ANALYTICAL MECHANICS

FOR STUDENTS OF PHYSICS AND ENGINEERING

BY

HAROUTUNE M. DADOURIAN, M.A., Ph.D.

Instructor of Physics in the Sheffield Scientific School of Yale University

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ANALYTICAL MECHANICS

INTRODUCTION.

1. Scope and Aim of Mechanics. Mechanics is the science of motion. It has a twofold object:

First, to describe the motions of bodies and to interpret them by means of a few laws and principles, which are generalizations derived from observation and experience.

Second, to predict the motion of bodies for all times when the circumstances of the motion for any one instant are given, in addition to the special laws which govern the motion.

The present tendency in science is toward regarding all physical phenomena as manifestations of motion. Complicated and apparently dissimilar phenomena are being explained by the interactions and motions of electrons, atoms, molecules, cells, and other particles. The kinetic theory of heat, the wave theories of sound and light, and the electron theory of electricity are examples which illustrate the tendency toward a mechanical interpretation of the physical universe.

This tendency not only emphasizes the fundamental importance of the science of mechanics to other physical sciences and engineering but it also broadens the aim of the science and makes the dynamical interpretation of all physical phenomena its ultimate object. The aim of elementary mechanics is, however, very modest and its scope is limited to the discussion of the simplest cases of motion and equilibrium which occur in nature.

2. Divisions of Mechanics. It is customary to divide Mechanics into *Kinematics* and *Dynamics*. The former treats of the time and space relations of the motions of bodies without regard to the interactions which cause them. In other words, Kinematics is the geometry of motion. In Dynamics, on the other hand, motion and equilibrium are treated as the results of interactions between bodies; consequently not only *time* and *space* enter into dynamical discussions, but also *mass*, the third element of motion.

Dynamics in its turn is divided into *Statics* and *Kinetics*. Statics is the mechanics of bodies in equilibrium, while Kinetics is the mechanics of bodies in motion.

Chapters II, III, and IV of the present work are devoted to problems in statics, while the rest of the book with the exception of Chapters I, V, and VII, are given to discussions of problems in kinetics. The subject matter of Chapters I and VII is essentially of a mathematical nature. In the former the addition and resolution of vectors are discussed, while in the latter the Calculus is applied to finding centers of mass and moments of inertia. Chapter V is devoted mainly to kinematical problems.

CHAPTER I.

ADDITION AND RESOLUTION OF VECTORS.

3. Scalar and Vector Magnitudes.— Physical magnitudes may be divided into two classes according to whether they have the property of orientation or not. Magnitudes which have direction are called *vectors*, while those which do not have this property are called *scalars*. Displacement, velocity, acceleration, force, torque, and momentum are vector magnitudes. Mass, density, work, energy, and time are scalars.

4. Graphical Representation of Vectors. — Vectors are represented by directed lines or arrows. The length of the directed line represents the magnitude of the vector, while its direction coincides with that of the vector. For brevity the directed lines as well as the physical quantities which they represent are called vectors. The head and the tail of the directed line are called, respectively, the *terminus* and the *origin* of the vector. In Fig. 1, for instance, P is the origin and Q the terminus of the vector $\mathbf{a}$.

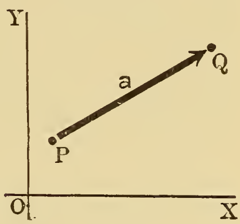


FIG. 1.

5. Notation. — Vectors will be denoted by letters printed in Gothic type, while their magnitudes will be represented by the same letters printed in italic type. Thus in Fig. 1 the vector PQ is denoted by $\mathbf{a}$, but if it is desired to represent the length PQ without regard to its orientation a is used.

6. Equal Vectors.—Two vectors are said to be equal if they have the same length and the same direction. It follows, therefore, that the value of a vector is not changed when it is moved about without changing its direction and magnitude.

7. Addition of Two Vectors.— Let the vectors a and b, Fig. 2, represent two displacements, then their sum is another vector, c, which is equivalent to the given vectors. In order to find c let us apply to a particle the operations indicated by a and b. Each vector displaces the particle along its direction through a distance equal to its length. There-

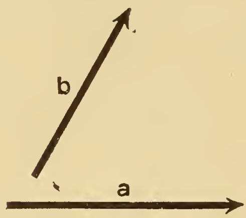


Fig. 2.

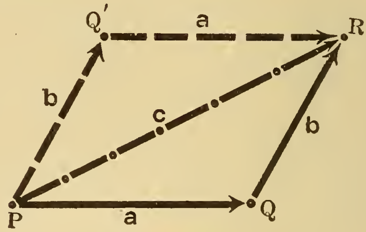


Fig. 3.

fore applying a to the particle at P, Fig. 3, the particle is brought to the point Q. Then applying the operation indicated by b the particle is brought to the point R. Therefore the result of the two operations is a displacement from P to R. But this is equivalent to a single operation represented by the vector c, which has P for its origin and R for its terminus. Therefore c is called the sum, or the resultant, of a and b. This fact is denoted by the following vector equation,

$$\mathbf{a} + \mathbf{b} = \mathbf{c}. \qquad \text{(I)}$$

8. Order of Addition. — The order of addition does not affect the result. If in Fig. 3 the order of the operations indicated by a and b is reversed the particle moves from P to Q' and then to R. Thus the path of the particle is changed but not the resultant displacement.

9. Simultaneous Operation of Two Vectors.—The operations indicated by a and b may be performed simultaneously without affecting the final result. In order to illustrate the simultaneous operation of two vectors suppose the particle to be a bead on the wire *AB*, Fig. 4. Move the wire, keeping it parallel to itself, until each of its particles is given a displacement represented by **b**. Simultaneously with the motion of the wire move the bead along the wire giving it a displacement equal to **a**. At the end of

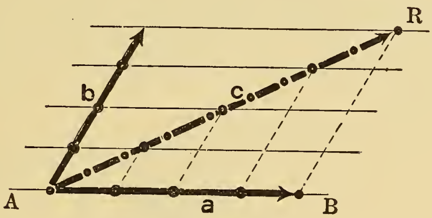


Fig. 4.

these operations the bead arrives at the point *R*. If both the wire and the bead are moved at constant rates the resultant vector **c** represents not only the resulting displacement but also the path of the particle.

10. Rules for Adding Two Vectors.—The results of the last three paragraphs furnish us with the following methods for adding two vectors graphically.

Triangle Method.—*Move one of the vectors, without changing its direction, until its origin falls upon the terminus of the other vector, then complete the triangle by drawing a vector the origin of which coincides with that of the first vector. The new vector is the resultant of the given vectors.*

Parallelogram Method.—*Move one of the vectors until its origin falls on that of the other vector, complete the parallelogram, and then draw a vector which has the common origin of the given vectors for its origin and which forms a diagonal of the parallelogram. The new vector is the resultant of the given vectors.*

11. Analytical Expression for the Resultant of Two Vectors.—Let **a** and **b**, Fig. 5, be two vectors and **c** their resultant. Then, solving the triangle formed by these vectors, we obtain

$$c^2 = a^2 + b^2 + 2\,ab\cos\phi \tag{II}$$

and
$$\tan\theta = \frac{b\sin\phi}{a + b\cos\phi}, \tag{III}$$

where a, b, and c are the magnitudes of **a**, **b**, and **c**, respectively, while ϕ and θ are the angles **b** and **c** make with **a**. Equation (II) determines the magnitude and equation (III) the direction of **c**.

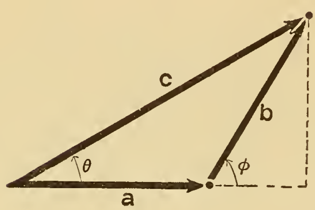


Fig. 5.

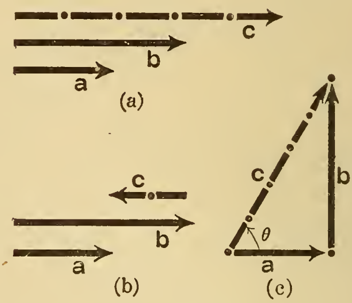


Fig. 6.

Special Cases. (a). If **a** and **b** have the same direction, as in Fig. 6a, then $\phi = 0$. Therefore

$$c^2 = a^2 + b^2 + 2\,ab, \quad \therefore \quad c = a + b,$$

and
$$\tan\theta = 0, \quad \therefore \quad \theta = 0.$$

Thus **c** has the same direction as **a** and **b**, while its magnitude equals the arithmetical sum of their magnitudes.

(b). When **a** and **b** are oppositely directed, as in Fig. 6b, $\phi = \pi$. Therefore

$$c^2 = a^2 + b^2 - 2\,ab, \quad \therefore \quad c = a - b,$$

and
$$\tan\theta = 0, \quad \therefore \quad \theta = 0.$$

Thus the magnitude of **c** equals the algebraic sum of the magnitudes of **a** and **b**, while its direction is the same as that of the larger of the two. It is evident that if the magnitudes of **a** and **b** are equal **c** vanishes. Therefore two vectors of equal magnitude and opposite directions are the negatives of each other. In other words, *when the direction of a vector is reversed its sign is changed.*

(c). When **a** and **b** are at right angles to each other, as in Fig. 6c, $\phi = \frac{\pi}{2}\cdot$ Therefore

$$c^2 = a^2 + b^2$$

and

$$\tan\theta = \frac{b}{a}\cdot$$

12. Difference of Two Vectors.—Subtraction is equivalent to the addition of a negative quantity. Therefore, to subtract **b** from **a** we add $-$**b** to **a**. Thus we have the following rule for subtracting one vector from another.

In order to subtract one vector from another reverse the one to be subtracted and add it to the other vector.

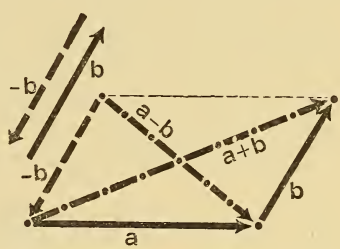


Fig. 7.

It is evident from Fig. 7 that the sum and the difference of two vectors form the diagonals of the parallelogram determined by them.

ILLUSTRATIVE EXAMPLES.

A particle is displaced 10 cm. N. 30° E., then 10 cm. E. Find the resulting displacement.

Representing the displacements and their resultant by the vectors **a**, **b**, and **c**, Fig. 8, we obtain

$$c^2 = a^2 + b^2 + 2\,ab\cos\phi$$
$$= (10\text{ cm.})^2 + (10\text{ cm.})^2 + 2\times 10\text{ cm.}\times 10\text{ cm.}\cos(60°)$$
$$= 300\text{ cm.}^2$$
$$\therefore\quad c = 10\sqrt{3}\text{ cm.}$$
$$\doteq 17.3\text{ cm.*}$$
$$\tan\theta = \frac{b\sin\phi}{a + b\cos\phi}$$
$$= \frac{10\text{ cm. }\sin(60°)}{10\text{ cm.} + 10\text{ cm. }\cos(60°)}$$
$$= \tfrac{1}{3}\sqrt{3}.$$
$$\therefore\quad \theta = 30°.$$

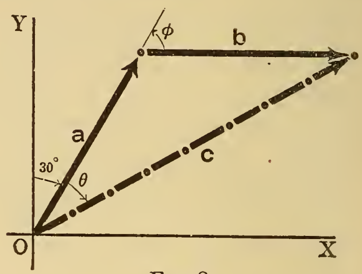


FIG. 8.

Therefore the resultant displacement is about 17.3 cm. along the direction N. 60° E.

PROBLEMS.

1. A vector which points towards the East has a length of 16 cm., and another vector which points towards the Southeast is 25 cm. long. Find the direction and the magnitude of their sum.

2. Find the direction and the magnitude of the difference of the vectors of the last problem.

3. The sum of two vectors is perpendicular to their difference. Show that the vectors are equal in magnitude.

4. The sum and the difference of two vectors are equal. Show that the vectors are at right angles to each other.

13. Resolution of Vectors into Components. — The projection of a vector upon a line is called the *component* of the vector along that line. The vectors $\mathbf{a}_x$ and $\mathbf{a}_y$ in Fig. 9, for instance, are the components of **a** along the x-axis and the y-axis, respectively. The following relations are evident from the figure and do not need further explanation.

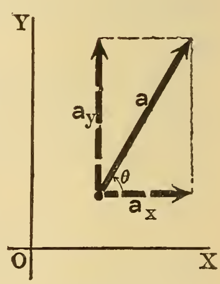


FIG. 9.

* The symbol " $\doteq$ " will be used to denote approximate equality. Therefore " $\doteq$ " should be read " equals approximately," or "equals about," or "equals nearly." See p. .

$$\mathbf{a} = \mathbf{a}_x + \mathbf{a}_y, \tag{IV}$$

$$\left.\begin{aligned} a_x &= a \cos \theta, \\ a_y &= a \sin \theta, \end{aligned}\right\} \tag{V}$$

$$a = \sqrt{a_x^2 + a_y^2}, \tag{VI}$$

$$\tan \theta = \frac{a_y}{a_x}. \tag{VII}$$

When **a** has components along all three axes of a rectangular system, Fig. 10, the following equations express the vector in terms of its components.

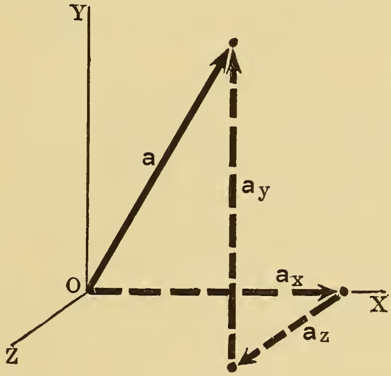


Fig. 10.

$$\mathbf{a} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z. \tag{IV'}$$

$$\left.\begin{aligned} a_x &= a \cos \alpha_1 \\ a_y &= a \cos \alpha_2 \\ a_z &= a \cos \alpha_3 \end{aligned}\right\} \tag{V'}$$

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}, \tag{VI'}$$

where α_1, α_2, and α_3 are the angles **a** makes with the coördinate axes.

14. Resultant of Any Number of Vectors. Graphical Methods. — The resultant of a number of vectors **a**, **b**, **c**, etc., may be obtained by either of the following methods.

First: move **b**, without changing either its direction or its magnitude, until its origin falls on the terminus of **a**, then move c until its origin falls on the terminus of b, and so on until all the vectors are joined. This gives, in general, an open polygon. Then the resultant is obtained by drawing a vector which closes the polygon and which has its origin at the origin of a. The validity of this method will be seen from Fig. 11, where r represents the resultant vector. Evidently the resultant vanishes when the given vectors form a closed polygon.

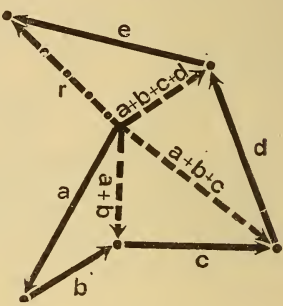


FIG. 11.

Second: draw a system of rectangular coördinate axes; resolve each vector into components along the axes; add the components along each axis geometrically, beginning at the origin. This gives the components of the required vector. Then draw the rectangular parallelopiped determined by these components. The resultant is a vector which has the origin of the axes for its origin and forms a diagonal of the parallelopiped. This method is based upon the following analytical method.

