PRACTICAL ASTRONOMY

AND

GEODESY:

INCLUDING

THE PROJECTIONS OF THE SPHERE

AND

SPHERICAL TRIGONOMETRY.

For the Use of the Royal Military College.

BY

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ADVERTISEMENT.

THE following Treatise on Practical Astronomy and Geodesy, including Spherical Trigonometry, is the fifth of a series which is to constitute a General Course of Mathematics for the use of the gentlemen cadets and the officers in the senior department of this Institution. The course, when completed, will comprehend the subjects whose titles are subjoined:—I. Arithmetic and Algebra. * II. Geometry. * III. Plane Trigonometry with Mensuration. † IV. Analytical Geometry with the Differential and Integral Calculus and the Properties of Conic Sections. V. Practical Astronomy and Geodesy, including Spherical Trigonometry. VI. The Principles of Mechanics, and VII. Physical Astronomy.

Royal Military College,
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PREFACE.

THE courses of study pursued at the military and naval seminaries of this country have, within a few years, been greatly extended, in order that they might be on a level with the improved state of the sciences relating to those branches of the public service, and also that they might meet the necessity of qualifying officers to conduct the scientific operations which have been undertaken, both at home and abroad, under the authority of the government. An education comprehending the higher departments of mathematics and natural philosophy has also been found necessary for the qualification of such as have been, or may be, appointed to superintend Institutions established in the remoter parts of the British empire for the purpose of promoting the advancement of physical science, and of preserving or extending its benefits among the European, and, in certain cases, the native inhabitants of the countries.

An attempt to supply the want of a treatise on the elements of Practical Astronomy and Geodesy is what is proposed in the present work; which, it is hoped, will be found useful to the scientific traveller, and to persons employed in the naval or military service of the country who may accompany expeditions to distant regions, where, with the aid of portable instruments, they may be able to make observations valuable in themselves, and possessing additional importance from their local character. The extent to which the subjects are here carried will probably be found sufficient for the proposed end, and may be useful in preparing the student for the cultivation of the highest branches of astronomical science.

A brief notice of the phenomena of the heavens forms the commencement of the work; but no more of the merely descriptive part of astronomy has been given than is necessary for a right understanding of the subjects to which the processes employed in determining the elements are applied. A tract on spherical trigonometry constitutes the third chapter;

and that tract has been introduced in this work because the general theorems are few in number, and have their chief applications in propositions relating to astronomy: the transformations which the theorems are made to undergo are, almost always, investigated for the purpose of rendering them convenient in computations; and, as both the investigations and the examples chosen for illustrating the formulæ generally refer to astronomical subjects, it is evident that facilities are afforded and repetitions avoided by comprehending the tract in a work for which it is immediately required.

There is given a description of the principal instruments employed in making observations; and, after investigations of the formulæ for refraction and parallax, there follow outlines of the methods by which the elements of the solar, lunar, and planetary orbits are determined: these are succeeded by formulæ for computing the apparent displacements of celestial bodies, produced by the actions of the sun, moon, and planets on the earth, and by the motions of the latter. In a chapter on Nautical Astronomy there is given a series of propositions relating to the geographical positions of places on the earth, the determination of local time and the declination of the needle; and it may be right to state, here, that the examples which illustrate the several propositions are taken from the book of sextant-observations made by the students at the Observatory belonging to the Institution. Each computation is made from a single observation, and the instruments used are graduated so as to give thirds or, at best, quarters of minutes: these circumstances will account for the discrepancies, generally amounting however to a few seconds only, in the results.

After an outline of the methods of computing eclipses of the moon and sun, and the occultations of stars by the moon, there are given formulæ and examples for determining terrestrial longitudes by those phenomena; and, in the chapter on geodesy, there are given the methods of executing trigonometrical surveys for the purpose of determining the figure of the earth, with propositions relating to spheroidal arcs and angles, also the manner of making pendulum experiments for a like purpose, together with notices of the principal formulæ relating to terrestrial magnetism.

CONTENTS.

CHAPTER I.

THE EARTH.—PHENOMENA OF THE CELESTIAL BODIES.

FORM OF THE EARTH AND ITS ROTATION ON ITS AXIS.—APPARENT MOVEMENT OF THE STARS.—REVOLUTION OF THE MOON ABOUT THE EARTH.—HYPOTHESIS OF THE EARTH'S ANNUAL MOTION.—PHASES OF THE MOON.—APPARENT MOVEMENTS OF THE PLANETS.—THE CIRCLES OF THE SPHERE.

CHAP. II.

PROJECTIONS OF THE SPHERE.

NATURE OF THE DIFFERENT PROJECTIONS EMPLOYED IN PRACTICAL ASTRONOMY AND GEOGRAPHY. — PROPOSITIONS RELATING TO THE STEREOGRAPHICAL PROJECTION IN PARTICULAR.—EXAMPLES OF THE ORTHOGRAPHICAL, GNOMONICAL, GLOBULAR, AND CONICAL PROJECTIONS.—MERCATOR'S DEVELOPMENT.

CHAP. III.

SPHERICAL TRIGONOMETRY.

DEFINITIONS AND THEOREMS.

CHAP. IV.

DESCRIPTION OF THE INSTRUMENTS EMPLOYED IN PRACTICAL ASTRONOMY.

THE SIDEREAL CLOCK. — MICROMETER. — TRANSIT INSTRUMENT. — MURAL CIRCLE. — AZIMUTH AND ALTITUDE CIRCLE. — ZENITH SECTOR. — EQUATORIAL INSTRUMENT. — COLLIMATOR. — REPEATING CIRCLE. — REFLECTING INSTRUMENTS.

CHAP. V.

REFRACTION: LATITUDE OF A STATION: PARALLAX.

CHAP. VI.

DETERMINATION OF THE EQUINOCTIAL POINTS AND THE OBLIQUITY OF THE ECLIPTIC BY OBSERVATION.

CHAP. VII.

TRANSFORMATION OF THE CO-ORDINATES OF CELESTIAL BODIES FROM ONE SYSTEM TO ANOTHER.

CHAP. VIII.

THE ORBIT OF THE EARTH.

ITS FIGURE SHOWN TO BE ELLIPTICAL. — SITUATION AND MOVEMENT OF THE PERIHELION POINT. — THE MEAN, TRUE, AND EXCENTRIC ANOMALIES. — EQUATION OF THE CENTRE.

CHAP. IX.

THE ORBIT OF THE MOON.

THE FIGURE OF THE MOON'S ORBIT.—PERIODICAL TIMES OF HER REVOLUTIONS.—THE PRINCIPAL INEQUALITIES OF HER MOTION.—HER DISTANCE FROM THE EARTH EMPLOYED TO FIND APPROXIMATIVELY THE DISTANCE OF THE LATTER FROM THE SUN.

CHAP. X.

APPARENT DISPLACEMENTS OF THE CELESTIAL BODIES ARISING FROM THE FIGURE AND MOTION OF THE EARTH.

THE EFFECTS OF PRECESSION, ABERRATION, AND NUTATION.

CHAP. XI.

THE ORBITS OF PLANETS AND THEIR SATELLITES.

THE ELEMENTS OF PLANETARY ORBITS. — VARIABILITY OF THE ELEMENTS. — KEPLER'S THREE LAWS. — PROCESSES FOR FINDING THE TIME OF A PLANET'S REVOLUTION ABOUT THE SUN. — FORMULÆ FOR THE MEAN MOTIONS, ANGULAR VELOCITIES AND TIMES OF DESCRIBING SECTORAL AREAS IN ELLIPTICAL ORBITS. — MODIFICATIONS OF KEPLER'S LAWS FOR BODIES MOVING IN PARABOLICAL ORBITS. — MOTIONS OF THE SATELLITES OF JUPITER, SATURN, AND URANUS. — IMMERSIONS ETC. OF JUPITER'S SATELLITES. — THE ORBITS OF SATELLITES ARE INCLINED TO THE ECLIPTIC. — SATURN'S RING.

CHAP. XII.

ROTATION OF THE SUN ON ITS AXIS, AND THE LIBRATIONS OF THE MOON.

CHAP. XIII.

THE FIXED STARS.

REDUCTION OF THE MEAN TO THE APPARENT PLACES.—PROPER MOTIONS.—ANNUAL PARALLAX.

CHAP. XIV.

TIME.

SIDEREAL, SOLAR, LUNAR, AND PLANETARY DAYS. — EQUATION OF TIME. — EQUINOCTIAL TIME. — REDUCTIONS OF ASTRONOMICAL ELEMENTS TO THEIR VALUES FOR GIVEN TIMES. — HOUR ANGLES.

CHAP. XV.

INTERPOLATIONS — PRECISION OF OBSERVATIONS.

CHAP. XVI.

NAUTICAL ASTRONOMY.

PROBLEMS FOR DETERMINING THE GEOGRAPHICAL POSITION OF A SHIP OR STATION, THE LOCAL TIME, AND THE DECLINATION OF THE MAGNETIC NEEDLE.

* This example has been accidently placed at the end of art. 357.

CHAP. XVII.

ECLIPSES.

OUTLINES OF THE METHODS OF COMPUTING THE OCCURRENCE OF THE PRINCIPAL PHENOMENA RELATING TO AN ECLIPSE OF THE MOON, AN ECLIPSE OF THE SUN FOR A PARTICULAR PLACE, THE OCCULTATION OF A STAR OR PLANET BY THE MOON, AND THE TRANSITS OF MERCURY AND VENUS OVER THE SUN'S DISK.— THE LONGITUDES OF STATIONS FOUND BY ECLIPSES AND OCCULTATIONS.

CHAP. XVIII.

GEODESY.

METHOD OF CONDUCTING A GEODETICAL SURVEY.—MEASUREMENT OF A BASE.—FORMULÆ FOR VERIFYING THE OBSERVED ANGLES AND COMPUTING THE SIDES OF THE TRIANGLES.—MANNER OF DETERMINING THE POSITION, AND COMPUTING THE LENGTH OF A GEODETICAL ARC.—PROPOSITIONS RELATING TO THE VALUES OF TERRESTRIAL ARCS ON THE SUPPOSITION THAT THE EARTH IS A SPHEROID OF REVOLUTION.—THE EMPLOYMENT OF PENDULUMS TO DETERMINE THE FIGURE OF THE EARTH.—INSTRUMENTS USED IN FINDING THE ELEMENTS OF TERRESTRIAL MAGNETISM.—FORMULÆ FOR COMPUTING THOSE ELEMENTS AND THEIR VARIATIONS.—EQUATIONS OF CONDITION EXEMPLIFIED.

PRACTICAL ASTRONOMY

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CHAPTER I.

THE EARTH.—PHENOMENA OF THE CELESTIAL BODIES.

FORM OF THE EARTH, AND ROTATION ON ITS AXIS.—APPARENT MOVEMENT OF THE STARS.—REVOLUTION OF THE MOON ABOUT THE EARTH.—HYPOTHESIS OF THE EARTH'S ANNUAL MOTION.—PHASES OF THE MOON.—APPARENT MOVEMENTS OF THE PLANETS.—THE CIRCLES OF THE SPHERE.

1. THAT the surface of the earth is of a form nearly spherical may be readily inferred from the appearance presented at any point on the ocean by a ship when receding from thence; for, on observing that the line which bounds the view on all sides is accurately or nearly the circumference of a circle, and that when a ship has reached any part of this line she seems to sink into the water, the spectator recognizes the fact that she is moving on a surface to which the visual rays from that circumference are tangents. These rays may be imagined to constitute the surface of a cone of which the eye of the spectator is the vertex; and the solid with which, at every part of its surface, a cone is in contact on the periphery of a line which is accurately or nearly a circle (that is, the solid whose section when cut any where by a plane is accurately or nearly a circle) is (Geom. 1. Prop. Cylind.) accurately or nearly a sphere. The like inference may be drawn from the appearance presented on all sides of a spectator on land, the curve line which bounds his view being the circumference of a circle except where inequalities of the ground destroy its regularity.

2. The plane of the circle which terminates the view of a spectator is designated his visible or sensible horizon. A plane

conceived to pass through the spectator and the sun at noon, perpendicularly to the horizon, is called his meridian; and, on the supposition that the earth is a sphere or spheroid, this plane will pass through its centre. Its intersection with the surface of the earth or with a horizontal plane, which, to the extent of a few yards in every direction about the spectator, may be considered as coincident with that surface, is called a meridian line: of this line, the extremity which is nearest to the Arctic regions of the earth is called the north point, and that which is opposite to it, the south point. A line imagined to pass through the spectator perpendicularly to the plane of the horizon, and to be produced above and below it towards the heavens, is denominated a vertical line; its upper and lower extremities are designated, respectively, the zenith and nadir. Every plane which may be conceived to pass through this line is said to be a vertical plane, but that which is at right angles to the plane of the meridian is called the prime vertical: it cuts the plane of the horizon in a line whose extremities are called the east and west points; the former being that which is on the right hand of the spectator when he looks towards the Arctic regions of the earth, and the latter, that which is on his left hand when in the same position.

3. Now, if a spectator were at any season of the year to land on the shores of Spitzbergen, the stars which are visible would appear to describe about him circles nearly parallel to his horizon. In the British Isles certain stars towards the north indicate by their movements that they describe during a day and a night the circumferences of circles whose planes are very oblique to the horizon and wholly above it, while others describe arcs which are easily seen to become smaller portions of a circumference as they rise more remotely from the northern part of the horizon; and a few may be observed which rise and set near the southern point, describing, during the time they are visible, curves which ascend but little above that plane. About the mouth of the Amazon, and in the islands of the Indian Ocean, the spectator would see the stars rise and set perpendicularly to the horizon, each of them describing half the circumference of a circle above it. If the spectator were to transfer himself to the southern regions of the earth he would see phenomena similar to those above mentioned exhibited by the stars which are situated in that part of the heavens; while on directing his eye towards the north, the stars which before were seen to ascend to considerable heights above the southern part of the horizon, would be either invisible or would be seen but for a short time, the places of rising and setting being near the northern point.

4. These circumstances indicate a general revolution of all the stars about an axis passing through the earth perpendicularly to the planes of the circular arcs apparently described by them: but it is very improbable that so many different bodies should perform the revolution in the same time, preserving their apparent distances unchanged; and, on the ground that the like phenomena would result from a rotation of the earth about the same axis, the latter hypothesis is adopted by astronomers. In the present age a certain star (α Polaris) nearly indicates the northern extremity of the axis in the heavens, and an arc supposed to be traced through two bright stars (γ and α of the southern cross) tends directly towards the opposite extremity.

5. The stars appearing to be at the same distance from the spectator, they are, for convenience, imagined to be attached to the concave surface of a hollow sphere of which his eye is the centre; and hence the general movement of the stars about the earth is sometimes called the diurnal rotation of the celestial sphere. The sun appears daily to rise and set: the moon also, when visible, exhibits the like phenomena; but these two celestial bodies, and certain stars which are not always visible, have proper movements which prove them to be quite distinct from the other bodies of the universe.

6. At a certain season, in any part of the world, a very distinguishable cluster of stars, which may have been observed to rise very near the eastern part of the horizon, will be on the meridian of the observer a short time before the rising of the sun: on continuing to observe the cluster, its apparent elongation from the sun will constantly increase, and, in two or three months from the time of the first observation, it will appear at morning dawn to be setting in the west. During several months it will not be visible, but it will afterwards appear near the east a little while before the sun is there; and at the end of one year from the first observation it will be again seen on the meridian near the time of his rising. The like phenomena are exhibited by all the stars which rise in or near the east; and from this change in the sun's position with respect to such stars, it may be inferred either that those stars, and with them all the others, have been carried towards the west independently of the diurnal movement, or that the sun has, during the year, moved eastward about the earth. The moon is observed to change her place in like manner; for if on any night her distance eastward from some star be remarked, on the following night the distance in that direction will be sensibly increased: the distance

will go on continually increasing and, in about a month, the moon will be in the same position as at first with respect to the star. That an annual and monthly, as well as a diurnal revolution of all the stars should take place simultaneously, is highly improbable; and it may therefore be inferred that the moon revolves about the earth from west to east: the phenomena above mentioned may seem to indicate a like movement of the sun; but since they will be the same whether the sun revolve about the earth in one year, or the earth about the sun in the same time, the latter hypothesis, which alone is consistent with the laws of general attraction among the great bodies of the universe, may be immediately adopted.

7. It is well known that the sun's angular elevation above the horizon at noon experiences, from mid-winter to mid-summer, a continual increase, and, from mid-summer to mid-winter, a continual decrease. Now, by observing what groups of stars appear to rise and set very near the sun during a year, the trace of his apparent route in the heavens may be distinguished; and hence it may be readily ascertained in what groups the sun is at the times when his elevations above the horizon are the least and the greatest. These groups, or constellations, are on opposite sides of the celestial sphere with respect to the earth; and it may be perceived that a mean between the greatest and least elevations of the sun at noon is nearly equal to the elevation of that luminary at the noon of the day which is the middle of the interval between mid-winter and mid-summer, or between mid-summer and mid-winter: it may be observed also, that the greatest elevation of any star, which, rising in the east and setting in the west, appears to describe daily about the earth a path whose plane is perpendicular to the axis of the diurnal rotation, is equal to that mean elevation of the sun; and it may from these circumstances be presumed, that the path of the sun in his apparent yearly revolution about the earth, or the path of the earth about the sun, is a plane curve having a certain inclination to the plane last mentioned. The angular elevations of the points which the moon appears to occupy in the celestial sphere are sometimes greater than the greatest, and sometimes less than the least elevations of the sun; and it may from thence be inferred that the moon's orbit has a certain inclination to that in which the earth revolves about the sun.

8. Since the angular distance of the sun below the plane passing through the earth perpendicularly to the axis of rotation is at noon, on the day of mid-winter, equal to the angular

distance of the sun above that plane at noon on the day of midsummer, and that on the two days of the year which are equally distant from those days the sun is in the plane perpendicular to the axis of rotation; it follows, if the earth be supposed to revolve annually about the sun, that the axis of the diurnal rotation must on those four days be in positions parallel to one another; also, since the changes in the angular distance of the sun from that plane take place gradually, it may be inferred that the axis of rotation continues always strictly or nearly parallel to itself.

9. If the moon be at first observed near the west about the time that the sun is setting, her angular distance from that luminary will on every succeeding evening be found to have increased, and at the end of about fourteen days, she may be observed rising near the east, when the sun is on the western side of the horizon: she is then said to be in opposition to the sun: from that time, she appears to approach the sun towards the east, the intervals between the times at which the two luminaries rise continually diminishing; and, at the end of about a month from the first observation, the moon is in conjunction with the sun. After remaining invisible for a few days, she re-appears in the west at a small distance from the setting sun, and the like phenomena are repeated.

10. During this revolution of the moon about the earth the form exhibited by the outline of her face gradually changes. At the time of new moon, when, in the western part of the horizon, she first emerges from the sun's rays, she assumes the form of a slender crescent of light, and this crescent daily increases in breadth, till, at the time of opposition, it becomes a complete circle: afterwards the breadth diminishes till the moon is about to become invisible from her proximity to the sun in the east, when the figure is again that of a slender crescent. These *phases* are exactly such as are presented by a globular body enlightened on half its surface by the sun, the circle bounding the light and shadow being in different positions with respect to the observer; and hence it is inferred that the figure of the moon is exactly or nearly that of a sphere.

It happens occasionally that, at the time of opposition, the centre of the moon is exactly or nearly in the direction of a line drawn from the centre of the sun through that of the earth, and produced towards her; and then, the shadow of the earth falling on the moon, the inhabitants of the side of the earth which is nearest to her observe her to be eclipsed, or deprived of the light which she would have received from the sun: it happens also, occasionally, that at the time of conjunction the centre of the moon is exactly or nearly in a line joining those of the earth and sun; in which case, the moon intercepting the rays of light coming from the sun towards the earth, the sun, to an inhabitant of the earth who may be situated near the direction of the line, on the side nearest to the luminary, is observed to suffer an eclipse.

11. Attentive and continued observations of the heavens show that some of the stars have movements independent of that general revolution which all of them appear to perform daily about the earth. These are the planets, which are ten in number, though only six can be seen by the unassisted eye, and their designations in the order of their distances from the sun are as follow: — Mercury, Venus, Mars, Vesta, Juno, Pallas, Ceres, Jupiter, Saturn, and Uranus; the Earth, which is also a planet, and is situated between Venus and Mars, being omitted in the enumeration. Two of the planets, Mercury and Venus, when visible, appear on the same side of the meridian as the sun is; and the former but a short time before his rising or after his setting: if first seen nearly in conjunction with the sun in the west, these planets then gradually recede from him towards the south, Mercury to an angular distance not exceeding $28\frac{1}{2}$ degrees, and Venus to a distance not exceeding 48 degrees: they afterwards appear to return towards him, and after having been for some days invisible, they may be seen in the east before sunrise; at first they appear to recede from that luminary towards the south; and subsequently, the greatest angular distances or elongations being equal to those which were attained in their former positions, they return towards it. After being again for a time invisible, they re-appear in the west as before, and the like phenomena are repeated. In the interval between the disappearance in the west and the next appearance in the east, both planets are occasionally, by the aid of the telescope, seen to pass like dark spots across the disk of the sun; the telescope moreover shows that each of these, like the moon, assumes the form of a crescent, a semicircle, an ellipse, and nearly a complete circle; the several phases succeeding each other in regular order. The inferences are that these planets are globular, and that they revolve about the sun within the orbit of the earth, Mercury being that which is nearest to him.

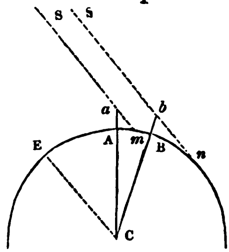
12. The other planets are seen at times nearly in conjunction with the sun, and at other times diametrically opposite to him in the heavens, and it is therefore inferred that they revolve about the sun in orbits, on the exterior of that which is described by the earth. The comets also, which occasionally

appear in the heavens, are observed to have such movements as indicate that they, like the planets, revolve about the sun. All the planets, moreover, are seen to move in different directions with respect to the fixed stars: sometimes they appear to recede from certain of these towards the west, sometimes towards the east, and, again, to remain for a time stationary, or nearly so. The telescope shows that their disks are nearly circular, or segments of circles, and from the movements of the spots which have been observed on most of their surfaces, it is inferred that they are globular bodies, which, like the earth, constantly turn on axes of rotation. The motions of the spots observed on the sun show that this luminary has a similar movement on an axis. The planets Jupiter, Saturn, and Uranus are, by the aid of the telescope, observed to be accompanied by satellites, which revolve about them as the moon revolves about the earth; and Saturn is, moreover, accompanied by a ring which revolves in its own plane about the planet.

13. The stars called fixed have, from the earliest ages, been reduced into groups under the figures chiefly of men and animals; and representations of such groups, or constellations as they are called, may be seen on any celestial globe. A certain zone of the sphere of stars, extending several degrees northward and southward of the sun's apparent annual path, is called the *zodiac*, and twelve groups of stars immediately about that path bear the name of the zodiacal constellations. The designations of these are as follow: — Aries (♈), Taurus (♉), Gemini (♊), Cancer (♋), Leo (♌), Virgo (♍), Libra (♎), Scorpio (♏), Sagittarius (♐), Capricornus (♑), Aquarius (♒), and Pisces (♓).

14. An approximate knowledge of the magnitude of the earth may be, and very early was, obtained by the aid of a simple trigonometrical proposition, from the measured length of the shadow cast at noon by a column or obelisk erected at each of two places, lying in a direction nearly due north and south of each other. Thus, it being assumed that the earth is a sphere, and the sun so remote that the rays of light which fall upon the earth at the two stations A and B may be considered as parallel to one another, let the plane of the paper represent that of a terrestrial meridian, whose circumference passes through the stations, and let A*a*, B*b*, be the obelisks there set up. Then, if S*am*, S*bn* be two parallel rays proceeding from the sun at noon, A*m*, B*n*, which may be considered as straight lines, will denote the lengths of the shadows; and in the triangles *a*A*m*, *b*B*n*, right angled at A and B, the lengths of A*a* and A*m*, B*b* and B*n* being known,

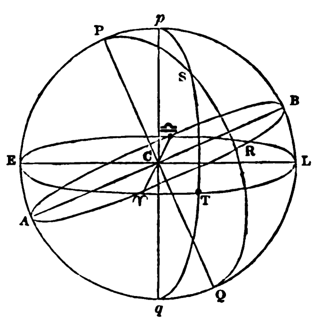
the angles A*am*, B*bn* may be computed. But C representing the centre of the earth, if EC be drawn parallel to S*a* or S*b*, the angles ACE, BCE will be respectively equal to A*am*, B*bn*; therefore the difference between these last angles is equal to the angle ACB. Hence, the arc AB being measured, the following proportion will give the length of the earth's circumference: —



ACB (in degrees) : 360° :: AB : circumference (=24850 miles, nearly.)

15. The processes by which the earth's form and magnitude are with precision determined, as well as those which are employed in finding the magnitudes of the sun, moon, and planets, the distances of the moon from the earth, and of the earth and planets from the sun, will be presently explained. It is sufficient to observe here, that since the planetary bodies, when viewed through a telescope, present the appearance of well-defined disks, subtending, at the eye of the spectator, angles of sensible magnitude, while the stars called fixed, though examined with the most powerful instruments, are seen only as lucid points; it will follow that the sun, the earth, and the planets constitute a particular group of bodies, and that a sphere supposed to encompass the whole of the planetary system may be considered as infinitely small when the imaginary sphere of the fixed stars is represented by one of any finite magnitude. Hence, in describing the systems of circles by which the apparent places of celestial bodies are indicated, it is permitted to imagine that either the earth or the sun is a point in the centre of a sphere representing the heavens: and, in the latter case, the earth and all the planets must be supposed to revolve in orbits whose peripheries are at infinitely small distances from the sun.

16. If the earth's orbit (supposed to be a plane) be produced to the celestial sphere, it will there form the circumference of a circle which is called the trace of the ecliptic (let it be E ♈ L ♎): this is represented on the common celestial globes; and on those machines, the representations of the zodiacal stars near which it appears to pass will serve, when the stars are recognised, as indications of its position in



the heavens. A line passing through the sun at C (the centre of the sphere) perpendicularly to the plane of the ecliptic, meets the heavens in the points designated p and q, which are called the poles of the ecliptic. Now, if planes be supposed to pass through p and q, these planes will be perpendicular to that of the ecliptic, and they will cut the celestial sphere in the circumferences of circles which are called circles of celestial longitude: these are also represented on the celestial globes.

