_2 - d_2'}{2} n^2 + \left(d_1 - d_1' + \frac{d_2' - d_2}{2}\right) n = L' - L, \qquad (c)$$

from which n is found by solution; the corresponding portion of the constant tabular interval is then added to T, thus giving the Greenwich time of conjunction. The time on any meridian to the west of Greenwich is found by subtracting the longitude. The value of n should be carried to three places of decimals to obtain the time to the nearest minute.

The method of finding the time of opposition is obvious from the above, noting that (c) becomes

$$\frac{d_2 - d'_2}{2} n^2 + \left(d_1 - d_1' + \frac{d'_2 - d_2}{2}\right) n = 180° + L' - L. \qquad (d)$$

Except when the moon is involved, the use of first differences will usually be found sufficient.

The times of conjunction and opposition in right ascension are found in accordance with the same principles.

TIME OF MERIDIAN PASSAGE.

To determine the local mean solar time of a given body coming to the meridian, it is to be noted that this time (P) is simply the hour angle of the mean sun at that instant, and that this hour angle is, by the general formula, P = sidereal time − R. A. of the mean sun.

Now the sidereal time at the instant is equal to the R. A. of the body on meridian, and this is equal to its R. A. at the preceding Greenwich mean noon (α) plus its increase of R. A. since that epoch, which is equal to $m\ (P + \lambda)$, λ being the longitude from Greenwich, and m the body's hourly increase in R. A. Or, sidereal time $= \alpha + m\,(P + \lambda)$.

Similarly we have, denoting the hourly increase of mean sun's R. A. by s, R. A. of mean sun $= \alpha_s + s\,(P + \lambda)$.

Therefore by the preceding formula,

$$P = [\alpha + m\,(P + \lambda)] - [\alpha_s + s\,(P + \lambda)].$$

Solving,

$$P = \frac{\alpha - \alpha_s + \lambda\,(m - s)}{1 - (m - s)}.$$

In this equation α and α_s are given directly in the Ephemeris, λ is supposed to be known, and s is constant and equal to 9.8565 seconds; m is obtained from the column adjacent to the one giving value of α, and should be taken so that its value will denote the change at the middle instant between the Greenwich mean noon and the instant under discussion, viz., $\frac{1}{2}\,(P + \lambda)$, as near as can be determined.

For the moon, whose motion in R. A. is varied, and for an inferior planet, a second approximation may be necessary. If the planet have a retrograde motion, m becomes negative. If the body be a star, m becomes zero.

If the *sidereal* time of culmination be required, the above formula holds, substituting for the mean sun the vernal equinox, whose R. A. and hourly motion in R. A. are zero.

Hence,

$$P' = \frac{\alpha + \lambda\, m}{1 - m}.$$

For a star, $P' = \alpha$.

AZIMUTHS.

Definitions.—In surveys and geodetic operations it often becomes necessary to determine the "azimuth" of lines of the survey; *i. e.*, the angle between the vertical plane of the line and the plane of the true meridian through one of its extremities; or, in other words, the *true bearing* of the line.

For reasons given under the head of Latitude, the geodetic may differ slightly from the astronomical azimuth of a line. Only the latter will be referred to here, and it is manifestly the angle at the astronomical zenith included between two vertical circles, one coinciding with the astronomical meridian, and the plane of the other containing the line in question.

Outline.—In outline, the method consists in measuring with the "Altazimuth" or "Astronomical Theodolite" the horizontal angle which is included between the line and *some celestial body* whose place is well known. Then having ascertained by computation the *true* azimuth of the *body* at the instant of its bisection by the vertical wire, the sum of the two will be the *true* azimuth of the line.

Instruments.—The "Astronomical Theodolite" is provided with both horizontal and vertical circles. In geodetic work the latter is used largely as a mere finder, but the former is often of great size—usually from one to two feet in diameter, and very accurately

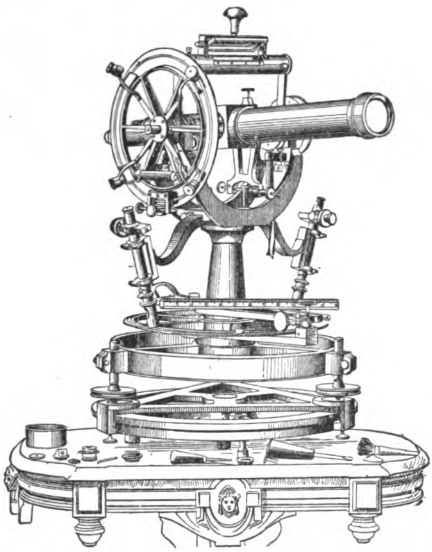


FIG. 31.

graduated throughout. For reading the circle, it is provided with several reading-microscopes fitted with micrometers, in lieu of verniers; and in order that any angle may be measured with different parts of the circle, the latter is susceptible of motion around the vertical axis of the instrument. Eccentricity and errors of graduation are thus in a measure eliminated.

To mark the direction of the line at night a bull's-eye lantern in a small box firmly mounted on a post is ordinarily used; the light being thrown through an aperture of such size as to present about the same appearance as the star observed. To avoid refocusing for the star, the lantern should be distant not less than a mile. If it is impracticable to place the lantern exactly on the line whose azimuth is required it may be placed at any convenient point, its azimuth determined at night, and the angle between it and the line measured by day; the aperture being then covered symmetrically by a target of any approved pattern. For convenience in the following discussion the target will be supposed to be on the line.

Classification of Azimuths.—Azimuths of the line with reference to the star are taken in "sets," the number of measurements of the angle in each set being dependent upon whether the final result is to be a *primary* or *secondary* azimuth. Primary azimuths are employed in determining the direction of certain lines connected with the fundamental or primary triangulation of a survey, and each set consists of from 4 to 6 measurements of the angle in each position of the instrument. The final result is required to depend upon several sets, with stars in different positions (generally not less than five, and often many more). The error of the chronometer (required in the reductions), together with its rate, are determined by very careful time observations with a transit.

Secondary azimuths are employed in determining the direction of certain lines connected with the secondary or tertiary triangles of a survey. The number of measurements in a set is about one half or one third that in a set for a primary azimuth; the number of sets is also reduced, and the time observations are usually made with a sextant. The sun is used in connection with secondary azimuths only.

Selection of Stars.—In order to make a proper selection of stars for the determination of azimuths, we have from the Astronomical Triangle,

$$\tan A = \frac{\sin P}{\cos \phi \tan \delta - \sin \phi \cos P} \qquad (178)$$

Errors in the assumed values of P, ϕ, or δ will produce errors in the computed azimuth, those in δ being for obvious reasons usually insignificant and least likely to occur.

Taking the reciprocal of (178), differentiating and reducing the first term of the resulting second member by

$$\cos a \cos \psi = \sin \phi \cos \delta - \cos \phi \sin \delta \cos P,$$

the second by

$$\sin a = \sin \phi \sin \delta + \cos \phi \cos \delta \cos P,$$

and the third by

$$\sin A : \sin P :: \cos \delta : \cos a,$$

we have

$$d\,A = -\frac{\cos \delta \cos \psi}{\cos a}\, d\,P + \tan a \sin A\, d\,\phi - \frac{\sin \psi}{\cos a}\, d\,\delta.$$

From this equation it is seen that if we select a close circumpolar star, any error ($d\,P$) in the clock correction or in the star's R. A., or any error ($d\,\phi$) in the assumed latitude, will produce but slight effect on the computed azimuth, since $\cos \delta$ and $\sin A$ will each be very small. If in addition the star be at elongation ($\psi = 90°$), the first mentioned error will produce *no* effect, while $\sin A$, although at a maximum for the star, will still be very small. (In latitude of West Point the azimuth of Polaris does not exceed 1° 40′.) At elongation the effect of errors ($d\,\delta$) in δ will be a maximum, although insignificant if δ be taken from the Ephemeris.

But if the star be observed at both east and west elongations, the effect of $d\,\delta$ and $d\,\phi$ will disappear in the mean result, since the computed azimuth (reckoned from the north through the east to 360°), if erroneous, will be as much too large in one case as too small in the other.

Circumpolar stars at their elongations (both) are most favorably situated, therefore, for the determination of azimuths; and since experience gives a decided preference to stars in these positions, other cases will not be considered, except to remark that the Astronomical Triangle then ceases to be right angled.

The stars α (Polaris), δ, and λ, Ursæ Minoris, and 51 Cephei, are those almost exclusively used (although the latter two cannot be used with small instruments). Their places are given in a special table of the Ephemeris, pp. 302–13, for every day in the year, and they are so distributed around the pole that one or more will usually be available for observation at some convenient hour. Of these four, λ Ursæ Minoris is both the smallest and nearest to the pole. For the large instruments it therefore presents a finer and steadier object than any of the others. For the small instruments suitable stars may be selected from the Ephemeris.

Measurements of Angles with Altazimuth.—In order to understand the measurement of the difference of azimuth of two points *at unequal altitudes,* let us suppose that the horizontal circle of the "Altazimuth" has its graduations increasing to the right (or like those of a watch-face), and that absolute azimuths are reckoned from the north point through the east to 360°, the origin of the graduation being at the point 0, Figure 32.

The angle $N L 0$ will then be the absolute azimuth of the origin

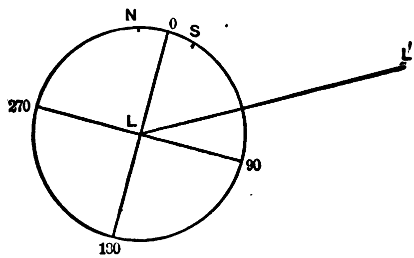


FIG. 32.

of graduations $= O$, and if the instrument be in adjustment and A_s and A_l denote the absolute azimuths of the star and line respectively, we shall have

$$\left.\begin{array}{l} A_s = O + R \\ A_l = O + R' \end{array}\right\} \text{ in which } R \text{ and } R' \text{ denote}$$

angles $0 L S$ and $0 L L'$ respectively, and may be considered as the readings of the instrument when pointed upon the star and over the line. These equations will be somewhat modified if the instrument be not in perfect adjustment. This will usually be the case. Let us suppose that the end of the telescope axis to the observer's left is *elevated* so that the axis has an inclination of b seconds of *arc.* Then if the telescope be horizontal and pointing in the direction

$L\,S$, it will, when moved in altitude, sweep to the right of the star, and the whole instrument must be moved to the left to bring the line of collimation on the star. The reading of the instrument will thus be *diminished* to r, and we shall have the *proper* reading, $R = r +$ a correction. The amount of this correction is readily seen, from the small right-angled spherical triangle involved (of which the required distance is the base), to be $b \cot z$. In the same way it is seen from the principles explained under "Equatorial Intervals," etc., that if the middle wire be to the left of the line of collimation by c seconds of arc, r must receive the correction $c \operatorname{cosec} z$. Hence when both these errors exist together, we shall have, z' denoting the zenith distance of the target,

$$A_s = O + r + b \cot z + c \operatorname{cosec} z, \tag{179}$$

$$A_l = O + r' + b' \cot z' + c \operatorname{cosec} z', \tag{180}$$

since c remains unchanged, while b is subject to changes.

Subtracting,

$$A_l - A_s = (r' + b' \cot z') - (r + b \cot z) + c\,(\operatorname{cosec} z' - \operatorname{cosec} z).$$

Since by reversing the instrument the sign of c is changed, but not altered numerically, we may, if an equal number of readings in the two positions be taken, drop the last term as being eliminated in the mean result. With this understanding, the equation will be

$$A_l - A_s = (r' + b' \cot z') - (r + b \cot z). \tag{181}$$

which gives the azimuth of the line *with reference to the star*, free from all instrumental errors. b is positive when the left end is higher, and its value, heretofore explained, is obtained by direct and reversed readings of both ends of the bubble, and is $\frac{d}{4}\left[\,(w + w') - (e + e')\,\right]$, d being the value of one division in seconds of *arc*. For stars at, or very near, elongation, it is evident that $\cot z$ may be replaced by $\tan \phi$, without material error; c is positive when middle wire is to the *left* of its proper position.

For very precise work the above result requires a small correction for diurnal aberration, the effect of which is to displace (apparently) *a star toward the east point.* **For stars at elongation, this** correction is $0''.311 \cos A_e$. (See Note 1.)

In using the reading-microscopes, care should be taken to correct for "error of runs." When a microscope is in perfect adjustment, a *whole* number of turns of the micrometer screw carries the wire exactly over the space between two consecutive graduations of the circle. Due to changes of temperature, etc., the distance between the micrometer and circle may change, thus altering the size of the image of a "space." The excess of a circle division over a whole number of turns is called the "Error of Runs." This error is determined by trial, and a proportional part applied to all readings of minutes and seconds made with the microscope.

Observations and Preliminary Computations.—The observations and the preliminary computations are as follows: The error and rate of the chronometer, error of runs of the micrometers, collimation error and latitude are supposed to have been obtained with considerable accuracy. The apparent R. A. and declination for the time of elongation of the star to be used must be taken from the Ephemeris, or, if not given there, reduced from the mean places given in the catalogue employed, as explained under Zenith Telescope.

Then for the star's hour-angle at elongation, $\cos P_e = \dfrac{\tan \phi}{\tan \delta}$.

" " " azimuth " " $\sin A_e = \dfrac{\cos \delta}{\cos \phi}$.

" " " zenith distance at " $\cos z_e = \dfrac{\sin \phi}{\sin \delta}$.

" " sidereal time " " $T_o = \alpha \pm P_e$.

" " chronometer " " " $T_c = T_o + E$,

α being the R. A., and E the chronometer error.

The instrument is then placed accurately over the station and levelled, so that everything will be in readiness to begin observations at about 20^m before the time of elongation as above computed. In the actual measurement of the angle several different methods have been followed. First, five or six pointings are made on the target, and for each pointing, the circle and all the microscopes are read; also if the angle of elevation of the target differ sensibly from zero (as would not usually be the case with the base-line of a survey) readings of the level, both direct and reversed, are made. If the target be on the same level as the instrument, cot z' will be zero, and the level correction will disappear. Then five or six pointings are made on the star, and in addition to the above readings the chronometer time of each bisection is noted. The instrument is then reversed to eliminate error of collimation, and the above operations repeated, beginning with the star. In the second method alternate readings are made on the mark and star, star and mark, until five or six measurements of the angle have been made, the chronometer being read at each bisection of the star; the circle, microscopes and level as before. The instrument is then reversed, and the same operations repeated in the reverse order. The middle of the time occupied by the whole set should correspond very nearly to the time of elongation. Similar observations are then made, on the same or following nights, on other stars, combining both eastern and western elongations, and using different parts of the horizontal circle for the measurement.

Reduction of Observations.—Since the observations on the star have been made at different times, and since these correspond to different though nearly equal azimuths, the first step in the reduction is to ascertain what each reading on the star would have been had the observation been made exactly at elongation. For this purpose find the difference between the chronometer time of each observation and the chronometer time of elongation as computed, applying the rate if perceptible. Let the sidereal interval between these two epochs be denoted by τ seconds. Then the elongation reading of the star would have been

$$\text{actual reading} \pm \text{the expression } 112.5\,\tau^2 \sin 1'' \tan A_e,$$

which denote by C. (See Note 2.)

[The quantity $112.5\,\tau^2 \sin 1''$ is almost exactly equal to the tabulated values of "m" in the "Reduction to the Meridian," and may if desired be taken directly from those tables.] With a circle graduated as assumed, this correction would manifestly be negative for a western, and positive for an eastern, elongation. Hence Eq. (181) becomes,

$$A_l - A_e = (r' + b' \cot z') - (r + b \cot z \pm C). \qquad (182)$$

Each pair of observations (on the line and star) with the telescope "direct" gives a value of $A_l - A_e$. If n_d be the number of such pairs, the mean will be $\frac{\Sigma(A_l - A_e)}{n_d}$, to which if A_e (positive for eastern, negative for western, elongations) be added as heretofore computed $\left(\sin A_e = \frac{\cos \delta}{\cos \phi}\right)$, we shall have the *true bearing of the line* for instrument "*direct.*"

Similarly, for instrument "reversed," we shall have

$$\frac{\Sigma(A_l - A_e)}{n_r},$$

from which by adding A_e we obtain the *true bearing of the line* for instrument *reversed.*

The mean of the two is the *true bearing of the line* as given by the star employed.

[For the greatest precision, this must be corrected by *adding* the diurnal aberration, $0''.311 \cos A_e$.]

The adopted value of the azimuth of the line should rest upon at least five such determinations.

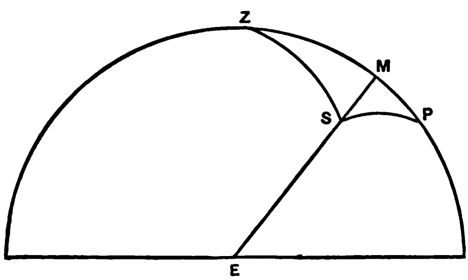


FIG. 33.

✠ 1. **Diurnal Aberration in Azimuth.**

It has already been shown when treating of the transit instrument and in Art. 225 Young, that due to diurnal aberration all stars are apparently displaced toward the east point of the horizon by $0''.319 \cos \phi \sin \theta$ of a great circle, where θ is the angle made by the direction of a ray of light from the star with an east and west line (measured by $S\,E$, Fig. 33).