15. Analytical Method.—Expressing the given vectors and their resultant in terms of their rectangular components, we have

$$\left.\begin{array}{l} \mathbf{a} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z, \\ \mathbf{b} = \mathbf{b}_x + \mathbf{b}_y + \mathbf{b}_z, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \mathbf{r} = \mathbf{r}_x + \mathbf{r}_y + \mathbf{r}_z. \end{array}\right\} \tag{1}$$

Substituting from (1) in the vector equation

$$\mathbf{r} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \cdots \tag{2}$$

and collecting the terms we obtain

$$\begin{aligned} \mathbf{r}_x + \mathbf{r}_y + \mathbf{r}_z = (\mathbf{a}_x + \mathbf{b}_x + \cdots) + (\mathbf{a}_y + \mathbf{b}_y + \cdots) \\ + (\mathbf{a}_z + \mathbf{b}_z + \cdots). \end{aligned} \tag{3}$$

But since the directions of the coördinate axes are independent, the components of r along any one of the axes must equal the sum of the corresponding components of the given vectors. Therefore (3) can be split into the following three separate equations.

$$\left.\begin{aligned} \mathsf{r}_x &= \mathsf{a}_x + \mathsf{b}_x + \mathsf{c}_x + \cdots, \\ \mathsf{r}_y &= \mathsf{a}_y + \mathsf{b}_y + \mathsf{c}_y + \cdots, \\ \mathsf{r}_z &= \mathsf{a}_z + \mathsf{b}_z + \mathsf{c}_z + \cdots. \end{aligned}\right\} \quad (4)$$

It was shown in § 11 that when two vectors are parallel the algebraic sum of their magnitudes equals the magni-

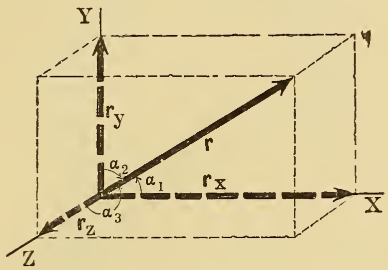


FIG. 12.

tude of their resultant. This result may be extended to any number of parallel vectors. Therefore we can put the vector equations of (4) into the following algebraic forms.

$$\left.\begin{aligned} r_x &= a_x + b_x + c_x + \cdots, \\ r_y &= a_y + b_y + c_y + \cdots, \\ r_z &= a_z + b_z + c_z + \cdots. \end{aligned}\right\} \quad (5)$$

Equations (5) determine r through the following relations

$$r = \sqrt{r_x^2 + r_y^2 + r_z^2}, \quad (6)$$

$$\cos\alpha_1 = \frac{r_x}{r}, \quad \cos\alpha_2 = \frac{r_y}{r}, \quad \cos\alpha_3 = \frac{r_z}{r}, \quad (7)$$

where α_1, α_2, and α_3 are the angles r makes with the axes.

16. Multiplication and Division of a Vector by a Scalar.— When a vector is multiplied or divided by a scalar the result is a vector which has the same direction as the original vector. If, in the equation b = ma, m be a scalar then b has the same direction as a but its magnitude is m times that of a.

ILLUSTRATIVE EXAMPLE.

A man walks 3 miles N. 30° E., then one mile E., then 3 miles S. 45° E., then 4 miles S., then one mile N. 30° W. Find his final position.

Representing the displacements by vectors we obtain the graphical solution given in Fig. 13, where **r** represents the resultant displacement.

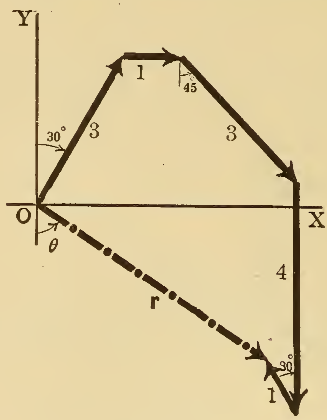


FIG. 13.

In order to find **r** analytically we first determine its components. Thus

$$\begin{aligned} r_x &= [3 \cos (60°) + \cos (0°) + 2 \cos (-45°) + 4 \cos (-90°) \\ &\qquad + \cos (120°)] \text{ miles} \\ &= (2 + \sqrt{2}) \text{ miles} \\ &\doteq 3.41 \text{ miles.} \\ r_y &= [3 \sin (60°) + \sin (0°) + 2 \sin (-45°) + 4 \sin (-90°) \\ &\qquad + \sin (120°)] \text{ miles} \\ &= (2\sqrt{3} - \sqrt{2} - 4) \text{ miles} \\ &\doteq -1.95 \text{ miles.} \\ \therefore \quad r &= \sqrt{r_x^2 + r_y^2} \\ &\doteq 3.93 \text{ miles.} \end{aligned}$$

The direction of **r** is given by the following relation.

$$\tan \theta = \frac{r_y}{r_x} \doteq \frac{-1.95}{3.41},$$

$$\therefore \quad \theta \doteq -37°.1.$$

Therefore the final position of the man is about 3.93 miles S. 52°.9 E. from his starting point.

PROBLEMS.

1. The resultant of two vectors which are at right angles to each other is twice the smaller of the two. The magnitude of the smaller vector is a; find the magnitude of the greater vector.

2. In the preceding problem find the resultant vector.

3. Find analytically the sum of three equal vectors which point in the following directions — East, N. 30° W., and S. 30° W.

4. In the preceding problem make use of the first graphical method.

5. In problem 3 make use of the second graphical method.

6. A vector which is 15 cm. long points N. 30° E. Find its components in the following directions.

(a) N. 30° W.	(c) W.	(e) S. 60° E.
(b) N. 60° E.	(d) S. 30° W.	(f) E.

7. A vector $\mathbf{a}$ is in the xy-plane. If 3 is added to a_x and 4 to a_y the magnitude of the new vector equals $a_x + a_y$. Find the magnitude and direction of $\mathbf{a}$.

8. Three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ lie in the xy-plane. Find their resultants analytically, taking the magnitudes of their components from the following tables:

	a_x	a_y	b_x	b_y	c_x	c_y
(1)	6,	9,	−5,	2,	0,	10.
(2)	−3,	7,	5,	0,	6,	−8.
(3)	0,	−10,	8,	5,	3,	−2.
(4)	2,	0,	−6,	4,	0,	8.

9. In the preceding problem make use of the second graphical method.

10. Straight horizontal tunnels in a mine connect the points P_1, P_2, P_3, and P_4, in the given order. The length of each tunnel and the angle it makes with the meridian are given in the following tables. Find the lengths and directions of the tunnels which have to be dug in order to connect P_1 with P_3 and P_4.

P_1P_2 = 200 feet, and makes 30° with the meridian.
P_2P_3 = 100 feet, and makes 120° with the meridian.
P_3P_4 = 400 feet, and makes 300° with the meridian.

11. Work out the preceding problem by the first graphical method.

12. Work out problem 10 by the second graphical method.

13. Find the direction and magnitude of the force experienced by an electrical charge of five units placed at one vertex of an equilateral triangle due to two unlike charges of 10 units each placed at the other vertices. The sides of the triangle are 2 cm.

CHAPTER II.

EQUILIBRIUM OF A PARTICLE.

ACTION AND REACTION. FORCE.

17. Particle. — A body whose dimensions are negligible is called a *particle*. In a problem any body may be considered as a particle so long as it does not tend to rotate. Even when the body rotates it may be considered as a particle if its rotation does not enter into the problem. For instance, in discussing the motion of the earth in its orbit the earth is considered as a particle, because its rotation about its axis does not enter into the discussion.

18. Degrees of Freedom.—The number of independent ways in which a body can move is called the number of *degrees of freedom* of its motion. It equals the number of coördinates which are necessary in order to specify completely the position of the body. A free particle can move in three independent directions, that is, along the three axes of a system of rectangular coördinates, therefore it has three degrees of freedom. When the particle is constrained to move in a plane its freedom is reduced to two degrees, because it can move only in two independent directions. When it is constrained to move in a straight line it has only one degree of freedom.

19. Force.—While considering the motion or the equilibrium of a body our attention is claimed not only by that body but also by others which act upon it. In order to insure concentration of attention problems in Dynamics are simplified in the following manner. All bodies are eliminated, except the one the motion of which is being discussed, and their actions upon the latter are represented by certain vector magnitudes known as *forces*. As an illustration consider the equilibrium of the shaded part of the rope in Fig. 14a. The shaded part is acted upon by the adjoining sections of the rope. Therefore we consider the shaded part alone and represent the actions of the adjoining parts by the forces F and F′, as shown in Fig. 14b.

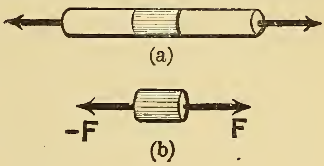


FIG. 14.

20. Definition of Force.—*Force is a vector magnitude which represents the action of one body upon another.* The interaction between two bodies takes place across an area, while the forces which represent them are supposed to be applied at one point. Therefore the introduction of the idea of force presupposes the simplification of dynamical problems which is obtained by considering bodies as single particles, or as a system of particles.

21. Internal Force.—A force which represents the action of one part of a body upon another part of the same body is called an internal force.

22. External Force.—A force which represents the action of one body upon another body is called an external force.

23. Unit Force.—The engineering unit of force among English speaking people is the *pound.* The pound is the weight, in London, of a certain piece of platinum kept by the British government.

24. The Law of Action and Reaction.—The fundamental law of Mechanics is known as the *law of action and reaction.* Newton (1692–1727), who was the first to formulate it, put the law in the following form.

"*To every action there is an equal and opposite reaction, or the mutual actions of two bodies are equal and oppositely directed.*"

Let us apply this law to the interaction between a book and the hand in which you hold it. Your hand presses upward upon the book in order to keep it from falling, while the book presses downward upon your hand. The law states that the action of your hand equals the reaction of the book and is in the opposite direction. The book reacts upon your hand because the earth attracts it. When your hand and the earth are the only bodies which act upon the book, the action of your hand equals and is opposite to the action of the earth. In other words the sum of the two actions is nil. Generalizing from this simple illustration we can put the law into the following form:

To every action there is an equal and opposite reaction, or the sum of all the actions to which a body or a part of a body is subject at any instant vanishes:

$$\Sigma \mathbf{A} = 0. \tag{A}$$

25. Condition for the Equilibrium of a Particle. — The condition of equilibrium of a particle is obtained by replacing the term "action" by the term "force" in the last form of the fundamental law and then stating it in the form of a condition. Thus — *in order that a particle be in equilibrium the sum of all the forces which act upon it must vanish.*

In other words if $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots, \mathbf{F}_n$ are the forces which act upon a particle, then the vector equation

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots + \mathbf{F}_n = 0 \tag{I}$$

must be satisfied in order that the particle be in equilibrium. Equation (I) is equivalent to stating that when the forces are added graphically they form a closed polygon. But when the sum of a number of vectors vanishes the sum of their components also vanishes. Therefore we must have

$$\left.\begin{aligned} \mathsf{X}_1 + \mathsf{X}_2 + \cdots + \mathsf{X}_n &= 0, \\ \mathsf{Y}_1 + \mathsf{Y}_2 + \cdots + \mathsf{Y}_n &= 0, \\ \mathsf{Z}_1 + \mathsf{Z}_2 + \cdots + \mathsf{Z}_n &= 0, \end{aligned}\right\} \tag{II$'$}$$

where X_i, Y_i, and Z_i are the components of $\mathbf{F}_i$.* Since the

* The subscript "$_i$" is used to denote "any one," thus $\mathbf{F}_i$ denotes any one of $\mathbf{F}_1$, $\mathbf{F}_2$, etc.

vectors in each of the equations of (II′) are parallel we can write them as algebraic equations. Therefore we have the following equations for the analytical form of the condition of equilibrium of a particle.

$$\left.\begin{aligned} \Sigma X &\equiv X_1 + X_2 + \cdots + X_n = 0,^* \\ \Sigma Y &\equiv Y_1 + Y_2 + \cdots + Y_n = 0, \\ \Sigma Z &\equiv Z_1 + Z_2 + \cdots + Z_n = 0. \end{aligned}\right\} \qquad \text{(II)}$$

The condition of equilibrium may, therefore, be stated in the following form.

In order that a particle be in equilibrium the algebraic sum of the components of the forces along each of the axes of a rectangular system of coördinates must vanish.

The following rules will be helpful in working out problems on the equilibrium of a particle.

First. Represent the particle by a point and the action of each body which acts upon it by a properly chosen force-vector. Be sure that all the bodies which act upon the particle are thus represented.

Second. Set the sums of the components of the forces along properly chosen axes equal to zero.

Third. If there are not equations enough to determine the unknown quantities, obtain others from the geometrical connections of the problem.

Fourth. Solve these equations for the required quantities.

Fifth. Discuss the results.

ILLUSTRATIVE EXAMPLES.

1. A particle suspended by a string is pulled aside by a horizontal force until the string makes an angle α with the vertical. Find the tensile force in the string and the magnitude of the horizontal force in terms of the weight of the particle.

The particle is acted upon by three bodies, namely, the earth, the string, and the body which exerts the horizontal force. Therefore, we

* The relation $\Sigma X \equiv X_1 + X_2 + \cdots + X_n$ is not an equation. It merely states that ΣX is identical with and is an abbreviation for $X_1 + X_2 + \cdots + X_n$.

represent the actions of these bodies by three force-vectors, **W**, **T**, and **F**, Fig. 15, and then apply the conditions of equilibrium. Setting equal to zero the sums of the components of the forces along the x- and y-axes, we get

$$\Sigma X \equiv F - T \sin \alpha = 0. \quad \text{(a)}$$

$$\Sigma Y \equiv -W + T \cos \alpha = 0. \quad \text{(b)}$$

Solving equations (a) and (b) we have

$$T = \frac{W}{\cos \alpha},$$

and

$$F = T \sin \alpha$$
$$= W \tan \alpha.$$

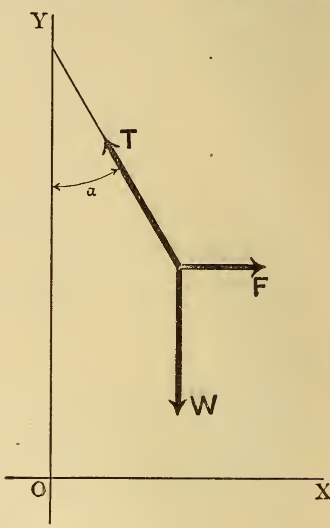


FIG. 15.

DISCUSSION. — When $\alpha = 0$, $T = W$ and $F = 0$. When $\alpha = \frac{\pi}{2}$, $T = \infty$ and $F = \infty$. Therefore no finite horizontal force can make the string perfectly horizontal.

2. A uniform bar, of weight W and length a, is suspended in a horizontal position by two strings of equal length l. The lower ends of the strings are fastened to the ends of the bar and the upper ends to a peg. Find the tensile force in the strings.

The bar is acted upon by three bodies, namely the earth and the two strings. We represent their actions by the forces **W**, $\mathbf{T}_1$, and $\mathbf{T}_2$, Fig. 16a. The tensile forces of the strings act at the ends of the bar. On the other hand the weight is distributed all along the rod. But we may consider it as acting at the middle point, as in Fig. 16a, or we may replace the rod by two particles of weight $\frac{W}{2}$ each, as shown in Fig. 16b. In the last case the rigidity of the bar which prevents its ends from coming together is represented by the forces **F** and $-$**F**.

Considering each particle separately and setting equal to zero the sums of the components of the forces along the axes, we obtain

$$\Sigma X \equiv T_1 \cos \alpha - F = 0,$$

$$\Sigma Y \equiv T_1 \sin \alpha - \frac{W}{2} = 0,$$

for the first particle, and

$$\Sigma X \equiv -T_2 \cos \alpha + F = 0,$$

$$\Sigma Y \equiv T_2 \sin \alpha - \frac{W}{2} = 0,$$

for the second particle. It follows from these equations that

$$T_1 = T_2 = \frac{W}{2 \sin \alpha} = \frac{l}{\sqrt{4l^2 - a^2}} W.$$

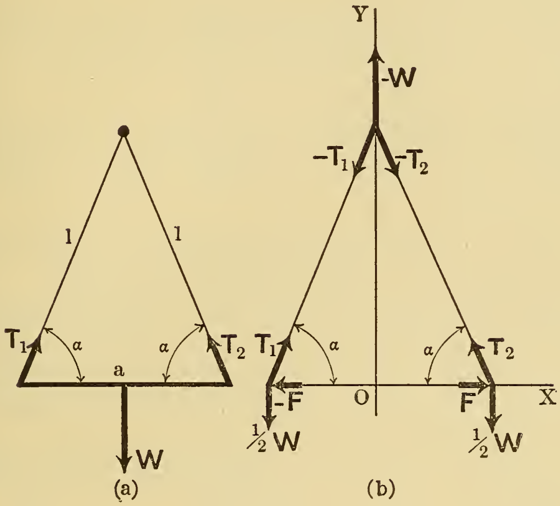


Fig. 16.