17. The trace of the ecliptic in the heavens is imagined to be divided into twelve equal parts called *signs*, which bear the names of the zodiacal constellations before mentioned. They follow one another in the same order as those constellations, that is, from the west towards the east; and the movement of any celestial body in that direction is said to be *direct*, or according to the order of the signs: if the movement take place from the east towards the west, it is said to be *retrograde*, or contrary to the order of the signs.

18. It has been shown in art. 7. that the path (the ecliptic) of the earth about the sun is inclined to the plane which is perpendicular to the axis of the diurnal rotation; the axis $p\,q$ must, therefore, be inclined to the latter axis. Now, the centre of the earth being at C infinitely near, or in coincidence with that of the sun, agreeably to the above supposition, let P Q be the axis of the diurnal rotation; then the plane A ♈ B passing through C perpendicularly to P Q and produced to the heavens will be the plane last mentioned: it will cut the surface of the earth (which is assumed to be a sphere or spheroid) in the circumference of a circle called the terrestrial equator, and that of the sphere of the fixed stars in the circumference A ♈ B of the celestial equator. The plane of this circle will cut that of the ecliptic in a line, as ♈ C ♎, which is called the line of the equinoxes, of which one extremity ♈ in the heavens is called the point of the vernal equinox. If planes pass through P Q they will be perpendicular to the equator, and their circumferences in the celestial sphere form what are called circles of declination: such is the circle P R Q which is made to pass through S, the supposed place of a star. Of these circles that which passes through the line of the equinoxes is called the equinoctial colure, and that which, being at right angles to the former, passes through p, is called the solsticial colure. The last-mentioned planes will cut the surface of the earth in the circumferences of circles, if the earth be a sphere, or in the perimeters of ellipses if it be a spheroid; and these are called the meridians of the stations, or remarkable points which they pass through on

the earth. This is, however, only the popular definition of a terrestrial meridian: if from every point in the circumference of a circle of declination in the celestial sphere lines be let fall in the directions of normals, or perpendiculars, to the earth's surface, a curve line supposed to join the points in which the normals meet that surface will be the correct terrestrial meridian; and if the earth be not a solid of revolution this meridian is a curve of double curvature. If any point, as S, be the place where a perpendicular raised from any station on the surface of the earth meets the celestial sphere, S will be the zenith of that station; ♈ R will express its geographical longitude, and R S its geographical latitude. The arc ♈ T or the angle ♈ C T on the plane of the ecliptic is designated the longitude of any star S, through which and the axis $p\,q$ the plane of a circle is supposed to pass; and the arc T S, or the angle T C S, is called the latitude of such star: p S or q S is called its ecliptic polar distance. A plane passing through any point, parallel to the ecliptic E ♈ L, will cut the celestial sphere in the circumference of a circle which is called a parallel of celestial latitude. The arc ♈ R, or the angle ♈ C R on the plane of the equator, is designated the right-ascension of any star S, through which and the axis P Q the plane of a circle of declination is supposed to pass; and the arc R S or the angle R C S is called the declination of such star: P S or Q S is called the polar distance. A plane passing through any point, parallel to that of the equator A ♈ B, will cut the celestial sphere in the circumference of a small circle which is called a parallel of declination.

19. The ecliptic and the circular arcs perpendicular to it form one system of co-ordinates: the equator and its perpendicular arcs form another system; and the knowledge of the number of degrees in the arcs ♈ T and T S, ♈ R and R S, whether obtained by direct observation, or from astronomical tables, is sufficient to determine the place S of a star in the celestial sphere. It may be necessary to observe that the system of the equator and its perpendiculars is continually changing its position by the annual movement of the earth about the sun; but on account of the smallness of the orbit when compared with the magnitude of the celestial sphere, and the axis P Q being always parallel to itself (art. 8.), omitting certain deviations which will be hereafter mentioned, that change of position creates no sensible differences, except such as depend on the deviations alluded to, in the situations of the stars with respect to these co-ordinates.

20. A third system of co-ordinates is formed by a plane

supposed to pass through the centre of the earth parallel to the plane of the horizon of a spectator, as before mentioned, and by planes intersecting one another in the line drawn through that centre and the station of the observer, that is, perpendicularly to his horizon. The first plane, or its trace in the heavens, is called the rational horizon of the observer, and the circles in which the perpendicular planes cut the celestial sphere are called azimuthal or vertical circles: their circumferences evidently intersect each other in the zenith and nadir. An arc of the rational horizon intercepted between the meridian of a station and a vertical circle passing through a celestial body is called the azimuth, and an arc of a vertical circle between the horizon and the celestial body is called the altitude of that body: also the arc between the body and the zenith point is called the zenith distance; and a plane passing through any point, parallel to the horizon, will cut the celestial sphere in a small circle which is called a parallel of altitude. This last system of co-ordinates is that to which the places of celestial bodies are immediately referred by such observations as are made at sea; and it is also generally employed by scientific travellers who have occasion to make celestial observations on land.

21. An arc of the horizon intercepted between the east or west point and the place of any celestial body at the instant when, by the diurnal rotation, it comes to the circumference of the horizon, (that is, the instant of rising or setting,) is called the amplitude of that celestial body.

CHAP. II.

PROJECTIONS OF THE SPHERE.

NATURE OF THE DIFFERENT PROJECTIONS EMPLOYED IN PRACTICAL ASTRONOMY AND GEOGRAPHY. — PROPOSITIONS RELATING TO THE STEREOGRAPHICAL PROJECTION IN PARTICULAR. — EXAMPLES OF THE ORTHOGRAPHICAL, GNOMONICAL, GLOBULAR, AND CONICAL PROJECTIONS. — MERCATOR'S DEVELOPMENT.

22. THE trigonometrical operations which occur in the investigations of formulæ for the purposes of practical astronomy, require the aid of diagrams in order to facilitate the discovery of the steps by which the proposed ends may be most readily gained; and representations of the visible heavens or of the surface of the earth, with the circles by which the astronomical positions of celestial bodies or the geographical positions of places are determined, are particularly necessary for the purpose of exhibiting in one view the configurations of stars or the relative situations of terrestrial objects. The immediate objects of research in practical astronomy are usually the measures of the sides or angles of the triangles formed by circles which, on the surface of the celestial sphere, connect the apparent places of stars with each other, and with certain points considered as fixed: and as, for exhibiting such triangles, the formation of diagrams on the surface of a ball would be inconvenient, mathematicians have invented methods by which the surface of a sphere with the circles upon it can be represented on a plane, so that the remarkable points on the former may be in corresponding positions on the latter; and so that with proper scales, when approximative determinations will suffice for the purpose contemplated, the values of the arcs and angles may be easily ascertained.

23. These are called projections of the sphere, and they constitute particular cases of the general theory of projections. The forms assumed by the circles of the sphere on the plane of projection depend upon the position of the spectator's eye, and upon that of the plane; but of the different kinds of projection which may be employed for the purposes of astronomy and geography, it will be sufficient to notice only those which follow.

The first is that in which the eye is supposed to be upon the surface of the sphere, and the plane of projection to pass through the centre perpendicularly to the diameter at the extremity of which the eye is situated. This projection is described by Ptolemy in his tract entitled "The Planisphere," and its principles are supposed to have been known long before his time: it was subsequently called the *Stereographical* Projection, from a word signifying the representation of a solid body. A modification of this projection was made by La Hire, in supposing the eye of the spectator to be at a distance beyond the surface of the sphere equal to the sine of 45 degrees (the radius being considered as unity); and this method, which has been much used in the formation of geographical maps, is sometimes called the *Globular* projection. The second is that in which the eye is supposed to be infinitely distant from the sphere, and the plane of projection to be any where between them, perpendicular to the line drawn from the eye to the centre of the sphere: it is employed by Ptolemy in his tract entitled "The Analemma," and it has been since called the *Orthographical* Projection. It may be here observed, that when from any given point, or when from every point in a given line or surface, a straight line is imagined to be drawn perpendicularly to any plane, the point, line, or surface, supposed to be marked on the plane by the extremities of the perpendiculars, is said to be an orthographical projection of that point, line, or surface. In the third projection the eye is supposed to be at the centre of the sphere, and the plane of projection to be a tangent to its surface: this projection is called *Gnomonical,* from a correspondence of the projecting point, or place of the eye, to the summit of the gnomon or index of a sun-dial. It was not used by the ancients. A modification of this projection was proposed by Flamstead, but it has not been adopted; the method which is now distinguished by the name of that astronomer consists in placing the projecting point at the centre of the sphere, and projecting a zone of its surface on the concave surface of a hollow cone in contact with the sphere on the circumference of a parallel of latitude or declination, or on the concave surface of a hollow cylinder in contact with the sphere on the circumference of the equator.

The demonstrations of the principal properties relating to the different projections, and the rules for constructing the representations of circles of the sphere on a plane surface, constitute the subjects of this chapter.

DEFINITION.

24. The eye of the spectator being upon or beyond the surface of the sphere, the circle on whose plane, produced if necessary, that surface is represented, is called the *Primitive Circle;* its plane is called the plane of projection.

Cor. 1. The projecting point being in the direction of a diameter of the sphere perpendicular to the plane of projection, either extremity of the diameter is one of the poles of the primitive circle, and it is evident that the centre of the latter is the point in which the pole opposite to the projecting point is projected.

Cor. 2. Since every two great circles of the sphere intersect one another in a diameter of the sphere, it follows that the projection of any great circle intersects the circumference of the primitive circle in two points which are in the direction of a diameter of the latter; and that when any two great circles are projected, the line which joins their points of intersection will pass through the centre of the primitive circle.

Note. The chords, sines, tangents, and secants of the angles or arcs, which, in the projections of the sphere, are to be formed or measured, may be most conveniently taken from the scales on a sector; the arms of the instrument being opened, so that the distance between the chord of 60°, the sine of 90°, the tangent of 45°, or the secant of 0°, may be equal to the radius of the primitive circle.

PROPOSITION I.

25. In any projection, if the plane of a circle of the sphere pass through the projecting point or the eye of the spectator, the representation of its circumference on the plane of projection will be a straight line.

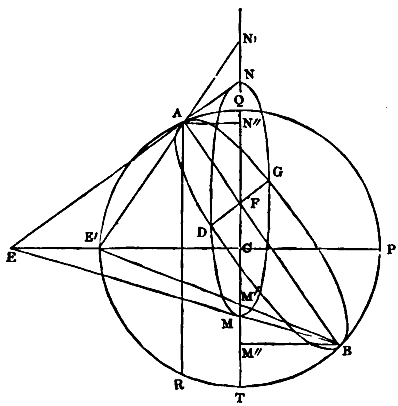
For imagine lines to be drawn from all points in that circumference to the projecting point, and to be produced in directions from thence, if necessary, they will be in the plane of the circle (Geom. 1. Planes); and the plane of projection is cut by the same lines, that is, by the plane in which they are. But the intersection of two planes is a straight line, therefore the representation of the circumference is a straight line. Q. E. D.

PROPOSITION II.

26. If from a point on the exterior of a sphere lines be drawn to the circumference of any circle of the sphere, whose plane does not pass through the point, and those lines, pro-

duced if necessary, cut a plane passing through the centre of the sphere perpendicular to the diameter in which produced is the given point; the figure projected on the plane will, except when the circle is parallel to the plane of projection, be an ellipse.

Let C be the centre, and let A G B D, supposed to be perpendicular to the plane of the paper, and having its centre in the line AFB, represent the circle of the sphere; also let the plane of projection passing through the line Q C T, perpendicularly to the plane of the paper, cut the plane of the circle in some line, as DFG: this line will be perpendicular to AB and to QT. (Geom., Planes, Prop. 19.) Let E, in the produced diameter PCE at right angles to QT, be the given point; and through that diameter imagine a great circle A P T, in the plane of the paper, to be described: this circle will be perpendicular to the planes of projection and of the circle A G B D.



Now, if lines be drawn from E to every point in the circumference A G B D, they will constitute the curve surface of a cone whose base A G B D is a circle making any angle with its axis, that is, with a line drawn from E to the centre of A G B D: let this surface be produced if necessary, and let it be cut by the plane of projection; the section N G M D of the oblique cone is the projection of the circle A G B D, and it is required to prove that it is an ellipse.

In the triangles AFN, MFB (Plane Trigonometry, Art. 57.),

FN : FA :: sin. FAN : sin. FNA,

and FM : FB :: sin. FBM : sin. FMB.; whence

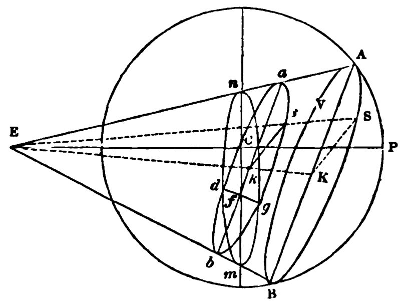
FN.FM : FA.FB :: sin. FAN sin. FBM : sin FNA sin. FMB.

Let the two last terms of the proportion be represented by p and q respectively; then

FN.FM : FA.FB :: p : q, and FA.FB $= \frac{q}{p}$ FN.FM:

but A G B D being a circle, FA.FB = FG^2 (Euc. 35. 3.); therefore $\frac{q}{p}$ FN.FM = FG^2. But again, with the same circle of the sphere, the four angles above mentioned are constant; therefore $\frac{q}{p}$ is constant, and the ratio of FN.FM to FG^2 is constant; which, by conic sections, is a property of an ellipse: consequently the figure NGMD is an ellipse. Q. E. D.

If the given circle of the sphere be in the position ASBV, so that it does not intersect the plane of projection, and is not parallel to it; the projected figure is an ellipse. Let a plane parallel to that of the circle cut, in *a s b d*, the oblique cone formed by lines drawn from E to the circumference of ASBV, so that it may intersect the plane of projection; this section of the cone will be a circle. For, K being the centre of the circle ASBV, let any plane



EKS passing through the axis EK cut the section *asbd* in *ks*; this line (Geom. Planes, Prop. 14.) will be parallel to KS, and the triangles EKS, E*ks* will be similar to one another: therefore EK : E*k* :: KS : *ks*.

But the ratio of EK to E*k* is constant, and KS is constant; therefore *ks* is constant, and *a s b d* is a circle. Now if *n g m d* represent the projection of ASBV, and *dfg* be the intersection of this projected figure with the plane of the circle *a s b d*, we shall have, reasoning as in the former case, a constant ratio between $nf \cdot fm$ and fg^2: consequently, the projected figure *n g m d* is an ellipse.

Cor. 1. When the projecting point is on the surface of the sphere, as at E′ in the preceding figure, the projection of any circle of the sphere, as AGBD not passing through E′, is a circle. For draw AR parallel to QT; then (Euc. 30. 3.) the arc E′A will be equal to the arc E′R, and consequently (Euc. 26. 3.) the angle E′AR to the angle ABE′; but E′AR = AN′T (Euc. 29. 1.); therefore FN′A = FBM′, and the opposite angles at F being equal to one another, the angle FAN′ = FM′B. It follows that, in the present case, the products corresponding to those represented by p and q above are equal to one another, and FN′. FM′ becomes equal to FG^2, which is a property of a circle (Euc. 35. 3.); therefore when E is on the surface of the sphere, &c.

Scholium. — If the plane of a circle of the sphere were parallel to the plane of projection, it is evident, whether E be on the surface of the sphere, or within it, or on its exterior, that lines drawn from thence to every part of the circumference would form the convex surface of an upright cone having a circular base; and the surface of this cone, produced if necessary, being cut by the plane of projection, the section, that is, the projected figure, will (Geom., Cor. 7. Def. Cyl.) be a circle. The observation will evidently hold good whether the plane of projection pass through the centre of the sphere or be elsewhere situated.

Cor. 2. When the point E is infinitely remote, so that all lines drawn to it from the circumference of any circle of the sphere may be considered as parallel to one another, those lines will constitute the convex surface of a cylinder whose base is a circle forming any angle whatever with the axis, that is, with a line drawn through the centre of the circle parallel to the lines before mentioned. In this situation of the eye the section of the cylinder made by the plane of projection, that is, the projected figure, will, except when the circle of the sphere is perpendicular to the axis, be an ellipse.

For, in the first figure to this proposition, if A G B D be the given circle of the sphere, the points A and B, the extremities of its diameter, will be projected respectively in the points N″, M″, where perpendiculars from A and B meet Q T: therefore, in the triangles A F N″, B F M″, the angles at N″ and M″ are right angles; and since the opposite angles at F are equal to one another, we have (Pl. Trigon., Art. 56.)

$$\text{AF} \cos. \text{F} = \text{FN}'', \text{ and } \text{FB} \cos. \text{F} = \text{FM}'';$$

consequently, $\text{AF}.\text{FB} = \dfrac{\text{FN}''.\text{FM}''}{\cos.^2 \text{F}}$.

But $\text{AF}.\text{FB} = \text{FG}^2$; therefore $\dfrac{\text{FN}''.\text{FM}''}{\cos.^2 \text{F},} = \text{FG}^2$,

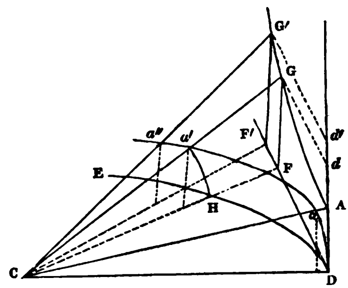
that is, $\text{FN}''.\text{FM}''$ has a constant ratio to FG^2, which is a property of an ellipse. Therefore the section N″ G M″ D is an ellipse.

27. *Scholium.* The orthographical projection of an ellipse, on any plane, is also an ellipse. For let it be supposed that the figure D A G B is an ellipse: then by conic sections, t being the semi-transverse, and c the semi-conjugate axis, $\dfrac{c^2}{t^2}\text{AF}.\text{FB} = \text{FG}^2$; or the rectangle A F. F B has a constant ratio to FG^2. Now from the second corollary we have $\text{AF}.\text{FB} = \dfrac{\text{FN}''.\text{FM}''}{\cos.^2 \text{F}}$; therefore $\dfrac{c^2}{t^2}\,\text{AF}.\text{FB}\,(= \text{FG}^2) = \dfrac{c^2}{t^2}\dfrac{\text{FN}''.\text{FM}''}{\cos.^2 \text{F}}$: hence the rectangle $\text{FN}''.\text{FM}''$ has still a constant ratio to FG^2, or the projected figure N″ G M″ D is an ellipse.

PROPOSITION III.

28. If a plane touch a sphere at any point, and from the centre of the sphere lines be drawn through the circumference of a small circle whose plane is perpendicular to the tangent plane; those lines if produced will meet the latter in points which will be in an hyperbola.

Let C be the centre of the sphere, and D the point at which the tangent plane DG is in contact with it: let aa'' be part of the circumference of a small circle whose plane is perpendicular to DG, and let DE be part of the circumference of a great circle parallel to the small circle; also let the plane of this great circle meet the tangent plane, to which it is perpendicular, in DF.



Again, let a plane pass through CD perpendicular to the planes of the great and small circle, it will also be perpendicular to the tangent plane; and in it let the point a be situated: then a line drawn from C through a will meet the tangent plane in some point as A. Let, also, a plane pass through C and any other point a' in the circumference of the small circle, perpendicularly to CDE; it will meet the tangent plane in the line FG which (Geom., 19. Planes) will be perpendicular to the plane CDE, and consequently (Geom. 6. Planes) parallel to DA. Draw a line from C through a' to meet FG in G; and from G in the tangent plane draw Gd parallel to FD; Gd will be equal to DF, and FG to Dd.

Let CD, the semidiameter of the sphere, be represented by unity; then (Euc. 47. 1.) $CF^2 = 1 + DF^2$, and (Pl. Trigon. Art. 56.) $FG = CF \tan. GCF$. But the angle $GCF = ACD$, the arcs Da and Ha' which measure those angles being equal to one another (Geom., 2. Cyl.), and therefore

$$\tan. GCF, \text{ or } \tan. ACD, = AD.$$

It follows that $FG = CF. AD$ and $FG^2 = CF^2. AD^2$, that is, $FG^2 = AD^2 + AD^2. DF^2$, or $Dd^2 = AD^2 + AD^2. Gd^2$, and

$$Dd^2 - AD^2 = AD^2. Gd^2.$$

In like manner, for any other point, as a'', in the same small circle we should have $Dd'^2 - AD^2 = AD^2. G'd'^2$; and consequently $Gd^2 : G'd'^2 :: Dd^2 - AD^2 : Dd'^2 - AD^2$.

This, by conic sections, is a property of an hyperbola of which AD is one of the semi-axes; therefore, &c.

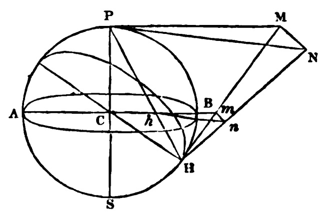
If C be the projecting point, or the eye of the spectator, it is manifest that the representation of a portion of a small circle whose plane is perpendicular to the tangent plane, or plane of projection, will be an hyperbola consisting of two similar branches, one on each side of the plane CDA, which passes through the centre C and the point D of contact perpendicularly to the plane of the circle.

Note. The fifteen following propositions relate to the stereographical projection only.

PROPOSITION IV.

29. The angle contained between the tangents to two circles of the sphere, which intersect each other, when drawn from one of the points of section, is equal to the angle contained between the projections of those tangents.

Let PS and AB be two diameters of the sphere at right angles to one another, and let the plane of projection pass through AB perpendicularly to that of the paper: then P may be the projecting point.



Let HM, HN be tangents to two circles of the sphere which intersect each other at H, and let them be produced till, in M and N, they meet a plane touching the sphere at P, which plane is consequently parallel to the plane of projection. Draw PM, PN, and PH, and imagine planes to pass through PM, HM, and PN, HN: these will intersect each other in PH, and the plane of projection in hm and hn; therefore hm and hn, produced if necessary, are the projections of the tangents HM, HN; and it is required to prove that the angle mhn is equal to MHN.

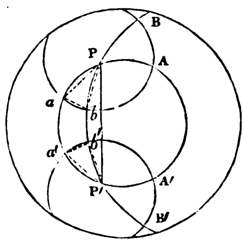
Since NP, NH are tangents drawn to the sphere, they are tangents drawn from the point N to the circle formed by the intersection of the plane PNH with the sphere; therefore (Euc., A. Cor. 36. 3.) they are equal to one another: for the like reason MP, MH are equal to one another. Therefore the triangle MPN is equal to the triangle MHN, and the angle MPN to the angle MHN. But the triangle mhn is a section parallel to MPN, in the pyramid HMPN: therefore mhn is similar to MPN (Geom., 1. Prisms, &c.), and the angle mhn is equal to MPN, that is, to MHN.

Since the angle made by two circles which intersect one another on the surface of the sphere, or on a plane, is expressed by the angle contained between the tangents at the point of intersection (Spher. Geom., 2. Cor. 2. Def.), it may be said that the angle contained between the planes of two circles, which intersect one another on a sphere, is equal to the angle contained between the projections of those circles.

PROPOSITION V.

30. If a circle of the sphere be described about each of the intersections of two circles of the sphere as a pole, at equal distances from the intersections; the arcs intercepted upon the circles so described will be equal to one another.

Let the straight line PP′ be the intersection of any two circles of the sphere, and let abAB, $a'b'$A′B′ be parts of two circles of the sphere described about P and P′ as poles at equal distances from each; the intercepted arcs ab, $a'b'$ will be equal to one another.



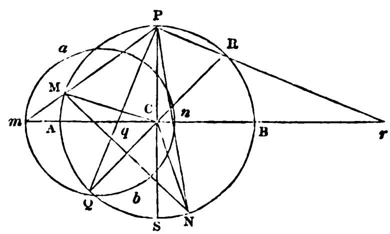
Imagine arcs of great circles to pass through the points P and a, P and b, P′ and a', P′ and b'; all those arcs will (Geom., 2. Cylind.) be equal to one another, and consequently (Euc., 29. 3.) all their chords, viz. the straight lines Pa, Pb, P′a', P′b', will be equal to one another: also (Euc., 28. 3.) the arcs Pa, Pb, P′a', P′b', in the diagram, will be equal to one another. Then, since the arc Pb is equal to the arc P′b', the arcs P′b and Pb' are equal to one another, and therefore the angle P′Pb is equal to PP′b' (Euc., 27. 3.). In like manner the angle P′Pa is equal to PP′a': therefore the solid angles at P and P′ have two plane angles of the one equal to two plane angles of the other, and the inclinations of the plane angles to one another are equal, each of them being the inclination of the circles PaP′, PbP′, to one another; consequently the third plane angles aPb and a'P′b' are equal to one another (Geom., 23. Planes). It follows that in the two triangles aPb, a'P′b', the chord ab is equal to the chord $a'b'$ (Euc., 4. 1.); and the circles described about P and P′ being equal, the arcs ab and $a'b'$ are equal to one another.

Scholium. It is obvious that the arcs AB and A′B′ are equal to the arcs ab and $a'b'$ and to one another. It is obvious also that the demonstration would be the same if the circle abA described about P should inclose the point P′: the equal circle about P′ consequently inclosing P. Thus, if P and P′ were, respectively, a pole of the primitive and of any other great circle of the sphere, the arcs intercepted on the primitive and on that great circle by the circles APA′, BPB′, would be equal to one another.

PROPOSITION VI.

31. The projected poles of any circle of the sphere, and the centre of the projection of the same circle, are in the representation of a great circle passing through the poles of the primitive and of the given circle.

Let APBS be a circle of the sphere perpendicular to the primitive and to the given circle; and let AB and MN be the diameters of these circles in the plane of the paper: then P and S, Q and R, the extremities of the two diameters of the sphere which are perpendicular to the planes of the circles, will be the poles of those circles, and P may be the projecting point.



On drawing the lines PQ, PR, PM, PN, producing any of them if necessary, they will intersect AB, or AB produced, in the points q, r, m, and n; then q and r will be the projected poles of the given circle whose diameter is MN, and the line mn will be the projection of that diameter. Now, since the projection of any circle of the sphere, which does not pass through the projecting point, is (1 Cor., Prop. 2.) a circle, the projection of the semicircle, perpendicular to the paper, on each side of MN will be a semicircle on the corresponding side of mn, also perpendicular to the paper; and the centre of the projected circle will be at the middle of the line mn. Thus the projected poles and the centre of the projected circle are in the diameter AB (produced if necessary) which is the projection of the circle PASB passing through the poles of the primitive and of the given circle.

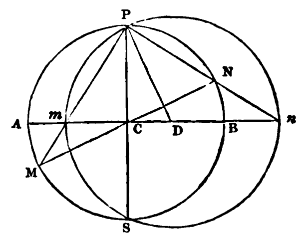
Scholium. If the primitive circle be supposed to revolve about AB till it lie in the plane of the paper, it will coincide with the circle APBS, and the projected circle with $ambn$. The projecting point is, in this case, at a distance from C equal to CP, measured perpendicularly to the plane of the paper.