To determine the effect of this small displacement on the azimuth of a star, the right-angled triangle $Z\,S\,M$ gives, denoting $Z\,M$ by b,

$$\sin A = \frac{\cos\theta}{\sin z}$$

$$\sin z \cos A = \sin\theta \sin b.$$

Hence

$$\sin A = \frac{\cos\theta \cos A}{\sin\theta \sin b},$$

$$\tan A = \frac{\cot\theta}{\sin b}.$$

Differentiating

$$d\,A = -\frac{\cos^2 A}{\sin b \sin^2\theta}\,d\,\theta = -\frac{\cos A}{\sin\theta \sin z}\,d\,\theta.$$

Substituting $-0''.319 \cos\phi \sin\theta$ for $d\,\theta$ (since θ is a decreasing function of A),

$$d\,A = \frac{0''.319 \cos A \cos\phi}{\sin z}.$$

For a close circumpolar star at elongation

$$\cos\phi = \sin z, \quad \text{sensibly.}$$

Hence,

$$d\,A = 0''.319 \cos A_e. \qquad (183)$$

✠ 2. To Reduce an Azimuth Observed Shortly Before or After the Time of Elongation, to its Value at Elongation.

If we conceive the meridian to be revolved to the position of the declination circle passing through the point of elongation, evidently the arc of this circle intercepted between the vertical wire of the instrument and the point of elongation will have the same numerical value as the "Reduction to the Meridian" deduced in connection with the Zenith Telescope, viz.:

$$\tfrac{1}{4}(15\tau)^2 \sin 1'' \sin 2\,\delta = 112.5\,\tau^2 \sin 1'' \sin\delta \cos\delta.$$

The angle at the zenith subtended by this arc, *i.e.*, the correction to azimuth, is seen from the small right-angled triangle to be

$$112.5\,\tau^2 \sin 1'' \,\frac{\sin\delta\cos\delta}{\sin z_e}. \qquad (184)$$

Substituting $\cos d$ and $\sin d$ for $\sin\delta$ and $\cos\delta$ ($d =$ polar distance), making $\cos p = 1$, and $\sin p = \tan p$ (since the star is a close circumpolar), the last factor becomes

$$\frac{\tan p}{\sin z_e} = \tan A_e.$$

Hence

$$C = 112.5\,\tau^2 \sin 1'' \tan A_e.$$

DECLINATION OF THE MAGNETIC NEEDLE.

The Declination of the Magnetic Needle may be found in accordance with the same principles, regarding the magnetic meridian pointed out by the needle, as the line whose azimuth is to be found. Or, note the reading of the needle when the instrument carrying it is pointed accurately along a line whose true bearing or azimuth is known. Or, take the magnetic bearing of some known celestial body, and note the time T. Then $P = T - \alpha$. This value of P in Eq. (178) gives the true azimuth, and the difference between this and the magnetic bearing gives the declination of the needle. Or, if the time be not known, measure the altitude of the body and solve the $Z\,P\,S$ triangle for A, knowing ϕ, δ, and α. Then having noted the magnetic bearing of the body at the instant of measuring the altitude, the difference is the declination of the needle.

SUN-DIALS.

A sun-dial is a contrivance for indicating *apparent solar time* by means of the shadow of a wire or straight-edge cast on a properly graduated surface. The wire or straight-edge, called the *style* or *gnomon*, must be parallel to the earth's axis; i.e., it must be inclined to the horizontal by an angle equal to the latitude, and be in the meridian. The graduated surface, called the dial-face, is usually a plane, and made either of metal or smoothed stone. It may have any position with reference to the style (consistent with receiving its shadow throughout the day), although it is usually either horizontal or placed in the prime vertical. The two varieties are shown in Fig. *a*, the first being by far the more common.

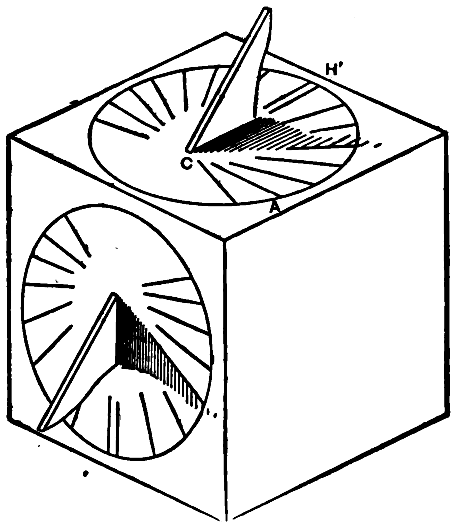


FIG. *a*.

The principle of the horizontal dial will be readily understood from an inspection of Fig. *b*.

Let PP' be the axis of the celestial sphere, Z the zenith, $A\ Q\ B$ the equinoctial, and $A\ H\ B$ perpendicular to $C\ Z$ the plane of the dial face, the style extending from C in the direction of P. Then if a plane be passed through the style and the position of the sun, S, at any instant, it will cut from the celestial sphere the sun's hour-circle, and from the dial-face the line $C\ IX$, which is therefore the shadow of the style on the dial-face. The direction of this line is thus seen to be independent of the sun's declination (season of the year), and dependent only on his hour angle. If, therefore, we mark on the dial-face the various positions of this line corresponding to assumed hour angles which differ from each other by, for example, 3° 45′ or 15 minutes, instants of apparent solar time will be indicated by the arrival of the style's shadow at the corresponding line. This construction may be made as follows, noting that

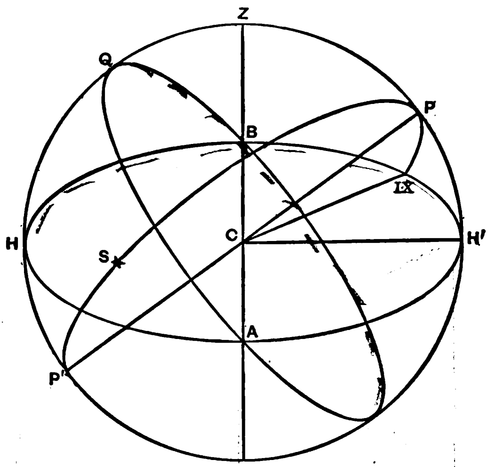


FIG. *b*.

the 12-o'clock line is the intersection of the dial-face with the vertical plane through the style.

Suppose, for example, it were required to construct the 9-o'clock line. In the spherical triangle $P\,H'\,IX$ right-angled at H' we have $P\,H' = \phi$, and the angle at $P = Z\,P\,S = 45°$, to determine the side $H'\,IX = x$, given by the formula

$$\tan x = \sin \phi \tan 45°.$$

Then with C as a center lay off an angle from CH' equal to the computed value of x, and draw the line $C\,IX$.

Generally,

$$\tan x = \sin \phi \tan P,$$

P denoting the hour angle assumed.

Values of x corresponding to intermediate values of P may be laid off with a pair of dividers.

The dial-face may have any convenient form,—circular, rectangular, or elliptical. The last is the best form (shown in Fig. *a*), since the axis can be so proportioned that the spaces along the edge will be nearly equal, thus greatly facilitating any subdivision. For the latitude of West Point, CH' should be about $2\frac{1}{2}$ times CA (Fig. *a*). If the plate be 18 or 20 inches long the subdivisions can be readily carried to minutes.

Usually the style is a triangular-shaped piece of metal of a sufficient thickness to avoid deformation by accident—say $\frac{1}{4}$ or $\frac{1}{8}$ inch. In this case one edge will cast the shadow in the A.M., and the other in the P.M. Hence the graduations on either side of the 12-o'clock line must be constructed using as a center the point where the shadow-casting edge pierces the plane of the dial-face. The plane of the style must be accurately perpendicular to the dial-face.

Having been graduated, the sun-dial is mounted on a firm pedestal, accurately levelled by a spirit-level, and turned till the plane of the style is in the meridian. For an approximation we may use a pocket compass, the declination of the needle being known within moderate limits. By day the orientation may be effected by means of a watch whose error is known. Compute the watch time of apparent noon = 12-o'clock − error + equation of time, and turn the dial slowly, keeping the shadow of style on the 12-o'clock mark until the time computed. The levelling must be carefully attended to. If the watch error be not known, it may be found by means of a sextant.

If no means of determining time are at hand, the dial may still be oriented by a determination of the meridian plane, either by day or night. At night advantage may be taken of the fact that Polaris and ζ Ursae Majoris (the middle star in the tail of the Great Bear or handle of the Dipper) cross the meridian at almost exactly the same instant. Therefore if two plumb-lines be suspended from firm supports as nearly in the meridian as may be, one touching the style and the other a few feet to the south arranged for lateral shifting, we may by sliding the latter cover both stars by both lines at the moment of meridian passage. These lines then define the meridian plane, into which the style is easily turned. The polar distance of ζ being between 34° and 35°, it is evident that for latitudes above about 40° the star must be observed at lower culmination, and for lower latitudes at the upper.

By day the meridian plane may be determined as follows: Suspend a plumb-line over the south end of a perfectly level table or other suitable surface. With the point A as a center describe an arc, CD. The shadow of a knot or bead at B will describe during the day a curve $CEFGD$. Mark the points C and D where it crosses the arc before and after noon. A line from A bisecting the chord CD will then be in the meridian, and its extremities may be projected to the earth by plumb-lines and the points marked. Stretch a fine cord from one point to the other, and note the instant

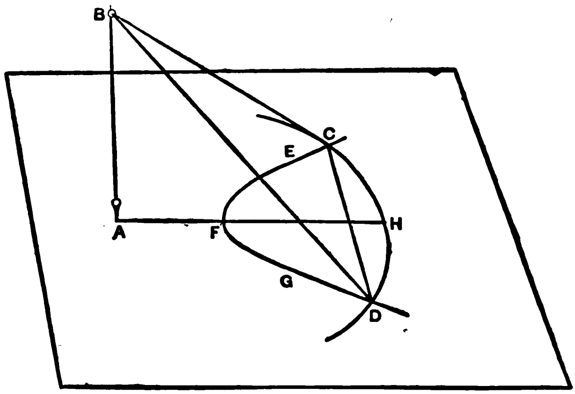


FIG. c.

when the shadow of the south plumb-line exactly coincides with that of the cord. This is evidently apparent noon; and if the dial be so turned that the shadow of the style falls on the 12-o'clock line at the same instant, it will be duly oriented.

Evidently this method supposes the sun's declination to be constant; its change may, however, for this purpose be neglected, except for a month at about the time of the equinoxes.

The meridian line may also be determined with a theodolite, as described in works on Surveying.

The dial-face may if desired be graduated after orientation by noting where the shadow of the style falls at 1 hour, 2 hours, etc., from the time of apparent noon.

The indications of all sun-dials must be corrected by the Equation of Time in order to give local mean time. This correction is practically constant for the corresponding days of all years, and its value at suitable intervals may either be engraved on the dial-plate, or taken from the annexed table.

Refraction, varying with the sun's altitude, is evidently a source of error, although too small to require consideration in the present connection.

The indications of a sun-dial with the solid style (Fig. *a*) will be one minute too great in the forenoon and one minute too small in the afternoon, since the shadow line will in each case be formed by the limb of the sun toward the meridian, and the sun requires about one minute to advance through an arc equal to its semi-diameter.

A dial constructed for a given latitude may be used without appreciable error in any latitude not differing therefrom by more than one third of a degree—say 25 miles.

Vertical dials are usually placed on the south fronts of buildings. Their construction is readily understood from what precedes, the graduations being computed by the formula

$$\tan x = \cos \phi \tan P.$$

EQUATION OF TIME—TO BE ADDED TO SUN-DIAL TIME.

Day.	Jan.	Feb.	March.	April.	May.	June.
1	$+4^m$	$+14^m$	$+12^m$	$+4^m$	-3^m	-2^m
8	7	14	11	2	-4	-1
16	10	14	9	0	-4	0
24	12	13	6	-2	-3	$+2$

Day.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
1	$+3^m$	$+6^m$	0^m	-10^m	-16^m	-10^m
8	5	5	-2	-12	-16	-7
16	6	4	-5	-14	-15	-4
24	6	2	-8	-15	-13	0

SOLAR ECLIPSE.

A solar eclipse can only occur at conjunction—that is, at new moon, and then only when the moon is near enough to the plane of the ecliptic to throw its shadow or penumbra upon the earth. The following discussion, abbreviated from that found in Chauvenet's Practical Astronomy, Vol. I, will suffice to give the student such a knowledge of the theory of eclipses as to enable him to project a solar eclipse, with the aid of the eclipse data found in the Ephemeris.

Solar Ecliptic Limits.—Let $N S$ Fig. 34 be the Ecliptic, $N M$

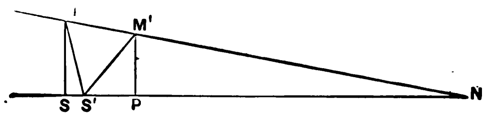


FIG. 34.

the intersection of the plane of the moon's orbit with the celestial sphere, N the moon's node, S and M the sun's and moon's center at conjunction, and S' and M' the same points at the instant of nearest angular distance of the moon from the sun. Assume the following notation, viz.:—

$\beta = S M$, the moon's latitude at conjunction.

$i = S N M$, the inclination of the moon's orbit to the ecliptic.

$\lambda =$ the quotient of the moon's mean hourly motion in longitude at conjunction, divided by that of the sun.

$\Delta = S' M'$, the least true distance.

$\gamma = S M S'$.

Considering $N M S$ as a plane triangle, and drawing the perpendicular $M' P$ from M' to $S N$, we have

$$S S' = \beta \tan \gamma. \qquad S P = \lambda \beta \tan \gamma.$$

$$S' P = \beta (\lambda - 1) \tan \gamma. \qquad M' P = \beta - \lambda \beta \tan \gamma \tan i.$$

$$\Delta^2 = \beta^2 [(\lambda - 1)^2 \tan^2 \gamma + (1 - \lambda \tan i \tan \gamma)^2].$$

Differentiating the last equation and placing $\dfrac{d\Delta}{d\gamma} = 0$, we find Δ will be a minimum for

$$\tan \gamma = \frac{\lambda \tan i}{(\lambda - 1)^2 + \lambda^2 \tan^2 i}.$$

This value gives

$$\Delta^2 = \frac{\beta^2 (\lambda - 1)^2}{(\lambda - 1)^2 + \lambda^2 \tan^2 i}, \qquad (185)$$

or

$$\Delta^2 = \beta^2 \cos^2 i', \tag{186}$$

when tan i' is placed equal to $\dfrac{\lambda}{\lambda - 1}$ tan i.

The least apparent distance of the sun's and moon's center as viewed from the surface of the earth may be less than Δ by the difference of the horizontal parallaxes of the two bodies. Call this distance Δ', then

$$\Delta' = \Delta - (\pi - P).$$

Now when Δ' is less than the sum of the apparent semi-diameters of the sun and moon there will be an eclipse; hence the condition is (denoting the semi-diameters of the moon and sun respectively by s' and s),

$$\Delta - (\pi - P) < s + s',$$

or

$$\beta \cos i' < \pi - P + s + s'. \tag{187}$$

To ascertain the probability of an eclipse, it is generally sufficient to substitute the mean values of the quantities in the above inequality. The extreme values, determined by observation are

	i	π	P
	5° 20′ 06″	61′ 32″	9″. 0
	4° 57′ 22″	52′ 50″	8″.70
mean	5° 8′ 44″	57′ 11″	8″.85

	s	s'	λ
	16′ 18″	16′ 46″	16.19
	15′ 45″	14′ 24″	10.89
mean	16′ 1″	15′ 35″	13. 5

The mean value of sec i', found from those of i and λ, is 1.00472 and hence,

$$\beta < (\pi - P + s + s') \sec i' = \beta < (\pi - P + s + s')\,(1 + 0.00472). \tag{188}$$

The fractional part of the second member of the inequality varies between 20″ and 30″; taking its mean 25″, we have for all but exceptional cases,

$$\beta < \pi - P + s + s' + 25''. \tag{189}$$

Substituting in this last form the greatest values of π, s, and s', and the least value of P; and then the least values of π, s, and s', and the greatest value of P, we have

$$\beta < 1^\circ\ 34'\ 27''.3,$$

and

$$\beta < 1^\circ\ 22'\ 50'',$$

respectively.

If, therefore, the moon's latitude at conjunction be greater than $1^\circ\ 34'\ 27''.3$ a solar eclipse is impossible; if less than $1^\circ\ 22'\ 50''$ it is certain; if between these values it is doubtful. To ascertain whether there will be one or not in the latter case, substitute the actual values of P, π, s and s' for the date, and if the inequality subsists there will be an eclipse, otherwise not.

PROJECTION OF A SOLAR ECLIPSE.

1. To find the Radius of the Shadow on any Plane perpendicular to the Axis of the Shadow.

In Fig. 35 let S and M be the centers of the sun and moon; V the vertex of the umbral or penumbral cone; FE the *fundamental* plane through the earth's center perpendicular to the axis of the shadow; and CD the parallel plane through the observer's position. It is required to find the value of CD at the beginning or ending of an eclipse.