Discussion. — The tensile force of the strings increases indefinitely as their total length approaches that of the bar. On the other hand as the length of the strings becomes very large compared with that of the bar the tensile force approaches $\frac{W}{2}$ as a limit.

The problem can be solved also by considering the forces acting on the peg, as shown in Fig. 16b.

PROBLEMS.

1. Show that when a particle is in equilibrium under the action of two forces, the forces must lie in the same straight line.

2. Show that when a particle is in equilibrium under the action of three forces the forces lie in the same plane.

3. Find the horizontal force which will keep in equilibrium a weight of 150 pounds on a smooth inclined plane which makes 60° with the horizon.

4. A ring of weight W is suspended by means of a string of length l, the ends of which are attached to two points on the same horizontal line. Find the tensile force of the string if the distance between its ends is d. Also discuss the limiting cases in which l approaches d or becomes very large compared with it.

5. A body of weight W is suspended by two strings of lengths l_1 and l_2. The upper end of each string is attached to a fixed point in the same horizontal line. Find the tensile forces in the strings if the distance between the two points is d.

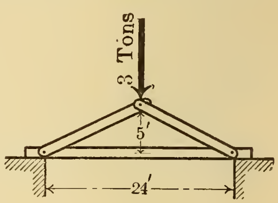
6. A weight is suspended by four equal strings, the upper ends of which are attached to the vertices of a horizontal square. Find the tensile forces in the strings.

7. A particle is in equilibrium on a smooth inclined plane under the action of two equal forces, the one acting along the plane upwards and the other horizontally. Find the inclination of the plane.

8. Apply the conditions of equilibrium to find the magnitude and direction of the resultant of a number of forces acting upon a particle.

9. Two spheres of equal radius and equal weight are in equilibrium in a smooth hemispherical bowl; find the reactions between the two spheres and between the spheres and the bowl.

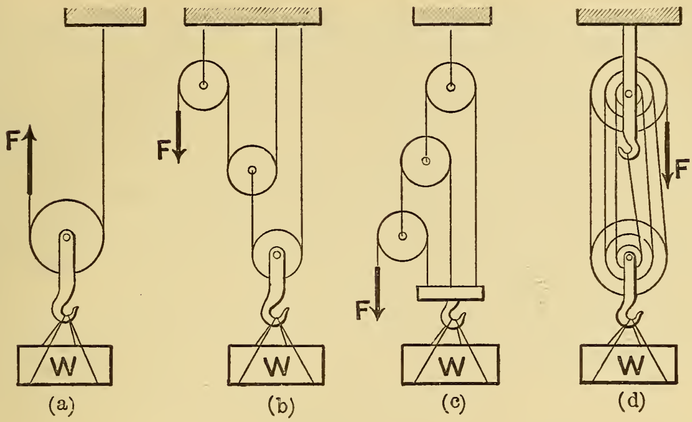
10. The ends of a string, 60 cm. long, are fastened to two points in the same horizontal line and at a distance of 40 cm. apart; two weights are hung from points in the string 25 cm. and 20 cm. from the ends. Find the ratio of the weights if the part of the string between them is horizontal.



11. A single triangular truss of 24 feet span and 5 feet depth supports a load of 3 tons at the apex. Find the forces acting on the rafters and the tie rod.

12. A particle of weight W can be kept in equilibrium upon a smooth inclined plane by a force F_1 acting horizontally; it can also be kept in equilibrium by a force F_2 acting parallel to the plane. Express W in terms of F_1 and F_2.

13. In the following arrangements of pulleys find the relation between **F** and **W**.



SLIDING FRICTION.

26. Frictional Force.—Consider the forces acting upon a body which is in equilibrium on a rough inclined plane, Fig. 17. The body is acted upon by two forces, namely, its weight, W, and the reaction of the plane, R. The reaction of the plane is the result of two distinct and independent forces. One of these, N, is perpendicular to the plane and is called the *normal reaction*. The other, F, is along the plane and is called the *frictional force*. The normal reaction is due to the rigidity of the plane. It resists the tendency of the body to go through the plane. The frictional force is due to the roughness of the contact between the body and the plane. It prevents the body from sliding down the plane.

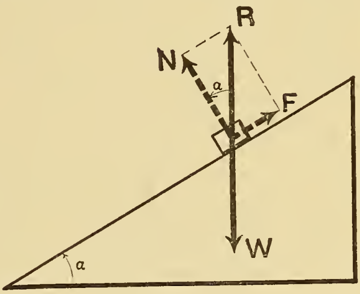


Fig. 17.

27. Angle of Friction. — As we increase the angle of elevation of the inclined plane a certain definite angle will be reached when the equilibrium is disturbed and the body begins to slide down the plane. This angle is called the *angle of friction.* This definition for the angle of friction does not hold when the body is acted upon by other forces besides its weight and the reaction of the plane. The following definition, however, is valid under all circumstances: *The angle of friction equals the angle which the total reaction makes with the normal to the surface of contact when the body is on the point of motion.*

28. Coefficient of Friction. — Denoting the angle of friction by ϕ, we obtain

$$F = R \sin \phi,$$

$$N = R \cos \phi.$$

Therefore
$$F = N \tan \phi$$

$$= \mu N, \qquad \text{(III)}$$

where $\mu = \tan \phi$ and is called the *coefficient of friction.* The angle of friction and consequently the coefficient of friction are constants which depend upon the surfaces in contact. *The last four equations hold true only when the body is on the point of motion.*

29. Static and Kinetic Friction. — The friction which comes into play is called static friction if the body is at rest and kinetic friction if it is in motion.

30. Laws of Friction. — The following statements, which are generalizations derived from experimental results, bring out the important properties of friction. They hold true within certain limits and are only approximately true even within these limits.

1. Frictional forces come into play only when a body is urged to move.

2. Frictional forces always act in a direction opposite to that in which the body is urged to move.

3. Frictional force is proportional to the normal reaction, $F = \mu N$.

4. Frictional force is independent of the area of contact.

5. The static frictional force which comes into play is not greater than that which is necessary to keep the body in equilibrium.

6. Kinetic friction is smaller than static friction.

Laws 1 to 4 hold true for both static and kinetic friction. The coefficient of friction between two bodies depends upon the condition of surfaces in contact. Therefore the value of μ is not a perfectly definite constant for a given pair of substances in contact.

The values given in the following table are averages of values obtained by several experimenters.

Materials in contact.	Condition of surfaces in contact.	Coefficient of friction.	
		Static.	Kinetic.
Wood on wood................	Dry	.50	.36
Wood on wood................	Wet	.68	.25
Wood on wood................	Polished and greased	.35	.12
Heavy rope on wood..........	Dry	.60	.40
Heavy rope on wood..........	Wet	.80	.35
Cast iron on cast iron.........	Dry	.24	.18
Cast iron on cast iron.........	Greased	.15	.13
Cast iron on oak..............	Wet	.65	—
Leather on cast iron	——	.30	—

ILLUSTRATIVE EXAMPLES.

1. A body which is on a rough horizontal floor can be brought to the point of motion by a force which makes an angle α with the floor. Find the reaction of the floor and the coefficient of friction.

The body is acted upon by three forces, Fig. 18,

P, the given force,
W, the weight of the body,
R, the reaction of the floor.

Replacing **R** by its components **F** and **N**, and applying the conditions of equilibrium, we obtain

$$\Sigma X \equiv P \cos \alpha - F = 0,$$
$$\Sigma Y \equiv P \sin \alpha + N - W = 0.$$

Therefore

$$F = P \cos \alpha,$$
$$N = W - P \sin \alpha,$$

and $$R = \sqrt{F^2 + N^2}$$
$$= \sqrt{P^2 + W^2 - 2\,PW \sin\alpha}.$$

But since the body is on the point of motion the relation $F = \mu N$ holds. Therefore

$$\mu = \frac{F}{N} = \frac{P\cos\alpha}{W - P\sin\alpha}.$$

Discussion. — (a) When $\alpha = 0$, $R = \sqrt{P^2 + W^2}$ and $\mu = \frac{P}{W}$. (b) When $\alpha = \frac{\pi}{2}$, $R = P - W = 0$, therefore $P = W$, and μ is indeterminate. (c) When $P = 0$, $\mu = 0$, and $R = W$.

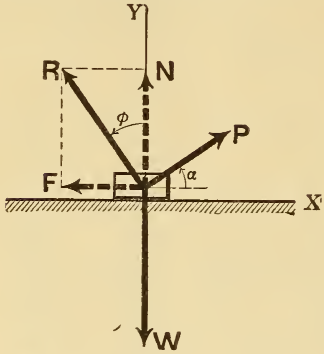


Fig. 18.

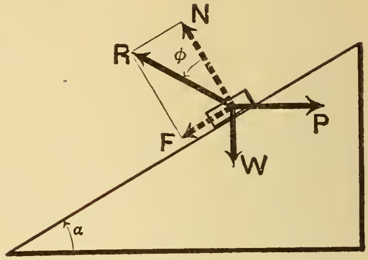


Fig. 19.

2. A body which rests upon a rough inclined plane is brought to the point of motion up the inclined plane by a horizontal force. Find μ and R.

The body is acted upon by three forces, Fig. 19,

P, the horizontal force,
W, the weight,
R, the reaction of the plane.

Replacing **R** by its components **F** and **N**, and taking the axes along and at right angles to the plane, we obtain

$$\Sigma X \equiv P\cos\alpha - F - W\sin\alpha = 0,$$
$$\Sigma Y \equiv -P\sin\alpha + N - W\cos\alpha = 0.$$

Therefore

$$F = P\cos\alpha - W\sin\alpha,$$
$$N = P\sin\alpha + W\cos\alpha,$$
$$R = \sqrt{F^2 + N^2}$$
$$= \sqrt{P^2 + W^2},$$

and
$$\mu = \frac{F}{N} = \frac{P \cos \alpha - W \sin \alpha}{P \sin \alpha + W \cos \alpha}.$$

Discussion. — (a) When $\alpha = 0$, $\mu = \frac{P}{W}$, and $R = W\sqrt{\mu^2 + 1}$.

(b) When $P = 0$, $\mu = -\tan \alpha$; therefore $\alpha = -\phi$, that is, the inclined plane must be tipped in the opposite direction and must be given an angle of elevation equal to the angle of friction in order that motion may take place towards the positive direction of the x-axis.

PROBLEMS.

1. A body which weighs 100 pounds is barely started to move on a rough horizontal plane by a force of 150 pounds acting in a direction making 30° with the horizon. Find R and μ.

2. A body placed on a rough inclined plane barely starts to move when acted upon by a force equal to the weight of the body. Find the coefficient of friction, (a) when the force is normal to the plane; (b) when it is parallel to the plane.

3. A horizontal force equal to the weight of the body has to be applied in order to just start a body into motion on a horizontal floor. Find the coefficient of friction.

4. A weight W rests on a rough inclined plane, which makes an angle α with the horizon. Find the smallest force which will move the weight if the coefficient of friction is μ.

5. How would you determine experimentally the coefficient of friction between two bodies?

6. A weight of 75 pounds rests on a rough horizontal floor. Find the magnitude of the least horizontal force which will move the body if the coefficient of friction is 0.4; also find the reaction of the plane.

7. A particle of weight $\mathbf{W}$ is in equilibrium on an inclined plane under the action of a force $\mathbf{F}$, which makes the magnitude of the normal pressure equal W. The coefficient of friction is μ and the angle of elevation of the inclined plane is α. Find the magnitude and direction of the force.

8. An insect starts from the highest point of a sphere and crawls down. Where will it begin to slide if the coefficient of friction between the insect and the sphere is $\frac{1}{3}$?

9. The greatest force, which can keep a particle at rest, acting along an inclined plane, equals twice the least force. Find the coefficient of friction. The angle of elevation of the plane is α.

31. Resultant of a System of Forces. — The resultant of a number of forces which act upon a particle is a force which is equivalent to the given forces. There are two criteria by which this equivalence may be tested. First: The resultant force will give the particle the same motion, when applied to it, as that imparted by the given system of forces. We cannot use this test just now because we have not yet studied motion. Second: When the resultant force is reversed and applied to the particle simultaneously with the given forces the particle remains in equilibrium.

According to the second criterion, therefore, the resultant, $\mathbf{R}$, of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . ., $\mathbf{F}_n$, must satisfy the equation

$$\left.\begin{aligned} -\mathbf{R}+(\mathbf{F}_1+\mathbf{F}_2+\cdots+\mathbf{F}_n)&=0,\\ \text{or}\qquad \mathbf{R}=\mathbf{F}_1+\mathbf{F}_2+\cdots+\mathbf{F}_n&.\end{aligned}\right\}\qquad\text{(IV}'\text{)}$$

Splitting the last equation into three algebraic equations, we obtain

$$\left.\begin{aligned} X&=X_1+X_2+\cdots+X_n,\\ Y&=Y_1+Y_2+\cdots+Y_n,\\ Z&=Z_1+Z_2+\cdots+Z_n,\end{aligned}\right\}\qquad\text{(IV)}$$

where $\mathbf{X}_i$, $\mathbf{Y}_i$, and $\mathbf{Z}_i$ are the components of $\mathbf{R}_i$.

The magnitude of $\mathbf{R}$ is given by the relation

$$R=\sqrt{X^2+Y^2+Z^2},\qquad\text{(V}'\text{)}$$

while the direction is obtained from the following expressions for its direction cosines.

$$\cos\alpha_1=\frac{X}{R},\quad\cos\alpha_2=\frac{Y}{R},\quad\cos\alpha_3=\frac{Z}{R}.\qquad\text{(VI}'\text{)}$$

Special Case.—When the forces lie in the xy-plane the z-components of each force equals zero. Therefore we have

$$R=\sqrt{X^2+Y^2},\qquad\text{(V)}$$

and

$$\tan\theta=\frac{Y}{X},\qquad\text{(VI)}$$

where θ is the angle $\mathbf{R}$ makes with the x-axis.

PROBLEMS.

1. Three men pull on a ring. The first man pulls with a force of 50 pounds toward the N. 30° W. The second man pulls toward the S. 45° E. with a force of 75 pounds, and the third man pulls with a force of 100 pounds toward the west. Determine the magnitude and direction of the resultant force.

2. Show that the resultant of two forces acting upon a particle lies in the plane of the given forces.

3. Show that the line of action of the resultant of two forces lies within the angle made by the forces.

4. Find the direction and magnitude of the resultant of three equal forces which act along the axes of a rectangular system of coördinates.

GENERAL PROBLEMS.

1. A particle is in equilibrium under the action of the forces **P**, **Q**, and **R**. Prove that

$$\frac{P}{\sin(\mathbf{Q}, \mathbf{R})} = \frac{Q}{\sin(\mathbf{P}, \mathbf{R})} = \frac{R}{\sin(\mathbf{P}, \mathbf{Q})},$$

where (**Q**, **R**), etc., denote the angles between **Q** and **R**, etc.

2. Two particles of weights W_1 and W_2 rest upon a smooth sphere of radius a. The particles are attached to the ends of a string of length l, which passes over a smooth peg vertically above the center of the sphere. If h is the distance between the peg and the center of the sphere, find (1) the position of equilibrium of the particles, (2) the tensile force in the string, and (3) the reaction of the sphere.

3. The lengths of the mast and the boom of a derrick are a and b respectively. Supposing the hinges at the lower end of the boom and the pulley at the upper end to be smooth, find the angle the boom makes with the vertical when a weight W is suspended in equilibrium.

4. Find the tensile force in the chain and the compression in the boom of the preceding problem.

5. Two rings of weights W_1 and W_2 are held on a smooth circular wire in a vertical plane by means of a string subtending an angle 2α at the center. Show that the inclination of the string to the horizon is given by

$$\tan\theta = \frac{W_1 - W_2}{W_1 + W_2}\tan\alpha.$$

6. A bridge, Fig. (a), of 60-foot span and 40-foot width has two queen-post trusses 9 feet deep. Each truss is divided into three equal parts by two

posts. What are the stresses in the different parts of the trusses when there is a load of 150 pounds per square foot of floor space?