PROPOSITION VII.

32. The radius of the circle constituting the projection of a great circle of the sphere which does not pass through

the projecting point, is equal to the secant of the inclination of the great circle to the plane of projection.

Let the plane of projection and that of the given circle be perpendicular to the plane of the paper, and let APBS be the great circle of the sphere which is perpendicular to the two first planes; also let the intersections of these planes with APBS be AB, produced either way, and MN: then AB and MN will be diameters of the primitive and of the given circle, and the angle ACM will be the inclination of the latter to the primitive; also the projecting point will be at P or at S, at one extremity of the diameter PS which is at right angles to A B. Draw PM intersecting AB in m, and through N draw PN, which being produced will meet AB produced, in n: then mn will be the projection of the diameter MN. Bisect mn in D; then D will be the centre of the projected circle.



Join P, D; then, since MN is a diameter of APBS, the angle MPN or mPn is a right angle, and PD is equal to Dm or Dn; also each of these lines is equal to a semidiameter of the projected circle. Now, since Dm is equal to DP, the angle DPm is equal to DmP: and since CM is equal to CP, the angle CPM is equal to CMP: but (Euc., 32. 1.) DmP = MCm + mMC; therefore,

$$\text{DP}m = \text{MC}m + m\text{MC, or DP}m = \text{MC}m + \text{CPM}.$$

Again, DPm = DPC + CPM; therefore DPC = MCm, that is, the angle DPC is equal to the inclination of the given great circle to the plane of projection. Now PD is the secant of CPD, that is, of the inclination; the radius CP being supposed to be unity. Therefore, &c.

Scholium. The line CD is evidently equal to the tangent of the inclination.

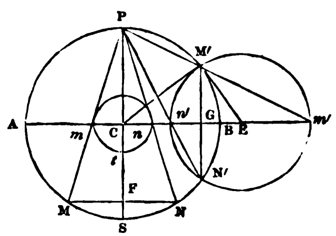
PROPOSITION VIII.

33. The distances of the extremities of the diameter of any projected circle of the sphere from the centre of the primitive circle are equal to the tangents of half the arcs which measure the least and greatest distances of the circle of the sphere from that pole of the primitive which is opposite the projecting point.

Let the plane of the paper be perpendicular to that of the primitive circle and of the given circle of the sphere, and (Fig. to Prop. VI.) let AB and MN be the diameters of these circles; then P, at the extremity of the diameter PS perpendicular to AB, will be the projecting point. Draw the lines PM and PN, and produce either of them if necessary; then the line *mn* on AB will be the diameter of the projected circle. Now SN measures the angle SCN of which the angle CP*n* is equal to the half (Euc., 20. 3.), and SM measures the angle SCM of which CP*m* is equal to the half: But C*n* and C*m*, are respectively the tangents of CP*n* and CP*m*; the radius of the primitive circle being supposed to be unity. Therefore, &c.

Cor. 1. If the circle of the sphere be parallel to the plane of the primitive, the centre of the projected circle will be at C and the radius will be equal to the tangent of half the distance of the circle from that pole of the primitive which is opposite the projecting point.

For, the plane of the paper being perpendicular to the plane of projection and of the given circle as before, let AB be a diameter of the primitive, and MN, parallel to AB, that of the given circle: then, P being the projecting point, the diameter PS is perpendicular to AB and to MN, and it bisects the latter in F (Euc., 3. 3.); therefore FM or FN is a radius of the given circle, S is its nearest pole, and is also one of the poles of the primitive.



On drawing the lines PM and PN cutting the primitive, produced if necessary, *mn* will be the diameter of the projected circle; and since MN is bisected in F, *mn* is bisected in C (Euc., 4. 6.): therefore C*m* or C*n* is a radius of the projected circle, and it is manifestly the tangent of the angle CP*n*, which is measured by the half of SN, the distance of the circle from the pole opposite the projecting point.

If the plane of the primitive circle be supposed to revolve upon AB till it coincide with that of the paper, the projected circle will also be in the plane of the paper and will be represented by *mtn*.

Cor. 2. If the circle of the sphere be perpendicular to the plane of projection, the distance of the centre of its projection from the centre of the primitive circle will be equal to the secant, and the radius of the projected circle, to the tangent of the circle's distance from its nearest pole.

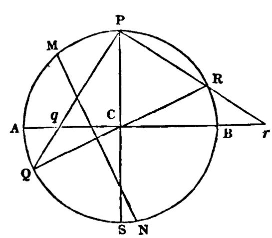
The plane of the paper being perpendicular to the plane of projection and to the given circle as before, let AB be a diameter of the primitive, and M′N′, perpendicular to AB, be a diameter of the given circle of the sphere: then M′N′ will be bisected in G and the arc M′BN′ in B; and B will be the nearest pole of the circle. On drawing the lines PN′, PM′, and producing the latter to meet AB in m', the former line intersecting it in n', the line $m'n'$ will be the diameter of the required projected circle; and, bisecting it in E, E will be its centre. Then, if the plane of the primitive circle be supposed to revolve upon AB till it lie in the plane of the paper, it will coincide with APBS, and the projected circle will coincide with n'M′m'N′; for m' and n' are in the axis of revolution, and the circumference will pass through M′ and N′, because, during the revolution, its two intersections with the surface of the sphere continue at a distance from G equal to a radius of the given circle, that is, equal to GM′ or GN′.

Now the angle PCM′ is equal to twice PN′M′ (Euc., 20. 3.), and for a like reason, M′En' is equal to twice M′N′n', that is, to twice PN′M′; therefore PCM′ = M′En' or M′EG. But PS being parallel to M′N′ the angle PCM′ = CM′G; therefore M′EG = CM′G. But again, each of the two triangles M′GE, M′GC having a right angle at G and the angle M′EG being equal to CM′G, those triangles are similar to one another and to the whole triangle CM′E (Euc., 8. 6.). Therefore the angle CM′E is a right angle; and consequently CE is the secant and M′E the tangent of the angle M′CB or of the arc BM′. That is, the distance of the centre, &c.

PROPOSITION IX.

34. The distances of the projected poles of any circle of the sphere from the centre of the primitive circle are equal to the tangent and cotangent of half the inclination of the plane of the circle to the primitive.

Let the plane of the paper be perpendicular to the plane of projection and to the circle of the sphere; and let AB and MN be diameters of the primitive and of the latter circle: through C draw PS perpendicular to AB, and QR perpendicular to MN; then P and S will be the poles of the primitive, and Q and R the poles of the circle whose diameter is MN. Join P,Q and P,R, and produce PR till it meets AB produced; then the intersections q and r will be the projected poles of the circle on

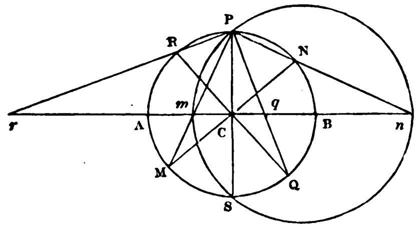


MN. Now the arc QS, the distance between the nearest poles, is equal (Sph. Geom., 3. Cor. 2. Def.) to the inclination of the circle of the sphere to the primitive: but QS measures the angle QCS, of which CPq is equal to half (Euc., 20. 3.); and CP being supposed to be unity, Cq is the tangent of CPq, that is, of half the inclination. Again, since QR is a diameter of the circle APBS, the angle QPR or qPr is a right angle, and the angle CPr is the complement of CPq: but Cr is the tangent of CPr, that is, of the complement of half the inclination. Therefore the distances of the projected poles q and r from the centre C of the primitive are equal to, &c.

PROPOSITION X.

35. To find the poles of a given projected great circle, and the converse.

Let APBS be the primitive circle, and PmSn the given projection of a great circle: then, since every projected great circle cuts the primitive at the extremities of a diameter of the latter, draw the line PS, which will pass through C the centre of the primitive; and draw AB through C at right angles to PS, producing it if necessary. Through m and n,



the points of intersection, draw PmM, PNn; then, because the arc mPn is a semicircumference, the angle mPn or MPN is a right angle, and MN is a diameter of the primitive; it is also a diameter of the circle of the sphere, of which mPnS is a projection. Again, drawing RCQ at right angles to MN, R and Q are the poles of the circle on MN; then joining P, Q and P, R and producing PR; the intersections q and r with AB or with AB produced will be the projected poles of the circle of which mPnS is the projection.

If the projected great circle be a diameter of the primitive, as PS, its poles are at the extremities of a diameter as AB, at right angles to PS.

In practice, having drawn the diameters PS and AB, join P, m, and produce Pm to M: make MQ and MR each equal to a quadrant, by a scale of chords or otherwise, and join PQ and PR; the intersections of PQ and PR produced, with AB, will give the points q and r which are the projected poles.

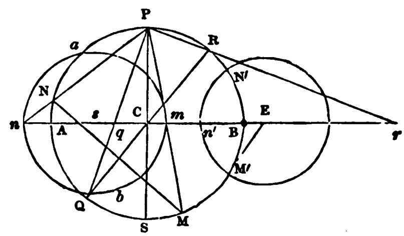
If about a given projected pole, as q, it were required to describe the projection of a great circle, a converse process must be used. Having drawn through q a diameter AB to the primitive circle, and produced it, draw the diameter PS at right angles to it; then join P, Q and make QM, QN each equal to a quadrantal arc. Join P, M and P, N, and produce PN; the intersections m and n will determine the diameter $m\,n$, and mPnS will be the projected circle.

When the point q is in the circumference of the primitive circle, the projection of the great circle is a diameter of the primitive at right angles to the line joining q and the centre.

PROPOSITION XI.

36. To find the poles of a given projected small circle, and the converse.

Let APBS be the primitive circle, and $namb$ the given projected small circle. Find s the centre of this circle, and through it draw AB, a diameter of the primitive circle; also through C, the centre of the primitive circle, draw PS at right angles to AB. Draw Pn intersecting the primitive in N; also draw Pm and produce the latter till it cuts the primitive in M. Then the line NM will be the diameter of the circle of which $namb$ is the projection; bisect the arc MN in Q, and draw QR through the centre of the primitive; then Q and R will be poles of the given circle. Draw PQ, PR, and produce the latter; then the intersections q and r will be the projected poles of the given circle.



If about a given pole, as q, it were required to describe the projection of a small circle at a distance expressed by a given arc or a given number of degrees, a converse operation must be performed.

Through q and C, the centre of the primitive circle, draw the diameter AB; and draw PS at right angles to it: from P draw the line PQ through q, and make each of the arcs QM and QN, by a scale of chords, equal to the given distance: draw PM and PN, and produce either of those lines if necessary; the intersections m and n will determine the diameter, and the circle $namb$ will be the required projection.

If the given pole were at C, the centre of the primitive, a circle described about C as a centre with a radius equal to the tangent of half the given distance of the circle from its pole would (1 Cor. art. 33.) be the required projected circle.

If the given pole were on the circumference of the primitive circle, as at B; on a diameter of the primitive, passing through B and produced, make CE equal to the secant of the given distance of the circle from its pole; then (2 Cor. art. 33.) E will be the centre, and the tangent of the given distance will be the radius of the required projection.

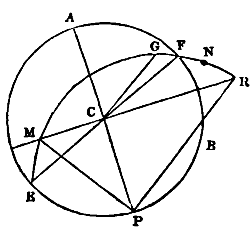
Or, by a scale of chords make the arc BM′ equal to the given distance, and from M′ draw a tangent to the primitive circle, meeting the radius CB produced in E; then E will be the centre, and EM′ the radius of the required projection.

Or again, having made BM′ and BN′ each equal to the given distance, make Cn' equal to the tangent of half the arc SM′, that is, of half the complement of the given distance: then a circle described through M′, n', and N′, will be that which is required.

PROPOSITION XII.

37. To describe the projection of a great circle of the sphere through two given points on the plane of projection.

Let ABP be the primitive circle, C its centre, and M and N the two given points. Through either of the points, as M, draw a line MR through C, and draw the diameter AP at right angles to MC. Join M, P and draw PR at right angles to MP meeting MC produced in R: then through M, N, R describing a circle; it will be the projection required.

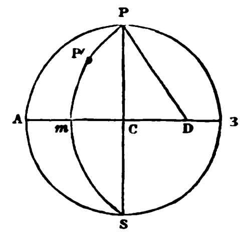


Let the circle intersect the primitive in E and F, and draw CE, CF. Since the angle MPR is a right angle, and CP is perpendicular to MR, the rectangle MC.CR is (Euc., 35. 3.) equal to CP^2; and since CE, CF are each equal to CP, MC.CR = CE.CF. Now if EC and CF be not in one straight line, let ECG be a straight line; then (Euc., 35. 3.) MC.CR = EC.CG; therefore EC.CG = EC.CF, and CG would be equal to CF, which is absurd. Therefore ECF is a straight line, and consequently EMNR is the projection of a great circle of the sphere.

PROPOSITION XIII.

38. Through a given point in the circumference of the primitive circle, to describe the projection of a great circle of the sphere making with the plane of projection a given angle.

Let APBS be the primitive circle in the plane of projection, C its centre, and P the given point. Through P draw the diameter PS, and at right angles to it the diameter AB: then make the angle CPD equal to the given angle. The point D is the centre, and the line DP the radius of the projection (art. 32.): thus the required circle P m S may be described.

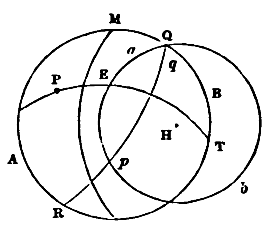


It is obvious that if the given point were, as at P′, not in the circumference of the primitive circle, the centre of the required projection might be found thus: describe an arc of a circle with P′ as a centre, and with a radius equal to the secant of the angle which the circle makes with the plane of the primitive; and also an arc from C the centre of the primitive, with a radius equal to the tangent of the angle (the radius of the primitive being supposed to be unity): the intersection of these arcs would be the centre of the required projected circle, and the secant of the angle would be its radius.

Proposition XIV.

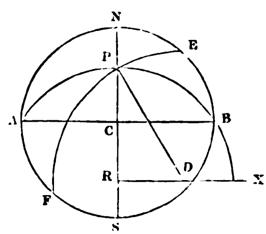
39. Through any given point to describe the projection of a great circle of the sphere, making a given angle with the projection of a given great circle.

Let MAB be the primitive circle, and MEN the projection of the given great circle, also let P be the given point. Find (art. 35.) H, a pole of MEN, and about H as a pole describe a small circle $a\,b$, (art. 36.) at a distance from H equal to the arc, or number of degrees by which the given angle is expressed; or at a distance equal to the supplement of that arc, if the latter were greater than a quadrant. About P as a pole, describe the projection QR of a great circle (art. 35.); and if the data be such that the construction is possible, this circle will touch or cut the small circle $a\,b$: let it cut the circle in p and q. Then, either from p or q (suppose from p) as a pole describe the projection PT of a great circle, it will pass through P and be the projection which is required.



For let it intersect the circle M E N in E: then, since p is a projected pole of P E T, and H is a projected pole of M E N, an arc of a great circle drawn from p to H would measure the distance between the poles of the circles M E N and P E T; therefore it would measure the angle M E P, or that at which the circles are inclined to one another. But the circle $a\,b$ was described about H at a distance equal to the measure of that angle; therefore either the angle M E P or the angle M E T is equal to the given angle, and P E T is the required projection.

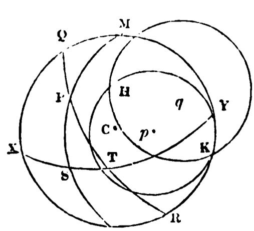
If through a given point, as P, it were required to describe the projection of a great circle, making with the projection N P S of a great circle perpendicular to the plane of projection any given angle, the construction might be very conveniently effected in the following manner. Through P describe the projection of a great circle at right angles to N S; this will pass through A and B, the poles of N S, at the extremities of the diameter A C B at right angles to N S, and let R be its centre. Then, considering A P B as a new primitive circle, through R draw the line R X at right angles to N S; and draw P D making the angle R P D equal to the complement of the given angle, that is, equal to the angle which the circle whose projection is required makes with the plane of the circle whose projection is A P B. The point D is the centre, and D P the radius (art. 32.) of the circle E P F, the required projection.



PROPOSITION XV.

40. To describe the projection of a great circle making, with two projected great circles, angles which are given.

Let M Q N be the primitive circle, C its centre, and M P N, Q P R, the projections of two given great circles. Find (art. 35.) p a projected pole of M P N, and q a projected pole of Q P R: about p as a pole (art. 36.) describe the projection of a circle at a distance equal to the angle which the required projection is to make with the circle M P N, and about q as a pole, a circle at a distance equal to the angle which the



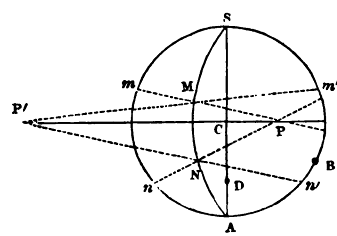
required projection is to make with Q P R. Then, if the data be such that the construction of the problem is possible, the circles about p and q will either touch in some point, or cut each other in two points; let them cut each other in H and K, and about either H or K (suppose H) as a pole describe (art. 35.) the projection X Y of a great circle. This will be the circle required, and the angles at S and T will be equal to the given angles.

For, since p is a pole of M P N and H a pole of X S Y, the arc of a great circle which measures the distance between p and H will measure the angle P S X or P S T, between the circles M P N and X S Y: and for a like reason the arc of a great circle between q and H will measure the angle P T S or P T Y. But the distances between p and H, q and H are by construction equal to arcs which measure those angles; therefore the angles at S and T are equal to those which were given.

PROPOSITION XVI.

41. To measure an arc of a projected great circle.

If the given arc, as A B, be anywhere on the circumference of the primitive circle, it may be measured by a scale of chords on which the chord of 60° is equal to the radius of the primitive. And if the given arc, as C D, be on the projection of a circle at right angles to the plane of projection, and one extremity of the arc be at the centre or pole of the primitive, it may be measured on a scale of tangents; the number of degrees found on the scale being doubled, because C D (art. 33. 1 Cor.) is equal to the tangent of half the arc which it represents.

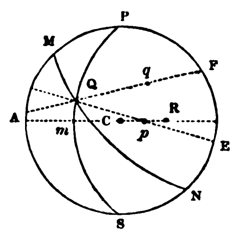


If the arc, as M N, be on the circumference of a projected circle which does not coincide with the primitive, it may be measured in the following manner. Find (art. 35.) P a projected pole of S M N A, and draw the lines P M m and P N n, cutting the primitive circle in m and n; the arc $m\ n$, measured by a scale of chords, is the value of the arc of which M N is the projection. For P M m and P N n being straight lines, are the projections of two circles which pass through the projecting point, or the pole of the primitive; and since they pass through P, a projected pole of S M N A, the circles which they represent must pass through a pole of the circle of which S M N A is the projection. Therefore (art. 30. Schol.) the arcs on the sphere, which are represented by M N and $m\,n$, are equal to one another. If the lines P′ M m', P′ N n' had been drawn from the exterior pole P′, the arc $m'\,n'$ would have been that which is equal to M N on the sphere.

PROPOSITION XVII.

42. To measure the angle contained by the projections of two great circles of the sphere.

If the given angle be at the centre of the primitive circle, it may be measured on that circle by the chord of the arc by which it is subtended. If it be at the circumference of the primitive and be contained between that circle and the projection P m S of a great circle which is inclined to it; find the centre R, or the pole p of the circle P m S; then (art. 32. Schol.) C R measured on a scale of tangents, or (art. 34.) C p measured on a scale of tangents and the number of degrees doubled, will give the number of degrees in the angle of which A P m is the projection: or again, C m measured on a scale of tangents, and the number of degrees doubled, will give the complement of that angle.



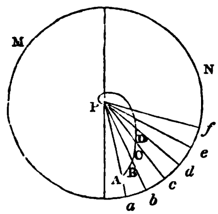
If the angle be contained between the projections, as P Q S, M Q N of two great circles, the angular point not being at the centre or circumference of the primitive; find the projected pole p of P Q S, and the projected pole q of M Q N, and draw the lines Q p E, Q q F, meeting the primitive circle in E and F; then E F measured by a scale of chords will give the value of the angle of which M Q P is the projection. For p and q being the projected poles of P Q S and M Q N, an arc of a great circle joining them will measure the angle P Q M between the circles; and arcs of projected great circles joining Q and p, Q and q, being the projections of quadrants, Q is the projected pole of a great circle passing through p and q. But, by the Scholium in art. 30., and as in the last proposition, E F measures the arc of which one joining p and q is the projection; therefore it measures the angle P Q M.

PROPOSITION XVIII.

43. A line traced on the surface of a sphere representing the earth so as to make constantly a right angle with the meridian circles is either the circumference of the equator or

of some parallel of latitude; but a line traced so as to make any constant acute angle with the meridian circles will be a spiral of double curvature: and if such a curve be represented on a stereographical projection of a hemisphere of the earth, the equator being the primitive circle, the projected curve will be that which is called a logarithmic spiral.

For let MN be the plane of the equator, and P its pole; and let P*a*, P*b*, P*c*, &c. be the projections of terrestrial meridians making equal angles *a*P*b*, *b*P*c*, &c. with one another; those angles being, by supposition, infinitely small. Let ABC &c. represent the projection of the curve line which on the sphere makes equal angles with the meridians: then the arcs AB, BC, &c., being infinitely small, may be considered as portions of great circles of the sphere; and consequently, by the principles of the stereographical projection, the angles PAB, PBC, &c. on the paper are also equal to one another. But all the angles at P are by construction equal to one another; therefore all the triangles, considered now as rectilineal, will be similar to one another. Hence

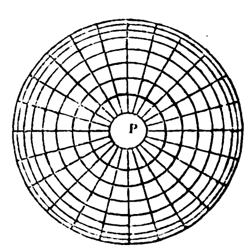


AP : PB :: PB : PC, PB : PC :: PC : PD, &c.

Thus all the radii PA, PB, PC, &c. are in geometrical progression, while the angles APB, APC, APD, &c., or the arcs *ab*, *ac*, *ad*, &c. are in arithmetical progression: hence, by the nature of logarithms, the arcs *ab*, *ac*, *ad*, &c. may be considered as logarithms of the radii PB, PC, PD, &c., and the curve line ABCD, &c. may be considered as a logarithmic spiral.

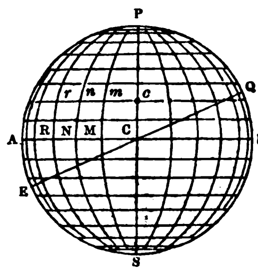
Scholium. The curve on the sphere, or in the projection, is usually called a loxodromic line, and on the earth it is that which would be traced by a ship if the latter continued to sail on the same course, provided that course were neither due east and west, nor due north and south.

44. If a hemisphere be projected orthographically on the plane of the equator, that circle will be the primitive, and its pole will be the centre of the projection; the terrestrial meridians, or the declination circles, the planes of which pass through the projecting point, will be straight lines diverging, as in the annexed diagram,



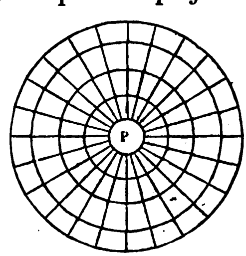
from the centre or pole P, and the parallels of terrestrial latitude, or of declination, the planes of which are parallel to the plane of projection, will be the circumferences of circles whose radii, or distances from the centre, will evidently be equal to the sines of their distances on the sphere, in latitude or declination, from the pole.

45. If a hemisphere be projected orthographically upon the plane of a terrestrial meridian, or of a circle of declination, such circle will be the primitive; the projecting point will be upon the produced plane of the equator in a line passing through the centre of the projection perpendicularly to its plane. The equator A B, and the declination circle P S, which pass through that centre, as well as the parallels of latitude or of declination, being all perpendicular to the plane of projection, will be straight lines; and if, as in the figure, the primitive circle represent the solsticial colure, the ecliptic will also be a straight line, as E Q, crossing the equator in C, the centre of the projection, which will then represent one of the equinoctial points. The straight line, P C S, will be the equinoctial colure; and, except this and the primitive circle, the meridians or declination circles, being the projections of circles inclined to the primitive, will (2 Cor. art. 26.) be ellipses.



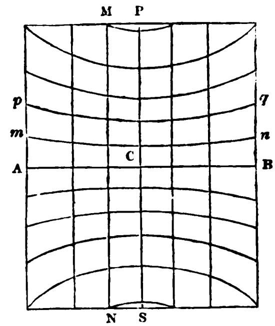
The distances of the parallel circles from the equator will be evidently equal to the sines of their latitudes or declinations; and the distances C M, C N, &c., *cm*, *cn*, &c., of the declination ellipses measured on the equator from its centre, or on each parallel of latitude or declination from its middle point, in the line P S, will be the sines of their longitudes or right ascensions; the radius of the equator and of each parallel being considered as unity.

46. When a portion of the surface of a sphere is projected gnomonically, and the plane of projection is a tangent to the sphere at one of the poles of the equator, the terrestrial meridians, or the circles of declination, and it may be added, every great circle of the sphere, since all pass through the projecting point, are represented by straight lines; the declination circles intersecting each



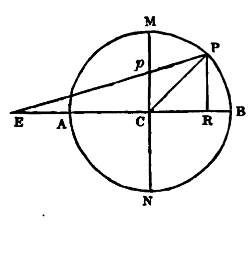
other, as at P, in the centre of the projection. The parallels of latitude or declination being the bases of upright cones of which the projecting point is the common vertex, are represented by circles having P for their common centre; and their radii are evidently equal to the tangents of their distances on the sphere from the pole of the equator.

47. When the plane of projection is in contact with the sphere at some point, as C, on the equator, the terrestrial meridians or circles of declination are represented by straight lines, as P S, M N, &c., since their planes pass through the projecting point; their distances from the centre, C, of the projection being equal to the tangents of the longitudes, or right ascensions, reckoned from that point on A B, which represents a portion of the equator. The parallels of declination, *mn*, *pq*, &c. are (art. 28.) hyperbolic curves; and their distances from the equator, measured on a declination circle, P C S, passing through the centre of the projection, are equal to the tangents of their latitudes, or declinations, on the sphere.



48. By placing the projecting point at a distance from the surface of the sphere equal to the sine of 45°, the radius of the primitive circle being unity, La Hire has diminished the distortion to which the surface of the sphere is subject in the stereographical projection, and on that account it is more convenient than the latter for merely geographical purposes. The most important circumstance in the globular projection, as it is called, is, that on a great circle whose plane passes through the projecting point, an arc equal in extent to 45 degrees, or half a quadrant, when measured from the pole opposite the projecting point, is represented by half the radius of the primitive circle.