Take the earth's mean distance from the sun to be unity, and let $ES = r$, $EM = r'$, $MS = r - r'$. Place $\frac{r - r'}{r'} = g$, and let k be the ratio of the earth's equatorial radius to the moon's radius $= 0.27227$. Then P_0 being the sun's mean horizontal parallax, we have

$$\text{Earth's radius} = \sin P_0.$$

$$\text{Moon's radius} = k \sin P_0 = 0.27227 \sin P_0.$$

$$\text{Sun's radius} = \sin s.$$

s being the apparent semi-diameter of the sun at mean distance. From the figure we have

$$\sin FVE = \sin f = \frac{\sin s \pm k \sin P_0}{r' g}, \qquad (190)$$

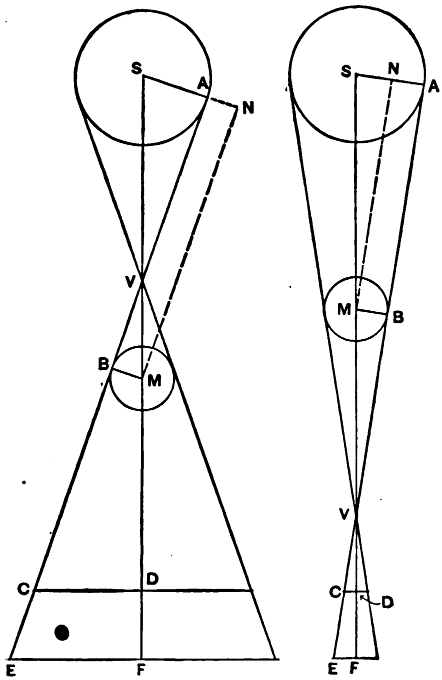


FIG. 35.

in which the upper sign corresponds to the penumbral and the lower to the umbral cone. The numerator of the second member is constant, and since $s = 959''.758$, $P_0 = 8''.85$, we have

$$\log [\sin s + k \sin P_0] = 7.6688033 \text{ for exterior contact.}$$

$$\log [\sin s - k \sin P_0] = 7.6666913 \text{ for interior contact.}$$

If the equatorial radius of the earth be taken as unity, we have

$$VM = \frac{k}{\sin f}, \quad MF = z.$$

Whence the distance c of the vertex of the cone from the fundamental plane is

$$c = z \pm \frac{k}{\sin f}. \qquad (191)$$

If l and L be radius of the shadow on the fundamental and on the observer's plane respectively, and ζ be their distance apart, we have

$$l = c \tan f = z \tan f \pm k \sec f. \qquad (192)$$

$$L = (c - \zeta) \tan f = l - \zeta \tan f. \qquad (193)$$

2. To find the distance of the Observer at a given time from the Axis of the Shadow in terms of his Co-ordinates and those of the Moon's Center, referred to the Earth's Center as an Origin.

Let O, Figure 36, be the earth's center, and $X\ Y$ the fundamental plane. Take $Z\ Y$ to be the plane of the declination circle

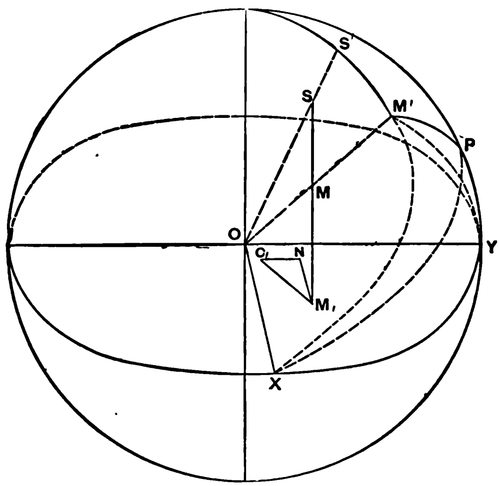


FIG. 36.

passing through the point Z in which the axis of the moon's shadow pierces the celestial sphere; $X\ Z$ being perpendicular to the other two coördinate planes. Let M and S be the centers of the moon and sun, M', S', their geocentric places on the celestial sphere, $M_{,}$ their projections on the fundamental plane, and $C_{,}$ the projection of the observer's place on the same plane. Let P be the north pole. The axis $O Z$, being always parallel to the axis of the shadow, will pierce the celestial sphere in the same point, as $S M$. Assume the following notation:

α, δ, r = the R. A., Dec., and distance from the earth's center, respectively, of the moon's center.
α', δ', r' = the corresponding coördinates of the sun's center.
a, d, = the R. A. and Dec. of the point Z.
x, y, z = the coördinates of the moon's center.
ξ, η, ζ = the coördinates of the observer's position.
ϕ, ϕ' = the latitude and reduced latitude respectively.
λ = the longitude of the observer's station west from Greenwich.
ρ = the earth's radius at the observer's station in terms of the earth's equatorial radius taken as unity.
$\mu_{,}$ = the Greenwich hour angle of the point Z.
μ = the sidereal time at which the point Z has the R. A. a.
$\varDelta$ = the required distance of the place of observation from the axis of the shadow at the time μ.

From the conditions, we have

$$\text{R. A. of } Z = a,$$

$$\text{R. A. of } M' = \alpha,$$

$$\text{R. A. of } X = 90^\circ + a,$$

and therefore

$$Z P M' = \alpha - a, \quad \text{and } P M' = 90^\circ - \delta.$$

Through $M_{,}$ and $C_{,}$ draw $M_{,} N$ and $C_{,} N$ parallel to the axis of X and Y respectively; then $M_{,} C_{,} N = P Z M' = P$, the position angle of the point of contact, and we have

$$\left.\begin{aligned} \varDelta \sin P &= x - \xi, \\ \varDelta \cos P &= y - \eta. \end{aligned}\right\} \qquad (194)$$

From the spherical triangles $M'\,P\,X$, $M'\,P\,Y$, and $M'\,P\,Z$, we have

$$\left.\begin{aligned} x &= r\cos M'X = r\cos\delta\sin(\alpha - a) \\ y &= r\cos M'Y = r\,[\sin\delta\cos d - \cos\delta\sin d\cos(\alpha - a)] \\ z &= r\cos M'Z = r\,[\sin\delta\sin d - \cos\delta\cos d\cos(\alpha - a)]. \end{aligned}\right\} \quad (195)$$

Similarly the coördinates of the place of observation are

$$\left.\begin{aligned} \xi &= \rho\cos\phi'\sin(\mu - a) \\ \eta &= \rho\,[\sin\phi'\cos d - \cos\phi'\sin d\cos(\mu - a)] \\ \zeta &= \rho\,[\sin\phi'\sin d + \cos\phi'\cos d\cos(\mu - a)]. \end{aligned}\right\} \quad (196)$$

The hour angle $(\mu_{,} - a)$ of the point Z for the meridian of the observer can be found from

$$\mu - a = \mu_{,} - \lambda,$$

in which $\mu_{,}$ is the hour angle of the point Z for the Greenwich meridian and λ is the longitude of the observer's meridian.

The distance of the observer from the axis of the moon's shadow $\Delta_{,} = C_{,}M_{,}$ can be found from the above formulas,

since, $$\Delta^2 = (x - \xi)^2 + (y - \eta)^2. \quad (197)$$

3. To Find the Time of Beginning or Ending of the Eclipse at the Place of Observation.

For the assumed Greenwich mean time of computation take from the Besselian table of elements given in the Ephemeris for each eclipse the values of $\sin d$, $\cos d$, and μ. The values of $\rho\cos\phi'$ and $\rho\sin\phi'$ are found on page 505, computed from the formulas,

$$\left.\begin{aligned} \rho\cos\phi' &= \frac{a\cos\phi}{\sqrt{1 - e^2\sin^2\phi}} = F\cos\phi \\ \rho\sin\phi' &= \frac{\sin\phi\sqrt{1 - e^2\sin^2\phi}}{a(1 - e^2)} = \frac{\sin\phi}{G}. \end{aligned}\right\} \quad (198)$$

The variations of ξ and η in one minute of mean time are obtained by differentiating the first two of Eqs. (196), and give

$$\left.\begin{aligned}\xi' &= [7.63992]\,\rho\cos\phi'\cos(\mu-\lambda)\\ \eta' &= [7.63992]\,\rho\cos\phi'\sin d\sin(\mu-\lambda)\\ &= [7.63992]\,\xi\sin d.\end{aligned}\right\} \quad (199)$$

The variations of x and y for one minute of mean time are represented by x', and y', and their logarithms are given in the lower table of the Ephemeris elements for the eclipse. Now, if the time chosen for computation be exactly the instant of beginning or ending of the eclipse, then $\Delta = L$; but as this is scarcely possible a correction τ in minutes must be made to the assumed Ephemeris time T.

We may then write,

$$L\sin P = x - \xi + (x' - \xi')\,\tau, \quad (200)$$

$$L\cos P = y - \eta + (y' - \eta')\,\tau. \quad (201)$$

Assume the auxiliary quantities m, M, n, N, given by the equations,

$$\left.\begin{aligned}m\sin M &= x - \xi,\\ m\cos M &= y - \eta,\\ n\sin N &= x' - \xi',\\ n\cos N &= y' - \eta'.\end{aligned}\right\} \quad (202)$$

From these we have

$$\left.\begin{aligned}L\sin(P-N) &= m\sin(M-N),\\ L\cos(P-N) &= m\cos(M-N) + n\,\tau.\end{aligned}\right\} \quad (203)$$

Hence putting $\psi = P - N$, we have

$$\sin\psi = \frac{m\sin(M-N)}{L}, \quad (204)$$

$$\tau = -\frac{m\cos(M-N)}{n} \pm \frac{L\cos\psi}{n}, \quad (205)$$

the lower sign of the second term in the second member of the last equation corresponding to the time of beginning and the upper to the time of ending of the eclipse.*

4. **The Position Angle of the Point of Contact.**—The angle required is $P = N + \psi$ for the end and $P = N - \psi \pm 180°$ for the beginning of the eclipse.

5. We now have all the equations, and the Ephemeris gives us the Besselian table of elements from which the circumstances of an eclipse can be computed at any place. These equations are here arranged in the order in which they would be used, and the student is referred to the type problem worked out in the Ephemeris as a guide.

1. Constants for the given place,

$\left.\begin{matrix}\rho \sin \phi' \\ \rho \cos \phi'\end{matrix}\right\}$ Found from table page 505, Ephemeris, knowing the observer's latitude.

2. Coördinates of observer, referred to center of earth.

$$\xi = \rho \cos \phi' \sin (\mu - a).$$

$$\eta = \rho \sin \phi' \cos d - \rho \cos \phi' \sin d \cos (\mu - a),$$

$$\zeta = \rho \sin \phi' \sin d + \rho \cos \phi' \cos d \cos (\mu - a).$$

3. Variations of observer's coördinates in one minute of mean time,

$$\xi' = [7.63992]\ \rho \cos \phi' \cos (\mu - \lambda).$$

$$\eta' = [7.63992]\ \xi \sin d.$$

4. The values of m, M, n and N, given by

$$m \sin M = x - \xi',$$

$$m \cos M = y - \eta,$$

$$n \sin N = x' - \xi',$$

$$n \cos N = y' - \eta'.$$

---

* See page 506, Ephemeris.

5. The radius L of the shadow or penumbra on a plane passing through the observer, parallel to the fundamental plane, and at a distance ζ from it.

$$L = l - \zeta \tan f.$$

6. The value of the angle ψ,

$$\sin \psi = \frac{m \sin (M - N)}{L},$$

7. The value of the time τ in minutes

$$\tau = -\frac{m \cos (M - N)}{n} \pm \frac{L \cos \psi}{n},$$

8. The position angle P, from

$$P = N + \psi.$$

or

$$P = \mathrm{N} - \psi \pm 180°.$$

TABLES.

TABLE I. Reduction of Latitude, and Logarithm of Earth's Radius.

Argument, φ = Geographical Latitude, ρ = Earth's Radius.

φ	$\varphi-\varphi'$	$\log\rho$	φ	$\varphi-\varphi'$	$\log\rho$
° ′	′ ″		° ′	′ ″	
0 0	0 0.00	0.000 0000	35 0	10 48.25	9.999 5248
1 0	0 24.02	9.999 9996	10	49.63	5208
2 0	0 48.02	9982	20	50.98	5169
3 0	1 11.95	9961	30	52.31	5129
4 0	1 35.80	9930	40	53.62	5089
5 0	1 59.54	9891	50	54.90	5049
6 0	2 23.12	9.999 9843	36 0	10 56.16	9.999 5009
7 0	2 46.54	9786	10	57.41	4969
8 0	3 9.76	9721	20	58.63	4929
9 0	3 32.74	9648	30	59.82	4888
10 0	3 55.47	9566	40	11 1.00	4848
11 0	4 17.93	9476	50	2.15	4807
12 0	4 40.06	9.999 9377	37 0	11 3.28	9.999 4767
13 0	5 1.85	9271	10	4.39	4726
14 0	5 23.28	9157	20	5.47	4686
15 0	5 44.33	9035	30	6.54	4645
16 0	6 4.95	8905	40	7.58	4604
17 0	6 25.14	8768	50	8.59	4563
18 0	6 44.86	9.999 8624	38 0	11 9.59	9.999 4522
19 0	7 4.09	8472	10	10.56	4481
20 0	7 22.80	8314	20	11.51	4440
21 0	7 40.99	8149	30	12.44	4399
22 0	7 58.61	7977	40	13.34	4358
23 0	8 15.66	7799	50	14.22	4317
24 0	8 32.10	9.999 7614	39 0	11 15.08	9.999 4276
25 0	8 47.93	7424	10	15.92	4234
26 0	9 3.12	7228	20	16.73	4193
27 0	9 17.65	7027	30	17.52	4152
28 0	9 31.50	6820	40	18.29	4110
29 0	9 44.66	6608	50	19.04	4069
30 0	9 57.12	9.999 6392	40 0	11 19.76	9.999 4027
10	9 59.12	6355	10	20.46	3985
20	10 1.11	6319	20	21.13	3944
30	3.07	6282	30	21.79	3902
40	5.02	6245	40	22.42	3860
50	6.94	6208	50	23.02	3819
31 0	10 8.85	9.999 6171	41 0	11 23.61	9.999 3777
10	10.73	6134	10	24.17	3735
20	12.59	6096	20	24.70	3693
30	14.44	6059	30	25.22	3651
40	16.26	6021	40	25.71	3609
50	18.06	5984	50	26.18	3567
32 0	10 19.84	9.999 5946	42 0	11 26.62	9.999 3525
10	21.60	5908	10	27.04	3483
20	23.34	5870	20	27.44	3441
30	25.05	5832	30	27.82	3399
40	26.75	5794	40	28.17	3357
50	28.43	5755	50	28.50	3315
33 0	10 30.08	9.999 5717	43 0	11 28.80	9.999 3273
10	31.71	5678	10	29.08	3230
20	33.32	5640	20	29.34	3188
30	34.91	5601	30	29.58	3146
40	36.48	5562	40	29.79	3104
50	38.03	5523	50	29.98	3062
34 0	10 39.55	9.999 5484	44 0	11 30.14	9.999 3019
10	41.06	5445	10	30.29	2977
20	42.54	5406	20	30.41	2935
30	44.00	5367	30	30.50	2892
40	45.44	5327	40	30.57	2850
50	46.86	5288	50	30.62	2808

TABLE I. Reduction of Latitude, and Logarithm of Earth's Radius.

Argument, φ = Geographical Latitude, ρ = Earth's Radius.

φ ° ′	$\varphi-\varphi'$ ′ ″	log ρ
45 0	11 30.65	9.999 2766
10	30.65	2723
20	30.63	2681
30	30.58	2639
40	30.51	2596
50	30.42	2554
46 0	11 30.31	9.999 2512
10	30.17	2470
20	30.01	2427
30	29.82	2385
40	29.61	2343
50	29.38	2300
47 0	11 29.12	9.999 2258
10	28.85	2216
20	28.54	2174
30	28.22	2132
40	27.87	2089
50	27.50	2047
48 0	11 27.10	9.999 2005
10	26.69	1963
20	26.24	1921
30	25.78	1879
40	25.29	1837
50	24.78	1795
49 0	11 24.24	9.999 1753
10	23.69	1711
20	23.11	1669
30	22.50	1627
40	21.87	1586
50	21.22	1544
50 0	11 20.25	9.999 1502
10	19.85	1460
20	19.13	1419
30	18.39	1377
40	17.63	1335
50	16.84	1294
51 0	11 16.02	9.999 1252
10	15.19	1211
20	14.33	1170
30	13.45	1128
40	12.55	1087
50	11.62	1046
52 0	11 10.67	9.999 1005
10	9.70	0963
20	8.71	0922
30	7.69	0881
40	6.66	0840
50	5.60	0800
53 0	11 4.51	9.999 0759
10	3.40	0718
20	2.27	0677
30	1.12	0637
40	10 59.94	0596
50	58.74	0556
54 0	10 57.52	9.999 0515
10	56.28	0475
20	55.02	0435
30	53.73	0395
40	52.42	0355
50	51.09	0315

φ ° ′	$\varphi-\varphi'$ ′ ″	log ρ
55 0	10 49.74	9.999 0275
10	48.36	0235
20	46.97	0195
30	45.55	0155
40	44.11	0116
50	42.65	0076
56 0	10 41.16	9.999 0037
10	39.65	9.998 9998
20	38.13	9958
30	36.58	9919
40	35.01	9880
50	33.41	9841
57 0	10 31.80	9.998 9802
10	30.16	9764
20	28.50	9725
30	26.83	9686
40	25.13	9648
50	23.40	9610
58 0	10 21.66	9.998 9571
10	19.90	9533
20	18.11	9495
30	16.31	9457
40	14.48	9419
50	12.63	9382
59 0	10 10.77	9.998 9344
10	8.88	9307
20	6.97	9269
30	5.04	9232
40	3.08	9195
50	1.11	9158
60 0	9 59.12	9.998 9121
61 0	9 46.74	8902
62 0	9 33.65	8688
63 0	9 19.85	8479
64 0	9 5.36	8275
65 0	8 50.21	8077
66 0	8 34.40	9.998 7884
67 0	8 17.97	7697
68 0	8 0.92	7517
69 0	7 43.29	7342
70 0	7 25.08	7174
71 0	7 6.33	7013
72 0	6 47.06	9.998 6859
73 0	6 27.28	6713
74 0	6 7.03	6573
75 0	5 46.33	6441
76 0	5 25.20	6317
77 0	5 3.67	6201
78 0	4 41.77	9.998 6093
79 0	4 19.53	5993
80 0	3 56.96	5901
81 0	3 34.10	5818
82 0	3 10.98	5743
83 0	2 47.63	5676
84 0	2 24.07	9.998 5619
85 0	2 0.33	5570
86 0	1 36.44	5530
87 0	1 12.43	5498
88 0	0 48.34	5476
89 0	0 24.18	5463
90 0	0 0.00	9.998 5458

Table II. Mean Refraction.