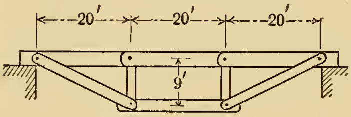


FIG. (a).

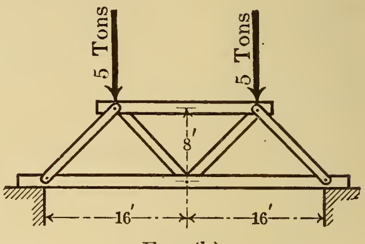


FIG. (b).

7. Find the force in one of the members of the truss of figure (b).

8. A weight rests upon a smooth inclined plane, supported by two equal strings the upper ends of which are fastened to two points of the plane in the same horizontal line. Find the tensile force in the strings and the reaction of the plane.

9. In the preceding problem suppose the plane to be rough.

10. A particle is suspended by a string which passes through a smooth ring fastened to the highest point of a circular wire in a vertical plane. The other end of the string is attached to a smooth bead which is movable on the wire. Find the position of equilibrium supposing the bead and the suspended body to have equal weights.

11. A particle is in equilibrium on a rough inclined plane under the action of a force which acts along the plane. If the least magnitude of the force when the inclination of the plane is α equals the greatest magnitude when it is α_2, show that $\phi = \frac{\alpha_1 - \alpha_2}{2}$, where ϕ is the angle of friction.

12. Two weights W_1 and W_2 rest upon a rough inclined plane, connected by a string which passes through a smooth pulley in the plane. Find the greatest inclination the plane can be given without disturbing the equilibrium.

13. Two equal weights, which are connected by a string, rest upon a rough inclined plane. If the direction of the string is along the steepest slope of the plane and if the coefficients of friction are μ_1 and μ_2, find the greatest inclination the plane can be given without disturbing the equilibrium.

14. In the preceding problem find the tensile force in the string.

15. One end of a uniform rod rests upon a rough peg, while the other end is connected, by means of a string, to a point in the horizontal plane which contains the peg. When the rod is just on the point of motion it

is perpendicular to the string. Show that $2l = \mu a$, where l is the length of the string, a that of the rod, and μ the coefficient of friction.

16. A particle resting upon an inclined plane is at the point of motion under the action of a force **F**, which acts downward along the plane. If the angle of elevation of the plane is changed from α_1 to α_2 and the direction of the force reversed the particle will barely start to move up the plane. Express μ in terms of α_1 and α_2.

17. A string, which passes over the vertex of a rough double inclined plane, supports two weights. Show that the plane must be tilted through an angle equal to twice the angle of friction, in order to bring it from the position at which the particles will begin to move in one direction to the position at which they will begin to move in the opposite direction.

18. Three equal spheres are placed on a smooth horizontal plane and are kept together by a string, which surrounds them in the plane of their centers. If a fourth equal sphere is placed on top of these, prove that the tensile force in the string is $\frac{W}{3\sqrt{6}}$, where W is the weight of each sphere.

19. Three equal hemispheres rest with their bases upon a rough horizontal plane and are in contact with one another. What is the least value of μ which will enable them to support a smooth sphere of the same radius and material?

20. If the center of gravity of a rod is at a distance a from one end and b from the other, find the least value of μ which will allow it to rest in all positions upon a rough horizontal ground and against a rough vertical wall.

21. A string, which is slung over two smooth pegs at the same level, supports two bodies of equal weight W at the ends, and a weight W at the middle by means of a smooth ring through which it passes. Find the position of equilibrium of the middle weight.

CHAPTER III.

EQUILIBRIUM OF RIGID BODIES.

TRANSLATION AND ROTATION.

32. Rigid Body. — There are problems in which bodies cannot be treated as single particles. In such cases they are considered to be made up of a great number of discrete particles. A body is said to be *rigid* if the distances between its particles remain unchanged whatever the forces to which it may be subjected. There are no bodies which are strictly rigid. All bodies are deformed more or less under the action of forces. But in most problems discussed in this book ordinary solids may be treated as rigid bodies.

33. Motion of a Rigid Body. — A rigid body may have two distinct types of motion. When the body moves so that its particles describe straight paths it is said to have a *motion of translation.* Evidently the paths of the particles are parallel, Fig. 20. If the particles of the body describe circular paths it is said to have a *motion of rotation.* The planes of the circles are parallel, while their centers lie on a straight line perpendicular to these planes, which is called the *axis of rotation.* The motion of a flywheel is a well-known example of motion of rotation. Suppose A, Fig. 21, to be a rigid body which is brought from the position A to the position A' by a motion of rotation about an axis through the point O perpendicular to the plane of the paper, then the paths of its particles

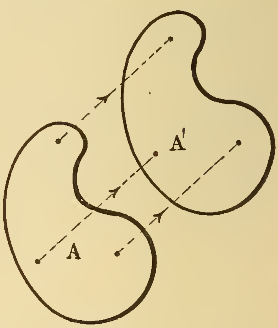


FIG. 20.

are arcs of circles whose planes are parallel to the plane of the paper and whose centers lie on the axis of rotation.

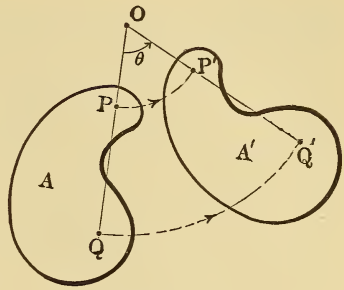


Fig. 21.

34. Uniplanar Motion. — When a rigid body moves so that each of its particles remains at a constant distance from a fixed plane the motion is said to be *uniplanar*. The fixed plane is called the *guide plane*.

35. Theorem I. — *Uniplanar motion of a rigid body consists of a succession of infinitesimal rotational displacements.*

Suppose the rigid body A, Fig. 22, to describe a uniplanar motion parallel to the plane of the paper and let A and A' be any two positions occupied by the body. Then it may be brought from A to A' by a rotational displacement about an axis the position of which may be found in the following manner. Let P and Q be the positions of any two particles of the body in a plane parallel to the plane of the paper when the body is at the position A, and P' and Q' be the positions of the same particles when the body occupies the position A'. Then the desired axis is perpendicular to the plane of the paper and passes through the point of intersection of the perpendicular bisectors of the lines PQ and $P'Q'$, drawn in the plane determined by these lines.

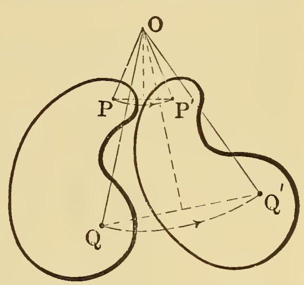


Fig. 22.

Therefore the body can be brought from any position A to any other position A' by a single rotational displacement. The actual motion between A and A' will be, in general, quite different from the simple rotation by which we accomplished the passage of the body from one of these positions to the other. But the result, which we have just obtained, is true not only for positions which are separated by finite distances but also for positions which are infinitely near each other. Therefore by giving the body infinitesimal rotational displacements about properly chosen axes it may be made to assume all the positions which it occupies during its actual motion.

36. Instantaneous Axis. — As the body is made to occupy the various positions of its actual motion the axis of rotation moves at right angles to itself and generates a cylinder whose elements are perpendicular to the guide plane. The elements of the cylinder are called *instantaneous axes*, because each acts as the axis of rotation at the instant when the body occupies a certain position. The curve of intersection of the cylinder and the guide plane is called the *centrode*.

The motion of a cylinder which rolls in a larger cylinder is a simple example of uniplanar motion. In this case the common element of contact is the instantaneous axis. As the cylinder rolls different elements of the fixed cylinder become the axis of rotation.

Motion of translation and motion of rotation are special cases of uniplanar motion. In motion of translation the axis of rotation is infinitely far from the moving body. In rotation the cylinder formed by the instantaneous axes reduces to a single line, i.e., the axis of rotation.

37. Theorem II. — *Rotation about any axis is equivalent to a rotation through the same angle about a parallel axis and a translation in a direction perpendicular to it.*

The truth of this theorem will be seen from Fig. 23, where the rigid body A is brought from the position A to the position A' by a single rotation about an axis through the point O perpendicular to the plane of the paper. This displacement may be produced also by rotating the body to the position A'' and then translating it to the position A'.

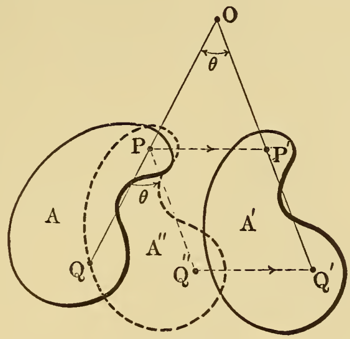


Fig. 23.

PROBLEMS.

1. Show that in theorem II the order of the rotation and of the translation may be changed.

2. Show that the converse of theorem II is true.

38. Theorem III.—*The most general displacement of a rigid body can be obtained by a single translation and a single rotation.*

Let A and A' be any two positions occupied by the rigid body and P and P' be the corresponding positions of any one

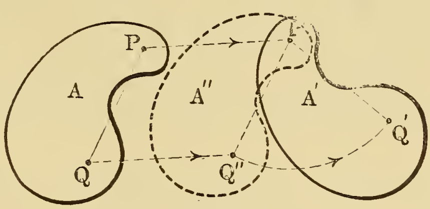


Fig. 24.

of its particles. Then the body may be brought from A to A' by giving it a motion of translation which will bring the particle from P to P' and then rotating the body about a properly chosen axis through P'. A special case of this theorem is illustrated in Fig. 24, where the direction of the translation is perpendicular to the axis of rotation.

39. Theorem IV.—*The most general displacement of a rigid body can be obtained by a displacement similar to that of a screw in its nut, that is, by a rotation about an axis and a translation along it.*

This theorem states that the axis of rotation of the last theorem can be so chosen that the translation is along the axis of rotation. Let PP', Fig. 25, be the path of any point of the body described during the translation and BB be the line about which the body is rotated. Draw CC through P parallel to BB and drop the perpendicular $P'P''$ upon CC. The displacement may be accomplished now in the following three stages. First: translate the body along the line CC until the point which was at P arrives at P''. Second: translate the body along $P''P'$ until the point arrives at P'. Third: rotate the body about BB until it comes to the desired position. But by theorem II the last two operations can be accomplished by a single rotation about CC. Therefore the desired displacement can be obtained by a translation along and a rotation about the line CC.

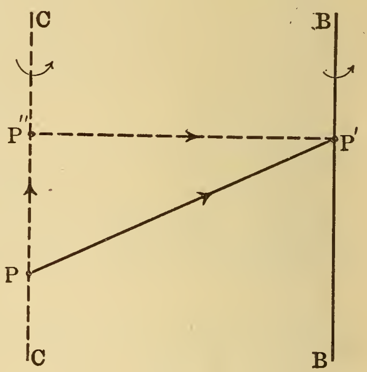


FIG. 25.

Evidently the last theorem holds for infinitesimal displacements as well as for finite displacements; therefore however complicated the motion of a rigid body it can be reproduced by a succession of infinitesimal *screw-displacements*, each displacement taking the body from one position which it has occupied during the motion to another position infinitely near it. Thus at every instant of its motion the rigid body is displaced like a screw in its nut. In general the pitch and the direction of the axis of the screw-motion change from instant to instant. In the case of the motion of a screw in its nut these do not change.

Translation and rotation are special cases of screw-motion. When the pitch of a screw is made smaller and smaller it advances less and less during each revolution. Therefore if the pitch is made to vanish the screw does not advance at all when it is rotated. Thus rotation is a special case of screw-motion in which the pitch is zero. On the other hand as the pitch of the screw is made greater and greater the screw advances more and more during each revolution. Therefore at the limit when the pitch is infinitely great the motion of the screw becomes a motion of translation. Thus translation is a special case of screw-motion in which the pitch is infinitely great.

LINEAR AND ANGULAR ACTION. TORQUE.

40. Two Types of Action.—We have seen that a rigid body may have two different and independent types of motion, namely, motion of translation and motion of rotation. These motions are the results of two independent and entirely different kinds of actions to which a rigid body is capable of being subjected. We will differentiate between these two types of action by adding the adjectives "linear" and "angular" to the term "action." Thus the action which tends to produce translation will be called *linear action* and that which tends to produce rotation *angular action.*

41. Torque.—The vector magnitude which represents the angular action of one body upon another is called *torque.*

42. Couple.—Although a single force is not capable of producing the effect of a torque upon a rigid body, two or more external forces will produce it when properly applied. The simplest system of forces which is capable of producing rotation is known as a *couple.* It consists of two equal and opposite forces which are not in the same line, Fig. 26.

It is evident from Fig. 26 that a couple is capable of giving a rigid body a motion of rotation. But this is not enough to show that the effect produced by a couple is the same as that produced by a torque. We must show also that the couple is not capable of producing a motion of translation. Consider the rigid body A, Fig. 27, which is acted upon by a couple. Suppose the couple did tend to

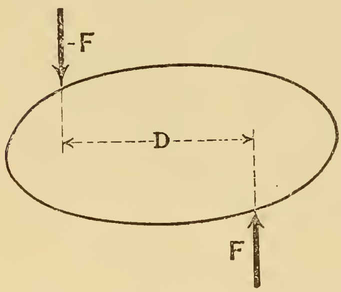


FIG. 26.

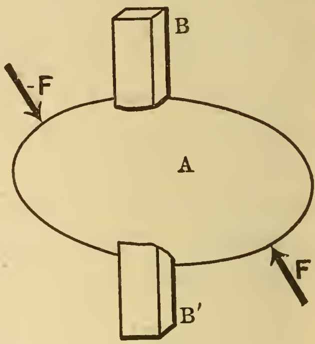


FIG. 27.

produce a translation in a direction BB'. Then pass through the body a smooth bar of rectangular cross-section in the direction of the supposed motion, so that the body is free to move along the bar but not free to rotate. When this constraint is imposed upon the rigid body it behaves like a particle and therefore cannot be given a motion by two equal and opposite forces. But since any motion in the direction BB' is not affected by the presence of the bar, the assumption that the couple produces a motion of translation along BB' must be wrong. Hence we see that when the bar is taken out the motion due to the couple will be one of pure rotation.

43. Measure of Torque. — When a rigid body is in equilibrium under the action of two couples it is always found that the product of one of the forces of one couple by the distance apart of the forces of the same couple equals the corresponding product for the other couple. In order, for instance, that the rigid body A, Fig. 28, be in equilibrium, we must have

$$FD = F'D'.$$

Therefore the product FD is the measure of the torque of the couple formed by the forces **F** and −**F**, the lines of action of which are separated by the distance D. Thus denoting the torque of a couple by G, we have

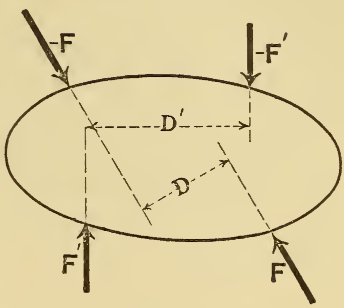


Fig. 28.

$$G = FD. \qquad \text{(I)}$$

The distance D is called the *arm* of the couple and the plane of the forces the *plane* of the couple.

44. Unit Torque. — The torque of a couple whose forces are one pound each and whose arm is one foot is the unit of torque. The symbol for the unit torque is the lb. ft.

45. Vector Representation of Torque.—Torque is a vector magnitude and is represented by a vector which is perpendicular to the plane of the couple. The vector points away from the observer when the couple tends to rotate the body in the clockwise direction and points towards the observer when it tends to rotate the body in the counterclockwise direction, Fig. 29. In the first case the torque is considered to be negative and in the second case positive.