In proof of this theorem, let the plane of projection pass through the diameter M N perpendicularly to the plane of the paper: let E be the projecting point in the direction of the diameter B A, perpendicular to M N, and at a distance from A, equal to the sine of 45° ($= \surd \frac{1}{2}$), the semidiameter of the sphere being unity; and let B P be half the quad-



rantal arc MPB. Draw the line EP cutting MC in p; then Cp will be equal to the half of CM, or Cp will be equal to $\frac{1}{2}$. Let fall PR perpendicularly on AB; then the angle PCB, being 45 degrees, PR and CR (the sine and cosine of 45°) are each equal to $\sqrt{\frac{1}{2}}$, and ER $= 2\sqrt{\frac{1}{2}} + 1$; also EC $= \sqrt{\frac{1}{2}} + 1$. Now the triangles ERP and ECp being similar to one another,

$$\text{ER} : \text{EC} :: \text{RP} : \text{C}p, \text{ that is}$$

$$2\sqrt{\tfrac{1}{2}} + 1 : \sqrt{\tfrac{1}{2}} + 1 :: \sqrt{\tfrac{1}{2}} : \text{C}p; \text{ therefore}$$

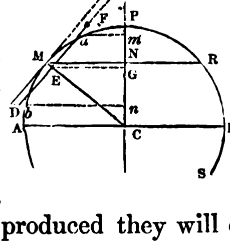
$$\text{C}p = \frac{\sqrt{\frac{1}{2}} + 1}{2\sqrt{\frac{1}{2}} + 1}\sqrt{\tfrac{1}{2}}.$$

Multiplying both the numerator and denominator of the fraction by $2\sqrt{\frac{1}{2}} - 1$, the value of Cp becomes $\frac{1}{2}$, or Cp is equal to half the radius of the primitive circle.

If lines were drawn from E to any other point in the arc MB so as to intersect CM, the distances of the intersections from C would be nearly, but not exactly, proportional to the corresponding arcs on MB; but, for ordinary purposes in geography, it is usual to consider them as such. Therefore, in representing a hemisphere of the earth on the plane of a meridian, the projections of the oblique meridians and of the parallels of latitude, which are respectively at equal distances from one another on the sphere, are usually made at equal distances from one another in the representation. The oblique meridians, and the parallels of latitude which, in the projection, should be portions of ellipses, are usually represented by portions of circles from which, in maps on a small scale, they do not sensibly differ.

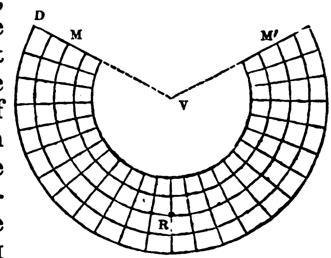
49. The conical projection is used only in representing a zone of the sphere; the concave surface of the cone being supposed to be in contact with its surface on the circumference of a parallel of latitude, and the geographical points to be projected upon it by lines drawn from the centre.

Thus let the segment APS represent part of any meridian of the sphere, P the pole of the equator, the latter passing through the diameter AB perpendicularly to the paper; and let MN be the radius of the parallel of latitude which is at the middle of the zone to be projected. Let VM produced be a tangent to the circle APBS at the point M, and let it be the side of the cone which touches the sphere on the circumference of the parallel circle MR. Then, the projecting point being at C, if the planes of the meridians be produced they will cut



the surface of the cone in straight lines converging to V; and if lines be drawn from C through the circumferences of the parallels of latitude on the sphere, to meet the surface of the cone, the projected parallels will be circles.

Now if the surface of the cone be cut in the direction of a line from V, as VMD, and laid on a plane surface, it will take the form of a sector of a circle, as MRM′, the meridians on the conical surface will still be straight lines diverging from V, and the projected parallels will be arcs of circles having V for their common centre: their distances from M are equal to the tangents of the corresponding arcs on the sphere, the tangents being measured from M both towards, and from V, and the radius of the sphere being considered as unity.



In order to find the angle subtended at V by MRM′ on the developed conical surface, let l represent the length of a degree on a circle whose radius is unity; then $360.l.\text{MN}$, in the preceding figure, will be equal to the circumference of the parallel of latitude MR on the sphere; and if d denote the number of degrees in the angle subtended by MRM′ in the development, $\text{MV}.l.d$ will express the length of the arc MRM′. But the circumference of the parallel and the arc of development were coincident when the cone encompassed the zone of the sphere: therefore, $360.l.\text{MN} = \text{MV}.l.d$,

or $$360.\text{MN} = \text{MV}.d;\ \text{whence } d = \frac{\text{MN}}{\text{MV}}\,360°.$$

But the triangle VMC in the preceding figure, being right angled at M, and MN being perpendicular to PC,

MV : MN :: CM : CN (Eucl., Cor. 8. 6.), that is
MV : MN :: radius : sine of the latitude of M:

consequently $d = 360° \times$ sine of the latitude of the parallel of contact, or of the middle of the zone to be developed.

Thus a map of a zone of the earth may be readily constructed. Such a map, however, though more correct than one formed by any of the projections before described, will become sensibly erroneous if it extend more than 15 or 20 degrees in latitude.

50. If the concave surface of a cylinder be in contact with a sphere on the circumference of the equator, and, the eye being at the centre, if the planes of the meridians be produced

to meet the cylinder, also if the circumferences of the parallels of latitude be projected as before upon the cylinder; then, the surface of the cylinder being extended on a plane, the projected meridians and parallels will be straight lines at right angles to one another. A map so formed will be nearly the same as that which is called a plane chart; and if it extend but a few degrees on each side of the equator, it will constitute, like the conical development above mentioned, a nearly accurate representation of the earth's surface.

51. Instead of making the convex surface of the cone a tangent to the sphere, it has been proposed to make a frustum of a cone cut the sphere on the circumferences of two parallels of latitude and extend on the exterior to certain distances northward and southward from those parallels: then, adopting the conditions assumed by the Rev. P. Murdoch (Phil. Trans., 1758) the distortion which exists in a projection upon the surface of a tangent cone, though small, will be diminished. These conditions are that the surface of the conical frustum shall be equal to that of the spherical zone, which it is intended to represent, and that its length shall be equal to the meridional arc between the extremities of the zone.

Let PMA (fig. to art. 49.) be a quadrant of a terrestrial meridian, a and b, the northern and southern extremities of the zone to be developed, M the middle point in latitude: join C, M, and imagine V′D, the line by whose revolution about PC the cone is to be formed, to be drawn perpendicularly to CM, meeting the polar semidiameter CP produced in V′. From E the intersection of V′D with CM make EF, ED each equal to the arc Ma or Mb, and let fall am, bn, and EG perpendicularly on PC: then (Geom., 10. Cyl.) the rectangle DF. circum. EG is equal to the convex surface of the conical frustum, and the rectangle mn. circum. CM is equal (1 Cor. 14. Cyl.) to the surface of the zone; therefore, conformably to the hypothesis,

$$\text{DF circum. EG} = mn \text{ circum. CM},$$

or (Euc. 16. 6.) $\text{DF} : mn :: \text{circum. CM} : \text{circum. EG}$;

or again (Geom. 10. Circ.) $\text{DF} : mn :: \text{CM} : \text{EG}$.

But DF is equal to the given arc ab; and, supposing the radius CM of the sphere to be unity, mn is the difference between the sines of the latitudes of a and b, the given extremities of the zone.

Let the difference between the sines of the latitudes be represented by d, then $\text{EG} = \dfrac{d}{\text{arc } ab}$. Again, in the right-angled triangle EGC, CG : CE (:: CN : CM) :: sin. lat. M : radius.

And the triangle V′EC being right angled at E is similar to CGE (Euc. 8. 6.); therefore

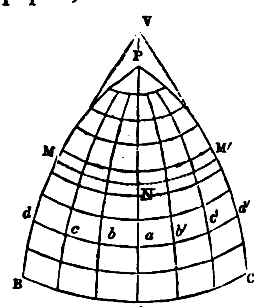
$$CG : CE :: EG : EV'.$$

or sin. lat. M : radius :: $\frac{d}{\text{arc } ab}$: V′E = $\left(\frac{d}{\text{arc } ab \text{ sin. lat. M}}\right)$.

Thus EG and EV′ being obtained, the angle subtended at V′ on the map, when the cone produced by the revolution of V′D is extended on a plane, may be found as before, and the map may be constructed.

52. A modification of Flamstead's principle has been adopted in France, for geographical maps, by which a portion of the earth's surface from one of the poles to the equator, and containing 90 degrees of longitude, is represented with great correctness.

The scale of the first of the two preceding diagrams being enlarged to any convenient magnitude, imagine M in that diagram to be a point at 45 degrees from the equator; then MV, a tangent at M on any meridian, will be the tangent of 45°. Let the conical surface produced by the revolution of VM about the axis PC be extended on the paper; and since it is proposed to exhibit a portion of that surface containing only 90 degrees of longitude, the expression $d = 90° \times$ sin. lat. M (d, art. 49., being the angle subtended at V by one quarter of the circumference of the parallel of contact when developed) gives $d = 63° \; 38 \cdot 4'$. Therefore make the angle M V M′ in the annexed figure equal to that value of d and bisect it by the line VA. With V as a centre, and a radius equal to that of the sphere (which may be considered as unity) describe the circular arc MNM′; this will be the parallel of the 45th degree of latitude on the map, and the arc will contain 90 degrees of longitude on that parallel. Make NP and NA each equal to the length of an arc of 45 degrees on the sphere; then P will represent the pole, and A will be a point on the equator. Divide PA into equal parts, each representing an interval of one, five, ten, or any convenient number of degrees, and through the points of division describe arcs of circles from V as a common centre; these will represent the parallels of latitude. On each of these parallels, as $d\,d'$, are to be set from its intersection with PA, spaces, as ab, bc, &c., ab', $b'c'$, &c., each equal to the chord of one, five, ten, or fifteen degrees, the length of the chord being

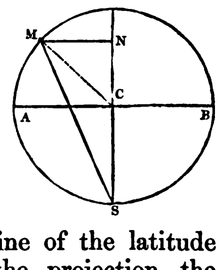


computed from the lengths, on the sphere, of the degrees of longitude on the respective parallels; then curves traced, as in the diagram, from P to the equator BC through the extremities of the chords, will represent the several meridians.

53. The facilities afforded by a map on which the circles of the sphere are represented by straight lines, in laying down on paper the course of a ship at sea, and in determining by a geometrical construction the differences of latitude and of longitude between the points of departure and arrival, on a given course, led to that modification of the plane chart which was proposed by Mercator about the year 1550, or by Wright in England some years later.

This consists in representing the meridians by straight lines parallel to one another, and at distances equal to the lengths of one, two, or any convenient number of degrees on the equator, the circumference of this circle of the sphere being conceived to be extended in a right line, which will be perpendicular to all the meridians: the parallels of latitude are also represented by straight lines, but their distances from the developed equator exceed the corresponding distances from that circle, on the sphere, in the same proportion as the lengths of the degrees of longitude on the several parallels are (in consequence of the parallelism of the meridians) increased on the chart, when compared with their lengths on the sphere. Thus the loxodromic curve, which cuts all the parallel meridians at equal angles, is a straight line, and, in the projection, the angle contained between a line joining two points representing the places of a ship, or of two stations on the earth, and the meridian line passing though one of them is correctly equal to the *angle of position*, or the bearing of one of those points from the other.

54. In order to determine the distance of any parallel of latitude from the equator, let it be observed that the radius MN of such parallel circle on the sphere is equal to the cosine of AM, its latitude (the radius of the sphere being unity), and that the length of a degree on the circumference of any circle varies with the radius of the circle; therefore the length of a degree on any parallel of latitude is to the length of a degree on a meridian circle as the cosine of the latitude of the parallel is to radius. But, in the projection, the length of a degree on each of the different parallels of latitude is equal to the length of an equatorial degree; therefore the cosine of the latitude of any parallel is to radius



as the length of an equatorial degree is to the length of a degree on the meridian, at the place of the parallel; or radius is to the secant of the latitude of the parallel as the length of an equatorial degree is to the length of a degree on the meridian, at the place of the parallel: hence it is evident, since radius and the length of the equatorial degree are constant, that, on the projection, the distance of any parallel of latitude from the equator may be denoted approximatively by the sum of the secants of all the degrees of latitude from the equator to the parallel (the length of an equatorial degree being considered as unity). But using the processes of the differential calculus, a correct expression for the distance of any parallel of latitude from the equator, on the projection, may be investigated in the following manner. Let l represent the distance of any parallel of latitude on the sphere from the equator, and L the corresponding distance on the projection: let also $d\,l$ represent an evanescent arc of the meridian, and $d\,$L the corresponding arc on the projection, then, $\cos. l$: radius $(= 1)$ :: $d\,l$: $d\,$L, and $d\,\mathrm{L} = \dfrac{d\,l}{\cos. l}$. But the arc being circular, if x represent an abscissa, as C N, and $y\ (= \sqrt{(1 - x^2)})$ the corresponding ordinate M N, we shall have

$$d\,l\ (= \sqrt{(d\,x^2 + d\,y^2)}) = \frac{d\,x}{\sqrt{(1 - x^2)}}, \text{ and } \cos. l = \sqrt{(1 - x^2)};$$

therefore $d\,\mathrm{L} = \dfrac{d\,x}{1 - x^2}$.

The second member being put in the form $\dfrac{\mathrm{A}\,d\,x}{1 + x} + \dfrac{\mathrm{B}\,d\,x}{1 - x}$, we have, on bringing to a common denominator, and equating the numerators,

$$\mathrm{A} - \mathrm{A}\,x + \mathrm{B} + \mathrm{B}\,x = 1;$$

from which, on equating like powers of x, we obtain $\mathrm{A} = \mathrm{B} = \frac{1}{2}$; consequently the last equation for $d\,$L becomes

$$d\,\mathrm{L} = \tfrac{1}{2}\left(\frac{d\,x}{1 + x} + \frac{d\,x}{1 - x}\right),$$

which, being integrated, gives

$$\mathrm{L} = \tfrac{1}{2}\left\{\text{hyp. log. } (1 + x) - \text{hyp. log. } (1 - x)\right\} + \text{const.}$$

or, $$\mathrm{L} = \tfrac{1}{2}\text{ hyp. log } \frac{1 + x}{1 - x} + \text{const.};$$

or again, $$\mathrm{L} = \text{hyp. log. } \left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}} + \text{const.}$$

But when $x = 0$, $\mathrm{L} = 0$; therefore there is no constant, and

$$\text{L} = \text{hyp. log.} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}.$$

Now, in the figure, $\text{MN} = \sqrt{(1-x^2)}$ and $\text{SN} = 1+x$; consequently,

$\frac{\text{SN}}{\text{MN}}$, or tang. $\angle$ SMN, is equal to $\frac{1+x}{\sqrt{(1-x^2)}}$, or $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$.

Therefore, L = hyp. log. tan. SMN, or = hyp. log. cotan. PSM; or again, L = hyp. log. cotan. $\frac{1}{2}$ PCM, that is, in the projection, the length of a meridional arc measured from the equator, northwards or southwards (the latitude of its extremity being expressed by l) is equal to the hyp. log. of the cotangent of half the complement of l, the radius of the sphere being unity. This theorem was first demonstrated by Dr. Halley. (Phil. Trans., No. 219.)

The numbers in the tables of meridional parts which are usually given in treatises of navigation may be obtained from the above formula; but in those tables the length of an equatorial minute is made equal to unity, and consequently the radius of the sphere is supposed to be 3437·75: therefore the value of L which is obtained immediately from the formula, must be multiplied by this number in order to have that which appears in the tables.

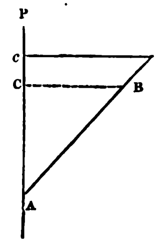
For example, let it be required to find the number in the table of meridional parts corresponding to the 80th degree of latitude:

Half the colatitude is 5°, whose log. cotan. = 1·05805, and the logarithm of this number is	0·02451
Subtract the log. of modulus (0·43429) . .	−1·63778
Logarithm of the hyp. log. cotan. 5° . . .	0·38673
Add log. of 3437·75	3·53627
Log. of 8375 (= L)	3·92300

And 8375 is the number in the tables.

55. As an example of the manner in which Mercator's projection is applied, let the distance which a ship has sailed be 100 miles, on a course making an angle of 50° with the meridian of her point of departure; the latitude of this point being 60°.

Let A be the point of departure, and draw the straight line PA to represent the meridian. Make the angle PAB equal to 50°; and, from any scale of equal parts representing geographical miles or equatorial minutes, take 100 for the length of AB, then draw BC perpendicular to PA. The line AC computed by plane trigonometry, or measured on the same scale, will express the difference of latitude (= 64·28, or



1° 4·28′), and in like manner C B (=76·6 or 1° 16·6′) the *Departure*. Now the latitude of A being 60°, that of B or C is 61° 4·28′, and from a table of meridional parts we have for 61° 4·28′, the number 4658 : for 60°, the number 4527 : the difference (=131) is the difference of latitude in the projection; therefore make A c, from the same scale, equal to 131, and draw $c\,b$ parallel to C B. Then $c\,b$ being computed, or measured on the scale, will be 156·15 or 2° 36·15′, the difference between the longitudes of A and B.

CHAP. III.

SPHERICAL TRIGONOMETRY.

DEFINITIONS AND THEOREMS.

56. THE objects principally contemplated in propositions relating to the elementary parts of practical astronomy, are the distances of points from one another on the surface of an imaginary sphere, to which the points are referred by a spectator at its centre, and the angles contained between the planes of circles cutting the sphere and passing through the points, it being understood that the plane of a circle passing through every two points is intersected in two different lines by the planes of the circles passing through those points and a third. Thus the circular arcs connecting three points are conceived to form the sides of a triangle on the surface of the sphere; and hence the branch of science which comprehends the rules for computing the unknown sides and angles is designated spherical trigonometry.

57. In general, each side of a triangle is expressed by the number of degrees, minutes, &c., in the angle which it subtends at the centre of the circle of which it is a part, and each angle of the triangle by the degrees, &c. in the angle at which the two circles containing it are inclined to one another; but it is frequently found convenient to express both sides and angles by the lengths of the corresponding arcs of a circle whose radius is unity, the trigonometrical functions (sines, tangents, &c.) of the sides and angles being also expressed as usual in terms of a radius equal to unity. The latter method is absolutely necessary when any such function is developed in a series of terms containing the side or angle of which it is a function.

It is obvious that, in order to render the measures of the sides of spherical triangles comparable with one another when expressed in terms of their radii, those radii must be equal to one another; and therefore such triangles are, in general, conceived to be formed by great circles of the sphere on whose surface they are supposed to exist. In the processes of practical astronomy, it is however often necessary to determine the lengths of the arcs of small circles as indirect means of finding the values of some parts of spherical triangles; but, before a final result is obtained, these must be converted into the corresponding arcs of great circles: occasionally also it is required to convert the arcs of great circles into the corresponding arcs of small circles, and the manner of effecting such conversions will be presently explained.

58. When the apparent places of celestial bodies are referred to what is frequently called the celestial sphere, whose radius may be conceived to be incalculably great when compared with the semidiameter of the earth, or with the distance of any planet from the sun, the arcs of great circles passing through such places may be supposed, at pleasure, to have their common centre at the eye of the spectator, or at the centre of the earth or of the sun; and the first supposition is generally adopted in computations relating to the positions of fixed stars: but, as the semidiameter of the earth is a sensible quantity, when compared with the distances of the sun, moon, and planets from its centre, and from a spectator on its surface, it is in general requisite, in determining the positions of the bodies of the solar system by the rules of trigonometry, to transfer those bodies in imagination to the surface of a sphere whose centre coincides with that of the earth. The planets, among which the earth may be included, are also conceived to be transferred to the surface of a sphere whose centre is that of the sun.

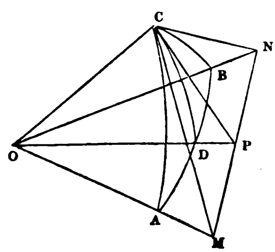
59. It is easy to perceive that there must be a certain resemblance between the propositions of spherical trigonometry and those which relate to plane triangles; and, in fact, any of the former, except one, may be rendered identical with such of the latter as correspond to them in respect of the terms given and required, by considering the rectilinear sides of the plane triangles as arcs of circles whose radii are infinitely great, or, which is the same, by considering them as infinitely small arcs of great circles of a sphere, whose radius is finite. For, in either case, on comparing the spherical triangles with the others, the sides of the former, if expressed as arcs in terms of the radius, may be substituted for their sines or tangents; also unity, or the radius, may be substituted for the cosines of the sides and the reciprocals of the sides for their cotangents.

The exception alluded to is that case in which the three angles of a spherical triangle are given, to find any one of the sides; for in the corresponding proposition of plane trigonometry, the ratio only of the sides to one another can be determined; it may be observed, however, that the sides of a spherical triangle, when computed by means of the angles, are also indeterminate unless it be considered that they appertain to a sphere whose diameter is given. With this exception, the propositions of plane trigonometry might be considered as corollaries to those of spherical trigonometry for the case in which the spherical angles become those which would be made by the intersections of three planes at right angles to that which passes through the angular points of the triangle, and in which, consequently, they are together equal to two right angles only; and it is evident that the sum of the angles of a spherical triangle of given magnitude approaches nearer to equality to two right angles as the radius of the sphere increases.

PROPOSITION I.

60. To express one of the angles of a spherical triangle in terms of the three sides.

Let ACB be a triangle on the surface of a sphere whose centre is O, and imagine its sides to lie in three planes passing through that centre intersecting one another in the lines OA, OB, OC. (The inclinations of the planes to one another, or the angles of the spherical triangle, are supposed to be less than right angles, and each of the sides to be less than a quadrant.) Imagine a plane to pass through OC perpendicularly to the plane AOB, cutting the surface of the sphere in CD: then the triangle ABC will be divided into two right-angled triangles ADC, BDC. Next let MCN be a plane touching the sphere at C, and bounded by the planes COA, COB, BOA produced, and let it meet the latter in MN; also imagine the plane ODC to be produced till it cuts MCN in CP.



The plane COP is by supposition at right angles to MON; and because CO is perpendicular to the tangent plane MCN, the plane COP is perpendicular to the same plane MCN; therefore (Geom. 19. Planes, and 1. Def. Pl.) MN is perpendicular to the plane COP and to the lines OP, CP: hence the plane triangles CPM, CPN, OPM, OPN are right angled at P. Now (Plane Trigon., art. 56.)

$$\frac{\text{CP}}{\text{CM}} = \cos.\,\text{MCP and } \frac{\text{CP}}{\text{CN}} = \cos.\,\text{NCP};$$

also $$\frac{\text{PM}}{\text{CM}} = \sin.\,\text{MCP and } \frac{\text{PN}}{\text{CN}} = \sin.\,\text{NCP}: \text{ therefore}$$

$$\frac{\text{CP}^2}{\text{CM.CN}} = \cos.\,\text{MCP}\cos.\,\text{NCP and } \frac{\text{PM.PN}}{\text{CM.CN}} = \sin.\,\text{MCP}\sin\,\text{NCP}:$$

consequently by subtraction,

$$\frac{\text{CP}^2-\text{PM.PN}}{\text{CM.CN}} = \cos.\ \text{MCP}\ \cos.\ \text{NCP} - \sin.\ \text{MCP}\ \sin.\ \text{NCP},$$

$=$ (Pl. Trigon., art. 32.) $\cos.\ (\text{MCP} + \text{NCP})$ or $\cos.\ \text{ACB}$: and in like manner,

$$\frac{\text{OP}^2 - \text{PM.PN}}{\text{OM.ON}} = \cos.\ \text{AOB}, \text{ or } \cos.\ \text{AB}.$$

Hence $\quad \text{CP}^2 - \text{PM.PN} = \text{CM.CN}\ \cos.\ \text{ACB},$

and $\quad \text{OP}^2 - \text{PM.PN} = \text{OM.ON}\ \cos.\ \text{AB}.$

But the radius of the sphere being unity, CP is the tangent and OP the secant of the angle DOC, or of the arc CD; likewise CM is the tangent and OM the secant of CA; also CN is the tangent and ON the secant of CB; therefore, subtracting the first of these equations from the last, observing that the difference between the squares of the secant and the tangent of any angle is equal to the square of radius, which is unity, we have

$$1 = \sec.\ \text{AC}\ \sec.\ \text{BC}\ \cos.\ \text{AB} - \tan.\ \text{AC}\ \tan.\ \text{BC}\ \cos.\ \text{ACB};$$

or, substituting $\frac{1}{\cos.}$ for secant, and $\frac{\sin.}{\cos.}$ for tangent,

$$1 = \frac{\cos.\ \text{AB}}{\cos.\ \text{AC}\ \cos.\ \text{BC}} - \frac{\sin.\ \text{AC}\ \sin.\ \text{BC}}{\cos.\ \text{AC}\ \cos.\ \text{BC}}\cos.\ \text{ACB};$$

whence $\cos.\ \text{AC}\ \cos.\ \text{BC} = \cos.\ \text{AB} - \sin.\ \text{AC}\ \sin.\ \text{BC}\ \cos.\ \text{ACB}$

and $\quad \cos.\ \text{ACB} = \dfrac{\cos.\ \text{AB} - \cos.\ \text{AC}\ \cos.\ \text{BC}}{\sin.\ \text{AC}\ \sin.\ \text{BC}} \ldots.\ (a)$

In like manner,

$$\cos.\ \text{CAB} = \frac{\cos.\ \text{BC} - \cos.\ \text{AB}\ \cos.\ \text{AC}}{\sin.\ \text{AB}\ \sin.\ \text{AC}} \ldots.\ (b)$$

and $\quad \cos.\ \text{ABC} = \dfrac{\cos.\ \text{AC} - \cos.\ \text{AB}\ \cos.\ \text{BC}}{\sin.\ \text{AB}\ \sin.\ \text{BC}} \ldots.\ (c)$

If the radius of the sphere and of the arcs which measure the angles of the triangle, instead of being unity, had been represented by r, we should have had, for the equivalent of any trigonometrical functions, as sin. A, cos. A, &c., the terms $\frac{\sin.\ \text{A}}{r}$, $\frac{\cos.\ \text{A}}{r}$ &c.; therefore, when any trigonometrical formula has been obtained on the supposition that the radius is unity, it may be transformed into the corresponding formula for a radius equal to r, by dividing each factor in the different terms by r, and then reducing the whole to its simplest form. Thus the above formula (a) would become $r\ \cos.\ \text{ACB}$ $= \dfrac{r\ \cos.\ \text{AB} - \cos.\ \text{AC}\ \cos.\ \text{BC}}{\sin.\ \text{AC}\ \sin.\ \text{BC}}$; and it is evident that r may be introduced in any formula by multiplying each term by such a power of r as will render all the terms homogeneous; that is, as will render the number of simple factors equal in all the terms.