Barometer 30 inches. Fahrenheit Thermometer 50°.

Apparent Altitude.		Mean Refraction.		Apparent Altitude.		Mean Refraction.		Apparent Altitude.		Mean Refraction.		Apparent Altitude.		Mean Refraction.	
°	′	′	″	°	′	′	″	°	′	′	″	°	′	′	″
				9	30	5	35.1	14	30	3	41.6	20	0	2	38.8
0	0	36	29		35	5	32.4		35	3	40.3		10	2	37.4
1	0	24	54		40	5	29.6		40	3	39.0		20	2	36.0
2	0	18	26		45	5	27.0		45	3	37.7		30	2	34.6
3	0	14	25		50	5	24.3		50	3	36.5		40	2	33.3
4	0	11	44		55	5	21.7		55	3	35.3		50	2	32.0
5	0	9	52.0	10	0	5	19.2	15	0	3	34.1	21	0	2	30.7
	5	9	44.0		5	5	16.7		5	3	32.9		10	2	29.4
	10	9	36.2		10	5	14.2		10	3	31.7		20	2	28.1
	15	9	28.6		15	5	11.7		15	3	30.5		30	2	26.9
	20	9	21.2		20	5	9.3		20	3	29.4		40	2	25.7
	25	9	14.0		25	5	6.9		25	3	28.2		50	2	24.5
	30	9	7.0		30	5	4.6		30	3	27.1	22	0	2	23.3
	35	9	0.1		35	5	2.3		35	3	25.9		10	2	22.1
	40	8	53.4		40	5	0.0		40	3	24.8		20	2	20.9
	45	8	46.8		45	4	57.8		45	3	23.7		30	2	19.8
	50	8	40.4		50	4	55.6		50	3	22.6		40	2	18.7
	55	8	34.2		55	4	53.4		55	3	21.5		50	2	17.5
6	0	8	28.0	11	0	4	51.2	16	0	3	20.5	23	0	2	16.4
	5	8	22.1		5	4	49.1		5	3	19.4		10	2	15.4
	10	8	16.2		10	4	47.0		10	3	18.4		20	2	14.3
	15	8	10.5		15	4	44.9		15	3	17.3		30	2	13.3
	20	8	4.8		20	4	42.9		20	3	16.3		40	2	12.2
	25	7	59.3		25	4	40.9		25	3	15.2		50	2	11.2
	30	7	53.9		30	4	38.9		30	3	14.2	24	0	2	10.2
	35	7	48.7		35	4	36.9		35	3	13.2		10	2	9.2
	40	7	43.5		40	4	35.0		40	3	12.2		20	2	8.2
	45	7	38.4		45	4	33.1		45	3	11.2		30	2	7.2
	50	7	33.5		50	4	31.2		50	3	10.3		40	2	6.2
	55	7	28.6		55	4	29.4		55	3	9.3		50	2	5.3
7	0	7	23.8	12	0	4	27.5	17	0	3	8.3	25	0	2	4.4
	5	7	19.2		5	4	25.7		5	3	7.3		10	2	3.4
	10	7	14.6		10	4	23.9		10	3	6.4		20	2	2.5
	15	7	10.1		15	4	22.2		15	3	5.5		30	2	1.6
	20	7	5.7		20	4	20.4		20	3	4.6		40	2	0.7
	25	7	1.4		25	4	18.7		25	3	3.7		50	1	59.8
	30	6	57.1		30	4	17.0		30	3	2.8	26	0	1	58.9
	35	6	53.0		35	4	15.3		35	3	1.9		10	1	58.1
	40	6	48.9		40	4	13.6		40	3	1.0		20	1	57.2
	45	6	44.9		45	4	12.0		45	3	0.1		30	1	56.4
	50	6	41.0		50	4	10.4		50	2	59.2		40	1	55.5
	55	6	37.1		55	4	8.8		55	2	58.3		50	1	54.7
8	0	6	33.3	13	0	4	7.2	18	0	2	57.5	27	0	1	53.9
	5	6	29.6		5	4	5.6		5	2	56.6		10	1	53.1
	10	6	25.9		10	4	4.1		10	2	55.8		20	1	52.3
	15	6	22.3		15	4	2.6		15	2	54.9		30	1	51.5
	20	6	18.8		20	4	1.0		20	2	54.1		40	1	50.7
	25	6	15.3		25	3	59.6		25	2	53.2		50	1	50.0
	30	6	11.9		30	3	58.1		30	2	52.4	28	0	1	49.2
	35	6	8.5		35	3	56.6		35	2	51.6		10	1	48.4
	40	6	5.2		40	3	55.2		40	2	50.8		20	1	47.7
	45	6	2.0		45	3	53.7		45	2	50.0		30	1	46.9
	50	5	58.8		50	3	52.3		50	2	49.2		40	1	46.2
	55	5	55.7		55	3	50.9		55	2	48.4		50	1	45.5
9	0	5	52.6	14	0	3	49.5	19	0	2	47.7	29	0	1	44.8
	5	5	49.6		5	3	48.1		10	2	46.1		20	1	43.4
	10	5	46.6		10	3	46.8		20	2	44.6		40	1	42.0
	15	5	43.6		15	3	45.5		30	2	43.1	30	0	1	40.6
	20	5	40.7		20	3	44.2		40	2	41.6		20	1	39.3
	25	5	37.9		25	3	42.9		50	2	40.2		40	1	38.0

TABLE II. Mean Refraction.

Barometer 30 inches. Fahrenheit Thermometer 50°.

Apparent Altitude.		Mean Refraction.		Apparent Altitude.		Mean Refraction.		Apparent Altitude.		Mean Refraction.		Apparent Altitude.		Mean Refraction.	
°	'	'	''	°	'	'	''	°	'	'	''	°	'	'	''
31	0	1	36.7	41	0	1	7.0	51	0	0	47.2	61	0	0	32.3
	20	1	35.5		20	1	6.2		20	0	46.6	62	0	0	31.0
	40	1	34.2		40	1	5.4		40	0	46.1	63	0	0	29.7
32	0	1	33.0	42	0	1	4.7	52	0	0	45.5	64	0	0	28.4
	20	1	31.8		20	1	3.9		20	0	45.0	65	0	0	27.2
	40	1	30.7		40	1	3.2		40	0	44.4	66	0	0	25.9
33	0	1	29.5	43	0	1	2.4	53	0	0	43.9	67	0	0	24.7
	20	1	28.4		20	1	1.7		20	0	43.4	68	0	0	23.6
	40	1	27.3		40	1	1.0		40	0	42.8	69	0	0	22.4
34	0	1	26.2	44	0	1	0.3	54	0	0	42.3	70	0	0	21.2
	20	1	25.1		20	0	59.6		20	0	41.8	71	0	0	20.1
	40	1	24.1		40	0	58.9		40	0	41.3	72	0	0	18.9
35	0	1	23.1	45	0	0	58.2	55	0	0	40.8	73	0	0	17.8
	20	1	22.0		20	0	57.6		20	0	40.3	74	0	0	16.7
	40	1	21.0		40	0	56.9		40	0	39.8	75	0	0	15.6
36	0	1	20.1	46	0	0	56.2	56	0	0	39.3	76	0	0	14.5
	20	1	19.1		20	0	55.6		20	0	38.8	77	0	0	13.5
	40	1	18.2		40	0	55.0		40	0	38.3	78	0	0	12.4
37	0	1	17.2	47	0	0	54.3	57	0	0	37.8	79	0	0	11.3
	20	1	16.3		20	0	53.7		20	0	37.3	80	0	0	10.3
	40	1	15.4		40	0	53.1		40	0	36.9	81	0	0	9.2
38	0	1	14.5	48	0	0	52.5	58	0	0	36.4	82	0	0	8.2
	20	1	13.6		20	0	51.9		20	0	35.9	83	0	0	7.2
	40	1	12.7		40	0	51.2		40	0	35.5	84	0	0	6.1
39	0	1	11.9	49	0	0	50.6	59	0	0	35.0	85	0	0	5.1
	20	1	11.0		20	0	50.0		20	0	34.5	86	0	0	4.1
	40	1	10.2		40	0	49.4		40	0	34.1	87	0	0	3.1
40	0	1	9.4	50	0	0	48.9	60	0	0	33.6	88	0	0	2.0
	20	1	8.6		20	0	48.3		20	0	33.2	89	0	0	1.0
	40	1	7.8		40	0	47.8		40	0	32.7	90	0	0	0.0

$$r'' = \alpha (\beta T)^A \gamma^\lambda \tan z$$

$z = 35^\circ$

$\log \alpha = 1.76156$

$$\log r'' = \log \alpha + A \log \beta + \lambda \log \gamma + \log \tan z,$$ } See Table III.

$\log \beta = \log B + \log T,$

α A & λ vary slowly with apparent z,

B depends on reading of barometer,

T reading of attached thermometer

γ function temperature of external air as shown by detached thermometer.

HEADQUARTERS U. S. MILITARY ACADEMY,

WEST POINT, N. Y., May 21, 1902.

CIRCULAR,
No. 29.

1. The officers named below, except the first three who constituted the original committee, are added to the committee on reception and entertainment, Centennial celebration, so that the committee is composed of: The Professor of Engineering, the Professor of Mathematics, the Instructor of Ordnance and Gunnery, Captains Landis, Hanson, Willcox, Davis, Robinson, Jones, Livermore, Palmer, Saxton, Parker, Jervey, Coe, Averill, Bigelow, A. Hamilton, Nolan, Sills, Callan, Guignard, McNeil, Hinkley, Hagood, Abernethy, Sarratt, Bowley, Gilbert, 1st Lieutenants Stuart, Oakes, Pope, Mitchell, Smither, Jewell.

2. Captains Franklin and Hero are added to the banquet committee, Centennial celebration, so that the committee now consists of the following officers: The Professor of Mathematics, the Commandant of Cadets, the Instructor of Practical Military Engineering, Captains Franklin, Hero.

3. The three officers last named below are added to the committee on exterior decoration and illumination, Centennial celebration, and the committee now consists of: The Professor of Philosophy, the Instructor of Ordnance and Gunnery, the Instructor of Practical Military Engineering, Captains Pierce, Jamieson, 1st Lieutenant Ladue.

4. The officers whose names appear below, except the first two who constituted the original committee, are added to the committee on sports and military ceremonies, Centennial celebration, so that the committee consists of: The Professor of Philosophy, the Commandant of Cadets, Captains Greble, G. F. Hamilton, Lewis, 1st Lieutenants Roberts, Koehler, 2nd Lieutenant Glade.

BY ORDER OF COLONEL MILLS:

W. C. RIVERS,
Captain 1st Cavalry,
Adjutant.

Zenith distance.		Arg. app. zen.-dist.				Zenith distance.		Arg. app. zen.-dist.			
		Log α.		A	λ			Log α.		A	λ
°	′					°	′				
0	0	1.76156	2			77	0	1.75229	24	1.0026	1.0252
10	0	1.76154	5				10	1.75205	25	1.0026	1.0258
20	0	1.76149	10				20	1.75180	25	1.0027	1.0264
30	0	1.76139	9				30	1.75155	26	1.0027	1.0272
35	0	1.76130	11				40	1.75129	28	1.0028	1.0281
40	0	1.76119	15				50	1.75101	29	1.0029	1.0290
45	0	1.76104	4		1.0018	78	0	1.75072	29	1.0030	1.0299
46	0	1.76100	4		1.0019		10	1.75043	30	1.0030	1.0308
47	0	1.76096	4		1.0019		20	1.75013	32	1.0031	1.0318
48	0	1.76092	5		1.0020		30	1.74981	34	1.0032	1.0328
49	0	1.76087	5		1.0021		40	1.74947	35	1.0033	1.0338
50	0	1.76082	5		1.0023		50	1.74912	36	1.0034	1.0347
51	0	1.76077	6		1.0025	79	0	1.74876	37	1.0035	1.0357
52	0	1.76071	6		1.0026		10	1.74839	40	1.0036	1.0367
53	0	1.76065	7		1.0027		20	1.74799	42	1.0037	1.0377
54	0	1.76058	8		1.0029		30	1.74757	43	1.0038	1.0387
55	0	1.76050	8		1.0031		40	1.74714	44	1.0039	1.0398
56	0	1.76042	9		1.0034		50	1.74670	47	1.0040	1.0409
57	0	1.76033	10		1.0037	80	0	1.74623	50	1.0041	1.0420
58	0	1.76023	11		1.0040		10	1.74573	52	1.0042	1.0431
59	0	1.76012	11		1.0043		20	1.74521	53	1.0043	1.0442
60	0	1.76001	13		1.0046		30	1.74468	56	1.0045	1.0454
61	0	1.75988	15		1.0049		40	1.74412	60	1.0046	1.0466
62	0	1.75973	16		1.0054		50	1.74352	64	1.0047	1.0479
63	0	1.75957	18		1.0058	81	0	1.74288	65	1.0049	1.0493
64	0	1.75939	20		1.0063		10	1.74223	68	1.0050	1.0508
65	0	1.75919	22		1.0068		20	1.74155	72	1.0052	1.0523
66	0	1.75897	26		1.0075		30	1.74083	76	1.0054	1.0540
67	0	1.75871	29		1.0083		40	1.74007	79	1.0056	1.0559
68	0	1.75842	33		1.0092		50	1.73928	83	1.0058	1.0579
69	0	1.75809	38		1.0101	82	0	1.73845	88	1.0060	1.0600
70	0	1.75771	45		1.0111		10	1.73757	94	1.0062	1.0622
71	0	1.75726	51		1.0124		20	1.73663	99	1.0065	1.0646
72	0	1.75675	60		1.0139		30	1.73564	105	1.0067	1.0671
73	0	1.75615	72		1.0156		40	1.73459	112	1.0070	1.0697
74	0	1.75543	86		1.0175		50	1.73347	118	1.0073	1.0725
75	0	1.75457	16		1.0197	83	0	1.73229	124	1 0075	1.0754
	10	1.75441	16		1.0200		10	1.73105	131	1.0978	1,0784
	20	1.75425	17		1.0204		20	1.72974	142	1.0081	1.0815
	30	1.75408	17		1.0208		30	1.72832	151	1.0084	1.0846
	40	1.75391	18		1.0212		40	1.72681	162	1.0082	1.0879
	50	1.75373	18		1.0216		50	1.72519	173	1.0098	1.0914
76	0	1.75355	19		1.0220	84	0	1.72346	186	1.0096	1.0951
	10	1.75336	20		1.0225		10	1.72160	199	1.0100	1.0992
	20	1.75316	21		1.0230		20	1.71961	212	1.0105	1.1036
	30	1.75295	21		1.0235		30	1.71749	227	1.0110	1.1082
	40	1.75274	22		1.0241		40	1.71522	243	1.0115	1.1130
	50	1.75252	23		1.0246		50	1.71279	259	1.0121	1.1178
77	0	1.75229		1.0026	1.0252	85	0	1.71020		1.0127	1.1229

TABLE III. Bessel's Refraction Table.

Factor depending upon the barometer.

Eng. ins.	Log B.
27.5	—0.03191
27.6	0.03033
27.7	0.02876
27.8	0.02720
27.9	0.02564
28.0	0.02409
28.1	0.02254
28.2	0.02099
28.3	0.01946
28.4	0.01793
28.5	0.01640
28.6	0.01488
28.7	0.01336
28.8	0.01185
28.9	0.01035
29.0	0.00885
29.1	0.00735
29.2	0.00586
29.3	0.00438
29.4	0.00290
29.5	—0.00142
29.6	+0.00005
29.7	0.00151
29.8	0.00297
29.9	0.00443
30.0	0.00588
30.1	0.00732
30.2	0.00876
30.3	0.01020
30.4	0.01163
30.5	0.01306
30.6	0.01448
30.7	0.01589
30.8	0.01731
30.9	0.01871
31.0	+0.02012

Factor depending upon the attached thermometer.