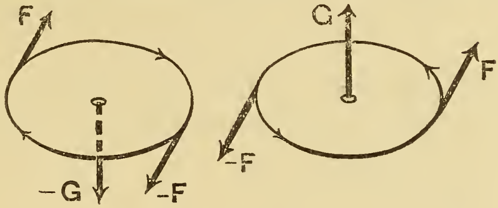


Fig. 29.

46. Equal Couples. — Two couples are equal when the vectors which represent their torques are equal in magnitude and have the same direction. The three couples in Fig. 30 are equal if $G_1 = G_2 = G_3$.

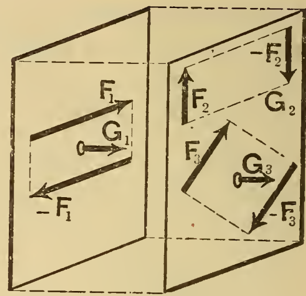


Fig. 30.

Resultant of two couples is a third couple, whose torque is the vector sum of the torques of the given couples.

PROBLEMS.

1. Find the direction and magnitude of the resultant torque of three equal couples the forces of which act along the edges of the bases of a right prism. The bases of the prism are equilateral triangles.

2. In the preceding problem let the forces have a magnitude of 15 pounds each, the length of the prism be 2 feet and the sides of the bases 10 inches.

3. In problem 1 suppose the prism to have hexagonal bases.

4. In problem 2 suppose the prism to be hexagonal.

5. A right circular cone, of weight W and angle 2α, is placed in a circular hole of radius r, cut in a horizontal table. Assuming the coefficient of friction between the cone and the table to be μ, find the least torque necessary to rotate the former about its axis.

47. Moment of a Force. — The most common method of giving a rigid body a motion of rotation is to put an axle through it and to apply to it a force which acts in a plane perpendicular to the axle. The rotation is produced by the couple formed by the applied force and the reaction of the axle. The torque due to the couple equals the product of the applied force by the shortest distance from the axle to the line of action of the force. It is often more con-

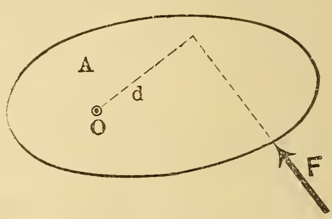


Fig. 31.

venient to disregard the reaction of the axle. When this is done the torque of the couple is called the *moment* of the force applied. Therefore the *moment of a force about an axis equals the product of the force by its lever-arm.* The lever-arm of a force is the shortest distance between the axis and the line of action of the force. In Fig. 31 the moment of **F** about the axis through the point O and perpendicular to the plane of the paper is

$$G = Fd, \tag{II}$$

where d is the lever-arm.

PROBLEMS.

1. Prove that the moment of a force about an axis equals the moment of its component which lies in a plane perpendicular to the axis.

2. Prove that the sum of the moments of the forces of a couple about any axis perpendicular to the plane of the couple is constant and equals the torque of the couple.

48. Degrees of Freedom of a Rigid Body.—A rigid body may have a motion of translation along each of the axes of a rectangular system of coördinates and at the same time it can have a motion of rotation about each of these axes. Therefore a rigid body has six degrees of freedom, three of translation and three of rotation. When one point in it is constrained to move in a plane the number of degrees of freedom is reduced to five. When the point is constrained to move in a straight line the number becomes four. When the point is fixed the body has only the three degrees of freedom of rotation. If two points are fixed the body can only rotate about the line joining the two points. Therefore its freedom is reduced to one degree. When a third point, which is not in the line determined by the other two, is fixed the body cannot move at all, that is, it has no freedom of motion.

49. The Law of Action and Reaction.—The law from which the conditions of equilibrium of a particle were obtained is a universal law applicable to all bodies under all conditions; therefore it is applicable to rigid bodies as well as to single particles. But since rigid bodies may be subject to two distinct types of action the law may be stated in the following form.

The sum of all the linear and angular actions to which a body or a part of body is subject at any instant vanishes.

$$\Sigma(\mathbf{A}_l + \mathbf{A}_a) = 0 \qquad (A')$$

But since the two types of action are independent of each other the sum of each type must vanish when the combined sum vanishes. Therefore we can split the law into the following two sections.

To every linear action there is an equal and opposite linear reaction, or, the sum of all the linear actions to which a body or a part of body is subject at any instant vanishes:

$$\Sigma\mathbf{A}_l = 0 \qquad (A_l)$$

To every angular action there is an equal and opposite angular reaction, or, the sum of all the angular actions to which a body or a part of body is subject at any instant vanishes:

$$\Sigma\mathbf{A}_a = 0 \qquad (A_a)$$

50. Conditions of Equilibrium of a Rigid Body.— If we replace the term "linear action" in the first section of the law by the word "force" and the term "angular action" in the second section of the law by the word "torque" we obtain the two conditions which must be satisfied in order that a rigid body be in equilibrium. Thus, in order that a rigid body be in equilibrium the following conditions must be satisfied.

First. *The sum of all the forces acting upon the rigid body must vanish,* that is, if $\mathbf{F}_1, \mathbf{F}_2, \cdots \mathbf{F}_n$ denote all the forces acting upon the body then the vector equation

$$\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = 0 \qquad (III)$$

must be satisfied.

Second. *The sum of all the torques acting upon the rigid body must vanish,* that is, if $\mathbf{G}_1$, $\mathbf{G}_2$, $\cdots$ $\mathbf{G}_n$ denote all the torques acting upon the body then the vector equation

$$\mathbf{G}_1 + \mathbf{G}_2 + \cdots + \mathbf{G}_n = 0 \qquad \text{(IV)}$$

must be satisfied.

The following forms of the statement of these two conditions are better adapted for analysis.

First. *The algebraic sum of the components of all the forces along each of the axes of a rectangular system of coördinates must vanish,* that is,

$$\left.\begin{aligned} \Sigma X &\equiv X_1 + X_2 + \cdots + X_n = 0. \\ \Sigma Y &\equiv Y_1 + Y_2 + \cdots + Y_n = 0. \\ \Sigma Z &\equiv Z_1 + Z_2 + \cdots + Z_n = 0. \end{aligned}\right\} \qquad \text{(V}'\text{)}$$

Second. *The algebraic sum of the components of all the torques about each of the axes of a system of rectangular coördinates must vanish,* that is,

$$\left.\begin{aligned} \Sigma G_x &\equiv G_x' + G_x'' + \cdots + G_x^{(n)} = 0. \\ \Sigma G_y &\equiv G_y' + G_y'' + \cdots + G_y^{(n)} = 0. \\ \Sigma G_z &\equiv G_z' + G_z'' + \cdots + G_z^{(n)} = 0. \end{aligned}\right\} \qquad \text{(VI}'\text{)}$$

51. Coplanar Forces.— If two or more forces act in the same plane they are said to be *coplanar.* If a system of coplanar forces act in the xy-plane then the conditions of equilibrium reduce to the following equations:

$$\left.\begin{aligned} \Sigma X &\equiv X_1 + X_2 + \cdots + X_n = 0, \\ \Sigma Y &\equiv Y_1 + Y_2 + \cdots + Y_n = 0, \end{aligned}\right\} \qquad \text{(V)}$$

$$\Sigma G_z \equiv F_1 d_1 + F_2 d_2 + \cdots + F_n d_n = 0, \qquad \text{(VI)}$$

where d_1, d_2, . . . , d_n are the lever-arms of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . . $\mathbf{F}_n$, respectively, about any axis which is perpendicular to the plane of the forces. The z-components of the forces and the x- and y-components of the moments vanish identically. Consequently they need not be considered.

52. Transmissibility of Force. — A force which acts upon a rigid body may be considered to be applied to any particle of the body which lies on the line of action of the force. In order to prove this statement consider the rigid body A, Fig. 32, which is in equilibrium under the action of the two

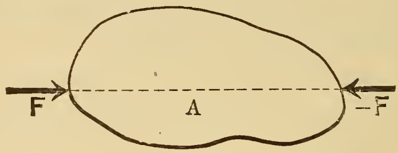


FIG. 32.

equal and opposite forces **F** and − **F**. Now suppose we change the point of application of **F**, without changing either its direction or its line of application. Evidently the equilibrium is not disturbed, because by moving **F** in its line of action we neither changed the sum of the forces nor the sum of their moments about any axis. Therefore the line of action of a force is of importance and not its point of application.

53. Internal Forces. — *Internal forces do not affect the equilibrium of a rigid body.* This is a direct consequence of the law of "action and reaction." Since by definition the internal forces are due to the interaction between the particles of the system these forces exist in equal and opposite pairs, therefore mutually annul each other.

ILLUSTRATIVE EXAMPLES.

1. A uniform beam rests with its lower end on a smooth horizontal ground and its upper end against a smooth vertical wall. The beam is held from slipping by means of a string which connects the foot of the beam with the foot of the wall. Find the tensile force in the string and the reactions at the ends of the beam.

There are four forces acting upon the beam, i.e., the two reactions, $\mathbf{R}_1$ and $\mathbf{R}_2$, the tensile force **T** and the weight **W**. Since both the ground and the wall are supposed to be smooth, $\mathbf{R}_1$ is normal to the ground, and $\mathbf{R}_2$

to the wall. Therefore denoting the lengths of the beam and the string by l and a respectively, we have

$$\Sigma X \equiv R_2 - T = 0,$$
$$\Sigma Y \equiv R_1 - W = 0,$$
$$\Sigma G_{o'} \equiv -R_2 l \sin\alpha + W\frac{l}{2}\cos\alpha = 0,$$

where $\Sigma G_{o'}$ denotes the sum of the moments of the forces about an axis through the point O' perpendicular to the xy-plane. Solving the last three equations we have

$$R_1 = W,$$
$$R_2 = \frac{W}{2}\cot\alpha = \frac{W}{2}\frac{a}{\sqrt{l^2-a^2}},$$

and

$$T = \frac{W}{2}\frac{a}{\sqrt{l^2-a^2}}.$$

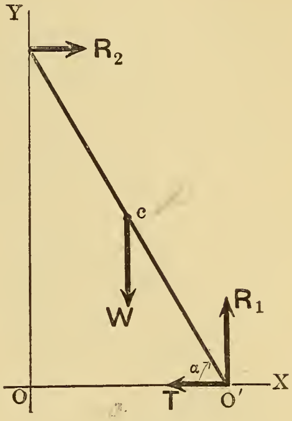


FIG. 33.

DISCUSSION. — It should be noticed that in taking the moments the axis was chosen through the point O' in order to eliminate the moments of as many forces as possible and thus to obtain a simple equation.

The reaction $\mathbf{R}_1$ is independent of the angular position of the beam and equals the weight **W**. On the other hand $\mathbf{R}_2$ and **T** vary with α. When $\alpha = \frac{\pi}{2}$ both $\mathbf{R}_2$ and **T** vanish. As α is diminished from $\frac{\pi}{2}$ to 0, $\mathbf{R}_2$ and **T** increase indefinitely.

2. A ladder rests on a rough horizontal ground and against a rough vertical wall. The coefficient of friction between the ladder and the ground is the same as that between the ladder and the wall. Find the smallest angle the ladder can make with the horizon without slipping.

There are three forces acting on the ladder, i.e., its own weight **W** and the two reactions $\mathbf{R}_1$ and $\mathbf{R}_2$. Replacing $\mathbf{R}_1$ and $\mathbf{R}_2$ by their components and writing the equations of equilibrium we obtain

$$\Sigma X \equiv F_1 - N_2 = 0,$$
$$\Sigma Y \equiv N_1 + F_2 - W = 0,$$
$$\Sigma G_{0'} \equiv F_2 l\cos\alpha + N_2 l\sin\alpha - W\frac{l}{2}\cos\alpha = 0,$$

where α is the required angle.

We have further

$$\mu = \frac{F_1}{N_1} = \frac{F_2}{N_2}.$$

Solving these we get

$$F_1 = \frac{\mu}{1+\mu^2} W,$$

$$N_1 = \frac{1}{1+\mu^2} W,$$

$$R_1 = \frac{1}{\sqrt{1+\mu^2}} W,$$

$$F_2 = \frac{\mu^2}{1+\mu^2} W,$$

$$N_2 = \frac{\mu}{1+\mu^2} W,$$

$$R_2 = \frac{\mu}{\sqrt{1+\mu^2}} W,$$

$$\tan\alpha = \frac{1-\mu^2}{2\mu}.$$

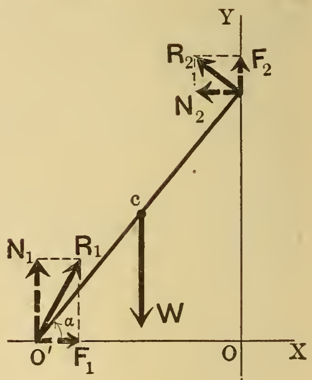


FIG. 34.

DISCUSSION. — The last expression gives the value of α for a given value of μ. When $\mu = 1$, $\alpha = 0$, therefore in this case the ladder will be in equilibrium at any angle between 0 and $\frac{\pi}{2}$ with the ground. Evidently this is true for any value of μ greater than unity.

3. Find the smallest force which, when applied at the center of a carriage wheel, will drag it over an obstacle.

The forces acting on the wheel are, its weight **W**, the required force **F**, and the reaction **R**. Since the first two meet at the center of the wheel, the direction of **R** must pass through the center also. Take the coördinate axes along and at right angles to **R**, as shown in Fig. 31, and let **F** make an angle θ with the x-axis. Then the equations of equilibrium become

$$\Sigma X \equiv F\cos\theta - R + W\cos\alpha = 0,$$
$$\Sigma Y \equiv F\sin\theta - W\sin\alpha = 0,$$
$$\Sigma G_{o'} \equiv W \cdot a\sin\alpha - F\sin\theta \cdot a = 0.$$

From either of the last two equations we get

$$F = \frac{\sin\alpha}{\sin\theta} W.$$

Since W and α are fixed F can be changed only by changing θ. Therefore the minimum value of F is given by the maximum value of $\sin\theta$, i.e., $\theta = \frac{\pi}{2}$, which makes

$$F = W\sin\alpha.$$

From the figure we obtain $\cos \alpha = \dfrac{a-h}{a}$,

therefore $$\sin \alpha = \frac{1}{a}\sqrt{h(2a-h)},$$

and $$F = \frac{W}{a}\sqrt{h(2a-h)}.$$

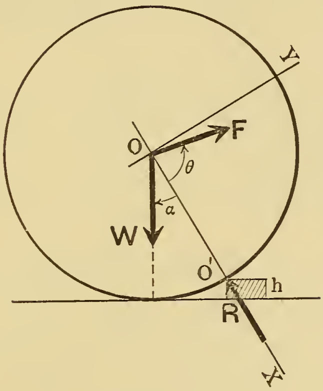


FIG. 35.

Since $\cos \theta = 0$ the first equation of equilibrium gives

$$R = W \cos \alpha$$
$$= \frac{a-h}{a} W.$$

DISCUSSION. — It will be observed that the first two of the equations of equilibrium are sufficient to solve the problem.

When h is zero, $F = 0$ and $R = W$. On the other hand when $h = a$, $F = W$ and $R = 0$.

PROBLEMS.

1. Prove that the true weight of a body is the geometric mean between the apparent weights obtained by weighing it in both pans of a false balance.

2. A uniform bar weighing 10 pounds is supported at the ends. A weight of 25 pounds is suspended from a point 20 cm. from one end. Find the pressure at the supports if the length of the bar is 50 cm.

3. A uniform rod which rests on a rough horizontal floor against a smooth vertical wall is on the point of slipping. Find the reactions at the two ends of the rod.

4. A body is suspended from the middle of a uniform rod which passes over two fixed supports 6 feet apart. In moving the body 6 inches nearer to one of the supports the pressure on the support increases by 100 pounds. What is the weight of the body if 5 pounds is the weight of the rod?

5. A uniform rod of length a and weight W is suspended by two strings having lengths l_1 and l_2. The lower ends of the strings are attached to the ends of the rod, while the upper ends are tied to a peg. Find the tensile force in the strings.