Cor. 1. When the angle ACB is a right angle, its cosine vanishes; and radius being unity, the formula (a) becomes

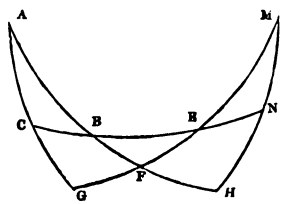
$$\cos. AB = \cos. AC \cos. BC.$$

If the radius be represented by r, this last expression becomes

$$r \cos. AB = \cos. AC \cos. BC; \ldots (d)$$

and corresponding equations may be obtained from (b) and (c).

Cor. 2. Let the terms in the formula (d) be supposed to appertain to the right-angled spherical triangle ABC; then, on substituting for them their equivalents in the complemental triangle BFE (Sph. Geom., 18.), AG, AF, CE, and GE being quadrants and the angles at C, G, and F, right angles; also the angles at A and E, being measured by the arcs GF and CG respectively, we shall have



$$r \sin. BF = \sin. BEF \sin. BE. \ldots (e)$$

Again, substituting for the terms in this formula their equivalents in the complemental triangle ENM, the arcs BH, BN, MF and MH being quadrants and the angles at F, H, and N right angles; also the angles at B and M being measured by the arcs HN and FH respectively, we have

$$r \cos. EMN = \sin. MEN \cos. EN. \ldots (f)$$

Cor. 3. In any oblique spherical triangle, as ABC (fig. to the Prop.), letting fall a perpendicular CD from one of the angles, as C, we have, from the equation (d), in the right-angled triangles ADC, BDC, $r \cos. AC = \cos. AD \cos. DC$ and $r \cos. BC = \cos. BD \cos. DC$; whence, by division, $\frac{\cos. AC}{\cos. BC} = \frac{\cos. AD}{\cos. BD}$, or $\cos. AC : \cos. BC :: \cos. AD : \cos. BD$; or, again, $\cos. AC \cos. BD = \cos. BC \cos. AD$.

PROPOSITION II.

61. The sines of the sides of any spherical triangle are to one another as the sines of the opposite angles.

Let ABC (fig. to the Prop., art. 60.) be a spherical triangle, and let CD be a perpendicular let fall from any one of the angles, as C, to the opposite side: then the terms which, in the right-angled triangles ACD, BCD correspond to those in formula

(e) above (for the right-angled triangle BEF) being substituted for the latter terms, we have

$$r \sin. \mathrm{CD} = \sin. \mathrm{CAD} \sin. \mathrm{AC},$$

and $$r \sin. \mathrm{CD} = \sin. \mathrm{CBD} \sin. \mathrm{BC}:$$

whence $$\sin. \mathrm{CAD} \sin \mathrm{AC} = \sin. \mathrm{CBD} \sin. \mathrm{BC},$$

or $$\sin. \mathrm{AC} : \sin. \mathrm{BC} :: \sin. \mathrm{CBD} : \sin. \mathrm{CAD}.$$

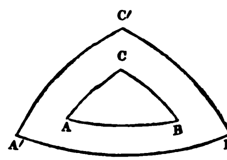
In like manner, $$\sin. \mathrm{AC} : \sin. \mathrm{AB} :: \sin. \mathrm{ABC} : \sin. \mathrm{ACB},$$

and $$\sin. \mathrm{BC} : \sin. \mathrm{AB} :: \sin. \mathrm{BAC} : \sin. \mathrm{ACB}.$$

PROPOSITION III.

62. To express one of the sides of a spherical triangle in terms of the three angles.

Let ACB be any spherical triangle, and A′C′B′ be that which is called (Sph. Geo., 1.) the supplemental triangle: then, substituting in the formulæ (a), (b), (c), art. 60., terms taken from the latter triangle which are the equivalents of the sines and cosines of the sides, and the cosines of the angles belonging to the first triangle; observing that while the sides, and the arcs which measure the angles of the triangle ABC, are less than quadrants, their supplemental arcs in the triangle A′B′C′ are greater; and therefore that in the substitution the signs of cosines must be changed from positive to negative, and the contrary, we have (radius being unity) for the equation preceding the formula (a)



$$\cos. \mathrm{B}' \cos. \mathrm{A}' = - \cos. \mathrm{C}' + \sin. \mathrm{B}' \sin. \mathrm{A}' \cos. \mathrm{A}'\mathrm{B}';$$

whence $$\cos. \mathrm{A}'\mathrm{B}' = \frac{\cos. \mathrm{C}' + \cos \mathrm{B}' \cos. \mathrm{A}'}{\sin. \mathrm{B}' \sin. \mathrm{A}'} \dots (a')$$

In like manner $$\cos. \mathrm{B}'\mathrm{C}' = \frac{\cos. \mathrm{A}' + \cos. \mathrm{B}' \cos. \mathrm{C}'}{\sin. \mathrm{B}' \sin. \mathrm{C}'}; \dots (b')$$

and $$\cos. \mathrm{A}'\mathrm{C}' = \frac{\cos. \mathrm{B}' + \cos. \mathrm{A}' \cos. \mathrm{C}'}{\sin. \mathrm{A}' \sin. \mathrm{C}'} \dots (c')$$

These expressions hold good for any spherical triangles, and therefore the accents may be omitted.

Cor. 1. When the angle C′ is a right angle its cosine is zero, and the formula (a') becomes, omitting the accents and introducing the radius, or r,

$$r \cos. \mathrm{AB} = \mathrm{cotan.}\, \mathrm{B}\ \mathrm{cotan.}\, \mathrm{A}. \dots (d')$$

Substituting for the terms in this formula their equivalents in the complemental triangle BFE (fig. to 2 Cor., art. 60.) we have

$$r \sin. \mathrm{BF} = \mathrm{cotan.}\, \mathrm{B} \tan. \mathrm{FE}. \dots (e')$$

Again, substituting for the terms in this last formula their equivalents in the complemental triangle ENM, we get

$$r \cos. \text{EMN} = \tan. \text{MN} \cot\text{an}. \text{EM}. \ldots (f').$$

COR. 2. The equation preceding ($\acute{a}$), omitting the accents, is

$$\cos. \text{C} = \sin. \text{B} \sin. \text{A} \cos. \text{AB} - \cos. \text{B} \cos. \text{A}.$$

Now, substituting in this equation the value of cos. B from the formula (c'); viz. cos. AC sin. A sin. C — cos. A cos. C, it becomes

$$\cos. \text{C} = \sin. \text{B} \sin. \text{A} \cos. \text{AB} - \cos. \text{AC} \sin. \text{A} \cos. \text{A} \sin. \text{C} + \cos.^2\text{A} \cos. \text{C},$$

or transposing the last term of the second member, and substituting sin. ²A for 1 — cos. ²A (Pl. Trigon., art. 19.), we get

$$\cos. \text{C} \sin.^2\text{A} = \sin. \text{B} \sin. \text{A} \cos. \text{AB} - \cos. \text{AC} \sin. \text{A} \cos. \text{A} \sin. \text{C};$$

whence $\cos. \text{C} \sin. \text{A} = \sin. \text{B} \cos. \text{AB} - \cos. \text{AC} \cos. \text{A} \sin. \text{C}.$

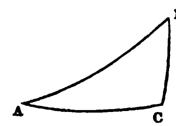
But from art. 61. we have $\sin. \text{C} = \dfrac{\sin. \text{AB} \sin. \text{B}}{\sin. \text{AC}}$; therefore, dividing the first of these equations by the other, viz. the first and last terms of the former by sin. C, and the middle term by the equivalent of sin. C, we have

$$\cot\text{an}. \text{C} \sin. \text{A} = \sin. \text{AC} \cot\text{an}. \text{AB} - \cos. \text{AC} \cos. \text{A}.$$

63. The corollaries (d), (e), (f), (d'), ($\acute{e}$), (f') contain the formulæ which are equivalent to what are called the *Rules* of Napier; and since these rules are easily retained in the memory, their use is very general for the solution of right-angled spherical triangles. The manner of applying them may be thus explained.

In every triangle, plane as well as spherical, three terms are usually given to find a fourth; and, in those which have a right angle, one of the known terms is, of course, that angle. Therefore, omitting for the present any notice of the right angle among the data, it may be said that in right-angled spherical triangles two terms are given to find a third. Now the three terms may lie contiguously to one another (understanding that when the right angle intervenes between two terms those terms are to be considered as joined together), or one of them, on the contour of the triangle, may be separated from the two others, on the right and left, by a side or an angle which is not among the terms given or required; and that term which is situated between the two others is called the middle part. The terms which are contiguous to it, one on the right and the other on the left, are called adjacent parts; and those which are situated on contrary sides of it, but are separated from it by a side or an angle, are called opposite parts.

Thus, in the solution of a spherical triangle, as ABC, right angled at C, and in which two terms are given besides the right angle to find a third, there may exist six cases according to the position of the middle term with respect to the two others, as in the following table, in which the middle part is placed in the third column between the extremes, the latter being in the second and fourth columns. The first column contains merely the numbers of the several cases, and the fifth denotes the corollaries to the preceding propositions, in which are given the formulæ for finding any one of the three parts, the two others being given. The first three cases are those in which the extreme parts are adjacent, and the other three those in which they are opposite to the middle part.



1.	angle A	hypot. AB	angle B	(d') Prop. 3.
2.	hypot. AB	angle A	side AC	(f') Prop. 3.
3.	angle A	side AC	side BC	(e') Prop. 3.
4.	side AC	hypot. AB	side BC	(d) Prop. 1.
5.	angle A	angle B	side AC	(f) Prop. 1.
6.	hypot. AB	side BC	angle A	(e) Prop. 1.

The two rules which were discovered by Napier for the solution of right-angled spherical triangles in these six cases may be thus expressed, M being put for the middle part, E and E′ for the adjacent extremes, and D, D′ for the opposite or disjoined extremes:

$$\text{Rad. sin. M} = \text{tan. E tan. E}',$$

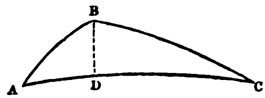
and

$$\text{Rad. sin. M} = \text{cos. D cos. D}'.$$

But, in using the rule, the following circumstance must be attended to: when the middle term or either of the extreme terms is the hypotenuse, or one of the angles adjacent to it, the complement of the value of that term must be substituted for the term itself. Thus if M, E, or D, &c. denote the hypotenuse or one of the angles; for sin. M must be written cos. M; for tan. E, cotan. E; for cos. D, sin. D, &c.

64. The problems which require for their solution the determination of certain parts of an oblique spherical triangle may conveniently, except when three sides or three angles are the data, be worked by the Rules of Napier, or by the formulæ in the six corollaries above mentioned, on imagining a per-

pendicular to be let fall from one of the angular points of the oblique triangle in such a manner that the latter may be divided into two right-angled spherical triangles, in one of which there shall be data sufficient for the application of the rules; and, as a general method, it may be observed that such perpendicular should fall upon one of the sides from an extremity of a known side and opposite to a known angle. Thus, as an example, let the sides AB and AC, with the included angle A, be given, in the oblique spherical triangle ABC; and let the side BC with the angles at B and C be required.



Imagine the arc BD to be let fall from the point B upon the side AC, then in the right-angled triangle ADB, the side AB and the angle at A are given besides the right angle at D; and with these data let the segment AD be found. If the rules of Napier be employed, it is to be observed that the angle at A will be the middle part, and that the hypotenuse AB with the side AD are adjacent extremes; therefore by the first of the two rules above (as in the formula (f') art. 62.)

$$r \,.\, \cos.\ BAD = \text{cotan.}\ AB\ \tan.\ AD;$$

thus AD may be found, and, subtracting it from AC, the segment DC may also be found. The side BC will be most readily obtained by forming in each of the triangles ADB, BDC, an equation corresponding to (d) in art. 60., and dividing one by the other; each of these equations may be formed by the method given in that article, or by the second of Napier's Rules. Using the latter method, and considering AB and BC in those triangles to be the middle parts, in the first triangle let AD, DB, and in the second let BD, DC be the opposite or disjoined extremes; then by the Rule,

$$r.\ \cos.\ AB = \cos.\ AD\ \cos.\ BD \text{ and } r\,.\,\cos.\ BC = \cos.\ DC\ \cos.\ BD,$$

and dividing the former by the latter, we have

$$\frac{\cos.\ AB}{\cos.\ BC} = \frac{\cos.\ AD}{\cos.\ DC}; \text{ whence } \cos.\ AD : \cos.\ DC :: \cos.\ AB : \cos.\ BC,$$

$$\text{or } \cos.\ DC\ \cos.\ AB = \cos.\ AD\ \cos.\ BC \ \ldots . \ (A),$$

and either from the equation or the proportion the value of BC may be found.

The angles ABC and ACB might now be found by Prop. 2., using the proportions

$$\sin.\ BC : \sin.\ A :: \sin.\ AB : \sin.\ C,$$

$$\sin.\ BC : \sin.\ A :: \sin.\ AC : \sin.\ B;$$

or the angle at C may be conveniently found from the triangle

BDC, by Napier's first rule, considering the angle at C as the middle part, and the sides BC, DC as the adjacent parts; therefore (as at (f') art. 62.)

$$\cos. \text{ACB} = \text{cotan. BC tan. DC}.$$

Thus all the unknown parts in the triangle ABC are determined.

65. The formulæ given in the first and third Propositions, and in the second corollary to the latter, are unfit to be immediately employed in logarithmic computations, since they contain terms which are connected together by the signs of addition and subtraction; and, therefore, before they can be rendered subservient to the determination of numerical values in the problems relating to practical astronomy, they must be transformed into others in which all the terms may be either factors or divisors; for then, by the mere addition or subtraction of the logarithms of those terms, the value of the unknown quantity may be obtained. The transformations which are more immediately necessary are contained in the two following Propositions; and those which may be required in the investigations of particular formulæ for the purposes of astronomy and geodesy will be given with the Propositions in which those formulæ are employed.

PROPOSITION IV.

66. To investigate a formula which shall be convenient for logarithmic computation, for finding any one angle of a spherical triangle in terms of its sides.

From Prop. I. we have

$$\cos. \text{ACB} = \frac{\cos. \text{AB} - \cos. \text{AC} \cos. \text{BC}}{\sin. \text{AC} \sin. \text{BC}};$$

and subtracting the members of this equation from those of the identical equation $1 = \frac{\sin. \text{AC} \sin. \text{BC}}{\sin. \text{AC} \sin. \text{BC}}$, we get

$$1 - \cos. \text{ACB} = \frac{\cos. \text{AC} \cos. \text{BC} + \sin. \text{AC} \sin. \text{BC} - \cos. \text{AB}}{\sin. \text{AC} \sin. \text{BC}},$$

or $$1 - \cos. \text{ACB} = \frac{\cos. (\text{AC} - \text{BC}) - \cos. \text{AB}}{\sin. \text{AC} \sin. \text{BC}}.$$

But (Pl. Trigon., art. 36.) $1 - \cos. \text{ACB}$, or the versed sine of ACB, is equivalent to $2 \sin.^2 \frac{1}{2} \text{ACB}$; and, substituting in the numerator of the second member the equivalent of the difference between the two cosines (Pl. Trigon., art. 41.), observing that AB is greater than AC $-$ BC, since two sides of a triangle are greater than the third, the last equation becomes

$$2 \sin.^2 \tfrac{1}{2} \text{ACB} = \frac{2 \sin. \frac{1}{2} (\text{AB} + \text{AC} - \text{BC}) \sin. \frac{1}{2} (\text{AB} - \text{AC} + \text{BC})}{\sin. \text{AC} \sin. \text{BC}}.$$

But if P represent the perimeter of the triangle,

$$\tfrac{1}{2} (\text{AB} + \text{AC} - \text{BC}) = \tfrac{1}{2} \text{P} - \text{BC},$$
$$\text{and } \tfrac{1}{2} (\text{AB} - \text{AC} + \text{BC}) = \tfrac{1}{2} \text{P} - \text{AC},$$

therefore the formula becomes $\sin.^2 \frac{1}{2} \text{ACB}$ (or $\frac{1}{2}$ vers. sin. ACB) =

$$\frac{\sin. (\frac{1}{2} \text{P} - \text{BC}) \sin. (\frac{1}{2} \text{P} - \text{AC})}{\sin. \text{AC} \sin. \text{BC}} \quad \ldots \text{(I)}.$$

Again, if to the members of the above equation for cos. ACB (Prop. I.) there be added those of the identical equation $1 = \frac{\sin. \text{AC} \sin. \text{BC}}{\sin. \text{AC} \sin. \text{BC}}$, there will be obtained

$$1 + \cos. \text{ACB} = \frac{\sin. \text{AC} \sin. \text{BC} - \cos. \text{AC} \cos. \text{BC} + \cos. \text{AB}}{\sin. \text{AC} \sin. \text{BC}},$$

or
$$1 + \cos. \text{ACB} = \frac{\cos. \text{AB} - \cos. (\text{AC} + \text{BC})}{\sin. \text{AC} \sin. \text{BC}}.$$

But (Pl. Trigon., art. 36.) $1 + \cos. \text{ACB}$ is equivalent to $2 \cos.^2 \frac{1}{2} \text{ACB}$; and, substituting in the numerator of the second member the equivalent of the difference between the two cosines (Pl. Trigon., art. 41.), the last equation becomes

$$2 \cos.^2 \tfrac{1}{2} \text{ACB} = \frac{2 \sin. \frac{1}{2} (\text{AC} + \text{BC} + \text{AB}) \sin. \frac{1}{2} (\text{AC} + \text{BC} - \text{AB})}{\sin. \text{AC} \sin. \text{BC}}:$$

putting, as before,

$\frac{1}{2}$ P for $\frac{1}{2} (\text{AC} + \text{BC} + \text{AB})$ and $\frac{1}{2} \text{P} - \text{AB}$ for $\frac{1}{2} (\text{AC} + \text{BC} - \text{AB})$,

we have
$$\cos.^2 \tfrac{1}{2} \text{ACB} = \frac{\sin. \frac{1}{2} \text{P} \sin. (\frac{1}{2} \text{P} - \text{AB})}{\sin. \text{AC} \sin. \text{BC}} \quad \ldots \text{(II)}.$$

Thirdly, dividing the formula (I) by (II), member by member, we get

$$\tan.^2 \tfrac{1}{2} \text{ACB} = \frac{\sin. (\frac{1}{2} \text{P} - \text{BC}) \sin. (\frac{1}{2} \text{P} - \text{AC})}{\sin. \frac{1}{2} \text{P} \sin. (\frac{1}{2} \text{P} - \text{AB})} \quad \ldots \text{(III)}.$$

In the above investigations it has been supposed that the radius is unity: if it be represented by r, the numerators of the formulæ (I), (II) and (III) must (art. 60.) be multiplied by r^2.

PROPOSITION V.

67. To investigate formulæ convenient for logarithmic computation in order to determine two of the angles of any spherical triangle when there are given the other angle and the two sides which contain it.

In a spherical triangle ABC, let AB and AC be the given sides, and BAC the given angle; it is required to find the angles at B and C.

From (a) and (b) respectively in Prop. I. we have

$$\cos. AB = \cos. ACB \sin. AC \sin. BC + \cos. AC \cos. BC \ldots. (h)$$

$$\cos. BC = \cos. BAC \sin. AC \sin. AB + \cos. AC \cos. AB \ldots. (k)$$

Multiplying both members of this last equation by cos. AC we get $\cos. BC \cos. AC =$

$$\cos. BAC \sin. AC \cos. AC \sin. AB + \cos.^2 AC \cos. AB,$$

and substituting the second member for its equivalent in (h) there results $\cos. AB =$

$$\cos. ACB \sin. AC \sin. BC + \cos. BAC \sin. AC \cos. AC \sin. AB + \cos.^2 AC \cos. AB,$$

or $\cos. AB (1 - \cos.^2 AC) = \cos. ACB \sin. AC \sin. BC + \cos. BAC \sin. AC \cos. AC \sin. AB.$

In this last substituting $\sin.^2 AC$ for $1 - \cos.^2 AC$, and dividing all the terms by sin. AC, we have $\cos. AB \sin. AC =$

$$\cos. ACB \sin. BC + \cos. BAC \cos. AC \sin. AB \ldots. (m).$$

In like manner, from the formulæ (c) and (b) in Prop. I., and from (k) above, we get $\cos. AC \sin. AB =$

$$\cos. ABC \sin. BC + \cos. BAC \cos. AB \sin. AC \ldots. (n);$$

and it may be perceived that this last equation can be obtained from that which precedes it by merely substituting B for C and C for B.

Adding together the equations (m) and (n), and afterwards subtracting (n) from (m) we have, respectively, putting A, B, and C for the angles BAC, ABC, and ACB,

$$\cos. AB \sin. AC + \sin. AB \cos. AC = \sin. BC (\cos. C + \cos. B) + \cos. A (\cos. AC \sin. AB + \sin. AC \cos. AB),$$

$$\cos. AB \sin. AC - \sin. AB \cos. AC = \sin. BC (\cos. C - \cos. B) + \cos. A (\cos. AC \sin. AB - \sin. AC \cos. AB).$$

But (Pl. Trigon., art. 41.) $\cos. C + \cos. B =$

$$2 \cos. \tfrac{1}{2} (B + C) \cos. \tfrac{1}{2} (B - C),$$

and $$\cos. C - \cos. B = 2 \sin \tfrac{1}{2} (B + C) \sin. \tfrac{1}{2} (B - C):$$

also (Pl. Trigon., art. 32.) $\sin. AC \cos. AB \pm \cos. AC \sin. AB = \sin. (AC \pm AB):$

therefore, after transposition and substitution, the last equations become

$$\sin. (AC + AB) (1 - \cos. A) = 2 \sin. BC \cos. \tfrac{1}{2} (B + C) \cos. \tfrac{1}{2} (B - C),$$

and $\sin.\ (\text{AC} - \text{AB})\ (1 + \cos.\ \text{A}) =$
$2 \sin.\ \text{BC} \sin.\ \frac{1}{2}(\text{B} + \text{C}) \sin.\ \frac{1}{2}(\text{B} - \text{C})$.

But again (Pl. Tr., arts. 35, 36.) $\sin.\ a = 2 \sin.\ \frac{1}{2} a \cos.\ \frac{1}{2} a$, $1 - \cos.\ \text{A} = 2 \sin.^2 \frac{1}{2} \text{A}$, and $1 + \cos.\ \text{A} = 2 \cos.^2 \frac{1}{2} \text{A}$: therefore the last equations may be put in the form

$$2 \sin.\ \tfrac{1}{2}(\text{AC} + \text{AB}) \cos.\ \tfrac{1}{2}(\text{AC} + \text{AB}) \sin.^2 \tfrac{1}{2} \text{A} = \sin.\ \text{BC} \cos.\ \tfrac{1}{2}(\text{B} + \text{C}) \cos.\ \tfrac{1}{2}(\text{B} - \text{C}) \ldots (p),$$

and $$2 \sin.\ \tfrac{1}{2}(\text{AC} - \text{AB}) \cos.\ \tfrac{1}{2}(\text{AC} - \text{AB}) \cos.^2 \tfrac{1}{2} \text{A} = \sin.\ \text{BC} \sin.\ \tfrac{1}{2}(\text{B} + \text{C}) \sin.\ \tfrac{1}{2}(\text{B} - \text{C}) \ldots (q).$$

Dividing (q) by (p) we have

$$\frac{\sin.\ \frac{1}{2}(\text{AC} - \text{AB}) \cos.\ \frac{1}{2}(\text{AC} - \text{AB})}{\sin.\ \frac{1}{2}(\text{AC} + \text{AB}) \cos.\ \frac{1}{2}(\text{AC} + \text{AB})} \cot an.^2 \tfrac{1}{2} \text{A} = \tan.\ \tfrac{1}{2}(\text{B}+\text{C}) \tan.\ \tfrac{1}{2}(\text{B} - \text{C}) \ldots (r).$$

Now from Prop. II. we have

$$\sin.\ \text{B} : \sin.\ \text{C} :: \sin.\ \text{AC} : \sin.\ \text{AB};$$

whence $\sin.\ \text{B} + \sin.\ \text{C} : \sin.\ \text{B} - \sin.\ \text{C} :: \sin.\ \text{AC} + \sin.\ \text{AB} : \sin.\ \text{AC} - \sin.\ \text{AB}$,

and $$\frac{\sin.\ \text{AC} + \sin.\ \text{AB}}{\sin.\ \text{AC} - \sin.\ \text{AB}} = \frac{\sin.\ \text{B} + \sin.\ \text{C}}{\sin.\ \text{B} - \sin.\ \text{C}},$$

or (Pl. Trigon., art. 41.),

$$\frac{\sin.\ \frac{1}{2}(\text{AC}+\text{AB}) \cos.\ \frac{1}{2}(\text{AC}-\text{AB})}{\cos.\ \frac{1}{2}(\text{AC}+\text{AB}) \sin.\ \frac{1}{2}(\text{AC}-\text{AB})} = \frac{\sin.\ \frac{1}{2}(\text{B}+\text{C}) \cos.\ \frac{1}{2}(\text{B}-\text{C})}{\cos.\ \frac{1}{2}(\text{B}+\text{C}) \sin.\ \frac{1}{2}(\text{B}-\text{C})};$$

or again,

$$\frac{\sin.\ \frac{1}{2}(\text{AC}+\text{AB}) \cos.\ \frac{1}{2}(\text{AC}-\text{AB})}{\cos.\ \frac{1}{2}(\text{AC}+\text{AB}) \sin.\ \frac{1}{2}(\text{AC}-\text{AB})} = \frac{\tan.\ \frac{1}{2}(\text{B}+\text{C})}{\tan.\ \frac{1}{2}(\text{B}-\text{C})} \ldots (s).$$

Multiplying (r) by (s) we have

$$\frac{\cos.^2 \frac{1}{2}(\text{AB} - \text{AC})}{\cos.^2 \frac{1}{2}(\text{AB} + \text{AC})} \cot an.^2 \tfrac{1}{2} \text{CAB} = \tan.^2 \tfrac{1}{2}(\text{B} + \text{C}),$$

whence

$$\frac{\cos.\ \frac{1}{2}(\text{AB} - \text{AC})}{\cos.\ \frac{1}{2}(\text{AB} + \text{AC})} \cot an.\ \tfrac{1}{2} \text{CAB} = \tan.\ \tfrac{1}{2}(\text{B} + \text{C}) \ldots (\text{I}).$$

Again, dividing (r) by (s) and extracting the roots,

$$\frac{\sin.\ \frac{1}{2}(\text{AB} - \text{AC})}{\sin.\ \frac{1}{2}(\text{AB} + \text{AC})} \cot an.\ \tfrac{1}{2} \text{CAB} = \tan.\ \tfrac{1}{2}(\text{B} - \text{C}) \ldots (\text{II}).$$

Thus from the formulæ (I) and (II) there may be obtained the values of the angles ABC and ACB in terms of the angle CAB and of the sides AB and AC, which contain it.