F.	Log T.
°	
—30	+0.00242
20	0.00203
—10	0.00164
0	0.00125
10	0.00086
+20	0.00047
30	+0.00008
40	—0.00031
50	0.00070
60	0.00109
70	0.00148
80	0.00186
90	0.00225
+100	—0.00264

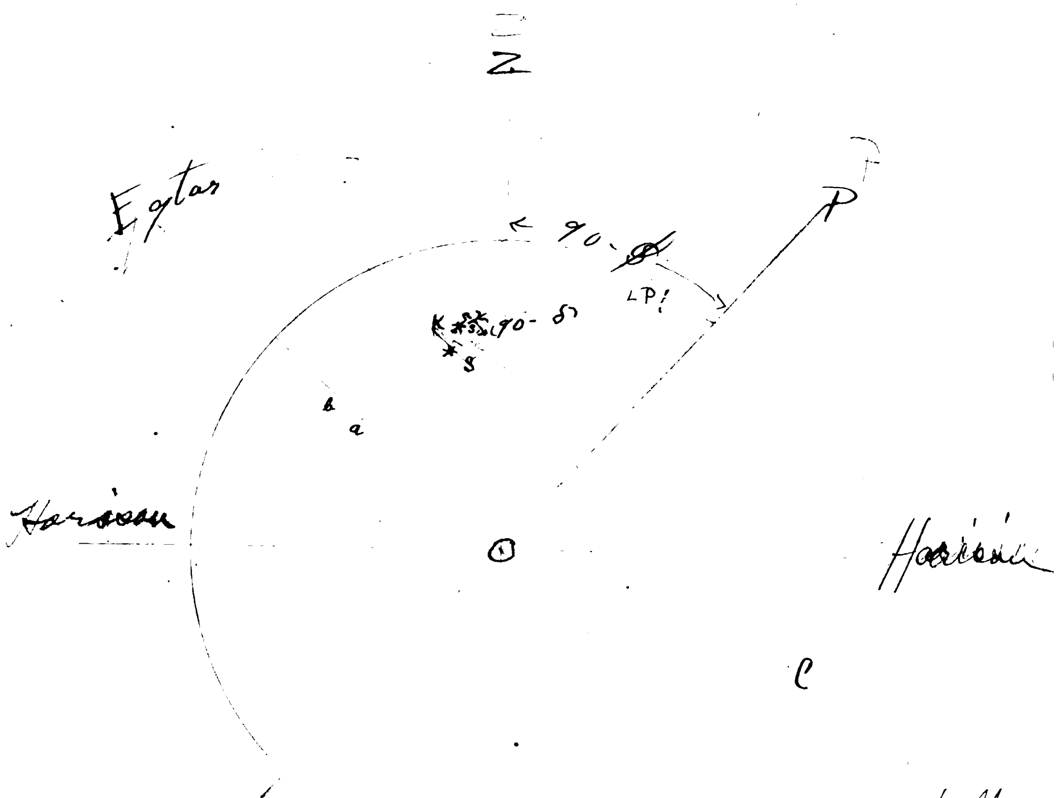
$\log\beta = \log B + \log T.$

Factor depending upon the external thermometer.

F.	Log γ.	F.	Log γ.
°		°	
—20	+0.06276	+35	+0.01185
19	0.06181	36	0.01098
18	0.06083	37	0.01011
17	0.05985	38	0.00924
16	0.05887	39	0.00837
15	0.05790	40	0.00750
14	0.05693	41	0.00664
13	0.05596	42	0.00578
12	0.05500	43	0.00492
11	0.05403	44	0.00406
10	0.05307	45	0.00320
9	0.05211	46	0.00234
8	0.05115	47	0.00149
7	0.05020	48	+0.00064
6	0.04924	49	—0.00021
5	0.04829	50	0.00106
4	0.04734	51	0.00191
3	0.04640	52	0.00275
2	0.04545	53	0.00360
—1	0.04451	54	0.00444
0	0.04357	55	0.00528
+1	0.04263	56	0.00612
2	0.04169	57	0.00696
3	0.04076	58	0.00780
4	0.03982	59	0.00863
5	0.03889	60	0.00946
6	0.03796	61	0.01029
7	0.03704	62	0.01112
8	0.03611	63	0.01195
9	0.03519	64	0.01278
10	0.03427	65	0.01360
11	0.03335	66	0.01443
12	0.03243	67	0.01525
13	0.03152	68	0.01607
14	0.03060	69	0.01689
15	0.02969	70	0.01770
16	0.02878	71	0.01852
17	0.02787	72	0.01933
18	0.02697	73	0.02015
19	0.02606	74	0.02096
20	0.02516	75	0.02177
21	0.02426	76	0.02257
22	0.02336	77	0.02338
23	0.02247	78	0.02419
24	0.02157	79	0.02499
25	0.02068	80	0.02579
26	0.01979	81	0.02659
27	0.01890	82	0.02738
28	0.01801	83	0.02819
29	0.01713	84	0.02898
30	0.01624	85	0.02978
31	0.01536	86	0.03057
32	0.01448	87	0.03136
33	0.01360	88	0.03216
34	0.01273	89	0.03294
+35	+0.01185	+90	—0.03373

$KS' = \Delta\delta$ ~~&~~ $\sin \Delta\delta = \sin r \times \sin S = \sin r \cos \psi$, KSS' being at $\angle S$ in our small ⊙ KSC being ⊥ to PS & PS'.

KS



place $SS' = r$, S being true place of Star, S' apparent place.

~~S~~ C' small ⊙ through S whose plane is ∥ to plane of Eqtor.

then δ is apparently increased by $KS' = \Delta\delta$ & α by ab, but $ab = \frac{SK}{\cos\delta}$, $SK = SS' \sin\psi = r \sin\psi$

$\therefore ab = \Delta\alpha = \frac{r \sin\psi}{\cos\delta}$; $KS' = S'S \cos\psi = r \cos\psi$

$\therefore KS' = \Delta\delta = r \cos\psi$

~~$\Delta\alpha$ is + when~~ $\Delta\alpha$ always carries the object towards the meridian $\therefore$ it increases the R.A. of objects ~~are~~ west of the meridian & decreases that of objects east of the meridian. $\alpha = T + P$ a.m. } P being always too small
$\alpha = T - P$ p.m. } α is too small for a.m. & too large for p.m. observation

TABLE IVa. Logarithms of A and B.

For computing Equation of Equal Altitudes when "Elapsed Time" is short.

Elapsed Time h.	Elapsed Time m.	Log. A.	Log. B.	Elapsed Time h.	Elapsed Time m.	Log. A.	Log. B.	Elapsed Time h.	Elapsed Time m.	Log. A.	Log. B.
0	0	9.4059	9.4059	0	40	9.4065	9.4048	1	20	9.4081	9.4015
	2	.4059	.4059		42	.4065	.4047		22	.4083	.4013
	4	.4059	.4059		44	.4066	.4046		24	.4084	.4010
	6	.4060	.4059		46	.4067	.4045		26	.4085	.4008
	8	.4060	.4059		48	.4067	.4043		28	.4086	.4006
	10	.4060	.4059		50	.4068	.4042		30	.4087	.4003
	12	.4060	.4058		52	.4069	.4041		32	.4089	.4001
	14	.4060	.4058		54	.4069	.4039		34	.4090	.3998
	16	.4060	.4058		56	.4070	.4038		36	.4091	.3995
	18	.4061	.4057	0	58	.4071	.4036		38	.4093	.3993
	20	.4061	.4057	1	0	.4072	.4034		40	.4094	.3990
	22	.4061	.4056		2	.4073	.4033		42	.4095	.3987
	24	.4061	.4055		4	.4074	.4031		44	.4097	.3984
	26	.4062	.4055		6	.4074	.4029		46	.4098	.3981
	28	.4062	.4054		8	.4075	.4027		48	.4100	.3978
	30	.4062	.4053		10	.4076	.4025		50	.4101	.3975
	32	.4063	.4052		12	.4077	.4023		52	.4103	.3972
	34	.4063	.4051		14	.4078	.4021		54	.4104	.3969
	36	.4064	.4050		16	.4079	.4019		56	.4106	.3965
0	38	9.4064	9.4049	1	18	9.4080	9.4017	1	58	9.4107	9.3962

TABLE IV. Logarithms of A and B.

For computing Equation of Equal Altitudes.

Elapsed Time. h.	m.	Log. A.	Log. B.	Elapsed Time. h.	m.	Log. A.	Log. B.	Elapsed Time. h.	m.	Log. A.	Log. B.
2	0	9.4109	9.3959	4	0	9.4260	9.3635	6	0	9.4515	9.3010
	2	.4111	.3955		2	.4263	.3627		2	.4521	.2996
	4	.4113	.3952		4	.4266	.3620		4	.4526	.2982
	6	.4114	.3948		6	.4270	.3612		6	.4531	.2968
	8	.4116	.3944		8	.4273	.3604		8	.4536	.2954
	10	.4118	.3941		10	.4277	.3596		10	.4542	.2940
	12	.4120	.3937		12	.4280	.3588		12	.4547	.2925
	14	.4121	.3933		14	.4284	.3580		14	.4552	.2911
	16	.4123	.3929		16	.4288	.3572		16	.4558	.2896
	18	.4125	.3925		18	.4291	.3564		18	.4563	.2881
	20	.4127	.3921		20	.4295	.3555		20	.4569	.2866
	22	.4129	.3917		22	.4299	.3547		22	.4574	.2850
	24	.4131	.3913		24	.4302	.3538		24	.4580	.2835
	26	.4133	.3909		26	.4306	.3530		26	.4585	.2819
	28	.4135	.3905		28	.4310	.3521		28	.4591	.2804
	30	.4137	.3900		30	.4314	.3512		30	.4597	.2788
	32	.4139	.3896		32	.4317	.3503		32	.4602	.2772
	34	.4141	.3892		34	.4321	.3494		34	.4608	.2756
	36	.4144	.3887		36	.4325	.3485		36	.4614	.2739
	38	.4146	.3882		38	.4329	.3476		38	.4620	.2723
	40	.4148	.3878		40	.4333	.3467		40	.4625	.2706
	42	.4150	.3873		42	.4337	.3457		42	.4631	.2689
	44	.4152	.3868		44	.4341	.3448		44	.4637	.2672
	46	.4155	.3863		46	.4345	.3438		46	.4643	.2655
	48	.4157	.3859		48	.4349	.3429		48	.4649	.2638
	50	.4159	.3854		50	.4353	.3419		50	.4655	.2620
	52	.4162	.3849		52	.4357	.3409		52	.4661	.2602
	54	.4164	.3843		54	.4361	.3399		54	.4667	.2584
	56	.4167	.3838		56	.4366	.3389		56	.4673	.2566
2	58	.4169	.3833	4	58	.4370	.3379	6	58	.4679	.2548
3	0	.4172	.3828	5	0	.4374	.3369	7	0	.4685	.2530
	2	.4174	.3822		2	.4378	.3358		2	.4691	.2511
	4	.4177	.3817		4	.4383	.3348		4	.4697	.2492
	6	.4179	.3811		6	.4387	.3337		6	.4704	.2473
	8	.4182	.3806		8	.4391	.3327		8	.4710	.2454
	10	.4184	.3800		10	.4396	.3316		10	.4716	.2434
	12	.4187	.3794		12	.4400	.3305		12	.4723	.2415
	14	.4190	.3789		14	.4405	.3294		14	.4729	.2395
	16	.4193	.3783		16	.4409	.3283		16	.4735	.2375
	18	.4195	.3777		18	.4414	.3272		18	.4742	.2355
	20	.4198	.3771		20	.4418	.3261		20	.4748	.2334
	22	.4201	.3765		22	.4423	.3249		22	.4755	.2313
	24	.4204	.3759		24	.4427	.3238		24	.4761	.2292
	26	.4207	.3752		26	.4432	.3226		26	.4768	.2271
	28	.4209	.3746		28	.4437	.3214		28	.4774	.2250
	30	.4212	.3740		30	.4441	.3203		30	.4781	.2228
	32	.4215	.3733		32	.4446	.3191		32	.4788	.2206
	34	.4218	.3727		34	.4451	.3178		34	.4794	.2184
	36	.4221	.3720		36	.4456	.3166		36	.4801	.2162
	38	.4224	.3713		38	.4460	.3154		38	.4808	.2140
	40	.4227	.3707		40	.4465	.3142		40	.4815	.2117
	42	.4231	.3700		42	.4470	.3129		42	.4821	.2094
	44	.4234	.3693		44	.4475	.3116		44	.4828	.2070
	46	.4237	.3686		46	.4480	.3103		46	.4835	.2047
	48	.4240	.3679		48	.4485	.3091		48	.4842	.2023
	50	.4243	.3672		50	.4490	.3078		50	.4849	.1999
	52	.4246	.3665		52	.4494	.3064		52	.4856	.1974
	54	.4250	.3657		54	.4500	.3051		54	.4863	.1950
	56	.4253	.3650		56	.4505	.3038		56	.4870	.1925
3	58	9.4256	9.3643	5	58	9.4510	9.3024	7	58	9.4877	9.1900

TABLE IV. Logarithms of A and B.

For computing Equation of Equal Altitudes.

Elapsed Time. h. m.	Log. A.	Log. B.	Elapsed Time. h. m.	Log. A.	Log. B.	Elapsed Time. h. m.	Log. A.	Log. B.
8 0	9.4884	9.1874	14 0	9.6841	—9.0971	16 0	9.7895	—9.4884
2	.4892	.1848	2	.6856	.1057	2	.7915	.4937
4	.4899	.1822	4	.6872	.1141	4	.7935	.4990
6	.4906	.1796	6	.6887	.1224	6	.7955	.5042
8	.4913	.1769	8	.6903	.1306	8	.7975	.5094
10	.4921	.1742	10	.6919	.1387	10	.7996	.5146
12	.4928	.1715	12	.6934	.1468	12	.8016	.5197
14	.4935	.1687	14	.6950	.1547	14	.8037	.5248
16	.4943	.1659	16	.6966	.1625	16	.8058	.5300
18	.4950	.1630	18	.6982	.1703	18	.8078	.5351
20	.4958	.1602	20	.6998	.1779	20	.8099	.5401
22	.4965	.1573	22	.7014	.1855	22	.8120	.5452
24	.4973	.1543	24	.7030	.1930	24	.8141	.5502
26	.4980	.1513	26	.7047	.2004	26	.8162	.5553
28	.4988	.1483	28	.7063	.2078	28	.8184	.5603
30	.4996	.1453	30	.7079	.2150	30	.8205	.5653
32	.5003	.1422	32	.7096	.2222	32	.8227	.5702
34	.5011	.1390	34	.7112	.2298	34	.8248	.5752
36	.5019	.1359	36	.7129	.2364	36	.8270	.5801
38	.5027	.1327	38	.7146	.2434	38	.8292	.5850
40	.5035	.1294	40	.7162	.2503	40	.8314	.5900
42	.5042	.1261	42	.7179	.2571	42	.8336	.5948
44	.5050	.1228	44	.7196	.2639	44	.8358	.5997
46	.5058	.1194	46	.7213	.2706	46	.8380	.6046
48	.5066	.1159	48	.7280	.2773	48	.8402	.6094
50	.5074	.1125	50	.7247	.2839	50	.8425	.6143
52	.5082	.1089	52	.7264	.2905	52	.8447	.6191
54	.5091	.1054	54	.7281	.2970	54	.8470	.6239
56	.5099	.1017	56	.7299	.3034	56	.8493	.6287
8 58	.5107	.0981	14 58	.7316	.3098	16 58	.8516	.6335
9 0	.5115	.0943	15 0	.7333	.3162	17 0	.8539	.6383
2	.5123	.0996	2	.7351	.3225	2	.8562	.6431
4	.5132	.0867	4	.7369	.3287	4	.8585	.6478
6	.5140	.0828	6	.7386	.3350	6	.8608	.6526
8	.5148	.0789	8	.7404	.3411	8	.8632	.6573
10	.5157	.0749	10	.7422	.3472	10	.8655	.6621
12	.5165	.0708	12	.7440	.3533	12	.8679	.6668
14	.5174	.0667	14	.7458	.3593	14	.8703	.6715
16	.5182	.0625	16	.7476	.3653	16	.8727	.6762
18	.5191	.0583	18	.7494	.3713	18	.8751	.6809
20	.5199	.0540	20	.7512	.3772	20	.8775	.6856
22	.5208	.0496	22	.7531	.3831	22	.8799	.6903
24	.5217	.0452	24	.7549	.3889	24	.8824	.6949
26	.5225	.0406	26	.7568	.3947	26	.8848	.6996
28	.5234	.0360	28	.7586	.4005	28	.8873	.7043
30	.5243	.0314	30	.7605	.4062	30	.8898	.7089
32	.5252	.0266	32	.7924	.4119	32	.8923	.7136
34	.5261	.0218	34	.7642	.4175	34	.8948	.7182
36	.5269	.0169	36	.7661	.4232	36	.8973	.7228
38	.5278	.0119	38	.7680	.4288	38	.8999	.7275
40	.5287	.0069	40	.7699	.4343	40	.9024	.7321
42	.5296	.0017	42	.7718	.4399	42	.9050	.7367
44	.5305	8.9965	44	.7738	.4454	44	.9075	.7413
46	.5315	.9911	46	.7757	.4509	46	.9101	.7459
48	.5324	.9857	48	.7776	.4563	48	.9127	.7505
50	.5333	.9802	50	.7796	.4617	50	.9154	.7552
52	.5342	.9745	52	.7815	.4671	52	.9180	.7598
54	.5351	.9688	54	.7835	.4725	54	.9206	.7644
56	.5361	.9630	56	.7855	.4778	56	.9233	.7690
9 58	9.5370	8.9570	15 58	9.7875	—9.4831	17 58	9.9260	—9.7736

TABLE IV. Logarithms of A and B.