6. A safety valve consists of a cylinder with a plunger attached to a uniform bar hinged at one end. The plunger has a diameter of $\frac{1}{2}$ inch and is attached to the bar at a distance of 1 inch from the hinge. The bar is 2 feet long and weighs 1 pound. How far from the hinge must a slide-weight of 2 pounds be set if the steam is to blow off at 120 pounds per square inch?

7. The two legs of a stepladder are hinged at the top and connected at the middle by a string of negligible mass. Find the tensile force in the string and the pressure on the hinges when the ladder stands on a smooth plane. The weight of the ladder is W, the length of its legs l, and the length of the string a.

8. A uniform rod rests on two inclined planes making angles of α_1 and α_2 with the horizon. Find the angle which the rod makes with the horizon and the pressure on the planes.

9. A rectangular block is placed on a rough inclined plane whose inclination is gradually increased. If the block begins to slide and to turn about its lowest edge simultaneously find the coefficient of friction.

10. A uniform rod rests with one end against a rough vertical wall and the other end connected to a point in the wall by a string of equal length. Show that the smallest angle which the string can make with the wall is $\tan^{-1}\left(\frac{3}{\mu}\right)$.

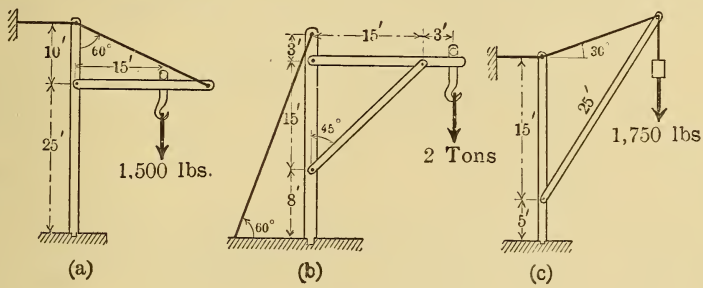
11. A uniform rod is suspended by a string which is attached to the ends and is slung over a smooth peg. Show that in equilibrium the rod is either horizontal or vertical.

12. A ladder 25 feet long and weighing 50 pounds rests against a vertical wall making 30° with it. How high can a man weighing 150 pounds climb up the ladder before it begins to slip? The coefficient of friction is 0.5 at both ends of the ladder.

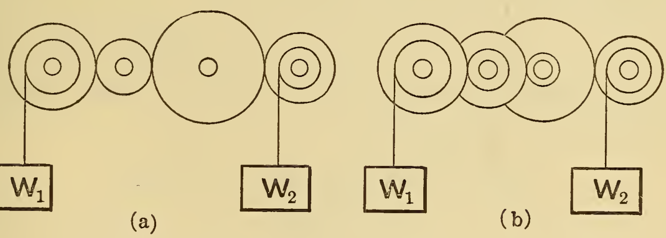
13. A rod of negligible weight rests wholly inside a smooth hemispherical bowl of radius r. A weight W is clamped on to the rod at a point whose distances from the ends are a and b. Show that the equilibrium position of the rod is given by $\sin\theta = \dfrac{a-b}{2\sqrt{r^2 - ab}}$, where θ is the angle it makes with the plane of the brim of the bowl which is horizontal.

14. Prove that when a rigid body is in equilibrium under the action of three forces their lines of action lie in the same plane and intersect at the same point.

15. Find the forces which tend to compress or extend the different members of the following cranes.



16. Supposing the weights of the following figures to be in equilibrium find their relative magnitudes. The circles which are tangent to other circles represent gears.



54. Resultant of a System of Forces Acting upon a Rigid Body.—We have already shown that the most general displacement of a rigid body consists of a translation along, and a rotation about, a certain line. Therefore such a displacement can be prevented by a single force opposed to the translation and a single torque opposed to the rotation. Thus a single force and a single torque can be found which will keep a rigid body in equilibrium against the action of any system of forces.

The resultant of a system of forces consists, therefore, of a single force and a single torque which, when reversed, will keep the rigid body in equilibrium against the action of the given system of forces.

55. Resultant of Coplanar Forces Acting upon a Rigid Body. — Let $\mathbf{F}_1, \mathbf{F}_2, \ldots \mathbf{F}_n$ denote the given forces and let the xy-plane be their plane of action. Then, if $\mathbf{R}$, $\mathbf{X}$, and $\mathbf{Y}$ denote the resultant force and its components, respectively, we have

$$\left.\begin{aligned} X &= X_1 + X_2 + \cdots + X_n, \\ Y &= Y_1 + Y_2 + \cdots + Y_n, \end{aligned}\right\} \qquad \text{(VII)}$$

$$R = \sqrt{X^2 + Y^2}, \qquad \text{(VIII)}$$

and

$$\tan\theta = \frac{Y}{X}, \qquad \text{(IX)}$$

where the terms in the right-hand members of the first two equations are the components of the given forces, and θ is the angle $\mathbf{R}$ makes with the x-axis.

On the other hand if $\mathbf{G}_o$ denotes the resultant torque and $d_1, d_2, \ldots, d_n$ denote the distances of the origin from the lines of action of the forces, then

$$G_o = F_1 d_1 + F_2 d_2 + \cdots + F_n d_n. \qquad \text{(X)}$$

If we represent this torque by the moment of the resultant force about the z-axis, then

$$\left.\begin{aligned} RD &= F_1 d_1 + F_2 d_2 + \cdots + F_n d_n, \\ \text{or} \qquad D &= \frac{\Sigma F d}{R}, \end{aligned}\right\} \qquad \text{(XI)}$$

gives the distance of the line of action of the resultant force from the origin.

ILLUSTRATIVE EXAMPLE.

Find the resultant of the six forces acting along the sides of the hexagon of Fig. 36.

Taking the sum of the components along the x and y directions, we have

$$X = 2F + 3F\cos\frac{\pi}{3} - 2F\cos\frac{\pi}{3} - F - 2F\cos\frac{\pi}{3} + F\cos\frac{\pi}{3}$$
$$= F.$$
$$Y = 0 - 3F\sin\frac{\pi}{3} - 2F\sin\frac{\pi}{3} + 0 + 2F\sin\frac{\pi}{3} + F\sin\frac{\pi}{3}$$
$$= -F\sqrt{3}.$$
$$\therefore \quad R = \sqrt{F^2 + 3F^2}$$
$$= 2F$$
and $\tan\theta = -\sqrt{3}.$

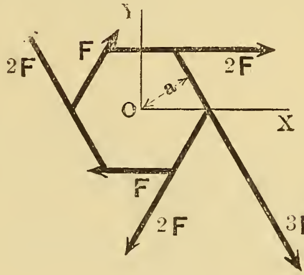


FIG. 36.

Therefore the resultant force has a magnitude $2F$ and makes an angle of $-60°$ with the x-axis.

Taking the moments about an axis through the center of the hexagon, we obtain

$$RD = (2F + 3F + 2F + F + 2F + F)a$$
$$= 11\,Fa,$$
therefore $D = 5.5a,$

where a is the distance of the center from the lines of action of the forces.

56. Resultant of a System of Parallel Forces. — Let **R** be the resultant of the parallel forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . ., $\mathbf{F}_n$, which act upon a rigid body. Then, since the forces are parallel, the resultant force equals the algebraic sum of the given forces. Thus

$$R = F_1 + F_2 + \cdots + F_n,$$
and
$$RD = F_1 d_1 + F_2 d_2 + \cdots + F_n d_n.$$

Now take the z-axis parallel to the forces and let x_i and y_i denote the distances of $\mathbf{F}_i$ from the yz-plane and the xz-plane, respectively. Then the last equation may be split into two parts, one of which gives the moments about the x-axis and the other about the y-axis. Thus,

$$\left.\begin{aligned} R\bar{x} &= F_1x_1 + F_2x_2 + \cdots + F_nx_n, \\ R\bar{y} &= F_1y_1 + F_2y_2 + \cdots + F_ny_n, \end{aligned}\right\} \qquad \text{(XII)}$$

where $\bar{x}$ and $\bar{y}$ are the coördinates of the point in the xy-plane through which the resultant force passes. In other words, $(\bar{x},\bar{y})$ is the point of application of the resultant force. The resultant force is evidently parallel to the given forces. The last two equations may be written in the following forms

$$\left.\begin{aligned}\bar{x} &= \frac{\Sigma Fx}{R}, \\ \bar{y} &= \frac{\Sigma Fy}{R}.\end{aligned}\right\} \qquad \text{(XIII)}$$

ILLUSTRATIVE EXAMPLE.

Find the resultant of two parallel forces which act upon a rigid body in the same direction.

Let the y-axis be parallel to the forces.

Then $$R = F_1 + F_2,$$

and $$\bar{x} = \frac{F_1x_1 + F_2x_2}{F_1 + F_2},$$

or $$\frac{F_1}{F_2} = \frac{x_2 - \bar{x}}{\bar{x} - x_1}.$$

But since $x_2 - \bar{x}$ and $\bar{x} - x_1$ are the distances of $\mathbf{F}_2$ and $\mathbf{F}_1$ from $\mathbf{R}$, we have

$$\frac{F_1}{F_2} = \frac{d_2}{d_1},$$

or $$F_1d_1 = F_2d_2.$$

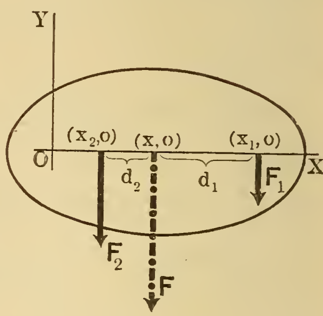


Fig. 37.

Therefore the distances of the resultant from the given forces are inversely proportional to the magnitudes of the latter.

PROBLEMS.

1. Find the resultant force and the resultant torque due to the forces P, $2\,P$, $4\,P$ and $2\,P$ which act along the sides of a square, taken in order.

2. Three forces are represented in magnitude and line of action by the sides of an equilateral triangle. Find the resultant force, taking the directions of one of the forces opposite to that of the other two.

3. The lines of action of three forces form a right isosceles triangle of sides a, a, and $a\sqrt{2}$. The magnitudes of the forces are proportional to the sides of the triangle. Find the resultant force.

4. The sum of the moments of a system of coplanar forces about any three points, which are not in the same straight line, are the same. Show that the system is equivalent to a couple.

5. Three forces are represented in magnitude, direction, and line of action by the sides of a triangle taken in order; prove that their resultant is a couple the torque of which equals, numerically, twice the area of the triangle.

6. Three forces act along the sides of an equilateral triangle, find the condition which will make their resultant pass through the center of the triangle.

FRICTION ON JOURNALS AND PIVOTS.

57. Friction on Journal Bearing. — If the horizontal shaft of Fig. 38 fits perfectly in its bearing the friction which comes into play is a sliding friction, therefore the laws of sliding friction may be assumed to hold good. The most important of these laws is: the frictional force which comes into play is proportional to the normal reaction, that is, in the relation

$$F = \mu N,$$

μ is independent of N. We will assume therefore that this law holds at each point of the surface of contact and thus reduce the problem under discussion to one of sliding friction. There is an important difference, however, between the problem of friction on journal bearing and the problems on friction which we have already discussed. In the present problem the normal reaction is not the same at the points of the surfaces in contact. We must apply, therefore, the laws of friction to small elements of surfaces of contact over which the normal reaction may be considered to be constant.

Let the element of surface be a strip, along the length of the shaft, which subtends an angle $d\theta$ at the axis of the shaft. Further let dN be the normal reaction over this element of surface, and dF be the corresponding frictional force; then we have

$$\begin{aligned} dF &= \mu \, dN \\ &= \mu p \cdot l \cdot a \, d\theta, \end{aligned}$$

where p is the normal reaction per unit area or the pressure, a is the radius of the shaft and l the length of the bearing.

Therefore the total frictional force and the total frictional torque are, respectively,

$$F = \mu a l \int_0^{\frac{\pi}{2}} p \, d\theta$$

and

$$G = \mu a^2 l \int_0^{\frac{\pi}{2}} p \, d\theta.$$

In order to carry out the integral of the foregoing expressions we have to make some assumption with regard to the nature

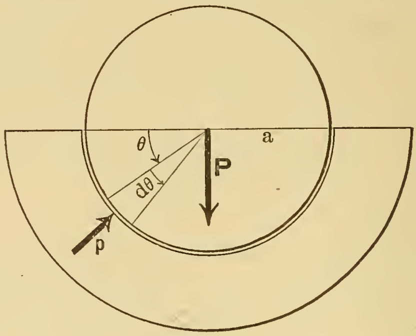


FIG. 38.

of dependence of p upon θ. But whatever the relation between p and θ it is obvious that the sum, over all the surface of contact, of the vertical component of the normal reaction must equal to the load which rests upon the bearing. If P denotes this load, then p must satisfy the condition

$$P = \int_0^A p \sin \theta \cdot dA$$

$$= a l \int_0^{\frac{\pi}{2}} p \sin \theta \, d\theta,$$

where A is the total area of contact.

ILLUSTRATIVE EXAMPLE.

The normal pressure on the bearing is given by the relation $p = p_0 \sin \theta$; find the total frictional force and the total frictional torque.

Substituting the given value of p in the expression for F we obtain

$$F = \mu alp_0 \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta$$
$$= \mu alp_0.$$

In order to determine p_0 in terms of the total load on the bearings we make p satisfy the condition

$$P = al \int_0^{\frac{\pi}{2}} p \sin\theta \cdot d\theta.$$

Substituting the given value of p in the right-hand member of the preceding equation we have

$$P = alp_0 \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$
$$= \frac{\pi alp_0}{4},$$

or
$$p_0 = \frac{4P}{\pi al}.$$

Therefore
$$F = \frac{4\mu}{\pi} P$$

and
$$G = \frac{4\mu}{\pi} aP.$$

It will be observed that the total frictional force varies with the load and is independent of the radius and of the length of the bearing; in other words it is independent of the area of contact.

PROBLEMS.

1. Supposing the normal pressure to be the same at every point of the surfaces of contact, derive the expressions for the total frictional force and the resisting torque due to friction.

2. Supposing the vertical component of the normal pressure at every point of the surfaces of contact to be constant, derive the expressions for the total frictional force and the resisting torque due to friction.

3. Derive expressions for the total frictional force and the resisting torque upon the assumption that the normal pressure is given by the relation $p = p_0 \sin^2\theta$.

58. Friction on Pivots. — The problem of friction on pivots also is a problem of sliding friction. The feature which distinguishes the pivot from the journal bearing is this: in the former the lever arm of the frictional force varies from point to point, while in the latter it is constant and equals the radius of the shaft.

Let dN be the normal reaction upon dA, an element of area at the base of the flat-end pivot of Fig. 39; then if dF denotes the corresponding frictional force, we have

$$\begin{aligned} dF &= \mu\, dN \\ &= \mu p \cdot dA, \end{aligned}$$

where p is the normal pressure. Evidently p is constant; therefore we can write

$$\begin{aligned} F &= \mu p \int_0^{\pi a^2} dA \\ &= \pi a^2 \mu p. \end{aligned}$$

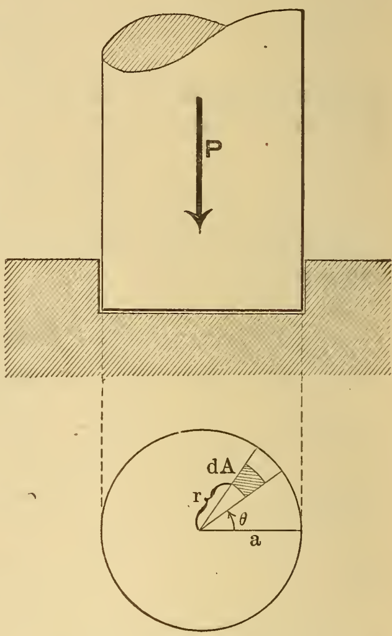


Fig. 39.