PROPOSITION VI.

68. To investigate formulæ convenient for logarithmic computation, for determining two sides of any spherical triangle when there are given the other side and the two adjacent angles.

In a spherical triangle ABC, the angles at B and C and the side BC being given; it is required to find the sides AB and AC.

From the formulæ (a') and (b') in Prop. III., omitting the accents, after multiplying both members of the latter by cos. A, and substituting as in the last Proposition, there will be obtained the equation

$$\cos. AB \sin. A = \cos. C \sin. B + \cos. BC \cos. B \sin. C \ldots . (m')$$

Also from the formulæ (c') and (b') in Prop. III., or, which is the same, writing C for B and B for C in the last equation, there is obtained

$$\cos. AC \sin. A = \cos. B \sin. C + \cos. BC \cos. C \sin. B. \ldots (n').$$

Adding together (n') and (m'), and afterwards subtracting (n') from (m') we get, on transforming, as in Art. 67.,

$$\sin. A \cos. \tfrac{1}{2} (AB + AC) \cos. \tfrac{1}{2} (AB - AC) = \sin. (B + C) \cos.^2 \tfrac{1}{2} BC \ldots . (p'),$$

$$\text{and } \sin. A \sin. \tfrac{1}{2} (AB + AC) \sin. \tfrac{1}{2} (AB - AC) = \sin (B - C) \sin.^2 \tfrac{1}{2} BC \ldots . (q').$$

Then, dividing (q') by (p') there is obtained

$$\frac{\sin. \frac{1}{2} (AB + AC) \sin. \frac{1}{2} (AB - AC)}{\cos. \frac{1}{2} (AB + AC) \cos. \frac{1}{2} (AB - AC)} = \frac{\sin. \frac{1}{2} (B - C) \cos. \frac{1}{2} (B - C)}{\sin. \frac{1}{2} (B + C) \cos. \frac{1}{2} (B + C)} \tan.^2 \tfrac{1}{2} BC \ldots . (r').$$

This last equation being first multiplied by (s), and afterwards divided by (s), in the last Proposition, there will be obtained after the necessary reductions

$$\tan. \tfrac{1}{2} (AB + AC) = \frac{\cos. \frac{1}{2} (B - C)}{\cos. \frac{1}{2} (B + C)} \tan. \tfrac{1}{2} BC. \ldots (\text{I}),$$

$$\text{and } \tan. \tfrac{1}{2} (AB - AC) = \frac{\sin. \frac{1}{2} (B - C)}{\sin. \frac{1}{2} (B + C)} \tan. \tfrac{1}{2} BC. \ldots (\text{II}).$$

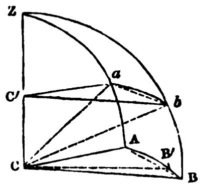
Thus the values of AB and AC may be separately found.

The formulæ (I) and (II) in Props. V. and VI., were discovered by Napier, and are frequently designated "Napier's Analogies."

69. The investigation of formulæ expressing the relations

between an angle and its orthographical projection on a plane inclined to that of the angle may be made as follows.

If the vertices of the given and projected angles are to be coincident, the sides containing the given angle and its projection may be conceived to be in the planes of two great circles of a sphere whose centre is the angular point, and which intersect each other in a line passing through that point. Thus, let $a\text{C}b$ be the given angle, then, ZC being any line passing through C, if the planes of two circles pass through ZC and the lines Ca, Cb, and meet a plane passing through C perpendicularly to ZC, the intersections of the circles with the latter plane will be in the lines CA, CB, and the angle ACB will be the orthographical projection of aCb.



Now the arc ab, of a great circle, measures the angle aCb; and the arc AB measures the projected angle ACB; therefore the arcs Aa, Bb, or the angles aCA, bCB being given, their complements Za, Zb are known; and in the spherical triangle aZb, with the three sides Za, Zb, ab, the angle aZb may be computed by one of the formulæ (a), (b), or (c), Prop. I., or by one of the formulæ (I), (II), or (III) Prop. IV., and consequently its equal ACB is found. If one of the sides, as Cb, were coincident with a side, as CB, of the reduced angle, since Zb would then be a quadrant, its sine would be equal to radius, and its cosine to zero: therefore, one of the formulæ (a), (b), or (c), Prop. I. would give

$$\cos.\ a\text{Z}b = \frac{\cos.\ ab}{\sin.\ \text{Z}a}, \text{ or } = \frac{\cos.\ ab}{\cos.\ \text{A}a}, \text{ (radius being unity)},$$

or $\qquad \cos.\ \text{AC}a : \cos.\ a\text{C}b :: \text{rad.} (= 1) : \cos.\ \text{ACB}.$

A particular formula for the reduction of the angle aCb, when Aa and Bb are small arcs, will be given in the chapter on Geodesy (art. 397.).

If the arcs Aa, Bb, or the angles ACa, BCb be equal to one another, each of them may be considered as the inclination of the plane aCb to ACB, and the reduction may be made thus: — imagine a plane, as aC$'b$, to pass through ab parallel to ACB, and the straight lines, or chords, ab, AB to be drawn; then the angle aC$'b$ will be equal to ACB, and the triangles aC$'b$, ACB will be similar to one another: therefore

$$\text{C}'b : \text{CB} :: \text{chord } ab : \text{chord AB, or as } 2 \sin.\ \tfrac{1}{2} \text{ arc. } ab : 2 \sin.\ \tfrac{1}{2} \text{ arc. AB}.$$

But $\qquad \text{C}'b : \text{CB} :: \sin.\ \text{Z}b : \text{rad., or as } \cos.\ \text{B}b : \text{radius};$

therefore $\quad \sin.\ \text{Z}b : \text{rad.} :: \sin.\ \tfrac{1}{2} \text{ arc. } ab : \sin.\ \tfrac{1}{2} \text{ arc. AB},$

or $\qquad \sin.\ \text{Z}b : \text{rad.} :: \sin.\ \tfrac{1}{2}\, a\text{C}b : \sin.\ \tfrac{1}{2}\, \text{ACB}.$

70. If it were required to find the length of an arc, as AB of a great circle from the given length of the corresponding arc ab, of a small circle, having the same poles (of which let Z be one), and consequently (Sph. Geom., 1 Cor. 1 Def.) parallel to it, the sectors ACB, $aC'b$ being similar to one another, we have

$$C'a : CA :: ab : AB.$$

But Za is the distance of the small circle from its pole Z; and, the radius of the sphere being unity, $C'a$ is the sine of that distance; therefore the above proportion becomes

$$\sin. Za : 1 \ (= \text{radius}) :: ab : AB;$$

whence $AB = \dfrac{ab}{\sin. Za} = \dfrac{ab}{\cos. Aa}$.

Conversely $ab = AB \sin. Za = AB \cos. Aa$.

If the chord of the arc ab were transferred from A to B′ on the arc AB, we should have

chord $ab = C'a \,.\, 2 \sin. \frac{1}{2}\, aC'b$, and chord $AB' = CA \,.\, 2 \sin. \frac{1}{2}\, ACB'$: but chord ab = chord AB′; therefore

$$C'a \,.\, 2 \sin. \tfrac{1}{2}\, aC'b = CA \,.\, 2 \sin. \tfrac{1}{2}\, ACB',$$

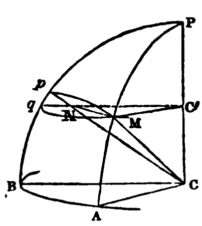
$$\text{and } C'a : CA :: \sin. \tfrac{1}{2}\, ACB' : \sin. \tfrac{1}{2}\, aC'b,$$

or $\quad \cos. Aa : \text{rad.} \ (= 1) :: \sin. \frac{1}{2}\, ACB' : \sin. \frac{1}{2}\, aC'b.$

When the arc ab is small, we have, nearly, the angle $ACB' = aC'b \cos. Aa$; or, conversely, $aC'b = \dfrac{ACB'}{\cos. Aa}$.

71. When two great circles, as PA, PB, make with each other a small angle at P, their point of intersection, and when from any point as M, in one of these, an arc Mp of a great circle is let fall perpendicularly on the other; also from the same point M, an arc Mq, of a small circle having P for its pole, is described; it may be required to find approximatively the value of pq, and the difference between the arcs Mq and Mp.

Let C be the centre of the sphere, and draw the radii CM, Cp; also imagine a plane MC′q to pass through Mq perpendicularly to PC, and let it cut the plane MCp in the line MN. Then, since the plane MC′q is perpendicular to PC, it is perpendicular to the planes PCB, PCA (Geom., 18 Planes); the plane MCp is also perpendicular to PCB; therefore the line of section MN is perpendicular to PCB and to the lines C′q, Cp, which it meets in that plane; and the angle MC′q is



equal to BPA or BCA, the inclination of the circles PA, PB to one another.

Now, if the radius of the sphere be considered as unity, we shall have

$$\text{MC}' = \sin.\ \text{PM, and CC}' = \cos.\ \text{PM; then}$$
$$\text{C}'\text{N}\ (= \text{MC}' \cos.\ \text{MC}'\text{N}) = \sin.\ \text{PM} \cos.\ \text{P, also}$$
$$\text{C}'\text{N}\ (= \text{CC}' \tan.\ \text{PC}p) = \cos.\ \text{PM} \tan.\ \text{P}p;$$

therefore $\sin.\ \text{PM} \cos.\ \text{P} = \cos.\ \text{PM} \tan.\ \text{P}p$,

and $\cos.\ \text{P} = \dfrac{\cos.\ \text{PM}}{\sin.\ \text{PM}} \tan.\ \text{P}p$, or $= \dfrac{\cos.\ \text{PM}}{\sin.\ \text{PM}}\ \dfrac{\sin.\ \text{P}p}{\cos.\ \text{P}p}$.

Subtracting the first and last members from unity, we have

$$1 - \cos.\ \text{P} = 1 - \frac{\cos.\ \text{PM} \sin.\ \text{P}p}{\sin.\ \text{PM} \cos.\ \text{P}p} = \frac{\sin.\ (\text{PM} - \text{P}p)}{\sin.\ \text{PM} \cos.\ \text{P}p}.$$

But $1 - \cos.\ \text{P} = 2 \sin.^2 \frac{1}{2}\text{P}$ (Pl. Trigon., art. 36.), and since P is supposed to be small, the arc which measures the angle may be put for its sine; therefore $1 - \cos.\ \text{P}$ becomes (P being expressed in seconds so that $\frac{1}{2}\,\text{P} \sin.\ 1''$ may represent $\sin.\ \frac{1}{2}\,\text{P}$) equal to $\frac{1}{2}\,\text{P}^2 \sin.^2 1''$. Also, for $\sin.\ (\text{PM} - \text{P}p)$ may be put $(\text{PM} - \text{P}p) \sin.\ 1''$, or $pq \sin.\ 1''$, and in the denominator, Pp may be considered as equal to PM; therefore, for the denominator there may be put $\sin.\ \text{PM} \cos.\ \text{PM}$, or its equivalent $\frac{1}{2} \sin.\ 2\,\text{PM}$. Thus we obtain, approximatively,

$$\tfrac{1}{2}\,\text{P}^2 \sin.^2 1'' = \frac{\text{PM} - \text{P}p}{\frac{1}{2} \sin.\ 2\,\text{PM}} \sin.\ 1'', \text{ or } \tfrac{1}{4}\,\text{P}^2 \sin.\ 2\,\text{PM} \sin.\ 1'' = pq$$
(in seconds).

Again, since $\text{MN} = \sin.\ \text{M}p$, on developing M$p$ in terms of its sine (Pl. Trigon., art. 47.), neglecting powers higher than the third, we have

$$\text{M}p = \text{MN} + \tfrac{1}{6}\,\text{MN}^3:$$

And, since $\dfrac{\text{MN}}{\text{MC}'} = \sin.\ \dfrac{\text{M}q}{\text{MC}'}$, or $\dfrac{\text{MN}}{\sin.\ \text{PM}} = \sin.\ \dfrac{\text{M}q}{\sin.\ \text{PM}}$, on developing as before, we get

$$\frac{\text{M}q}{\sin.\ \text{PM}} = \frac{\text{MN}}{\sin.\ \text{PM}} + \tfrac{1}{6}\,\frac{\text{MN}^3}{\sin.^3 \text{PM}}, \text{ or } \text{M}q = \text{MN} + \tfrac{1}{6}\,\frac{\text{MN}^3}{\sin.^2\ \text{PM}}.$$

Subtracting from this the above equation for Mp, we obtain

$$\text{M}q - \text{M}p = \tfrac{1}{6}\,\text{MN}^3 \left(\frac{1}{\sin.^2 \text{PM}} - 1\right) = \tfrac{1}{6}\text{MN}^3\, \frac{\cos.^2\text{PM}}{\sin.^2\text{PM}}$$

$= \frac{1}{6}\,\text{MN}^3 \text{ cotan.}^2\ \text{PM}$; or expressing M$p$ in seconds, and putting $\text{M}p \sin.\ 1''$ for MN, we have (in arc)

$$\text{M}q - \text{M}p = \tfrac{1}{6}\,\text{M}p^3 \sin.^3 1'' \text{ cotan.}^2\text{PM}, \text{ and, in seconds,}$$
$$= \tfrac{1}{6}\,\text{M}p^3 \sin.^2 1'' \text{ cotan.}^2\ \text{PM}.$$

CHAP. IV.

DESCRIPTIONS OF THE INSTRUMENTS EMPLOYED IN PRACTICAL ASTRONOMY.

THE SIDEREAL CLOCK. — MICROMETER. — TRANSIT INSTRUMENT. — MURAL CIRCLE. — AZIMUTH AND ALTITUDE CIRCLE. — ZENITH SECTOR. — EQUATORIAL INSTRUMENT. — COLLIMATOR. — REPEATING CIRCLE. — REFLECTING INSTRUMENTS.

72. THE longitudes and latitudes of celestial bodies are not, now, directly obtained from observations, on account of the difficulties which would attend the adjustments of the instruments requisite for such a purpose: but on land, particularly in a regular observatory, the positions of the sun, moon, planets, and fixed stars are generally determined by the method which was first practised by Römer or La Hire. This consists in observing the right ascensions by means of a transit telescope and a sidereal clock, and the declinations by means of a circular instrument whose plane coincides with that of the meridian: the longitudes and latitudes, when required, are then computed by the rules of trigonometry. It will be proper therefore, in this place, to explain the nature of the instruments just mentioned, their adjustments and verifications, and the manner of employing them. It is not intended, however, to describe at length the great instruments which are set up in a national observatory, but merely to indicate them, and to explain the natures and uses of such as, being similar to them and of more simple construction, may without great risk of injury be transported to foreign stations, where regular observatories do not exist, in order to be employed by persons charged with the duty of making celestial observations, either for the advancement of astronomy itself or in connection with objects of geodetical or physical inquiry.

73. The *Sidereal Clock* is one which is regulated so that the extremity of its hour-hand may revolve round the circumference of the dial-plate in the interval of time between the instants when, by the diurnal rotation of the earth, that intersection of the traces of the equator and ecliptic which is designated the vernal equinox, or the first point of Aries, appears successively in the plane of the geographical meridian, on the same side of the pole. This interval is called a sidereal day,

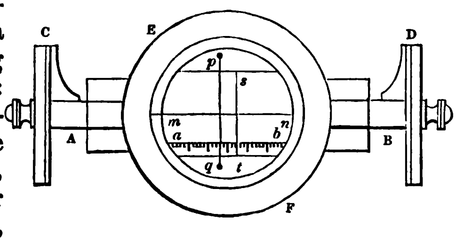
being very nearly that in which a fixed star appears to revolve about the earth. The dial of the clock having its circumference divided into twenty-four equal parts, the hour-hand is made to indicate xxiv, or 0 hours at the instant that the first point of Aries is in the meridian; and if the clock were duly adjusted, it is evident that the hour, minute, and second which it might express at any moment would be equivalent to the angle, at that moment, between the plane of the meridian and a plane passing the earth's axis and the equinoctial point above-mentioned. Thus, the clock would show the right ascension of the mid-heaven, or meridian, at that instant, right ascensions being generally expressed in time.

It follows, therefore, that if a telescope were accurately placed in the plane of the meridian, and a star were to appear to be bisected by a vertical wire in the middle of the field of view, the time shown by the sidereal clock at that instant would express the apparent right ascension of the star. On the other hand, if it were required to make the clock express sidereal time, it would be only necessary, at the moment that a fixed star whose apparent right ascension is given in the Nautical Almanac is observed to be bisected by the wire of the telescope, to set the hands to the hour, minute, &c. so given; for then, the clock being supposed to be duly regulated, 0 hours will be indicated by it when the first point of Aries is on the meridian.

The different species of time, and the processes which are to be used in reducing one species to another, will be explained further on. (Chap. XIV.)

74. The Micrometer is an instrument by which small angles are measured; and it is employed for the purpose of ascertaining the angle subtended at the observer's eye by the diameter of the sun, the moon, or a planet, or by the distance between a fixed star and the moon or a comet when they are very near each other, or for any like purpose in practical astronomy.

It consists of a brass tube with lenses, constituting the eye-piece of a telescope and carrying a frame, A B, containing two perforated plates: to one of these is attached a very fine wire, or spider thread, *pq*, and to the other a similar wire *st*,



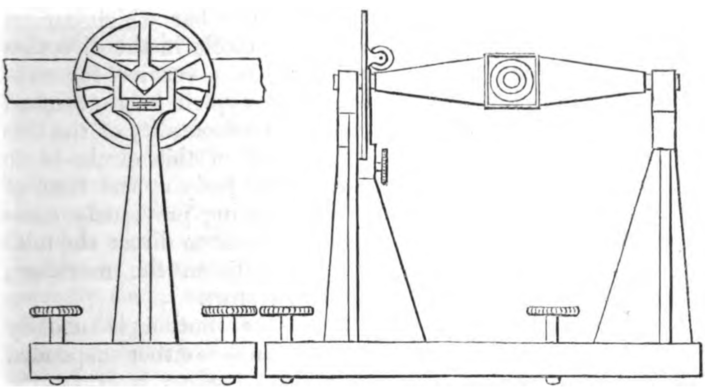
wire in the first, to the same wire in the second position, 15 t (since by the rotation of the earth on its axis in 24 hours every point in the heavens appears to describe about that axis an angle or arc equal to 15 degrees in an hour, 15 minutes in one minute of time, &c.) will express in seconds of a degree the angle subtended at the centre of the star's parallel of declination by the interval between the two places of the wire, and being multiplied by the cosine of the star's declination (art. 70.) it will express the angle subtended at the centre of the equator by an equal interval on that great circle. But, as will be hereafter explained, this as well as the other stars has constantly a movement in right ascension, that is, in the same direction as the earth revolves on its axis, the amount of which, during the time t, may be found from the Nautical Almanac; and, whether the star be above or below the pole, an increase of right ascension will increase the time in which, by the diurnal rotation, it appears to pass between the two positions of the wire: therefore that change of right ascension (in arc) must be subtracted from the above product in order to express the angle corresponding to the number of the screw's revolutions.

77. If the micrometrical apparatus be placed in a frame which is capable of being turned about the optical axis of the telescope, and be provided with a graduated circle, as EF, whose centre is in that axis, so that the position of the micrometer wires with respect to the plane of any vertical, or horary circle, may be known at the time that the micrometrical angle is observed, the instrument is called a *Position Micrometer.* It is usually attached to an equatorial instrument or to a telescope having an equatorial movement. The manner of using a micrometer for the measurement of small angles will be explained in the description of the equatorial (art. 119.).

78. The micrometer microscope, which is attached to the rim of an astronomical circle, consists of a system of lenses similar to those of an ordinary microscope, and the image of the graduations of the circle is by the disposition of the lenses made to fall at the place of a wire or of a pair of wires crossing each other at an acute angle near the eye-end of the tube. Now the smaller kind of astronomical circles are divided into spaces each equal to 15 minutes of a degree, but those of a larger kind into spaces equal to 10 minutes or 5 minutes; and across the field of view in the microscope is a scale or a plate divided into spaces, each of which is equal to one minute. The micrometer screw, by one complete revolution, moves the wire through one of the spaces on the plate, and its cir-

consisting of two conical arms which, at their larger extremities, are joined on opposite sides to the cubical or globular portion just mentioned: the smaller extremities of the cones are made cylindrical, and equal to one another; and when the telescope is mounted for service, each of these extremities lies in a notch cut at the top of a moveable vertical plate of brass, (corresponding to what in a common theodolite is called a Y,) which enters into a fixed plate of the like metal. The latter either rests on a stone pier or forms the head of the stand supporting the telescope; and each notch has the form of two inclined planes whose surfaces are tangents to the cylindrical pivot: one of the notched plates is capable, by means of a screw, of being elevated or depressed for the purpose of rendering the axis of motion horizontal; and the other may, in like manner, be moved a small way in azimuth, in order that the optical axis of the telescope, (the line joining the centres of all the lenses,) may be brought accurately into the plane of the meridian.

82. At the focus of the object glass are fixed three, five, or seven parallel wires, besides one which crosses them at



right angles: the latter is intended to be always in a horizontal position; and the others, when the telescope lies horizontally between its supports, are in vertical positions. The diaphragm, or perforated plate to which the wires are attached, is capable of a small movement by means of screws which pass through the sides of the telescope, in order that the intersection of the horizontal with the central wire at right angles to it may be made to fall exactly in the optical axis or line of collimation, as it is called, of the telescope. One of the arms of the axis of motion is hollow, and a lamp being attached to the pier, or side of the stand on which that arm

plicity may be the zero of the graduations, is vertically above the middle of the bubble or column of air, and to determine, by the difference of the readings at the extremities of the line, the error, if any exist, in the horizontality of the axis of motion.

When the level is in a horizontal position, the bubble of air is stationary at the middle of the upper surface of the spirit in the tube; but on giving the latter a small inclination to the horizon, the air will move towards the higher end, till the force of ascent is *in equilibrio* with the adhesion of the water to the glass; and it is assumed that the space which the centre or either extremity of the air bubble moves through is proportional to the angle of inclination. In such angle the number of seconds which correspond to each graduation of the scale may be determined by the maker of the instrument, or the astronomer may determine it himself by placing the level on the connecting bars of a graduated circle which is capable of turning in altitude, and reading the number of minutes of a degree through which the circle is turned, in order to make the air bubble move under any convenient number of graduations on the scale.

Now the length of the column of air in the tube will vary with changes in the temperature of the atmosphere, and the axis of the spirit tube may not be parallel to a plane passing through the points on which rest the feet of the level: therefore, though the latter plane were horizontal, the middle of the air might deviate from the zero of the scale. If the number of the graduation above each extremity of the air column be read, the excess of the greater number above half the sum of the two numbers will indicate the inclination of the axis of the spirit tube to a horizontal line, the axis being highest at that end of the scale towards which the air has moved; then, reversing the level on its points of support, and reading the numbers as before, if the excess of the greater number above half the sum of the two should be the same as before, it is evident that the points of support will be in a horizontal plane. But, should the said excess not be the same in thecontrary positions of the spirit level, half their sum, if the two excesses are on the same side of the centre of the scale, or half their difference, if on opposite sides, will be the number of graduations expressing the inclination of the plane passing through the points of support to one which is horizontal. Thus, for example, let the first excess be equal to 2.4 divisions of the scale (which, if each division correspond to 5 seconds of a degree, will be equivalent to an inclination of 12″) towards the west; and the second excess equal to 0.8 divisions, or 4″, also towards the west; then half their sum,

crometer till the wire which was previously in coincidence with the meridional wire of the telescope is made to bisect a well-defined terrestrial mark, ascertain the number of seconds by which that meridional wire deviates from the mark, and having reversed the axis of the telescope, ascertain, in like manner, the deviation in this position. Half the difference, if there be any, between the deviations, is the true deviation of the line of collimation from a line perpendicular to the axis on which the telescope turns. This deviation is supposed to be expressed in divisions of the micrometer scale, and if these denote seconds of a degree, on dividing by 15 as above mentioned, the deviation will be expressed in seconds of time. The observed time of the transit of an equatorial star must be corrected by the quantity of the deviation so found; but for a star having north or south declination, that deviation must (art. 70.) be divided by the cosine, or multiplied by the secant of the declination, in order to obtain the corresponding arc of the equator, or angle at the pole, which constitutes the error of the transit for such star. It must be observed, that if the line of collimation deviate from the south either eastward or westward, it also deviates (the telescope being turned on its axis) from the north towards the same part; and that when the telescope is reversed on its supports, the deviation is from the south or north towards the contrary parts: if the transit of a star below the pole be observed, the correction which, for a transit above the pole, north or south of the zenith, would be additive, must then be subtractive, and the contrary. At the Royal Observatory, Greenwich, the error of collimation is found by means of a telescope placed, for the purpose, in the north window of the Transit Room. See Collimator, art. 125.

89. When it is required to ascertain that the central or meridional wire is in a vertical position, the optical axis being horizontal, it is only necessary to observe that the wire exactly covers or bisects a terrestrial object, as the white disk before mentioned, while the telescope is turned on the pivots of its horizontal axis; or a star which passes the meridian near the pole may be made to serve the same purpose as the terrestrial mark. If the mark or star should appear to deviate from the wire near the upper and lower parts of the field of view, the eye-piece with the wires must be turned on the optical axis of the telescope till the object remains bisected in the whole length of the wire. The horizontality of the wire which should be at right angles to the meridional wire and those which are parallel to it is ascertained by placing it on an equatorial star when the latter enters the field of view, and observing that the star is bisected by the wire during the

time of its passage across that field. It is scarcely necessary to remark, that the wires in the eye-piece of the telescope must be adjusted by drawing them towards or from the eye till they appear well defined; and till, at the same time, a star which may be observed near one of them does not change its position on moving the eye towards the right or left hand. The coincidence of the horizontal wire with a diameter passing through the optical axis of the telescope, is of small importance when it is not intended to use the transit instrument for the purpose of obtaining altitudes; but, in the event of this being one of the objects contemplated, the adjustment must be carefully made, or the error accurately determined; the process which is to be employed will be explained in the description of the altitude and azimuth circle (art. 106.).