For computing Equation of Equal Altitudes.

Elapsed Time. *h.*	*m.*	Log. A.	Log. B.	Elapsed Time. *h.*	*m.*	Log. A.	Log. B.	Elapsed Time. *h.*	*m.*	Log. A.	Log. B.
18	0	9.9287	—9.7782	20	0	0.1249	—0.0625	22	0	0.4523	—0.4372
	2	.9314	.7827		2	.1290	.0676		2	.4601	.4455
	4	.9341	.7873		4	.1330	.0727		4	.4680	.4540
	6	.9368	.7919		6	.1371	.0779		6	.4761	.4625
	8	.9396	.7965		8	.1412	.0830		8	.4842	.4711
	10	.9424	.8011		10	.1454	.0882		10	.4926	.4799
	12	.9451	.8057		12	.1496	.0935		12	.5010	.4889
	14	.9479	.8103		14	.1538	.0987		14	.5097	.4980
	16	.9508	.8149		16	.1581	.1040		16	.5184	.5072
	18	.9536	.8195		18	.1623	.1093		18	.5274	.5165
	20	.9564	.8241		20	.1667	.1146		20	.5365	.5261
	22	.9593	.8287		22	.1711	.1200		22	.5458	.5358
	24	.9622	.8333		24	.1755	.1253		24	.5553	.5457
	26	.9651	.8379		26	.1799	.1308		26	.5649	.5557
	28	.9680	.8425		28	.1844	.1362		28	.5748	.5660
	30	.9709	.8471		30	.1889	.1417		30	.5848	.5764
	32	.9739	.8517		32	.1935	.1472		32	.5951	.5871
	34	.9769	.8563		34	.1981	.1527		34	.6056	.5979
	36	.9798	.8609		36	.2028	.1582		36	.6164	.6090
	38	.9829	.8655		38	.2075	.1638		38	.6273	.6204
	40	.9859	.8701		40	.2122	.1695		40	.6386	.6319
	42	.9889	.8748		42	.2170	.1751		42	.6501	.6438
	44	.9920	.8794		44	.2218	.1808		44	.6619	.6559
	46	.9951	.8840		46	.2267	.1866		46	.6740	.6684
	48	9.9982	.8887		48	.2316	.1924		48	.6865	.6811
	50	0.0013	.8933		50	.2366	.1982		50	.6993	.6942
	52	.0044	.8980		52	.2416	.2040		52	.7124	.7076
	54	.0076	.9026		54	.2467	.2099		54	.7259	.7214
	56	.0108	.9073		56	.2518	.2159		56	.7398	.7355
18	58	.0140	.9120	20	58	.2570	.2219	22	58	.7541	.7501
19	0	.0172	.9167	21	0	.2623	.2279	23	0	.7689	.7652
	2	.0204	.9213		2	.2676	.2339		2	.7842	.7807
	4	.0237	.9260		4	.2729	.2401		4	.8000	.7967
	6	.0270	.9307		6	.2783	.2462		6	.8163	.8133
	8	.0303	.9355		8	.2838	.2524		8	.8333	.8305
	10	.0336	.9402		10	.2893	.2587		10	.8508	.8483
	12	.0370	.9449		12	.2949	.2650		12	.8691	.8667
	14	.0403	.9497		14	.3005	.2714		14	.8882	.8860
	16	.0437	.9544		16	.3063	.2778		16	.9080	.9060
	18	.0472	.9592		18	.3120	.2843		18	.9288	.9270
	20	.0506	.9640		20	.3179	.2909		20	.9506	.9489
	22	.0541	.9687		22	.3238	.2975		22	.9734	.9719
	24	.0576	.9735		24	.3298	.3041		24	0.9975	—0.9961
	26	.0611	.9784		26	.3359	.3109		26	1.0228	—1.0216
	28	.0646	.9832		28	.3420	.3177		28	.0497	.0487
	30	.0682	.9880		30	.3482	.3245		30	.0783	.0774
	32	.0718	.9929		32	.3545	.3315		32	.1089	.1081
	34	.0754	—9.9977		34	.3609	.3385		34	.1416	.1409
	36	.0790	—0.0026		36	.3674	.3456		36	.1770	.1764
	38	.0827	.0075		38	.3739	.3527		38	.2154	.2149
	40	.0864	.0124		40	.3805	.3599		40	.2573	.2569
	42	.0901	.0173		42	.3873	.3673		42	.3037	.3033
	44	.0939	.0223		44	.3941	.3747		44	.3554	.3552
	46	.0976	.0272		46	.4010	.3822		46	.4140	.4138
	48	.1015	.0322		48	.4080	.3897		48	.4815	.4814
	50	.1053	.0372		50	.4151	.3974		50	.5613	.5612
	52	.1092	.0422		52	.4223	.4052		52	.6588	.6587
	54	.1131	.0478		54	.4297	.4130		54	.7844	.7843
	56	.1170	.0523		56	.4371	.4210		56	1.9610	—1.9610
19	58	0.1209	—0.0574	21	58	0.4446	—0.4291	23	58	2.2627	—2.2627

TABLE V. Reduction to the Meridian. $m = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}$

P	0^m	1^m	2^m	3^m	4^m	5^m	6^m	7^m	8^m
s	''	''	''	''	''	''	''	''	''
0	0.00	1.96	7.85	17.67	31.42	49.09	70.68	96.20	125.65
1	0.00	2.03	7.98	17.87	31.68	49.41	71.07	96.66	126.17
2	0.00	2.10	8.12	18.07	31.94	49.74	71.47	97.12	126.70
3	0.00	2.16	8.25	18.27	32.20	50.07	71.86	97.58	127.22
4	0.01	2.23	8.39	18.47	32.47	50.40	72.26	98.04	127.75
5	0.01	2.31	8.52	18.67	32.74	50.73	72.66	98.50	128.28
6	0.02	2.38	8.66	18.87	33.01	51.07	73.06	98.97	128.81
7	0.02	2.45	8.80	19.07	33.27	51.40	73.46	99.43	129.34
8	0.03	2.52	8.94	19.28	33.54	51.74	73.86	99.90	129.87
9	0.04	2.60	9.08	19.48	33.81	52.07	74.26	100.37	130.40
10	0.05	2.67	9.22	19.69	34.09	52.41	74.66	100.84	130.94
11	0.06	2.75	9.36	19.90	34.36	52.75	75.06	101.31	131.47
12	0.08	2.83	9.50	20.11	34.64	53.09	75.47	101.78	132.01
13	0.09	2.91	9.64	20.32	34.91	53.43	75.88	102.25	132.55
14	0.11	2.99	9.79	20.53	35.19	53.77	76.29	102.72	133.09
15	0.12	3.07	9.94	20.74	35.46	54.11	76.69	103.20	133.63
16	0.14	3.15	10.09	20.95	35.74	54.46	77.10	103.67	134.17
17	0.16	3.23	10.24	21.16	36.02	54.80	77.51	104.15	134.71
18	0.18	3.32	10.39	21.38	36.30	55.15	77,93	104.63	135.25
19	0.20	3.40	10.54	21.60	36.58	55.50	78.34	105.10	135.80
20	0.22	3.49	10.69	21.82	36.87	55.84	78.75	105.58	136.34
21	0.24	3.58	10.84	22.03	37.15	56.19	79.16	106.06	136.88
22	0.26	3.67	11.00	22.25	37.44	56.55	79.58	106.55	137.43
23	0.28	3.76	11.15	22.47	37.72	56.90	80.00	107.03	137.98
24	0.31	3.85	11.31	22.70	38.01	57.25	80.42	107.51	138.53
25	0.34	3.94	11.47	22.92	38.30	57.60	80.84	107.99	139.08
26	0.37	4.03	11.63	23.14	38.59	57.96	81.26	108.48	139.63
27	0.40	4.12	11.79	23.37	38.88	58.32	81.68	108.97	140.18
28	0.43	4.22	11.95	23.60	39.17	58.68	82.10	109.46	140.74
29	0.46	4.32	12.11	23.82	39.46	59.03	82.52	109.95	141.29
30	0.49	4.42	12.27	24.05	39.76	59.40	82.95	110.44	141.85
31	0.52	4.52	12.43	24.28	40.05	59.75	83.38	110.93	142.40
32	0.56	4.62	12.60	24.51	40.35	60.11	83.81	111.43	142.96
33	0.59	4.72	12.76	24.74	40.65	60.47	84.23	111.92	143.52
34	0.63	4.82	12.93	24.98	40.95	60.84	84.66	112.41	144.08
35	0.67	4.92	13.10	25.21	41.25	61.20	85.09	112.90	144.64
36	0.71	5.03	13.27	25.45	41.55	61.57	85.52	113.40	145.20
37	0.75	5.13	13.44	25.68	41.85	61.94	85.95	113 90	145.76
38	0.79	5.24	13.62	25.92	42.15	62.31	86.39	114.40	146.33
39	0.83	5.34	13.79	26.16	42.45	62.68	86.82	114.90	146.89
40	0.87	5.45	13.96	26.40	42.76	63.05	87.26	115.40	147.46
41	0.91	5.56	14.13	26.64	43.06	63.42	87.70	115.90	148.03
42	0.96	5.67	14.31	26.88	43.37	63.79	88.14	116.40	148.60
43	1.01	5.78	14.49	27.12	43.68	64.16	88.57	116.90	149.17
44	1.06	5.90	14.67	27.37	43.99	64.54	89.01	117.41	149.74
45	1.10	6.01	14.85	27.61	44.30	64.91	89.45	117.92	150.31
46	1.15	6.13	15.03	27.86	44.61	65.29	89.89	118.43	150.88
47	1.20	6.24	15.21	28.10	44.92	65.67	90.33	118.94	151.45
48	1.26	6.36	15.39	28.35	45.24	66.05	90.78	119.45	152.03
49	1.31	6.48	15.57	28.60	45.55	66.43	91.23	119.96	152.61
50	1.36	6.60	15.76	28.85	45.87	66.81	91.68	120.47	153.19
51	1.42	6.72	15.95	29.10	46.18	67.19	92.12	120.98	153.77
52	1.48	6.84	16.14	29.36	46.50	67.58	92.57	121.49	154.35
53	1.53	6.96	16.32	29.61	46.82	67.96	93.02	122.01	154.93
54	1.59	7.09	16.51	29.86	47.14	68.35	93.47	122.53	155.51
55	1.65	7.21	16.70	30.12	47.46	68.73	93.92	123.05	156.09
56	1.71	7.34	16.89	30.38	47.79	69.12	94.38	123.57	156.67
57	1.77	7.46	17.08	30.64	48.11	69.51	94.83	124.09	157.25
58	1.83	7.60	17.28	30.90	48.43	69.90	95.29	124.61	157.84
59	1.89	7.72	17.47	31.16	48.76	70.29	95.74	125.13	158.43

TABLE V. Reduction to the Meridian. $m = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}$

P	9^m	10^m	11^m	12^m	13^m	14^m	15^m	16^m
s	''	''	''	''	''	''	''	''
0	159.02	196.32	237.54	282.68	331.74	384.74	441.63	502.46
1	159.61	196.97	238.26	283.47	332.59	385.65	442.62	503.50
2	160.20	197.63	238.98	284.26	333.44	386.56	443.60	504.55
3	160.80	198.28	239.70	285.04	334.29	387.48	444.58	505.60
4	161.39	198.94	240.42	285.83	335.15	388.40	445.56	506.65
5	161.98	199.60	241.14	286.62	336.00	389.32	446.55	507.70
6	162.58	200.26	241.87	287.41	336.86	390.24	447.54	508.76
7	163.17	200.92	242.60	288.20	337.72	391.16	448.53	509.81
8	163.77	201.59	243.33	289.00	338.58	392.09	449.51	510.86
9	164.37	202.25	244.06	289.79	339.44	393.01	450.50	511.92
10	164.97	202.92	244.79	290.58	340.30	393.94	451.50	512.98
11	165.57	203.58	245.52	291.38	341.16	394.86	452.49	514.03
12	166.17	204.25	246.25	292.18	342.02	395.79	453.48	515.09
13	166.77	204.92	246.98	292.98	342.88	396.72	454.48	516.15
14	167.37	205.59	247.72	293.78	343.75	397.65	455.47	517.21
15	167.97	206.26	248.45	294.58	344.62	398.58	456.47	518.27
16	168.58	206.93	249.19	295.38	345.49	399.52	457.47	519.34
17	169.19	207.60	249.93	296.18	346.36	400.45	458.47	520.40
18	169.80	208.27	250.67	296.99	347.23	401.38	459.47	521.47
19	170.41	208.94	251.41	297.79	348.10	402.32	460.47	522.53
20	171.02	209.62	252.15	298.60	348.97	403.26	461.47	523.60
21	171.63	210.30	252.89	299.40	349.84	404.20	462.48	524.67
22	172.24	210.98	253.63	300.21	350.71	405.14	463.48	525.74
23	172.85	211.66	254.37	301.02	351.58	406.08	464.48	526.81
24	173.47	212.34	255.12	301.83	352.46	407.02	465.49	527.89
25	174.08	213.02	255.87	302.64	353.34	407.96	466.50	528.96
26	174.70	213.70	256.62	303.46	354.22	408.90	467.51	530.03
27	175.32	214.38	257.37	304.27	355.10	409.84	468.52	531.11
28	175.94	215.07	258.12	305.09	355.98	410.79	469.53	532.18
29	176.56	215.75	258.87	305.90	356.86	411.73	470.54	533.26
30	177.18	216.44	259.62	306.72	357.74	412.68	471.55	534.33
31	177.80	217.12	260.37	307.54	358.62	413.63	472.57	535.41
32	178.43	217.81	261.12	308.36	359.51	414.59	473.58	536.50
33	179.05	218.50	261.88	309.18	360.39	415.54	474.60	537.58
34	179.68	219.19	262.64	310.00	361.28	416.49	475.62	538.67
35	180.30	219.88	263.39	310.82	362.17	417.44	476.64	539.75
36	180.93	220.58	264.15	311.65	363.07	418.40	477.65	540.83
37	181.56	221,27	264.91	312.47	363.96	419.35	478.67	541.91
38	182.19	221.97	265.68	313.30	364.85	420.31	479.70	543.00
39	182.82	222.66	266.44	314.12	365.75	421.27	480.72	544.09
40	183.46	223.36	267.20	314.95	366.64	422.23	481.74	545.18
41	184.09	224.06	267.96	315.78	367.53	423.19	482.77	546.27
42	184.72	224.76	268.73	316.61	368.42	424.15	483.79	547.36
43	185.35	225.46	269.49	317.44	369.31	425.11	484.82	548.45
44	185.99	226.16	270.26	318.27	370.21	426.07	485.85	549.55
45	186.63	226.86	271.02	319.10	371.11	427.04	486.88	550.64
46	187.27	227.57	271.79	319.94	372.01	428.01	487.91	551.73
47	187.91	228.27	272.56	320.78	372.91	428.97	488.94	552.83
48	188.55	228.98	273.34	321.62	373.82	429.93	489.97	553.93
49	189.19	229.68	274.11	322.45	374.72	430.90	491.01	555.03
50	189.83	230.39	274.88	323.29	375.62	431.87	492.05	556.13
51	190.47	231.10	275.65	324.13	376.52	432.84	493.08	557.24
52	191.12	231.81	276.43	324.97	377.43	433.82	494.12	558.34
53	191.76	232.52	277.20	325.81	378.34	434.79	495.15	559.44
54	192.41	233.24	277.98	326.66	379.26	435.76	496.19	560.55
55	193.06	233.95	278.76	327.50	380.17	436.73	497.23	561.65
56	193.71	234.67	279.55	328.35	381.08	437.71	498.28	562.76
57	194.36	235.38	280.33	329.19	381.99	438.69	499.32	563.87
58	195.01	236.10	281.12	330.04	382.90	439.67	500.37	564.98
59	195.66	236.82	281.90	330.89	383.82	440,65	501.41	566.08

TABLE V. Reduction to the Meridian. $m = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}$