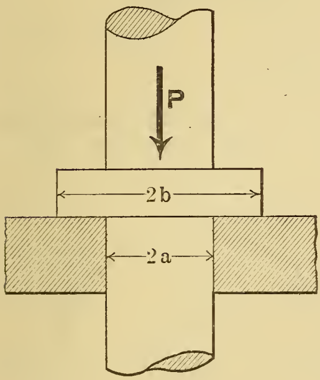
The expression for the resisting torque due to the friction is obtained as follows:

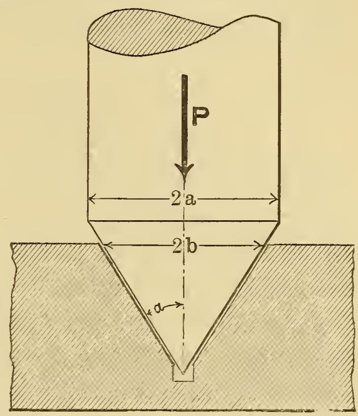
$$\begin{aligned} G &= \int_0^F r \cdot dF \\ &= \int_0^A r \cdot \mu p\, dA \\ &= \int_0^a \int_0^{2\pi} r \mu p \cdot \tfrac{1}{2} r\, d\theta \cdot dr \\ &= \pi \mu p \int_0^a r^2\, dr \\ &= \tfrac{2}{3} \pi a^3 \mu p \\ &= \tfrac{2}{3} a \mu P, \end{aligned}$$

where P is the total load on the pivot.

PROBLEMS.

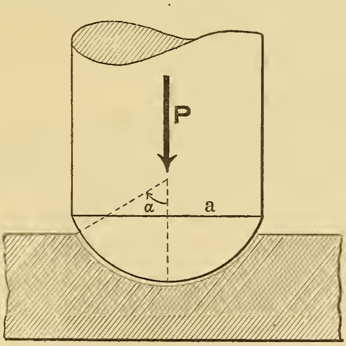
1. Derive an expression for the resisting torque due to friction in the collar-bearing pivot of the adjoining figure.





2. Supposing the normal pressure to be constant, derive an expression for the resisting torque due to friction in the conical pivot of the adjoining figure.

3. In the preceding problem suppose the vertical component of the normal pressure to be constant.



4. In problem 2 suppose the horizontal component of the normal pressure to be constant.

5. Taking the normal pressure to be constant derive an expression for the resisting torque, due to friction in the spherical pivot of the adjoining figure.

6. Prove that the resisting torque due to friction is greater for a hollow pivot than for a solid pivot, provided that the load and the load per unit area are the same in both cases.

7. Show that the resisting torque due to friction for a hemispherical pivot is about 2.35 times as large as that for a flat end pivot.

ROLLING FRICTION.

59. Coefficient of Rolling Friction. — Consider a cylinder, Fig. 38, which is in equilibrium on a rough horizontal plane under the action of a force **S**. In addition to this force the cylinder is acted upon by its weight and by the reaction of the plane. Applying the conditions of equilibrium we obtain

$$\Sigma X \equiv S - F = 0,$$
$$\Sigma Y \equiv -W + N = 0,$$
$$\Sigma G_0 \equiv ND - Sd = 0,$$

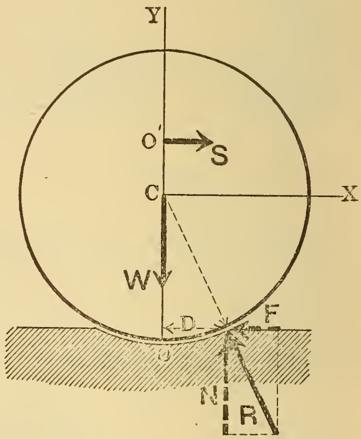


FIG. 40.

where **F** and **N** are the components of **R**, the reaction of the plane, while D and d are, respectively, the distances of the point of application of **R** and **S** from the point O, about which the moments are taken. These equations give us

$$R = \sqrt{F^2 + N^2}$$
$$= \sqrt{S^2 + W^2}, \quad (1)$$

and

$$D = \frac{S}{W} d. \quad (2)$$

If the cylinder is just on the point of motion then

$$F = \mu N,$$

and consequently

$$\mu = \frac{S}{W}. \quad (3)$$

Combining (2) and (3), we obtain

$$D = \mu d. \tag{XIV}$$

The distance D is called the *coefficient of rolling friction.* Equation (XIV) states, therefore, that the coefficient of the rolling friction equals the coefficient of the sliding friction times the distance of the point of contact from the line of action of the force which urges the body to roll.

60. Friction Couple. — It is evident from the above equations that a change in the value of d does not affect the values of N and F, consequently it does not change the value of μ. This is as it should be, since, according to the laws of sliding friction, μ depends only upon the nature of the surfaces in contact. A change in d, however, changes the value of D; in other words, it changes the point of application of **R**. When $d = 0$, that is, when **S** is applied at the point of contact, $D = 0$, in which case the body is urged to slide only. But when d is not zero the force **S** not only urges the body to slide but also to roll; therefore, in addition to the resisting force **F**, a resisting torque comes into play. This torque, which is due to the couple formed by N and W, is called *friction couple.*

PROBLEMS.

1. A gig is so constructed that when the shafts are horizontal the center of gravity of the gig is over the axle of the wheels. The gig rests on a perfectly rough horizontal ground. Find the least force which, acting at the ends of the shafts, will just move the gig.

2. Find the smallest force which, acting tangentially at the rim of a flywheel, will rotate it. The weight and the radius of the flywheel, the radius of the shaft, and the coefficient of friction between the shaft and its bearings are supposed to be known.

3. A flywheel of 500 pounds weight is brought to the point of rotation by a weight of 10 pounds suspended by a string wound around its rim. Find the coefficient of friction between the axle and its bearings. The diameters of the wheel and the axle are 10 feet and 8 inches, respectively.

4. A wheel of radius a and weight W stands on a rough horizontal ground. If μ is the coefficient of friction between the wheel and the ground find the smallest weight which must be suspended at one end of the horizontal diameter in order to move the wheel.

GENERAL PROBLEMS.

1. A table of negligible weight has three legs, the feet forming an equilateral triangle. Find the proportion of the weight carried by the legs when a particle is placed on the table.

2. A rectangular board is supported in a vertical position by two smooth pegs in a vertical wall. Show that if one of the diagonals is parallel to the line joining the pegs the other diagonal is vertical.

3. A uniform rod rests with its two ends on smooth inclined planes making angles α and β with the horizon. Where must a weight equal to that of the rod be clamped in order that the rod may rest horizontally?

4. A uniform ladder rests against a rough vertical wall. Show that the least angle it can make with the horizontal floor on which it rests is given by $\tan\theta = \dfrac{1 - \mu\mu'}{2\mu}$, where μ and μ' are the coefficients of friction for the floor and the wall respectively.

5. A uniform rod is suspended by two equal strings attached to the ends. In position of equilibrium the strings are parallel and the bar is horizontal. Find the torque which will turn the bar, about a vertical axis, through an angle θ and keep it in equilibrium at that position.

6. The line of hinges of a door makes an angle α with the vertical. Find the resultant torque when the door makes an angle β with its equilibrium position.

7. The lines of action of four forces form a quadrilateral. If the magnitude of the forces are a, b, c, d times the sides of the quadrilateral find the conditions of equilibrium.

8. A force acts at the middle point of each side of a plane polygon. Each force is proportional to the length of the side it acts upon and is perpendicular to it. Prove that the polygon will be in equilibrium if all the forces are directed towards the inside of the polygon.

9. A force acts at each vertex of a plane convex polygon in a direction parallel to one of the sides forming the vertex. Show that if the forces are proportional to the sides to which they are parallel and if their directions are in a cyclic order their resultant is a couple.

10. A uniform chain of length l hangs over a rough horizontal cylinder of radius a. Find the length of the portions which hang vertically when

the chain is on the point of motion under its own weight, (1) when a is negligible compared with l, (2) when it is not negligible compared with l.

11. Two equal weights are attached to the extremities of a string which hangs over a rough horizontal cylinder. Find the least amount by which either weight must be increased in order to start the system to move. The weight of the string is negligible.

12. Three cylindrical pegs of equal radius and roughness are placed at the vertices of a vertical equilateral triangle the two lower corners of which are in the same horizontal line. A string of negligible weight is attached to two weights and slung over the pegs. Find the ratio of the weights if they are on the point of motion.

13. A sphere laid upon a rough inclined plane of inclination α is on the point of sliding. Show that the coefficient of friction is $\frac{2}{7}\tan\alpha$.

14. A uniform ring of weight W hangs on a rough peg. A bead of weight w is fixed on the ring. Show that if the coefficient of friction between the ring and the peg is greater than $\frac{W}{\sqrt{W^2 + 2wW}}$ the ring will be in equilibrium whatever the position of the bead with respect to the peg.

15. A uniform rod is in equilibrium with its extremities on the interior of a rough vertical hoop. Find the limiting position of the rod.

16. A weight W is suspended from the middle of a cord whose ends are attached to two rings on a horizontal pole. If w be the weight of each of the rings, μ the coefficient of friction, and l the length of the cord, find the greatest distance apart between the rings compatible with equilibrium.

CHAPTER IV.

EQUILIBRIUM OF FLEXIBLE CORDS.

61. Simplification of Problems. — The simplest phenomenon in nature is the result of innumerable actions and reactions. The consideration of all the factors which contribute to any natural phenomenon would require unlimited analytical power. Fortunately the factors which enter into dynamical problems are not of equal importance. Often the influence of one or two predominate, so that the rest can be neglected without an appreciable departure from the actual problem. Any one who attempts to solve a physical problem must recognize this fact and use it to advantage by representing the actual problem by an ideal one which has only the important characteristics of the former. This was done in the last two chapters in which bodies were treated as single particles and rigid bodies, and thereby simplified the problems without changing their character.

The same procedure will be followed in discussing the equilibrium of flexible cords, such as belts, chains, and ropes. These bodies will be represented by an ideal cord of negligible cross-section and of perfect flexibility. The solution of the idealized problems gives us a close enough approximation for practical purposes. If, however, closer approximation is desired smaller factors, such as the effects of thickness and imperfect flexibility, may be taken into account.

62. Flexibility. — A cord is said to be *perfectly flexible* if it offers no resistance to bending; in other words, in a perfectly flexible cord there are no internal forces which act in a direction perpendicular to its length.

63. Suspension Bridge Problem. — The following are the important features of a suspension bridge which should be considered in order to simplify the problem:

1. The weights of the cables and of the chains are small compared with that of the road-bed.

2. The road-bed is practically horizontal.

3. The distribution of weight in the road-bed may be considered to be uniform.

We can, therefore, obtain a sufficiently close approximation if we consider an ideal bridge in which the cable and the chains have no weight and the distribution of weight in the road-bed is uniform in the horizontal direction. With these simplifications consider the forces acting upon that part of the cable which is between the lowest point and any point P, Fig. 41.

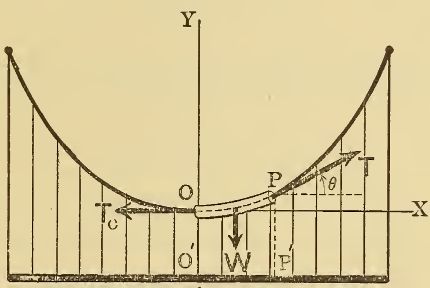


FIG. 41.

The forces are: The tensile force, $\mathbf{T}_0$, acting horizontally at O. The tensile force, $\mathbf{T}$, acting along the tangent to the curve at P. The weight of that part of the bridge which is between O and P. If $\mathbf{w}$ be the weight per unit length of the road-bed and x denotes the length OP, then the third force becomes $\mathbf{w}x$.

Therefore the conditions of equilibrium give

$$\Sigma X \equiv -T_0 + T\cos\theta = 0; \quad \therefore \quad T\cos\theta = T_0. \qquad (1)$$

$$\Sigma Y \equiv -wx + T\sin\theta = 0; \quad \therefore \quad T\sin\theta = wx. \qquad (2)$$

It is evident from equation (1) that the horizontal component of the tensile force is constant and equals T_0. Squaring equations (1) and (2) and adding we get

$$T^2 = T_0^2 + w^2x^2. \tag{3}$$

Thus we see that the smallest value of T corresponds to $x = 0$ and equals T_0, while its greatest value corresponds to the greatest value of x. If D denotes the span of the bridge then the greatest value of T, or the tensile force of the cable at the piers, is

$$T_m = \sqrt{T_0^2 + \frac{w^2D^2}{4}}.$$

In order to find the equation of the curve which the cable assumes we eliminate T between equations (1) and (2). This gives

$$\tan\theta = \frac{w}{T_0}x. \tag{4}$$

Substituting $\frac{dy}{dx}$ for $\tan\theta$ and integrating we get

$$y = \frac{1}{2}\frac{w}{T_0}x^2 + c,$$

where c is the constant of integration.

But with the axes we have chosen, $y = 0$ when $x = 0$, therefore $c = 0$. Thus the equation of the curve is

$$y = \frac{w}{2\,T_0}x^2, \tag{5}$$

which is the equation of a parabola.

Dip of the Cable. — Let H be the height of the piers above the lowest point of the cable. Then for $x = \frac{D}{2}$, $y = H$, therefore

$$H = \frac{w}{8\,T_0}D^2. \tag{6}$$

It is evident from the last equation that the greater the tension the less is the sag.

PROBLEM. A bridge is supported by two suspension cables. The bridge has a weight of 1.5 tons per horizontal foot and has a span of 400 feet. Supposing the dip of the bridge to be 50 feet find the values of the tensile force at the lowest and highest points of the cable.

64. Equilibrium of a Uniform Flexible Cord which is Suspended from Its Ends. — The problem is to determine the nature of the curve which a perfectly uniform and flexible cable will assume when suspended from two points. Let AOB, Fig. 42, be the curve. Consider the equilibrium of that part of the cable which is between the lowest point O and any other point P. The part of the cable which is under consideration is acted upon by the following three forces:

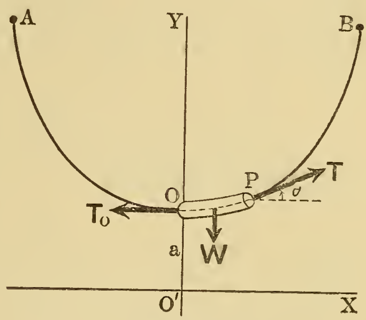


FIG. 42.

The tensile force at the point O, $\mathbf{T}_0$.

The tensile force at the point P, $\mathbf{T}$.

The weight of the cable between the points O and P.

Since the cable is perfectly flexible $\mathbf{T}_0$ and $\mathbf{T}$ are tangent to the curve. Therefore we have

$$\Sigma X \equiv -T_0 + T\cos\theta = 0, \quad \text{or} \quad T\cos\theta = T_0, \tag{1}$$

$$\Sigma Y \equiv -ws + T\sin\theta = 0, \quad \text{or} \quad T\sin\theta = ws, \tag{2}$$

where w is the weight per unit length of the cable and s is the length of OP.

Squaring equations (1) and (2) and adding we obtain

$$T^2 = T_0^2 + w^2s^2. \tag{3}$$

Eliminating T between equations (1) and (2) we get

$$s = \frac{T_0}{w}\tan\theta, \tag{4}$$

which is the intrinsic equation of the curve.

In order to express equation (4) in terms of rectangular coördinates we replace $\tan\theta$ by $\frac{dy}{dx}$ and obtain

$$s = \frac{T_0}{w} \cdot \frac{dy}{dx}. \tag{5}$$

But $ds^2 = dx^2 + dy^2$, therefore eliminating dx between this equation and equation (5) and separating the variables

$$dy = \frac{s\,ds}{\sqrt{s^2 + a^2}}, \tag{6}$$

and then integrating

$$y = \sqrt{s^2 + a^2} + c,$$

where $a = \frac{T_0}{w}$ and c is the constant of integration.