90. The advantage of having one wire or more parallel to, and on each side of that which is in the plane of the meridian, is, that the unavoidable inaccuracy in estimating the time at which a star, on its transit, appears to be bisected by a wire, may be almost wholly corrected by using a mean of the times at which the bisections take place on all the wires. If the distances between the parallel wires were precisely equal to one another, an arithmetical mean of the times (the sum of all the times divided by the number of wires) might be considered as the correct time of the transit at the middle wire; but, on account of the inequalities of those distances, a mean of the times at which any star appears to be bisected by the several wires is to be taken for the time of the transit at an imaginary wire situated near the central wire; and the differences between the times of the transit at the several wires and at this imaginary wire may then be taken. These differences being multiplied by the cosine of the star's declination (art. 70.), will give the corresponding distance for a star supposed to be in the plane of the equator.

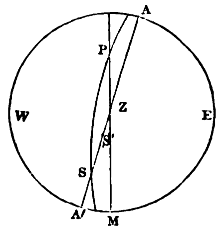
91. In a small transit telescope having five wires, it was found, for a star supposed to be in the plane of the equator, that the differences between the times of the transit at each wire, and at the imaginary mean wire, were as follow:—

First wire	-	-	$+ 51''.601$
Second	-	-	$+ 25''.590$
Third	-	-	$+ 0''.110$
Fourth	-	-	$- 25''.890$
Fifth	-	-	$- 51''.410$

When, therefore, the transit of a star has not been observed at all the wires of a telescope; if it be required to obtain, from such observations as have been made, the time of transit at the imaginary mean wire, one of the following processes

are commonly employed for this purpose, it being supposed that the azimuthal deviation does not exceed a few seconds of a degree: one of them depends on the observed transits of two stars which differ considerably in altitude, and the other upon the observed transits of a circumpolar star above and below the pole.

With respect to the first method, let a hemisphere of the heavens be projected on the plane of the observer's horizon, and let that horizon be represented by the circle A W M E, of which the centre Z is the projection of the zenith. Draw the diameter P Z M for the meridian, in which let P be the pole of the equator, and let A Z A′ be the projection of the vertical circle in whose plane the optical axis of the telescope moves: let also S be the place of a star when it is seen in the telescope, and through it draw the horary circle P S; then (art. 61.) we have in the triangle P Z S,



$$\sin.\ SPZ = \sin.\ PZS\ \frac{\sin.\ ZS}{\sin.\ PS};$$

or, since the angle at P and the supplement of the angle at Z are very small, we have, employing the number of seconds of a degree in P and Z instead of the sines of those angles,

$$P = Z\,\frac{\sin.\ ZS}{\sin.\ PS},\ \text{or}\ P = Z\,\frac{\sin.\ (PS \mp PZ)}{\sin.\ PS}\ \text{nearly}:$$

and, in order that the angle P may be expressed in seconds of sidereal time, the second member of the equation must be divided by 15. The formula may be represented by $P = Zn$ for the first of two stars S and S′, which enter the telescope, and by $P' = Zn'$ for the second.

Now, for the first star, let t be the time of the transit, observed on a sidereal clock, T the true time, or the right-ascension of the star, given in the Nautical Almanac; and for the second, let t' and T′ be the corresponding times. Let e represent the error (supposed to be unknown) of the clock by which the times were observed: then

$$t - e - T\ (= P,\ \text{in time}) = Z \,.\, n,$$

and, $$t' - e - T'\ (= P',\ \text{in time}) = Z \,.\, n':$$

hence $(t' - T') - (t - T) = Z\,(n' - n)$, and $Z = \dfrac{(t' - T') - (t - T)}{n' - n}$.

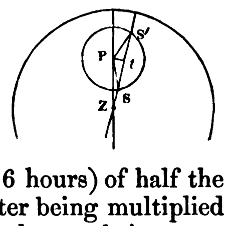
Thus the azimuthal deviation Z $(= A'ZM)$ is found. In using the formula, a deviation of the telescope, that is, of the circle

A Z A′, towards the south-west and north-east of the true meridian (as in the figure) is indicated by the value of z being positive: and a deviation towards the south-east and north-west by z being negative. This rule holds good whether the upper or the lower star culminate first, and whatever be the positions of the two stars with respect to the zenith and the pole.

The value of z thus determined is expressed in seconds of sidereal time, and it must be multiplied by 15 in order to reduce it to seconds of a degree. It may be observed also, that if the star pass the true meridian (as in the figure) before it is seen in the telescope, the value of Z n or of Z n' (the correction of the time of transit) must be subtracted from the observed time in order to give the time of the transit over the meridian P Z. On the contrary, the correction must be added if the star is seen in the telescope before it comes to the true meridian. The difference between this time of transit over the meridian, and the calculated right ascension of the star, in the Nautical Almanac, is the error of the sidereal clock.

Note. In the second equation for P, the lower sign is to be used when the star comes to the meridian below the pole.

96. For the second method, let a hemisphere be projected as before, and let S S′ be the places of a star at the times of its observed transits above and below the pole P. Then the angle S P S′, expressed in time, measures the least of the two intervals of time between the transits, and, letting fall P t perpendicularly on the vertical circle Z S S′, the angle S P t will be equal to half that interval, which consequently is known from the observed transits. Now (art. 62. (d')) we have in the right-angled spherical triangle P S t,



$r.\cos.\ \text{PS} = \text{cotan.}\ t\text{PS}\ \text{cotan.}\ \text{PS}t.$

But tP S being nearly equal to a right angle, we may, for cotan. tP S, put the number of seconds in the complement (to 6 hours) of half the least interval of time; let this number, after being multiplied by 15, be represented by D: also the angle P S t being very small, we may write for its cotangent, that is, for $\frac{\cos.\ \text{PS}t}{\sin.\ \text{PS}t}$, the term $\frac{1}{\text{PS}t}$ (P S t being expressed in seconds of a degree), and then the above equation will become $\cos.\ \text{PS} = \frac{\text{D}}{\text{PS}t}$, or $\text{PS}t = \frac{\text{D}}{\cos.\ \text{PS}}$. Again, in the spherical triangle P Z S, we have (art. 61.),

$$\sin. \text{PZ} : \sin. \text{PSZ} \ (= \sin. \text{PS}t) :: \sin. \text{PS} : \sin. \text{PZS},$$

or, $$\sin. \text{PZ} : \text{PS}t :: \sin. \text{PS} : \text{Z};$$

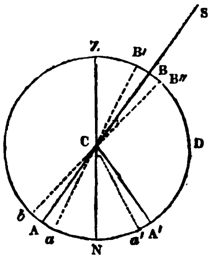
from which proportion, after substituting the above value of PSt, we obtain,

$$\text{Z (in seconds)} = \frac{\text{D} \sin. \text{PS}}{\cos. \text{PS} \sin. \text{PZ}}, \text{ or} = \frac{\text{D} \tan. \text{PS}}{\sin. \text{PZ}},$$

the azimuthal deviation required.

97. In a regular observatory, a large circle of brass attached to the east or west face of a wall, or stone pier, is used for obtaining the altitudes of celestial bodies above the horizon, or their distances from the zenith; or again, their distances from the pole of the equator. Such an instrument, called a mural circle, is generally of considerable dimensions (6 feet diameter), and it turns upon a horizontal axis, part of which enters into the supporting wall or pier; either its side or its edge is graduated, and six micrometer microscopes attached to the face of the wall at the circumference of the circle, and at nearly, or exactly, equal distances from each other, are used in reading the subdivisions of the degrees by which the required altitude or distance is expressed. A telescope is made to turn with the circle on the horizontal axis, in making the observation; but it is also capable of being turned independently, on the same axis, and of being made fast to the circle in several different positions, with respect to the zero of the graduations, in order that the angular admeasurement may be read on any part of the circumference at pleasure.

98. In general the horizontality of the axis of motion is verified by means of plumb-lines, or spirit-levels, or by observing both by direct view and by reflexion the transits of a star at the several wires in the eye-piece of the telescope as was mentioned in the account of the transit instrument; and the line of collimation is made perpendicular to the horizontal axis by the aid of meridian marks previously set up towards the north and south of the telescope by means of a transit instrument, which admits of being reversed on its supports. The deviation of the plane of the circle from the meridian may be found by observing the transits of two stars differing considerably in altitude, or of a circumpolar star when at its greatest and least elevation; but great accuracy of adjustment in this respect is evidently of less importance for an instrument which is intended to give altitudes or declinations only than for one by which transits are to be observed.

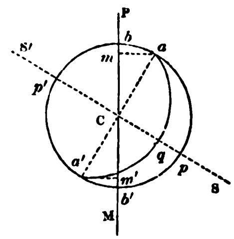


distance and the observed zenith distance of the same star on the second night be considered as a change in the zenith distance of either star on account of a derangement of the instrument. Let B B'' represent this change, and let it be applied by addition or subtraction to the zenith distance (suppose Z B) of the second star on the first night, so that Z B'' may be considered as the zenith distance of that star on the second night. Now let Z A' be the zenith distance of the same star when observed by reflexion on the second night; then B'' A' will represent the double altitude of that star, and the middle point between B'' and A' will be the place of the horizontal point on the circle. Several pairs of stars should be observed in like manner, and a mean of the results taken.

101. In using the mural circle for the purpose of obtaining by observation the declination of a star, if, when the star is bisected by the horizontal wire it is not on the central or meridional wire, but on one of those which are parallel to it, two corrections will be required: one of them depending on the change in the star's polar distance between the time of the observation, and the time at which the star was, or would be on that central wire; and the other on the distance between the parallel of declination apparently described by the star and the great circle passing through it, of which the horizontal wire is considered as a part. The first correction is readily found from the variation of the declination, or polar distance of the celestial body, given in the Nautical Almanac; and the second may be obtained from the formula $\frac{1}{4}P^2 . \sin. 2PM \sin. 1''$ (art. 71.), in which P is (in seconds of a degree) the equatorial distance from the middle wire to that at which the observation was made, and P M is the star's north polar distance.

102. If the declination or polar distance of the upper or lower limb of the moon is to be obtained from observation when that limb is not entirely enlightened, there will be required a correction which may be thus determined.

Let P M be the meridian, S the sun, C the centre of the moon at the time of culminating, and C S an arc of a great circle passing through the moon and sun: the moon's disk having a crescent form, $apa'q$, let a be the point which is in contact with the horizontal wire of the telescope at the time of the observation. Imagine am to be a horizontal line passing through a, and let b be the nearest extremity of a vertical diameter of the moon; then bm is the correction required: this is manifestly equal to $\text{C}a$ versin. $a\text{C}b$, in which expression $\text{C}a$ is the semidiameter of the moon, the angle $a\text{C}b$ is equal to the complement of MCS; and this last angle may be found sufficiently near the *truth* by means of a common celestial globe.



In a similar manner may the correction be found when the moon is oval, or gibbous, as $ap'a'q$: in that case S′ being the supposed place of the sun, the required correction $b'm'$ is equal to $\text{C}a'$ versin. $a'\text{C}b'$; and the angle $a'\text{C}b'$ is the complement of PCS′, which may be found as in the other case by means of a celestial globe.

103. The Greenwich mural instrument consists of two circles which, being on the same horizontal axis, act as counterpoises to each other, and neither plumb-lines nor spirit-levels are employed in their adjustment. They are brought as near as possible to the plane of the meridian, and each is provided with six microscopes nearly equally distant from one another at the circumference. Each also is provided with an artificial horizon of mercury, showing as much as possible of the reflected meridian.

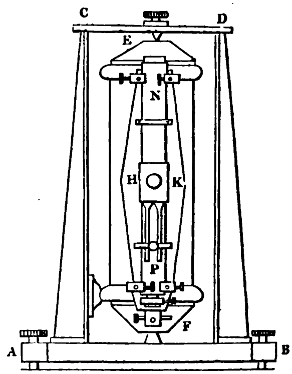
The meridian circle, which was made by Reichenbach for the observatory at Gottingen, in 1820, serves at once for a transit instrument, and for measuring altitudes: its telescope is five feet long, and it has, at the focus of the object-glass, seven vertical and two horizontal spider threads. The horizontal axis is about three feet long, and it carries, on one side, two concentric circles, three feet in diameter, whose outer surfaces are nearly in one plane: the exterior circle revolves with the telescope, and carries the graduations; and the interior circle would turn on the same axis, were it not that it may be made immovable by means of a clamp attached to the adjacent pier: on this circle are four indices, with verniers, each of which is at forty-five degrees from a vertical line passing through the centre. A suspended level serves to place the axis in a horizontal position; and the zero of the graduations is at the top, or zenith point. The instrument is capable of being reversed on the axis, in order, by making the observations with it in opposite positions, to eliminate the error of collimation: and in observing the circumpolar stars, the zenith distances, both at the superior and inferior culmination, are taken by direct view, and also by reflexion.

104. The most useful instrument in an observatory which

axis of the telescope: the position of the latter wire may be verified, as in the transit telescope, by making it bisect some well-defined mark; and then, having reversed the pivots of the vertical circle on their supports, so that the graduated face of the circle which, before, may have been towards the east, shall now be towards the west, observing that the wire is neither to the right nor left of the centre of the mark. Or the error of collimation in azimuth may be found by means of a micrometer, as explained in the description of the transit telescope (arts. 87, 88.). In order to verify the position of the horizontal wire, or find the deviation in altitude of the point of intersection from the optical axis (commonly called the error of collimation in altitude or the index error of the instrument), the apparent zenith distances of a star must be observed when on the meridian, with the graduated face of the vertical circle in contrary positions; and then half the difference between those distances will be the value of the error.

Let the circle ZDN (fig. to art. 100.) be in the plane of the meridian passing through the optical axis of the telescope, and let ZCN be a vertical line passing through C, the centre of the altitude circle in the instrument; also let A be the true or required place of the intersection of the wires at the focus of the object glass when a star on the meridian is at S in the line AB produced: then if aB′ be the observed position of the optical axis, or a be the place of the intersection when the latter appears to coincide with the star S, the arc aN or ZB′ will express the apparent distance of the star from the zenith. Next, let the instrument be turned half round in azimuth; then the line CA will be in the position CA′, so that NA is equal to NA′, and the line Ca will be in the position Ca'; and if the circle be turned on the horizontal axis passing through C till the star S appears as at first to coincide with the intersection of the wires, CA′ will be in the position CA and a' will be at b at a distance beyond A equal to A′a' or Aa; the apparent distance of the star from the zenith will then be bN or ZB″. Hence it follows that half the sum of the apparent zenith distances ZB′, ZB″ will be equal to ZB, the true zenith distance; or half the difference between the apparent zenith distances will be the index error of the instrument in altitude.

It will be most convenient to obtain the first apparent zenith distance when the star is on the central horizontal wire just before it comes to the intersection of the latter with the meridional wire, and the second apparent zenith distance when the star is on the same horizontal wire just



sunk a conical hole which receives a corresponding pivot attached to the base and to the upper horizontal bar, so that whichever of the two extremities is uppermost the frame can be made to revolve about a vertical axis. On one side of this frame is the bar HK, which carries the telescope NP; and both bar and telescope have a movement of small extent on each side of a vertical line, in a plane parallel to the front of the frame and on a horizontal axis which passes through the latter.

Near each extremity of the revolving frame is a graduated arc with two micrometer microscopes for the subdivisions.

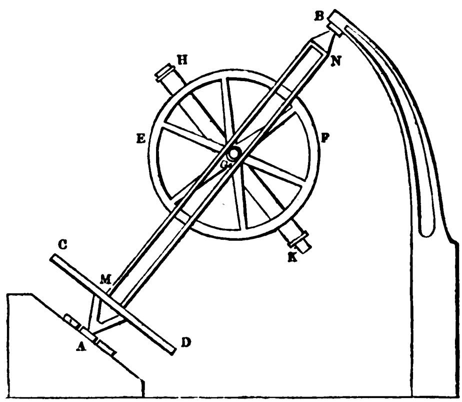
The verticality of the axis of the frame is ascertained by three spirit-levels which are attached horizontally at the back of the frame, and have the power of being reversed when the latter is inverted on the conical pivots. All the corrections due to the want of adjustment in the instrument are to be made by computation, after they have been ascertained by the observed transits of stars with the frame in direct and reversed positions; the reversions being with respect to the east and west sides and to the upper and lower extremities.

109. In making an observation there must be two observers; the first sets the telescope as nearly as possible to the zenith distance of the star, and reads the four microscopes; the second at the same time reads the scales on the levels. As soon as the star has arrived at a convenient part of the field, and before it comes to the centre, the first observer bisects it by the micrometer wire and reads the micrometers: he then turns the instrument half round in azimuth. The second observer now bisects the star by means of the tangent screw, or by the micrometer screw, and then reads the four microscopes; lastly, the first observer reads the levels. This completes the double observation; and the zenith distance may be obtained by the following rule: — Add together the mean readings of the microscopes, their corrections, the mean of the equivalents for the three levels and the equivalent for the micrometer reading: the mean between the two sums for the opposite positions in azimuth will be a quantity corresponding to a zenithal observation. Such a mean may be obtained for two or more stars in one night; and the difference between the number of the graduation expressing

the zenith point and the sum for each star is the true zenith distance of that star.

110. The Equatorial Instrument is occasionally used to obtain by direct observation the right ascensions and polar distances of celestial bodies when not in the plane of the meridian, but it has been of late chiefly employed in determining the differences between the right ascensions and declinations of stars which are visible at the same time in the field of the telescope, and in finding the positions and relative movements of double stars.

It consists of two graduated circles C D and E F, of which one is always parallel to the plane of the equator, being fixed at right angles to the mathematical axis A B of a bar, or system of bars, which axis is in the plane of the meridian, and inclined to the horizon in an angle equal to the latitude of the place; it is consequently parallel to the axis of the earth's rotation: the other, called the declination circle (to which in general the telescope H K is fixed), turns upon an axis which is always perpendicular to the former; and the circle is consequently in the plane of the meridian when its own axis, whose extremity is seen at G, is parallel to the horizon. But the bar or frame which carries the equatorial circle C D turns upon its axis A B, and carries with it the axis of the declination circle, so that the latter circle is in the plane of a horary circle inclined to the meridian when its axis is inclined to the horizon. The telescope being made to turn with the circle upon this axis, it is evident, when the telescope is directed to a celestial body, that an index fixed to the stand of the instrument may be made to show on the equatorial circle the horary angle of that body, or the angle which the plane of the declination circle makes with the plane of the meridian; and that an index fixed to the polar axis may be made to show on the declination circle the distance of the celestial body from the pole of the equator.



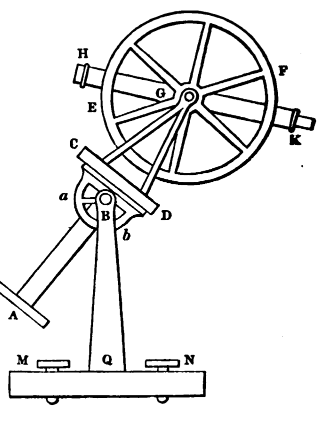
The equatorial circle may be turned round by hand, or by an endless screw working in teeth on the edge of the circle;

but it is now usual to connect that circle with a clock-work movement by which, when the telescope has been directed to a star, it may be made to revolve about the polar axis with a velocity equal to that of the earth's diurnal rotation; and thus the celestial body will remain in the field of the telescope during the time of making the observation. The hands of the observer are, therefore, left at liberty to turn the micrometer screw at the eye-piece of the telescope when it is required to ascertain, for example, the distances of two stars from each other.

The framework MN consists of four bars, or tubes, two on each side of the circle EF, to whose plane they are parallel; and those which are on opposite sides of the circle are at such a distance from it that the circle with the telescope can turn freely between them in its own plane. The upper extremities of the four bars which form the frame MN are let into a ring at N, and from this ring rise three arms which terminate in the conical pivot at B. The whole apparatus is supported at A and B in conical cups or sockets which receive the pivots; the lower socket being capable of a small movement in the plane of the meridian, and at right angles to that plane, for the purpose of allowing any derangement of the polar axis AB to be corrected.

111. An instrument like that which is above represented, being supported at the two extremities A and B, is only fit for a fixed observatory, but the following figure represents one in which the polar axis can be placed at any angle with the horizon, and consequently it may be set up in any part of the world.

The part MN represents the circular base of the whole instrument, and at each of the opposite extremities of one of its diameters is a pillar or stand BQ; the upper parts of these two pillars support a bar which turns on a horizontal axis, passing through B in an east and west direction; and, in the diagram, it is supposed to be perpendicular to the paper.



At right angles to this bar is fixed a tube in the direction AB, which, turning on the axis just mentioned, and carrying a graduated semicircle *ab*, its axis AB may be placed at any angle with a vertical line passing through it; consequently, at any station whose latitude is known, on bringing that vertical line and the axis A B in the plane of the meridian, the latter axis may be made parallel to that of the earth's rotation. Within this tube is a solid cylinder to the top of which is attached a rectangular brass plate at C D, perpendicular to the axis of the cylinder and tube, and capable of being turned upon that axis. The length of the plate, of which C D is one extremity, is about equal to the diameter of the circle M N; and the direction of the length is in the diagram supposed to be perpendicular to the paper.

At each of the opposite extremities of this plate, and perpendicularly to its plane, rises an arm, one of which appears from B to G, and the upper parts of these arms receive the extremities of a bar whose axis is parallel to the length of the plate at C D, or perpendicular to the polar axis A B (in the diagram the axis of the bar is also supposed to be perpendicular to the paper). The bar, near one of its extremities, as G, carries the declination circle E F, and, at the opposite extremity, the telescope H K: the circle and the telescope turn together on the axis passing through G; and at the same time the circle, telescope, and axis turn upon the polar axis A B: thus the circle with its telescope is constantly in a plane parallel to that of some horary circle, and the telescope may have any movement in declination.

112. If it be supposed that the polar axis of an equatorial instrument is already nearly parallel to the axis of the earth's rotation, its position with respect to the latter axis may be determined by the following processes. Direct the telescope to any one of the circum-polar stars whose apparent places are given in the Nautical Almanac, when the star is on, or nearly on the plane of the meridian, and (the index of the declination circle being at or near zero when the telescope is directed to the pole of the equator) observe by that circle the star's distance from the zero point; then turn the instrument half round on the polar axis, and having again directed the telescope to the star, read the distance of the index from zero. Take the mean of these distances, and find the refraction due to the star's altitude: the sum of these values is the corrected instrumental distance of the star from the pole; and, being compared with the polar distance of the star in the Nautical Almanac, the difference will be the error of the polar axis in altitude. This error, in an instrument constructed like that which is represented in the figure to art. 110., may be corrected by two of the screws at the foot of the axis (the angular movement of the axis depending on a

revolution of the screw having been previously found). Half the difference between the two polar distances read on the declination circle is the index error of that circle: and the index may, by its proper screws, be moved through a space equal to that half difference in order to render the readings equal, in the two positions of the circle.

The instrument must now be turned on the polar axis till the plane of the declination circle is at right angles to that of the meridian; and in this state the telescope must be directed to a star on the eastern or western side. The instrumental polar distance of the star must then be read, and corrected for the effects of refraction in polar distance (which may be computed by a formula proper for this purpose); and if the result should not agree with the north polar distance in the Nautical Almanac, the polar axis deviates from its true position, towards the east or west. This deviation must be corrected by means of the other two screws at the foot of the axis. In portable equatorials, like that which is represented in the figure to the preceding article, the adjustment of the polar axis is made by the vertical semicircle *ab*, and by a screw acting on the horizontal axis passing through B: the base MN having been previously levelled by means of the foot screws at M and N.

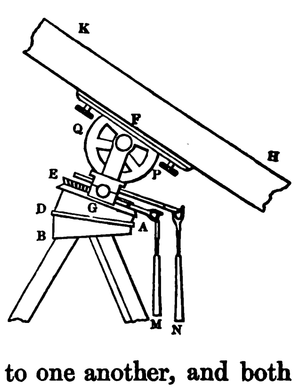
113. The position of the meridian wire in the telescope with respect to the optical axis is found by bringing the declination circle in, or near the plane of the meridian, and observing the transit of a star near the equator. Let both the time of the transit, by the sidereal clock, and the graduation at the index of the equatorial circle be read (it being supposed that this circle is divided into hours, &c. like the dial of the clock); then, turning the instrument half round on the polar axis in order that the meridional wire, if not correctly placed, may be on the opposite side of the optical axis, observe the transit of the same star: read the time by the clock, and the graduation on the equatorial circle as before. The difference between the times shown by the clock, and the difference between the readings on the circle (which would be equal if the wire passed through the optical axis), being supposed to disagree, the wires must be moved through a space equal to half the error by means of their proper screw at one side of the eye-piece. In order that this adjustment may be complete, several successive trials will probably be necessary.

114. The axis of the declination circle in the first of the two preceding figures may be made perpendicular to the polar axis by means of a spirit level: thus, bring the declination

nity of the comet, the telescope may be moved in declination till any star is found above or below the comet, and the difference of declination must then be read on the circle; the difference of right ascension being obtained as before from the times of the transits. If the transit and the declination of the star be afterwards observed on the meridian, the star's place, and consequently that of the comet, may be very correctly determined. But if, with the equatorial instrument, a star be observed above, and another below the parallel of the comet's declination; also, if, after having made the observations on the comet and star, the instrument be turned half round on the polar axis, and similar observations be again made, the mean results will give the place of the comet still more accurately.

117. An equatorial instrument enables an observer readily to find a star in the heavens during the day. For this purpose, the right ascension and declination being known, for a given time, by computation, or from the Nautical Almanac, the telescope is clamped in the position which it assumes when the indices of the equatorial and declination circles are set to the computed right ascension and declination respectively: then, at the given time, the star will be seen in the field of the telescope.

118. A telescope is frequently mounted on a stand having at its top a cylindrical block of wood, which is cut through obliquely to its mathematical axis, so that the plane of section makes angles with the upper and lower surfaces equal to half the colatitude of the place in which it is to be used; the upper portion being capable of turning upon that plane about a pin fixed in the lower portion perpendicularly to the same plane. Thus, let AED, ABD represent the two portions of the block, and let a plane passing through AD perpendicularly to the paper be the plane of separation: then, if the upper portion be placed in the position which it has in the figure, the superior surface of the block will be inclined to the inferior surface in an angle equal to the colatitude of the place. If the upper portion be turned till the point A comes to D, those surfaces will be parallel to one another, and both of them may be made parallel to the horizon.