P	17m	18m	19m	20m	21m	22m	23m	24m	25m
s	''	''	''	''	''	''	''	''	''
0	567.2	635.9	708.4	784.9	865.3	949.6	1037.8	1129.9	1225.9
1	568.3	637.0	709.7	786.2	866.6	951.0	1039.3	1131.4	1227.5
2	569.4	638.2	710.9	787.5	868.0	952.4	1040.8	1133.0	1229.2
3	570.5	639.4	712.1	788.8	869.4	953.8	1042.3	1134.6	1230.8
4	571.6	640.6	713.4	790.1	870.8	955.3	1043.8	1136.2	1232.5
5	572.8	641.7	714.6	791.4	872.1	956.7	1045.3	1137,8	1234.1
6	573.9	642.9	715.9	792.7	873.5	958.2	1046.8	1139.3	1235.7
7	575.0	644.1	717.1	794.0	874.9	959.6	1048.3	1140.9	1237.3
8	576.1	645.3	718.4	795.4	876.3	961.1	1049.8	1142.5	1239.0
9	577.2	646.5	719.6	796.7	877.6	962.5	1051.3	1144.0	1240.6
10	578.4	647.7	720.9	798.0	879.0	963.9	1052.8	1145.6	1242.3
11	579.5	648.9	722.1	799.3	880.4	965.4	1054.3	1147.2	1243.9
12	580.6	650.0	723.4	800.7	881.8	966.9	1055.9	1148.8	1245.6
13	581.7	651.2	724.6	802.0	883.2	968.3	1057.4	1150.4	1247.2
14	582.9	652.4	725.9	803.3	884.6	969.8	1058.9	1152.0	1248.9
15	584.0	653.6	727.2	804.6	886.0	971.2	1060.4	1153.6	1250.5
16	585.1	654.8	728.4	806.0	887.4	972.7	1062.0	1155.2	1252.2
17	586.2	656.0	729.7	807.3	888.8	974.1	1063.5	1156.8	1253.8
18	587.4	657.2	730.9	808.6	890.2	975.5	1065.0	1158.3	1255.5
19	588.5	658.4	732.2	809.9	891.6	977.0	1066.5	1159.9	1257.1
20	589.6	659.6	733.5	811.3	893.0	978.5	1068.1	1161.5	1258.8
21	590.8	660.8	734.7	812.6	894.4	979.9	1069.6	1163.1	1260.5
22	591.9	662.0	736.0	813.9	895.8	981.4	1071.1	1164.7	1262.2
23	593.0	663.2	737.3	815.2	897.2	982.9	1072.6	1166.3	1263.8
24	594.2	664.4	738.5	816.6	898.6	984.4	1074.2	1167.9	1265.5
25	595.3	665.6	739.8	817.9	900.0	985.8	1075.7	1169.5	1267.1
26	596.5	666.8	741.1	819.2	901.4	987.3	1077.2	1171.1	1268.8
27	597.6	668.0	742.3	820.5	902.8	988.8	1078.7	1172.7	1270.5
28	598.7	669.2	743.6	821.9	904.2	990.3	1080.3	1174.3	1272.1
29	599.9	670.4	744.9	823.2	905.6	991.8	1081.8	1175.9	1273.7
30	601.0	671.6	746.2	824.6	907.0	993.2	1083.3	1177.5	1275.4
31	602.2	672.8	747.4	825.9	908.4	994.7	1084.8	1179.1	1277.1
32	603.3	674.1	748.7	827.3	909.8	996.2	1086.4	1180.7	1278.8
33	604.5	675.3	750.0	828.6	911.2	997.6	1087.9	1182.3	1280.4
34	605.6	676.5	751.3	829.9	912.6	999.1	1089.5	1183.9	1282.1
35	606.8	677.7	752.6	831.2	914.0	1000.6	1091.0	1185.5	1283.8
36	607.9	678.9	753.8	832.6	915.5	1002.1	1092.6	1187.1	1285.5
37	609.1	680.1	755.1	833.9	916.9	1003.5	1094.1	1188.7	1287.1
38	610.2	681.3	756.4	835.3	918.3	1005.0	1095.7	1190.3	1288.8
39	611.4	682.6	757.7	836.6	919.7	1006.5	1097.2	1191.9	1290.5
40	612.5	683.8	759.0	838.0	921.1	1008.0	1098.8	1193.5	1292.2
41	613.7	685.0	760.2	839.3	922.5	1009.4	1100.3	1195.1	1293.8
42	614.8	686.2	761.5	840.7	923.9	1010.9	1101.9	1196.7	1295.5
43	616.0	687.4	762.8	842.0	925.3	1012.4	1103.4	1198.3	1297.2
44	617.2	688.7	764.1	843.4	926.8	1013.9	1105.0	1199.9	1298.9
45	618.3	689.9	765.4	844.7	928.2	1015.4	1106.5	1201.5	1300.5
46	619.5	691.1	766.7	846.1	929.6	1016.9	1108.1	1203.1	1302.2
47	620.6	692.4	768.0	847.5	931.0	1018.4	1109.6	1204.7	1303.9
48	621.8	693.6	769.3	848.9	932.4	1019.9	1111.2	1206.4	1305.6
49	623.0	694.8	770.6	850.2	933.8	1021·4	1112.7	1208.0	1307.3
50	624.1	696.0	771.9	851.6	935.2	1022.8	1114.3	1209.6	1309.0
51	625.3	697.3	773.1	852.9	936.6	1024.3	1115.8	1211.2	1310.7
52	626.5	698.5	774.5	854.3	938.1	1025.8	1117.4	1212.9	1312.4
53	627.6	699.7	775.7	855.7	939.5	1027.3	1118.9	1214.5	1314.1
54	628.8	701.0	777.1	857.1	940.9	1028.8	1120.5	1216.1	1315.7
55	630.0	702.2	778.4	858.4	942.3	1030.3	1122.0	1217.7	1317.4
56	631.2	703.5	779.7	859.8	943.8	1031.8	1123.6	1219.4	1319.1
57	632.3	704.7	781.0	861.1	945.2	1033.3	1125.1	1221.0	1320.8
58	633.5	705.9	782.3	862.5	946.6	1034.8	1126.7	1222.6	1322.5
59	634.7	707.1	783.6	863.9	948.1	1036.3	1128.3	1224.2	1324.2

TABLE V. Reduction to the Meridian.

$m = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}$

P (s)	26m (″)	27m (″)	28m (″)	29m (″)
0	1325.9	1429.7	1537.5	1649.0
1	1327.6	1431.4	1539.3	1650.9
2	1329.3	1433.2	1541.1	1652.8
3	1331.0	1434.9	1542.9	1654.7
4	1332.7	1436.7	1544.8	1656.6
5	1334.4	1438.5	1546.6	1658.5
6	1336.1	1440.3	1548.4	1660.4
7	1337.8	1442.1	1550.2	1662.3
8	1339.5	1443.9	1552.1	1664.2
9	1341.2	1445.6	1553.9	1666.1
10	1342.9	1447.4	1555.8	1668.0
11	1344.6	1449.2	1557.6	1669.9
12	1346.3	1451.0	1559.5	1671.9
13	1348.0	1452.8	1561.3	1673.8
14	1349.7	1454.5	1563.2	1675.7
15	1341.4	1456.3	1565.0	1677.6
16	1353.2	1458.1	1566.9	1679.5
17	1354.9	1459.9	1568.7	1681.4
18	1356.6	1461.6	1570.5	1683.3
19	1358.3	1463.4	1572.4	1685.2
20	1360.1	1465.2	1574.3	1687.2
21	1361.8	1466.9	1576.1	1689.1
22	1363.5	1468.7	1578.0	1691.0
23	1365.2	1470.5	1579.8	1692.9
24	1367.0	1472.3	1581.7	1694.8
25	1368.7	1474.1	1583.5	1696.7
26	1370.4	1475.9	1585.3	1698.6
27	1372.1	1477.7	1587.2	1700.5
28	1373.9	1479.5	1589.1	1702.5
29	1375.6	1481.3	1590.9	1704.4
30	1377.3	1483.1	1592.7	1706.3
31	1379.0	1484.9	1594.6	1708.2
32	1380.8	1486.7	1596.5	1710.2
33	1382.5	1488.5	1598.3	1712.1
34	1384.2	1490.3	1600.2	1714.0
35	1385.9	1492.1	1602.1	1715.9
36	1387.7	1493.9	1604.0	1717.9
37	1389.4	1495.7	1605.9	1719.8
38	1391.2	1497.5	1607.7	1721.7
39	1392.9	1499.3	1609.6	1723.6
40	1394.7	1501.1	1611.5	1725.6
41	1396.4	1502.9	1613.3	1727.5
42	1398.2	1504.7	1615.2	1729.5
43	1399.9	1506.5	1617.1	1731.5
44	1401.7	1508.4	1619.0	1733.4
45	1403.4	1510.2	1620.8	1735.3
46	1405.2	1512.0	1622.7	1737.2
47	1406.9	1513.8	1624.6	1739.2
48	1408.7	1515.6	1626.5	1741.2
49	1410.4	1517.4	1628.3	1743.1
50	1412.2	1519.2	1630.2	1745.1
51	1413.9	1521.0	1632.1	1747.0
52	1415.7	1522.9	1634-0	1749.0
53	1417.4	1524.7	1635.9	1750.9
54	1419.2	1526.5	1637.7	1752.8
55	1420.9	1528.3	1639.6	1754.8
56	1422.7	1530.2	1641.5	1756.8
57	1424.4	1532.0	1643.3	1758.7
58	1426.2	1533.8	1645.2	1760.7
59	1427.9	1535.6	1647.1	1762.6

$n = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}$

P (m)	P (s)	n (″)	P (m)	P (s)	n (″)
0	0	0.00	20	0	1.49
1	0	0.00		10	1.54
2	0	0.00		20	1.60
3	0	0.00		30	1.65
4	0	0.00		40	1.70
5	0	0.01		50	1.76
6	0	0.01	21	0	1.82
7	0	0.02		10	1.87
8	0	0.04		20	1.93
9	0	0.06		30	1.99
10	0	0.09		40	2.06
11	0	0.14		50	2.12
12	0	0.19	22	0	2.19
	10	0.20		10	2.25
	20	0.22		20	2.32
	30	0.23		30	2.39
	40	0.24		40	2.46
	50	0.25		50	2.54
13	0	0.26	23	0	2.61
	10	0.28		10	2.69
	20	0.30		20	2.77
	30	0.31		30	2.85
	40	0.33		40	2.93
	50	0.34		50	3.01
14	0	0.36	24	0	3.10
	10	0.38		10	3.18
	20	0.39		20	3.27
	30	0.41		30	3.36
	40	0.43		40	3.45
	50	0.45		50	3.55
15	0	0.47	25	0	3.64
	10	0.49		10	3.74
	20	0.52		20	3.84
	30	0.54		30	3.94
	40	0.56		40	4.05
	50	0.59		50	4.15
16	0	0.61	26	0	4.26
	10	0.64		10	4.37
	20	0.67		20	4.48
	30	0.69		30	4.60
	40	0.72		40	4.72
	50	0.75		50	4.83
17	0	0.78	27	0	4.96
	10	0.81		10	5.08
	20	0.84		20	5.20
	30	0.88		30	5.33
	40	0.91		40	5.46
	50	0.95		50	5.60
18	0	0.98	28	0	5.73
	10	1.02		10	5.87
	20	1.06		20	6.01
	30	1.09		30	6.15
	40	1.13		40	6.30
	50	1.18		50	6.44
19	0	1.22	29	0	6.59
	10	1.26		10	6.75
	20	1.30		20	6.90
	30	1.35		30	7.06
	40	1.40		40	7.22
	50	1.44		50	7.38
20	0	1.49	30	0	7.55

For rate.

Rate (s)	Log k
—30	9.999 6985
29	7085
28	7186
27	7286
26	7387
25	7487
24	7588
23	7688
22	7789
21	7889
20	7990
19	8090
18	8191
17	8291
16	8392
15	8492
14	8593
13	8593
12	8794
11	8894
10	8995
9	9095
8	9196
7	9296
6	9397
5	9497
4	9598
3	9698
2	9799
—1	9.999 9899
0	0.000 0000
+1	0101
2	0201
3	0302
4	0402
5	0503
6	0603
7	0704
8	0804
9	0905
10	1005
11	1106
12	1206
13	1307
14	1407
15	1508
16	1608
17	1709
18	1809
19	1910
20	2010
21	2111
22	2211
23	2312
24	2412
25	2513
26	2613
27	2714
28	2814
29	2915
+30	0.000 3015

TABLE VI.

Augmentation of Moon's Semi-Diameter on Account of Apparent Altitude.

Apparent Altitude.	Horizontal Semi-Diameter.					
	14′ 30″	15′ 0″	15′ 30″	16′ 0″	16′ 30″	17′ 0″
	″	″	″	″	″	″
0	0.10	0.12	0.13	0.14	0.15	0.17
2	0.58	0.62	0.66	0.71	0.76	0.81
4	1.05	1.12	1.20	1.28	1.37	1.46
6	1.51	1.62	1·74	1.86	1.98	2.10
8	1.98	2.12	2.27	2.42	2.58	2.75
10	2.44	2.62	2.80	2.99	3.18	3.39
12	2.90	3.11	3.33	3.56	3.78	4.02
14	3.36	3.61	3.86	4.11	4.37	4.66
16	3.82	4.10	4.38	4.67	4.97	5.28
18	4.28	4.58	4.89	5.22	5.56	5.90
20	4.72	5.06	5.41	5.76	6.14	6.52
22	5.16	5.53	5.91	6.30	6.71	7.13
24	5.60	5.99	6.41	6.83	7.27	7.72
26	6.03	6.45	6.90	7.35	7.83	8.31
28	6.45	6.91	7.38	7.87	8.37	8.89
30	6.86	7.35	7.85	8.37	8.91	9.46
32	7.27	7.78	8.32	8.87	9.44	10.02
34	7.67	8.21	8.77	9.35	9.95	10.57
36	8.06	8.62	9.22	9.83	10.46	11.11
38	8.43	9.03	9.65	10.29	10.95	11.63
40	8.80	9.42	10.07	10.74	11.43	12.14
42	9.16	9.80	10.48	11.17	11.89	12.63
44	9.51	10.17	10.88	11.60	12.34	13.11
46	9.84	10.54	11.26	12.01	12.78	13.57
48	10.16	10.88	11.63	12.40	13.20	14.02
50	10.48	11.22	11.99	12.78	13.60	14.45
52	10.78	11.54	12.33	13.15	13.99	14.86
54	11.07	11.84	12.65	13.50	14.36	15.25
56	11.34	12.14	12.97	13.83	14.72	15.63
58	11.60	12.42	13.27	14.15	15.05	15.99
60	11 84	12.68	13.55	14.44	15.37	16.32
62	12.07	12.93	13.81	14.73	15.67	16.64
64	12.29	13.16	14.06	14.99	15.95	16.94
66	12.49	13.37	14.29	15.24	16.21	17.22
68	12.68	13.58	14.50	15.46	16.45	17.47
70	12.85	13.76	14.70	15.67	16.67	17.71
72	13.00	13.92	14.88	15.86	16.88	17.92
74	13.14	14.07	15.04	16.03	17.06	18.12
76	13.27	14.21	15.18	16.18	17.22	18.29
78	13.38	14.32	15.30	16.31	17.36	18.43
80	13.47	14.42	15.40	16.42	17.47	18.56
82	13.54	14.50	15.49	16.51	17.57	18.66
84	13.60	14.56	15.56	16.59	17.65	18.74
86	13.64	14.61	15.60	16.64	17.70	18.80
88	13.67	14.63	15.63	16.67	17.73	18.83
90	13.67	14.63	15.64	16.68	17.74	18.85

FORMS.

$\frac{\omega + \omega' - (e + e')}{4}$ D   b D

Form No. 1.

ERROR OF SIDEREAL TIME-PIECE BY MERIDIAN TRANSIT OF STAR.

Station, West Point, N. Y. Latitude, 41° 23′ 22″.11. Chronometer No........, by ....

Date:				
Observer.				
Recorder.				
Transit.				
Illumination.				
Name of Star.				
Level. Direct.	**E. W.**	**E. W.**	**E. W.**	**E. W.**
Level. Reversed.	**E. W.**	**E. W.**	**E. W.**	**E. W.**
	h m s	*h m s*	*h m s*	*h m s*
Time of Passing Wires. I.				
II.				
III.				
IV.				
V.				
VI.				
VII.				
Sum.				
Mean.				
Red. to Mid. Wire.				
Chron. Time of Transit over Mid. Wire = *T*.				
Level Error = *b*.				
Level Correction = *Bb*.				
Collimation Error = *c*.				
Collimation Correction = *Cc*.				
Azimuth Error = *a*.				
Azimuth Correction = *Aa*.				
Chron. Time of Transit.				
App. R. A. of Star = *α*.				
Error of Chron. = *E*.				

FORM No. 2.

ERROR OF MEAN-SOLAR TIME-PIECE BY MERIDIAN TRANSIT OF SUN.

Date.	Station, WEST POINT, N. Y.
Latitude 41° 23′ 22″.11.	Longitude 4.93^h.
Observer.	Recorder.
Transit No........ By	Mean-Solar Chron. No....... By ....

		h m s
Chronometer Time of Transit of West Limb.	Wire I	
	" II	
	" III	
	" IV	
	" V	
	" VI	
	" VII	
Chronometer Time of Transit of East Limb.	" I	
	" II	
	" III	
	" IV	
	" V	
	" VI	
	" VII	
	SUM.	

Chron. Time of Transit of Center over Mean of Wires = Mean.	....
Reduction to Middle Wire.	....
Level Error............Level Correction.	....
Col. " ....Col. "	....
Azimuth " ....Azimuth "	
Chronom. Time of App. Noon.	....
Apparent " " " "	........12^h........0.0^m........0.0^s........
Eq. of Time.	....
Mean Time of Apparent Noon.	....
Error of Chronometer on Mean Solar Time at App. Noon.	....

FORM No. 3.

ERROR OF SIDEREAL TIME-PIECE BY SINGLE ALTITUDE OF STAR. NAME...............

Date.	Station, WEST POINT, N. Y.
Observer.	Recorder.
Sextant No........ By	Sidereal Chronom. No........ By........