Let the x-axis be so chosen that when $s = 0$, $y = a$, then $c = 0$. Therefore

$$y = \sqrt{s^2 + a^2}, \quad \text{or} \quad s = \sqrt{y^2 - a^2}. \tag{7}$$

Differentiating equation (7), squaring and replacing ds^2 by $(dx^2 + dy^2)$ we have

$$dx^2 + dy^2 = \frac{y^2\,dy^2}{y^2 - a^2}.$$

Solving for dx,

$$\left.\begin{aligned} dx &= -\frac{a}{\sqrt{y^2 - a^2}}\,dy \\ &= -\frac{a\,dy}{\sqrt{-1}\,\sqrt{a^2 - y^2}} \\ &= -\frac{a\,dy}{i\sqrt{a^2 - y^2}}, \end{aligned}\right\} \tag{8}$$

where $i = \sqrt{-1}$. Integrating equation (8) we get

$$\frac{ix}{a} = \cos^{-1}\frac{y}{a} + c'.$$

But $y = a$, when $x = 0$, therefore $c' = 0$. Thus we get

$$y = a \cos \frac{ix}{a}, \tag{9}$$

$$= a \cosh \frac{x}{a},^{*} \tag{10}$$

$$= \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right), \tag{11}$$

$$= \frac{T_0}{2\,w}\left(e^{\frac{wx}{T_0}} + e^{-\frac{wx}{T_0}}\right), \tag{12}$$

which are different forms of the equation of a *catenary*.

Discussion.—Expanding equation (12) by Maclaurin's Theorem† we obtain

$$y = a\left[1 + \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{24}\left(\frac{x}{a}\right)^4 + \cdot \cdot \cdot\right]. \tag{13}$$

In the neighborhood of the lowest point of the cable the value of x is small, therefore in (13) we can neglect all the terms which contain powers of x higher than the second. Thus the equation

$$y = a + \frac{x^2}{2\,a} \tag{14}$$

represents, approximately, the curve in the neighborhood of the lowest point. It will be observed that (14) is the equation of a parabola. This result would be expected since the curve is practically straight in the neighborhood of O and consequently the horizontal distribution of mass is very nearly constant, which is the important feature of the Suspension Bridge problem.

The nature of those parts of the curve which are removed from the lowest point may be studied by supposing x to be large. Then since $e^{-\frac{x}{a}}$ becomes negligible equation (11) reduces to

$$y = \frac{a}{2}e^{\frac{x}{a}}, \tag{15}$$

* See Appendix A. † See Appendix.

The curve, Fig. 43, defined by equation (15) is called an *exponential curve.* It has an interesting property, namely, its ordinate is doubled every time a constant value P is added to its abscissa. This constant is called the *half-value period* of the curve. The value of P may be determined in the following manner. By the definition of P and from equation (15) we have

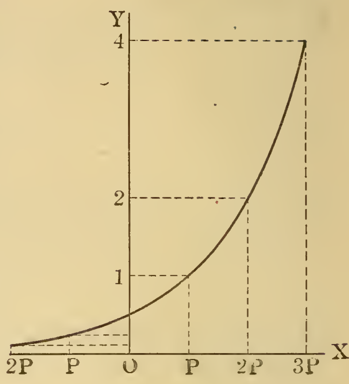


Fig. 43.

$$2\,y = \frac{a}{2} e^{\frac{x+P}{a}}. \tag{16}$$

Dividing (16) by (15) we get

$$2 = e^{\frac{P}{a}},$$

or

$$P = a \log_e 2.$$

Length of Cable.—In order to find the length in terms of the span eliminate y between equations (7) and (11). This gives

$$s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \tag{17}$$

$$= x + \frac{1}{2\cdot 3}\frac{x^3}{a^2} + \frac{1}{2\cdot 3\cdot 4\cdot 5}\frac{x^5}{a^3} + \cdots, \tag{18}$$

where the right member of equation (18) is obtained by expanding the right-hand member of equation (17) by Maclaurin's Theorem.

If D and L denote the span and the length of the cable, respectively, we have $s = \frac{1}{2} L$ when $x = \frac{1}{2} D$. Therefore substituting these values of s and x in (18) and replacing a by its value we obtain

$$L = 2\left(\tfrac{1}{2} D + \frac{1}{48}\frac{w^2}{T_0^{\,2}} D^3 + \cdots\right). \tag{19}$$

When the cable is stretched tight T_0 is large compared with w. Therefore the higher terms of the series may be neglected and equation (1) be put in the following approximate form.

$$L = D\left(1 + \frac{1}{24}\frac{w^2}{T_0{}^2}D^2\right). \tag{20}$$

Hence the increase in length due to sagging is $\frac{1}{24}\frac{w^2}{T_0{}^2}D^3$, approximately.

PROBLEMS.

1. A perfectly flexible cord hangs over two smooth pegs, with its ends hanging freely, while its central part hangs in a catenary. If the two pegs are on the same level and at a distance D apart, show that the total length of the string must not be less than De, in order that equilibrium shall be possible, where e is the natural logarithmic base.

2. In the preceding problem show that the ends of the cord will be on the x-axis.

3. Supposing that a telegraph wire cannot sustain more than the weight of one mile of its own length, find the least and the greatest sag allowable in a line where there are 20 poles to the mile.

4. Find the actual length of the wire per mile of the line in the preceding problem.

5. The width of a river is measured by stretching a tape over it. The middle point of the tape touches the surface of the water while the ends are at a height H from the surface. If the tape reads S, show that the width of the river is approximately $\sqrt{\frac{S^2 - H^2}{2}}$.

6. Show that the cost of wire and posts of a telegraph line is minimum if the cost of the posts is twice that of the additional length of wire required by sagging. The posts are supposed to be evenly spaced and large in number.

7. A uniform cable which weighs 100 tons is suspended between two points, 500 feet apart, in the same horizontal line. The lowest point of the cable is 40 feet below the points of support. Find the smallest and the greatest values of the tensile force.

8. In the preceding problem find the length of the cable.

65. Friction Belts. — The flexible cord AB, Fig. 44, is in equilibrium under the action of three forces, namely, $\mathbf{T}_0$

and **T**, which are applied at the ends of the cord, and the reaction of the rough surface of C, with which it is in contact. It is desired to find the relation between T_0 and T when the cord is just on the point of motion towards $\mathbf{T}_0$.

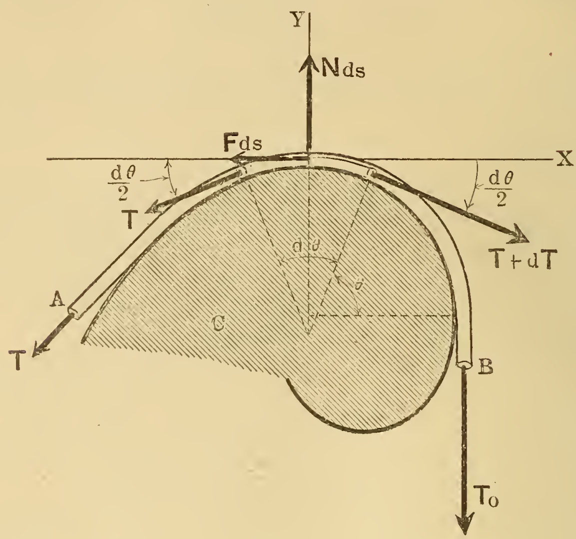


Fig. 44.

Consider the equilibrium of an element of that part of the cord which is in contact with the surface. The element is acted upon by the following three forces:

The tensile force in the cord to the right of the element.

The tensile force in the cord to the left of the element.

The reaction of the surface.

Let the tensile force to the left of the element be denoted by **T**, then the tensile force to the right may be denoted by $\mathbf{T} + d\mathbf{T}$. On the other hand if **R** denotes the reaction of the surface per unit length of the cord, the reaction on the element is **R** ds, where ds is the length of the element. We will, as usual, replace **R** by its frictional component **F** and its normal component **N**.

Taking the axes along the tangent and the normal through the middle point of the element and applying the conditions of equilibrium we obtain

$$\Sigma X \equiv (T + dT) \cos \frac{d\theta}{2} - T \cos \frac{d\theta}{2} - F(-ds) = 0,$$

$$\Sigma Y \equiv N\,ds - T \sin \frac{d\theta}{2} - (T + dT) \sin \frac{d\theta}{2} = 0,$$

$$\text{or} \quad dT \cos \frac{d\theta}{2} + F\,ds = 0,$$

$$N\,ds - 2\,T \sin \frac{d\theta}{2} - dT \sin \frac{d\theta}{2} = 0,$$

where $d\theta$ is the angle between the two tensile forces which act at the ends of the element. The negative sign in $F\,(-ds)$ indicates the fact that F and ds are measured in opposite directions. But since the cord is supposed to be perfectly flexible the tensile forces are tangent to the surface of contact. Therefore θ is the angle between the tangents, and consequently the angle between the normals, at the ends of the element. As an angle becomes indefinitely small its cosine approaches unity and its sine approaches the angle itself,* therefore we can make the substitutions

$$\cos \frac{d\theta}{2} = 1 \text{ and } \sin \frac{d\theta}{2} = \frac{d\theta}{2}$$

in the last two equations, and obtain

$$dT + F\,ds = 0, \tag{1}$$

and

$$N\,ds - T\,d\theta + \tfrac{1}{2}\,dT\,d\theta = 0. \tag{2}$$

Neglecting the differential of the second order in equation (2) and then eliminating ds between equations (1) and (2) we get

$$\frac{dT}{T} = -\frac{F}{N}\,d\theta = -\mu\,d\theta, \tag{3}$$

where μ is the coefficient of friction. Integrating the last

* See Appendix A.

equation and passing from the logarithmic to the exponential form, we have

$$T = ce^{-\mu\theta},$$

where c is the constant of integration. If θ is measured from the normal to the surface at the point where the right-hand side of the cord leaves contact we obtain the initial condition, $T = T_0$ when $\theta = 0$, which determines c. Applying this condition to the last equation we have

$$T = T_0 e^{-\mu\theta}. \tag{4}$$

Discussion. — Equation (4) gives the relation between the values of the tensile force at any two points of the cord. It must be observed that θ is measured in the same direction as **F**; in other words, opposite the direction towards which the cord is urged to move. Therefore T or T_0 has the larger value according to whether θ is positive or negative. As a concrete example suppose a weight W to be suspended from the right-hand end of the cord and to be held in equilibrium by a force F applied at the left-hand end. If F is just large enough to prevent W from falling then the cord will be on the point of moving to the right, therefore θ is measured in the counter-clockwise direction and is positive. In this case

$$F = We^{-\mu\theta}.$$

In case F is just large enough to start W to move up, then θ is measured in the clockwise direction and is negative. Therefore

$$F = We^{\mu\theta}.$$

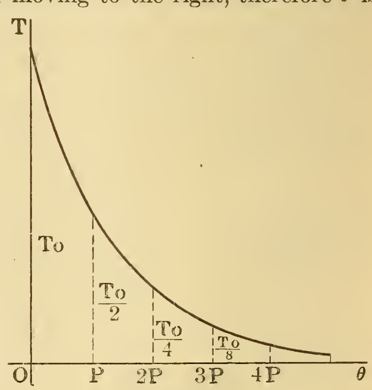


Fig. 45.

The value of T drops very rapidly with the increase of θ. This fact is made clear by drawing the graph of equation (4), Fig. 45. The graph may be constructed easily by making use of the half-value period of the curve. If P denotes the period, then, by definition, the ordinate is reduced to one-half its value every time P is added to θ.* We have therefore

$$\tfrac{1}{2}T = T_0 e^{-\mu(\theta+P)}.$$

* The difference between this definition of P and the one given in the preceding section is accounted for by the difference in the signs of the exponents in equation (4) and in equation (14) of the preceding section.

Dividing equation (4) by the last equation we get

$$\left.\begin{aligned} 2 &= e^{\mu P}, \\ P &= \frac{1}{\mu}\log_e 2 \\ &\doteq \frac{.7}{\mu}. \end{aligned}\right\} \qquad (5)$$

or

Thus if $\theta = nP$, then by equations (4) and (5)

$$T = \frac{T_0}{2^n}. \qquad (6)$$

Therefore taking 0.53 for hemp rope on oak and $\theta = 2\pi$, we obtain $n = 4.76$ and $2^n = 27.3$. Hence in this case T_0 is 27.3 times as great as T.

APPLICATION TO BELTS.

The tensile force on one side of a belt which transmits power is greater than that on the other side. The relation between the tensile forces on the two sides of the belt is given by equation (4). Thus if T_1 denotes the tensile force on the driving side and T_2 that on the slack side, then

$$T_2 = T_1 e^{-\mu\theta} \quad \text{or} \quad T_1 = T_2 e^{\mu\theta}. \qquad (4')$$

The difference between T_1 and T_2 is the effective force which drives the pulley. Denoting the effective force by F, we have

$$\left.\begin{aligned} F &= T_1 - T_2 \\ &= T_1 (1 - e^{-\mu\theta}) \\ &= T_2 (e^{\mu\theta} - 1). \end{aligned}\right\} \qquad (7)$$

We have neglected the cross-section of the cord in the solution of the foregoing problem. Therefore the results which we have obtained are applicable to actual problems only when the cross-section of the cord is negligible compared with that of the solid with which it is in contact.

PROBLEMS.

1. A weight of 5 tons is to be raised from the hold of a ship by means of a rope which takes $3\frac{1}{2}$ turns around the drum of a steam windlass. If $\mu = 0.25$ what force must a man exert at the other end of the rope?

2. By pulling with a force of 200 pounds a man just keeps from surging a rope, which takes 2.5 turns around a post. Find the tensile force at the other end of the rope. $\mu = 0.2$.

3. A weight W is suspended by a rope which makes $1\frac{1}{4}$ turns around a clamped pulley and goes to the hand of a workman. If $\mu \doteq 0.2$, find the force the man has to apply in order (a) to support the weight, (b) to raise it.

4. Two men, who can pull 250 pounds each, can support a weight by means of a rope which takes 2 turns around a post. On the other hand, one of the men can support it alone if the rope makes 2.5 turns. Find the weight.

5. In order to prevent surging a sailor has to exert a force of 150 pounds at the end of a hawser, which is used to keep the stern of a boat at rest while the bow is being turned by the engines. Find the pull exerted by the boat upon the hawser under the following conditions:

[*Hint.* — *Make use of equations* (5) *and* (6).]

(a) $\theta = \frac{\pi}{4}$, $\mu = 0.2$. (g) $\theta = 2\pi$, $\mu = 0.1$.

(b) $\theta = \frac{\pi}{4}$, $\mu = 0.5$. (h) $\theta = \frac{9\pi}{4}$, $\mu = 0.4$.

(c) $\theta = \frac{\pi}{2}$, $\mu = 0.5$. (i) $\theta = \frac{5\pi}{2}$, $\mu = 0.5$.

(d) $\theta = \pi$, $\mu = 0.4$. (j) $\theta = 3\pi$, $\mu = 0.3$.

(e) $\theta = \frac{5\pi}{4}$, $\mu = 0.3$. (k) $\theta = \frac{13\pi}{4}$, $\mu = 0.4$.

(f) $\theta = \frac{3\pi}{2}$, $\mu = 0.2$. (l) $\theta = \frac{7\pi}{2}$, $\mu = 0.5$.

6. A belt has to transmit an effective force of 500 pounds. Find the tensile force on both sides of the belt, under the following conditions:

(a) $\theta = 135°$, $\mu = 0.5$. (e) $\theta = 165°$, $\mu = 0.2$.
(b) $\theta = 135°$, $\mu = 0.4$. (f) $\theta = 180°$, $\mu = 0.3$.
(c) $\theta = 150°$, $\mu = 0.3$. (g) $\theta = 180°$, $\mu = 0.5$.
(d) $\theta = 165°$, $\mu = 0.5$. (h) $\theta = 195°$, $\mu = 0.4$.

7. In the preceding problem find the width of the belt, supposing the permissible safe tensile force to be 50 pounds per inch of its width.

ANALYTICAL MECHANICS

FOR STUDENTS OF PHYSICS AND ENGINEERING

BY

HAROUTUNE M. DADOURIAN, M.A., Ph.D.

Instructor of Physics in the Sheffield Scientific School of Yale University

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