	° ′ ″		h m s
Observed Double Altitude.		Chronom. Time.	
" " "		" "	
" " "		" "	
" " "		" "	
" " "		" "	
Mean " "	....	Sum	....
Index Error.	....	† Mean = t_0	
Eccentricity.	....	Barometer	....
Corrected Double Altitude.	....	Att. Thermom	....
" Altitude = a_0.	....	Ext. "	....
*Refraction = r.	....	Refraction	....
True Altitude = a	....		
Latitude = ϕ.	..41°..28′..22″.11..	$a.$ $c.$ log cos ϕ	....
N. Polar Dist. = d.	....	" " sin d	....
$m = \frac{a+\phi+d}{2}$.	....	" cos m	....
$m - a$.	....	" sin $(m-a)$	________
		" $\sin^2 \frac{1}{2} P$	....
		" $\sin \frac{1}{2} P$	....
		$\frac{1}{2} P$	....
		P	....
		P in Time	....
		Apparent R. A. of Star	________
		Sidereal Time = R. A. + P	....
		Mean of Chron. Times = t_0	________
		Error of Chronometer	....

* The correction to be added to this value of r, if desired (see Note 8, Text), is $2\frac{\sin r}{\sin^2 a_0}\Sigma\frac{2\sin^2\frac{1}{2}(a_0 - A)}{n \sin 1''}$, A denoting the different corrected altitudes, a_0 their mean, and n the number of observations. The values of $\frac{2\sin^2\frac{1}{2}(a_0 - A)}{\sin 1''}$ are taken from Tables (first converting $a_0 - A$ into its equivalent in time), as explained under "Latitude by Circum-Meridian Altitudes."

† The correction to be added algebraically to this value of t_0 if desired (see Note 3, Text) is, after computing P in arc, $\frac{1}{15}\left[\cot P - \frac{\sin P \cos\phi \sin d}{\cos a \cot a}\right]\Sigma\frac{2\sin^2\frac{1}{2}(T - t_0)}{n \sin 1''}$, T being the different chronometer times. The last factor is taken from Tables as before.

FORM No. 4.

ERROR OF MEAN-SOLAR TIME-PIECE BY SINGLE ALTITUDE OF SUN'S............LIMB.

Date.	**Station, WEST POINT, N. Y.**
Observer.	**Recorder.**
Sextant No....... By.......	**M. S. Chronom. No........ By. ....**

	° ′ ″		h m s
Observed Double Altitude.		Chronom. Time.	
" " "		" "	
" " "		" "	
" " "		" "	
" " "		" "	
Mean " "	....	Sum	....
Index Error.	....	†Mean = t_0	
Eccentricity.	....	Barometer	....
		Att. Thermom.	....
Corrected Double Altitude.	....	Ext. "	....
" Altitude = a_0.	....	Refraction	....
*Refraction = r.	....	Longitude = 4.931 hours.	
Semi-diameter.	....	Assumed Error of Chronom. =	....
Apparent Altitude = a'.	....	Resulting Greenwich Time of Obs.=	
Parallax in Altitude.	....	Log. Eq. Hor. Parallax	
		" ρ.	9.54726
True Altitude = a.	....	" $\cos a'$.	9.99936
Latitude = ϕ.	..41°..23′..22″.11..	Parallax in Altitude.	
N. Polar Dist. = d.	....	Dec. at Greenwich Mean Moon.	....
$m = \frac{a + \phi + d}{2}$.	....	Hourly Change × Greenwich Time	....
		Sun's Declination.	
$m - a$.	....	a. c. log $\cos \phi$	....
		" " $\sin d$	....
		" $\cos m$	....
		" $\sin (m - a)$	
		" $\sin^2 \frac{1}{2} P$	....
		P	....
		P in Time	... .
		Apparent Time	....
		Equation of Time.	
		Mean Time.	.. ..
		Mean of Chron. Times = t_0	
		Error of Chronometer.	....

* See foot-note to Form 3.
† " " " " " "

NOTE.—For correction of Semi-diameter due to difference of refraction between limb and center, see "Longitude by Lunar Distances."

$K = \alpha - \frac{T_1 + T_2}{2}$

FORM No. 5.

ERROR OF SIDEREAL TIME-PIECE BY EQUAL ALTITUDES OF A STAR.

Station, WEST POINT, N. Y.	Latitude, 41° 23′ 22″.11 = ϕ.
Observer......................	Recorder................
Sextant No........ By	Sid. Chronom. No ..... By ....
Name of Star.	App. Declination = δ..........................

	Observations East.		Date	
	Observed Double Altitudes.	Chronometer Times.		
	° ′ ″	h m s		
I.			Barom.	....
II.			Att. Thermom.	....
III.			Ext. "	....
			1st Refraction	....
Mean = 2a..............................		Sum		
(Correct this for index error, if correction for refraction be taken into account.)		1st Mean................................		......

	Observations West.		Date	
	Observed Double Altitudes.	Chronometer Times.		
	° ′ ″	h m s		
I.			Barom.	....
II.			Att. Thermom.	....
III.			Ext. "	......
			2d Refraction	......
			1st "	....
Mean = 2a................................		Sum	Difference	....
	(Same as above).	2d Mean..................	Log Difference	....
		1st "	Log cos a	....
			a. c. log 30	....
	Elapsed Time.	....	a. c. log cos ϕ	....
	½ Elapsed Time in arc = t.	....	a. c. log cos δ	....
			a. c. log sin t	....
	Middle Chronometer Time.	....	Log Correction	....
	Correction for Refraction.		Correction	....
	Chronom. Time of Transit.	....		
	App. R. A. of Star.			
	Error of Chronom. at Time of Transit.	....		

FORM No. 6.

ERROR OF MEAN-SOLAR TIME-PIECE BY EQUAL ALTITUDES OF SUN'S................LIMB.

Station, WEST POINT, N. Y. ϕ = Latitude, 41° 23′ 22″.11. Longitude 4.931^h, west.
Observer.................... Recorder....................
Sextant No........ By .... M. S. Chronom. No .... By ....
Sun's App. Dec. at local App. Noon (or midnight) = δ =... ....
Hourly change in δ at same time, = k = ...

Observations East. Date ....

	Observed Double Altitudes.	Chronometer Times.		
	° ′ ″	h m s		
I.			Barom.	....
II.			Att. Thermom.	....
III.			Ext. "	....
			1st Refraction	....

Mean = 2a.................................. Sum ____________
(Correct this for index error, if correction for refraction be taken into account). 1st Mean........................

Observations West. Date ....

	Observed Double Altitudes.	Chronometer Times.		
	° ′ ″	h m s		
I.			Barom.	....
II.			Att. Thermom.	....
III.			Ext. "	....
			2d Refraction	....
			1st "	______
Mean = 2a		Sum	Difference	....
	(Same as above).	2d Mean................	Log Difference	....
		1st " ____________	Log cos a	....
			$a.$ $c.$ log 30	....
	Elapsed Time..........................	....	$a.$ $c.$ log cos ϕ	....
	½ Elapsed Time in arc = t.	....	$a.$ $c.$ log cos δ	....
			$a.$ $c.$ log sin t	______
	Middle Chronometer Time.	... .	Log Correction	....
	Correction for Refraction.	____________	Correction	....
	Equation of Equal Altitudes.	____________	Log A	....
	Chronom. Time of App. Noon.	....	" k	....
	App. of Time at App. Noon.	....12^h...0^m...0^s...	" tan ϕ	
	Eq. of Time at App. Noon.	....	" 1st Part	....
	Mean Time of App. Noon.	....	1st Part.	....
	Error of Chronometer at App. Noon.	....	Log B.	....
			" k.	______
			" tan δ.	....
			" 2d Part.	... .
			2d Part.	

1st Part + 2d Part = Eq. of Equal Altitudes. ______

FORM No. 7.

LATITUDE BY CIRCUM-MERIDIAN ALTITUDES OF SUN'S................LIMB.

Date............Station, WEST POINT, N. Y. Longitude 4.93^h. Assumed Lat. $= \phi =$........
Observer............Recorder............ Barom.... .... Att. Th.... ... Ext. Th........
Sextant No........By................ M. S. Chronometer, No .... By................
Error of Chronometer $= E =$................ Rate of Chronometer $= r =$................

Observed Double Altitudes. ° ′ ″	Chronometer Times. h m s		Hour Angles. m. s.	m. s.	n. s.
I.		App. Time of App. Noon 12 0 0			
II.		Eq. of Time at App. Noon			
III.					
IV.		Mean Time of App. Noon			
V.					
VI.		Chron. Error			
VII.					
VIII.		Chron. Time of App. Noon			
IX.					
X.					
Sum.		Log. Eq. Hor. Px. .... Sums.	....	......	....
Mean.		" p	P_0	m_0	n_0
Eccentricity		" cos a* .... Means.	....	......	....

Index Error. ________ " Par in Alt. ....

Cor. D. Alt. .... " " " .... Eq. of Time at P_0...............

P_0...............

" S. " .... Longitude________

Refraction .... Correspond'g Greenwich Time ....

Semi-diam. .... Sun's Dec. at " " " δ_0.... ....

Par. in Alt. ________

True Alt. $= a_0 +$ .... $a_{\prime} = \delta_0 + 90° - \phi$. $\phi = \delta_0 + 90° - a_{\prime}$.

$A_0 m_0 +$ ....

$B_0 n_0 -$ ________ Change in Eq. of Time in $24h = e$......... ....

Sum $-$ .... Rate of Chronometer " $= r$________

$90° +$ .90°..0′.0..0″.0. $r - e$..........................

$\delta_0 +$ ________ log k...... ....

ϕ .... " cos ϕ..........................

" cos δ_0..... . ...

" sec $a_{\prime}$________

" A_0.........................

" m_0________

" $A_0 m_0$........... ....

$A_0 m_0$ .. . .

2 log A_0 ....

" tan $a_{\prime}$...........................

" n_0 ....

" $B_0 n_0$ ....

$B_0 n_0$ ... .

* a is obtained by applying refraction and semi-diameter to Corrected Single Altitude.

NOTE.—For correction to Semi-diameter due to difference of refraction between limb and center, see Longitude by Lunar Distances.

Form No. 8.

LATITUDE BY CIRCUM-MERIDIAN ALTITUDES OF (NAME OF STAR)..............

Date............Station, West Point, N. Y. Assumed Lat. = ϕ = ...

Observer..........Recorder..........Barom..........Att. Ther..........Ext. Ther..........

Sextant No........By............ Sidereal Chronom. No........By............

Error of Chronometer = E =........... Rate of Chronometer = r =...........

Observed Double Altitudes. ° ′ ″	Chronometer Times. h. m. s.		Hour Angles. m. s.	m. s.	n. s.
I.		App. R. A. of Star			
II.		Chron. Error			
III.		Chron. Time of Transit			
IV.					
V.					
VI.					
VII.					
VIII.					
IX.					
X.					

Sum.	....	Sums.	....	......	......
Mean.	....		P_0	m_0	n_0
Eccentricity.	....	Means.	....	......	......
Index Error.					

Cor. D. Alt. .... Star's App. Declination = δ_0 ....

" S. " .... $a_, = \delta_0 + 90° - \phi$. $\phi = \delta_0 + 90° - a_,$.

Refraction. ________

True Alt.=a_0+ .... Rate of Chronometer in 24^h = e...........................

A_0m_0 + ... log k..........................

B_0n_0 − ________ " cos ϕ..........................

Sum − .... " cos δ_0..........................

90° + .90°..0′.0..0″.0. " sec $a_,$ ________

δ_0 + ________ " A_0..........................

ϕ .... " m_0 ________

" A_0m_0..........................

A_0m_0 ....

2 log A_0..........................

" tan $a_,$..........................

" n_0 ________

" B_0n_0..........................

B_0n_0 ....

Form No. 9.

PROGRAMME FOR ZENITH TELESCOPE. (LATITUDE.)

Station, West Point, N. Y. Approximate Latitude........ Observer........

No.	Catalogue and No.	Mag.	Mean R. A.	Mean Dec.	Zenith Dist.	N. S.	Setting.
1.							
2.							
3.							
4.							
5.							
6.							
7.							
8.							
9.							
10.							

Form No. 9*a*.

OBSERVATIONS WITH ZENITH TELESCOPE.

Station, West Point, N. Y. Observer........ Recorder.... ... Sheet No........
Telescope No........ By........ Chronometer Error........

Date.	Star.		Micrometer.		Level.			Chronom.	Remarks
	Catalogue and Catalogue No.	N. S.	m_n & m_s	$m_n - m_s$	l_n & l'_n	l_s & l'_s	$(l_n+l'_n)$ $(l_s+l'_s)$	Time.	

Note.—Form No. 9 is for the observer's use.

Form 9*a* is for the recorder's use.

The records of the different nights at a given station are then collated, and the reductions made as per Form No. 9*b*, which is for the computer's use.

Form No. 9*b*.

REDUCTION OF OBSERVATIONS WITH ZENITH TELESCOPE.

Station, West Point, N. Y. Observation Sheet, No .... Observer............. Recorder .. .. Computer..... .. ....

Telescope No........... By.. .... One Div. Micrometer (R) = ... One Div. Level (D) = ...

Date.	Star.		Micrometer.		Level.			Hour Angle.	Declination.		Corrections.				Lati-tude.	v.	vv.
	Catalogue and Catalogue No.	N. S.	m_n & m_s	$m_n - m_s$	l_n & l'_n	l_s & l'_s	$(l_n + l'_n)$ $(l_s + l'_s)$	P.	δ_n & δ_s	$\frac{\delta_n + \delta_s}{2}$	Microm.	Level.	Ref.	Red. to Mer.			

Note.—Corrections.

$$\text{Micrometer} = R\frac{m_n - m_s}{2}.$$

$$\text{Level} = D\frac{(l_n + l'_n) - (l_s + l'_s)}{4}.$$

$$\text{Refraction} = \frac{R}{60}\frac{dr}{dz}\frac{m_n - m_s}{2}.$$

$$\text{Red. to Mer.} = [6.1347]\ P^2 \sin 2\delta.$$

Columns headed v and vv are for the determination of the probable error by the "Method of Least Squares."

FORM No 10.

LATITUDE BY POLARIS OFF THE MERIDIAN.

Station, WEST POINT, N. Y. Date.......... .... Observer.... .... Recorder.....................

Sextant No ....By............ Index Error............... Sid. Chron. No...........By... .... Error.........

Barometer............ Att. Thermom................ Ext. Thermom .... Refraction................

App. R. A. of Polaris. .... App. Dec. of Polaris...................

No.	Chron. Time of Observation.	True Time of Observation.	Hour Angle in Time.	Hour Angle in Arc.	Observed D. Altitudes.	True Altitudes.	REDUCTION TO POLE.			Latitude.
							$-d \cos P$.	$+\frac{1}{2}d^2 \sin 1'' \tan a \sin^2 P$.	$-\frac{1}{3}d^3 \sin^2 1'' \cos P \sin^2 P$.	

NOTE.—The column of True Altitudes is obtained from the preceding by applying instrumental errors and refraction.

Log. sin 1″= 4.685575−10

Sum.... ....

Mean... ....

Form No. 11.

LATITUDE BY EQUAL ALTITUDES OF TWO STARS.

Station, West Point, N. Y. Observer.................... Recorder..................

Date.................... Sextant, No......... .. By...........................

Sid. Chronom. No............ By............ ... Error............ Rate............

Name of the Star.	Chron. Time of Observation.	True Time of Observation.	App. R. A.	Hour Angle in Time.	Hour Angle in Arc. P & P'	App. Dec. δ & δ'	$\frac{\delta'-\delta}{2}$	$\frac{\delta'+\delta}{2}$	$\frac{P'-P}{2}$	$\frac{P'+P}{2}$
1.										
2.										
1.										
2.										
1.										
2.										
1.										
2.										
1.										
2.										

Log tan $\frac{P'-P}{2}$					
Log cot $\frac{\delta'-\delta}{2}$					
Log cot $\frac{\delta'+\delta}{2}$					
Log tan M					
M					
$a.c.$ log cos M					
Log cos $\frac{P'-P}{2}$					
Log tan $\frac{\delta'+\delta}{2}$					
Log m					
$\frac{P'+P}{2}$					
M					
$\frac{P'+P}{2}-M$					
Log cos $\frac{P'+P}{2}-M$					
Log m					
Log tan ϕ					
ϕ					
Mean					

FORM No. 12.

LONGITUDE BY ELECTRIC TELEGRAPH.

Stations. (1) Eastern. (2) Western.	Date.	Observer.	No. of Time Stars.	Chronom. Errors at Epoch. E & E'	$E - E'$	No. of Chronom. Comparisons.	$T - T'$ & $T''' - T''$	Dif. of Long. by E. & W. Signals. λ' & λ''	Mean. $\frac{\lambda' + \lambda''}{2}$	Retardation. $\frac{\lambda' - \lambda''}{2}$
1.										
2.										
1.										
2.										
1.										
2.										
1.										
2.										
1.										
2.										
								Sum...	....	....
								Mean...	....	....

FORM NO. 12*a*.

LONGITUDE BY ELECTRIC TELEGRAPH.

Reduction of Transits of Stars observed at .... Date.......................... With Transit

No........, By...................... to determine Correction to Sidereal Chronometer No...... By........................

h. *m.* *s.*

at.... ....Chronom. Time. Assumed Error at Epoch.... .... Rate........................

Names of Stars.	Illumination.	Observer.	Time of Transit over Middle Wire.	Diurnal Aberration.	Correction for Rate.	Correction for Level.	App. R. A.	n.	A.	C.	An.	Cn.	AC.	A^2.	C^2.
								$\Sigma(n)$	$\Sigma(A)$	$\Sigma(C)$	$\Sigma(An)$	$\Sigma(Cn)$	$\Sigma(AC)$	$\Sigma(A^2)$	$\Sigma(C^2)$

Normal Equations.

1. ....=.......... ....
2. ....=..........................
3. ....=..........................