To the upper surface is attached, perpendicularly, a brass

passes through both the stars, or is estimated to be parallel to such line (as in the figure), the index of the micrometer circle will show the value of the angle P*ap*, which, being corrected if necessary on account of the index error before mentioned, is the complement of the required angle *ba*S.

The wire *mn* is not in strictness necessary, for the micrometer may be turned so that one of the wires *pq* shall coincide with the line *ab* of the stars; then the number of the graduation at the index of the micrometer circle being read, that circle, and with it the whole micrometer, may be turned one quarter round upon the optical axis of the telescope, when the wires *pq* will be perpendicular to the line *ab* joining the stars.

In order to measure the angular distance *ab*, the whole equatorial instrument must be turned on the polar axis till one of the wires, as *pq*, passes through the star *a* (for example); then, turning the micrometer screw which moves the other wire, bring the latter wire to bisect the star *b* (the first wire continuing to pass through the star *a*), and read on the screw head belonging to the second wire the graduation which is in coincidence with the index. Again, move the whole equatorial till the first wire bisects the star *b*, when the second wire will be moved to *b′* so that *ab′* will be equal to twice the distance between the stars; and then, by turning the micrometer screw belonging to the second wire, bring the latter wire back to the star *a*: read the number of revolutions made by this wire, and the parts of a revolution on its screw head; then half the difference between this and the former reading will be the required distance, in terms of the screw's revolutions.

120. The knowledge of the distance *ab* in seconds of a degree, and of the angle P*ab*, or its supplement, will enable the observer, by letting fall *bc* perpendicularly on P*a*, produced if necessary, to compute *ac* and *be*: the former is the difference between the declinations of the stars *a* and *b*, and the latter, when reduced to the corresponding arc of the equator, that is, when divided by the cosine of the star's declination (art. 70.), is the difference (in arc) between their right ascensions.

If one of the celestial bodies were the moon, whose declination at times changes rapidly; on making the wire *mn* of the position micrometer a tangent to the path of the moon's upper or lower edge as she appears to move across the field of view, that wire, instead of being parallel to the equator, would, as in the last example, be in some other position, as *mn*, and

the lower part of which was an iron ring whose plane was perpendicular to the axis of the telescope, and this ring floated on mercury in an annular vessel, so that the telescope being in a vertical position, the observer looked towards the floor, or towards the cieling of the apartment, through the open space in the centre of the vessel. The former was called a horizontal, and the latter a vertical collimator.

124. In using the horizontal collimator, the vessel, with the telescope floating on the mercury, is placed either on the north or south side of the astronomical circle; and the telescope of this circle being brought by hand nearly in a horizontal position, the axes of the two telescopes are made nearly to coincide in direction, with the object-glass of one turned towards that of the other. The collimating telescope, which is about twelve inches long, is provided with two wires crossing each other at the focus of the object glass, and care is taken moreover that it shall be in the optical axis: now the rays of light in the pencils which diverge from those wires are, after refraction in passing through the object glass, made to proceed from thence parallel to one another; and consequently, falling in this state upon the object glass of the circle, they form in its focus a distinct image of the wires from whence they diverged. Thus the observer, on looking into the telescope of the circle, sees the wires of both telescopes distinctly; and, upon making the points of their intersections coincident, it will follow that a line joining those intersections is parallel to the horizon: consequently the graduation at which the index stands on the circle may be considered as the zero of the instrument when altitudes are to be observed.

The vertical collimator is used in a similar manner, and the wires of both telescopes being made coincident, a line joining them is in a vertical position: the index, therefore, stands then at the graduation which must be considered as zero when zenith distances are to be observed.

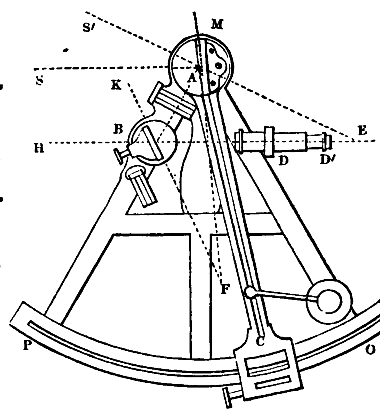
125. These floating collimators being unsteady, and the optical axes of their telescopes being seldom exactly horizontal or vertical, a telescope mounted on a stand, with its optical axis carefully adjusted, and furnished with a spirit-level which is capable of being reversed on its points of support, has been used for the like purpose. The disposition of the instrument is similar to that which has been described, and the axis of the telescope is made level by means of foot-screws.

In order to use a collimating telescope for the purpose of finding the error of collimation in a transit telescope the latter must be furnished with a micrometer: the moveable wire in the micrometer must be brought to coincide with the

or CS, the graduation at E will express twice the value of Ef or four times the required angle SCS′; and the above process may be repeated any number of times.

129. The reflecting octant, first brought into use by Mr. Hadley, as well as the reflecting sextant, quintant and circle, are now from their portability, and the convenience of being used when held in the hand, generally employed for the purpose of obtaining by observation the angular altitude of a celestial body above the horizon, or the angular distance between the moon and the sun, or a star. These instruments are, however, so well known as to render a minute description of them unnecessary; and therefore only a brief explanation of their nature and adjustments will be given.

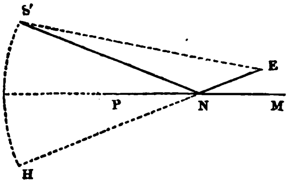
The subjoined figure represents the usual sextant; and the optical principle on which all such instruments are constructed is the same. Each carries on its surface, which is supposed to coincide with the plane of the paper, a mirror at A, and at B a glass which is in part transparent, and in part a mirror, the line of division being parallel to the plane of the instrument: both the glasses are perpendicular to the plane of the instrument; but B is fixed, and A is capable of being turned on an axis perpendicular to that plane by the motion of an index bar AC.



130. The arch OP of the instrument is graduated, and when the plane of the mirror A passes through AO, the index upon it is, or should be, at the zero of the graduations, and the glasses at A and B should be parallel to one another. In this state, if a pencil of light coming from a remote object at S fall upon A, it will be reflected to the quicksilvered part of B, and from thence be reflected in the direction BE parallel to SA; therefore if the eye of the observer be in the line BE, a pencil of light coming from the same object S will pass through the transparent part of B and enter the eye in the same direction as the reflected pencil; and the direct and reflected images of S will appear to coincide; the distance of S being great enough to render the angle at S between the directions of the pencils insensible.

is called the dip (art. 164.) of the horizon. On land the instrument is held so that the line E H may pass through the image of the sun when seen by reflection in quicksilver; and then the observed angle is equal to twice the altitude of the celestial body above the horizon.

For imagine a vertical plane to pass through S′ and E, and through the surface of the quicksilver, cutting the latter in M P; then the axis S′N of a pencil of light being reflected at N to the eye of the spectator at E, the reflected image of S′ will appear to be at H in the line E N produced; but the angle S′N P will by optics be equal to E N M or to the vertically opposite angle H N P; therefore S′N H = 2 S′N P. The line E N being very short compared with the distance of S′ from the spectator, the angle S′E N may be considered without error as equal to S′N H, or double the apparent altitude.



In taking the altitude of the sun or moon above the sea-line, it is customary to move the index till the lower edge, or *limb*, of the celestial body is a tangent to that line, because the contact of the limb with the sea-line can be more correctly distinguished than the coincidence of the centre of the disk with that line; in this case, there is obtained the altitude of the lower limb, and to this must be added the angle subtended by the visible semidiameter of the luminary, in order to have the altitude of its centre. On land, where an artificial horizon of quicksilver must be used, it is customary, for a like reason, to move the index till the upper or lower edge of the disk seen in the quicksilvered part of the horizon glass B is made to coincide with the lower or upper edge of the disk seen in the artificial horizon; and thus there is obtained twice the altitude of the upper or lower limb: the angular measure of the semidiameter must consequently be subtracted from, or added to half the angle obtained from the observation, in order to have the altitude of the centre.

134. A reflecting circle cannot measure an angle much exceeding the greatest which may be measured by a sextant (about 120 degrees); but it has an advantage over the latter instrument, since, by means of its three indexes, the value of the observed angle may be read on as many different parts of the circumference, and thus the errors of the graduation are diminished; the angle between two fixed objects may also be observed twice, by turning the mirror A in contrary directions, and thus an error in its position may be eliminated.

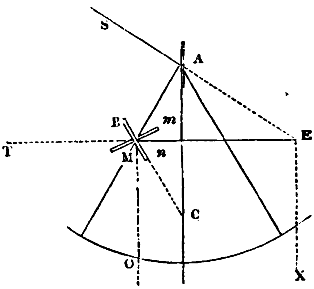
135. In the reflecting octant, sextant, or circle, the index glass A and the horizon glass B should be perpendicular to the plane of the instrument; and, with respect to the former glass, this position is verified by observing that, on looking obliquely into it, the parts of the arch which are seen by direct view and by reflection, are coincident. The perpendicularity of the horizon glass is proved by looking at the direct and reflected images of the sun when the index is at or near zero (the plane of the instrument being vertical), and observing that the eastern and the western limbs of both images are respectively coincident.

136. When the two images entirely coincide with one another, the index of the vernier should be at the zero point of the graduated arch: if this be not the case, the deviation of the index from that point is called the index error, and it may be determined by taking half the difference between the angles read on the arc, when the lower edge of the sun seen directly is placed in contact with the upper edge of the sun seen by reflection, and again when the upper edge of the former is placed in contact with the lower edge of the latter.

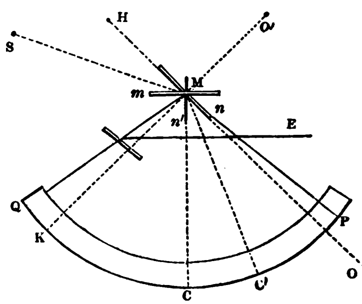
137. The remaining correction consists in placing the optical axis, or the line of collimation, in the telescope, parallel to the plane of the instrument. This condition is verified by bringing in contact, upon the wire nearest to the plane of the instrument, two celestial objects, as the sun and moon, when distant 90 or 100 degrees from each other; and then, by a general motion of the instrument bringing the objects upon the wire which is parallel to the former. If the contact remains good, the edges of the images having crossed in the centre of the field, the position of the telescope is correct: if the images separate on being brought to the second wire, the object glass is too near the plane of the instrument: a contrary error exists if the images overlap each other at the second wire. The correction is to be made by means of screws in the ring which carries the telescope.

138. The desire of measuring angles greater than those of 120 degrees, by means of reflecting instruments, has led to the construction of sextants in several different ways for the purpose of obtaining this end. Captain Fitzroy uses a sextant in which the quicksilvered part of the horizon glass is divided into two portions by a plane parallel to that of the instrument, the upper part being fixed so as to make a constant angle, suppose a right angle, with the lower: by this contrivance, if the instrument be held vertically, the image of an object above or below the horizon, or, if held horizontally,

on the right or left hand of another, is seen in the field of view when the index is at zero; so that on moving the index till the two objects appear to be in contact, the whole angle is obtained by adding the constant angle to that which is expressed on the graduated arch. Thus, let B*n* be that part of the divided horizon glass which is parallel to the index glass A when the index is at the zero of the graduations, and let M*m* be the part which makes a constant angle (90 degrees) with the former: then if E be the place of the observer's eye, O be one of the objects, and S the other, the former object will, after one reflection, appear in the mirror M, in the direction ET; and by moving the index to such a position, as AC, that the object S may appear after two reflections, as usual, to coincide with the image of O in the line ET, the arc of a great circle of the sphere between O and S will be measured by the sum of the angles OMT and SET. Therefore, if EX be supposed to be parallel to OM, and the distance of O be so remote that ME subtends at O no sensible angle, the angular distance of O from S will be equal to SEX.

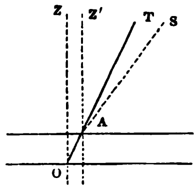


139. Captain Beechey has the horizon glass fixed in the usual way, but his index glass is divided into two parts M*n* and M*m* by a plane parallel to that of the instrument; one part as M*m* is fixed, and the other turns on its axis with the motion of the index bar. By placing the eye at a sight vane, or telescope, at E opposite the horizon glass, angles to the extent of about 120 degrees may be taken as usual, the degrees being read on a graduated arch from P towards Q. On the limb of the instrument, there is a second arc concentric with the former and graduated like it, but the numbers on this arc proceed in a contrary direction, or from Q towards P. There is a sight vane, or telescope, at K at the distance of about 95 degrees from P, the zero of the former arc; and to this the eye being applied, an object O on the right of the observer, if the instrument is



sum of several different angles: in this case, the time must be registered when each contact was made, that is, when each angle was taken; and then, the change of the angle being supposed to be uniform during the continuance of the observations, the whole arc passed over by the index being divided by the number of observations, gives an angle which corresponds to the mean of all the registered times of the observations.

142. An approximation to the law of refraction in terms of the apparent distance of the celestial body from the zenith of the observer is thus investigated: supposing the atmosphere to be of uniform density, the earth to be a plane, and the upper surface of the atmosphere parallel to that plane. Let O be the place of the observer, S the true place of a star, SA the incident ray, and AO the course of the ray in the atmosphere: produce OA to T; then T may represent the apparent place of the star. Let ZO be a vertical line passing through the spectator, and Z′A be parallel to it; then Z′AS is the angle of incidence, Z′AT, or its equal ZOT, the angle of refraction and also the apparent zenith distance of the star. The angle TAS is called the refraction. Let the angle ZOT be represented by z, and TAS by r; then, Z′AS$=z+r$. Now, by Optics, the sine of the angle of incidence is to the sine of the angle of refraction in a constant ratio, the medium remaining the same; therefore $\frac{\sin.(z+r)}{\sin. z}$ is constant: let it be represented by a. Then



$$a=\frac{\sin.(z+r)}{\sin. z}, \text{or (Pl. Tr. art. 32.)}\ a=\frac{\sin. z \cos. r+\cos. z \sin. r}{\sin z.}$$

whence $a \sin. z=\sin. z \cos. r+\cos. z \sin. r$, or

$$\frac{(a-\cos. r)\sin. z}{\cos. z.}=\sin. r;\ \text{or again, } (a-\cos. r)\tan. z=\sin. r.$$

But since r is a very small angle, let radius $(=1)$ be put for cos. r and r for sin. r; then, $(a-1)\tan. z=r$, or the refraction varies as the tangent of the apparent zenith distance of the celestial body.

Delambre, on developing the value of a in terms of tan. $\frac{1}{2}r$, found that tan. $\frac{1}{2}r$ may be represented, on the above hypothesis, by the series

$$\text{A} \tan. z+\text{B} \tan.^3 z+\text{C} \tan.^5 z+\&c.;$$

and he shows that all the formulæ which had before his time been employed, on the supposition that the surface of the earth and the strata of the atmosphere are spherical, might be brought to that form. The coefficients A, B, C, &c., are supposed to be determined by comparisons of observations.

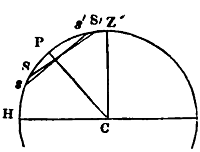
143. By direct experiments on the refractive power of the air we have $a=1.00028$ (Barom. = 30 in., Fahrenheit's Thermom. $=50°$): therefore,

$$a-1=0.00028, \text{ or, in seconds, } (=\frac{0.00028}{\sin. 1''}=57''.817;$$

and the formula $r=57''.817$ tan. z is very near the truth when the zenith distance does not exceed forty-five degrees.

144. Dr. Bradley, guided probably by a comparison of observations with the above formula, was led to adopt one which may be represented by $r=m$ tan. $(z-nr)$; and he found that the observations were represented by making $m=56''.9$, $n=3$. Subsequently MM. Biot and Arago, from numerous observations found $m=60''.666$ and $n=3.25$, the latitude of the observer being 45°, the temperature of the air $=32°$ Fahr., and the pressure of the atmosphere being represented by the weight of a column of mercury $=30$ in. Eng. And, lastly, Mr. Groombridge (Phil. Trans. 1824) by a mean of the refractions obtained from the observed meridian altitudes of sixteen circumpolar stars, found m (commonly called the constant of refraction) $=58''.133$ and $n=3.634$. In a paper on "Astronomical Refractions" in the Memoirs of the Astronomical Society (vol. ii. part 1.) Mr. Atkinson, by an investigation conducted on optical principles, found for the value of refraction the expression $57''.754$ tan. $(z-3.5937\ r)$, the barometer being at 29.6 inches and Fahrenheit's thermometer at 50°. The table of refractions in the Nautical Almanacs from 1822 to 1833 was computed from a formula given by Dr. Young in the "Philosophical Transactions" for 1819.

145. Refractions deduced from formulæ, without regard to the density and temperature of the air, are called mean refractions, and these differ from that which depends merely on tan. z, chiefly on account of the curvature of the earth and of the atmospherical strata: in order to obtain their values from observations in latitudes corresponding to those of the greater part of Europe, the circumpolar stars may be employed in the following manner: let ZHO, supposed to be in the celestial sphere, represent part of the meridian of the observer at C; let Z be his zenith, and P the pole of the equator; and let S, S′ be the apparent places of any star at the lower and upper culmination.



Also, let $ZS=z$, $ZS'=z'$, and let r and r' ($=Ss$ and $S's'$) be the corresponding refractions; then, $z+r$ and $z'+r'$ are the true zenith distances at the two times of culminating; and since at those times, after the corrections for refraction have been applied, the star is equally distant from the pole, $\frac{1}{2}(z+r+z'+r')$ or $\frac{1}{2}(z+z'+r+r')$ will be equal to PZ, the colatitude of the station; therefore $2PZ-(z+z')=r+r'$. Hence, PZ being accurately known, also z and z' being given by the observations, and the value of r' (the refraction at the upper culmination) being computed from the formula $r' = 57''.817$ tan. z', which will be sufficiently near the truth, the value of r may be obtained.

TABLE OF REFRACTIONS.

Alt.	Refract.	Var.	Bar.	Therm.	Alt.	Refract.	Var.	Bar.	Therm.
° ′	′ ″	″	′ ″	″	° ′	′ ″	″	″	″
0 0	33 51	11.7	1 14	8.1	6 0	8 32	1.2	17.2	1.15
5	32 53	11.3	1 11	7.6	10	8 20	1.2	16.8	1.11
10	31 58	10.9	1 9	7.3	20	8 9	1.1	16.4	1.09
15	31 5	10.5	1 7	7.0	30	7 58	1.1	16.0	1.06
20	30 13	10.1	1 5	6.7	40	7 47	1.0	15.7	1.03
25	29 24	9.7	1 3	6.4	50	7 37	1.0	15.3	1.00
30	28 37	9.4	1 1	6.1	7 0	7 27	1.0	15.0	.98
35	27 51	9.0	59	5.9	10	7 17	.9	14.6	.95
40	27 6	8.7	58	5.6	20	7 8	.9	14.3	.93
45	26 24	8.4	56	5.4	30	6 59	.8	14.1	.91
50	25 43	8.0	55	5.1	40	6 51	.8	13.8	.89
55	25 3	7.7	53	4.9	50	6 43	.8	13.5	.87
1 0	24 25	7.4	52	4.7	8 0	6 35	.7	13.3	.85
10	23 13	6.9	49	4.5	10	6 28	.7	13.1	.83
20	22 8	6.3	46	4.2	20	6 21	.7	12.8	.82
30	21 7	5.9	44	3.9	30	6 14	.7	12.6	.80
40	20 10	5.5	42	3.6	40	6 7	.7	12.3	.79
50	19 17	5.1	39	3.4	50	6 0	.6	12.1	.77
2 0	18 29	4.8	38	3.2	9 0	5 54	.6	11.9	.76
10	17 43	4.4	36	3.0	10	5 47	.6	11.7	.74
20	17 0	4.1	35	2.8	20	5 41	.6	11.5	.73
30	16 21	3.9	33	2.7	30	5 36	.6	11.3	.71
40	15 43	3.6	32	2.6	40	5 30	.5	11.1	.71
50	15 8	3.4	31	2.4	50	5 25	.5	11.0	.70
3 0	14 35	3.2	30	2.3	10 0	5 20	.5	10.8	.69
10	14 4	3.0	29	2.2	10	5 15	.5	10.6	.67
20	13 35	2.8	28	2.1	20	5 10	.5	10.4	.65
30	13 7	2.7	27	2.0	30	5 5	.5	10.2	.64
40	12 41	2.5	26	1.9	40	5 0	.5	10.1	.63
50	12 16	2.4	25	1.8	50	4 56	.4	9.9	.62
4 0	11 52	2.2	24.1	1.7	11 0	4 51	.4	9.8	.60
10	11 30	2.1	23.4	1.65	10	4 47	.4	9.6	.59
20	11 10	2.0	22.7	1.58	20	4 43	.4	9.5	.58
30	10 50	1.9	22.0	1.53	30	4 39	.4	9.4	.57
40	10 32	1.8	21.3	1.48	40	4 35	.4	9.2	.56
50	10 15	1.7	20.7	1.43	50	4 31	.4	9.1	.56
5 0	9 58	1.6	20.1	1.38	12 0	4 28	.4	9.0	.55
10	9 42	1.5	19.6	1.34	10	4 24	.4	8.9	.55
20	9 27	1.5	19.1	1.30	20	4 21	.4	8.8	.54
30	9 11	1.4	18.6	1.26	30	4 17	.3	8.6	.54
40	8 58	1.3	18.1	1.22	40	4 14	.3	8.5	.53
50	8 45	1.3	17.6	1.19	50	4 11	.3	8.4	.52

Alt.		Refract.	Var.	Bar.	Therm.	Alt.	Refract.	Var.	Bar.	Therm.
°	′	′ ″	″	″	″	°	″	″	″	″
13	0	4 7.5	0.30	8.30	0.51	48	52.3	0.031	1.75	0.104
	30	3 58.4	.29	8.0	.49	49	50.5	.030	1.69	.101
14	0	3 49.9	.28	7.7	.47	50	48.8	.029	1.63	.097
	30	3 41.8	.26	7.43	.45	51	47.1	.028	1.58	.094
15	0	3 34.3	.24	7.18	.44	52	45.4	.027	1.52	.090
	30	3 27.3	.22	6.95	.42	53	43.8	.026	1.47	.088
16	0	3 20.6	.21	6.73	.41	54	42.2	.026	1.41	.085
	30	3 14.4	.20	6.51	.40	55	40.8	.025	1.36	.082
17	0	3 8.5	.19	6.31	.39	56	39.3	.025	1.31	.079
	30	3 2.9	.18	6.12	.37	57	37.8	.025	1.26	.076
18	0	2 57.6	.17	5.98	.36	58	36.4	.024	1.22	.073
	30	2 52.5	.17	5.70	.35	59	35.0	.024	1.17	.070
19	0	2 47.7	.16	5.61	.34	60	33.6	.023	1.12	.067
	30	2 42.9	.16	5.46	.33	61	32.3	.022	1.08	.065
20	0	2 38.7	.15	5.31	.32	62	31.0	.022	1.04	.062
21	0	2 30.5	.13	5.04	.30	63	29.7	.021	.99	.060
22	0	2 23.2	.12	4.79	.29	64	28.4	.021	.95	.057
23	0	2 16.5	.11	4.57	.28	65	27.2	.020	.91	.055
24	0	2 10.1	.10	4.35	.26	66	25.9	.020	.87	.052
25	0	2 4.2	.09	4.16	.25	67	24.7	.020	.83	.050
26	0	1 58.8	.09	3.97	.24	68	23.5	.020	.79	.047
27	0	1 53.8	.08	3.81	.23	69	22.4	.020	.75	·045
28	0	1 49.1	.08	3.65	.22	70	21.2	.020	.71	.043
29	0	1 44.7	.07	3.50	.21	71	19.9	.020	.67	.040
30	0	1 40.5	.07	3.36	.20	72	18.8	.019	.63	.038
31	0	1 36.6	.06	3.23	.19	73	17.7	.018	.59	.036
32	0	1 33.0	.06	3.11	.19	74	16.6	.018	.56	.033
33	0	1 29.5	.06	2.99	.18	75	15.5	.018	.52	.031
34	0	1 26.1	.05	2.88	.17	76	14.4	.018	.48	.029
35	0	1 23.0	.05	2.78	.17	77	13.4	.017	.45	.027
36	0	1 20.0	.05	2.68	.16	78	12.3	.017	.41	.025
37	0	1 17.1	.95	2.58	.15	79	11.2	.017	.38	.023
38	0	1 14.4	.05	2.49	.15	80	10.2	.017	.34	.021
39	0	1 11.8	.04	2.40	.14	81	9.2	.017	.31	.018
40	0	1 9.3	.04	2.32	.14	82	8.2	.017	.27	.016
41	0	1 6.9	.04	2.24	.13	83	7.1	.017	.24	.014
42	0	1 4.6	.04	2.16	.13	84	6.1	.017	.20	.012
43	0	1 2.4	.04	2.09	.12	85	5.1	.017	.17	.010
44	0	1 0.3	.03	2.02	.12	86	4.1	.017	.14	.008
45	0	58.1	.03	1.94	.12	87	3.1	.017	.10	.006
46	0	56.1	.03	1.88	.11	88	2.0	.017	.07	.004
47	0	54.2	.03	1.81	.11	89	1.0	.017	.03	.002

The columns entitled " Alt." contain the apparent altitudes of a celestial body as they are obtained from the observations after being corrected for the index error; and immediately on their right hand are the columns containing the refractions corresponding to the altitudes, the barometer being at 30 inches, and Fahrenheit's thermometer at 50°. The columns headed " Var." contain the variations of the refraction for one minute of altitude; and the number opposite the nearest degree or minute in the column of altitudes is to be multiplied by the difference between the

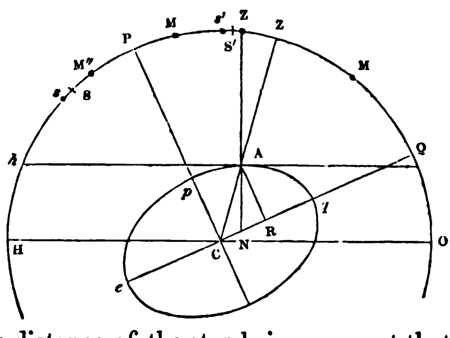
But when $\theta = 90°$, the decrement (CA—CB) is equal to the difference between the refraction of C and the refraction of B: let this be found from the table of refractions, and represented in seconds by a; then $a \sin.^2\theta$ will express the excess of CA above CM, or the quantity which must be subtracted from the horizontal semi-diameter of the sun or moon in order to have the inclined semi-diameter. The values of the said excess for every fifth or tenth degree of altitude, and every fifteenth degree of θ are given in the following table.

Inclination of the oblique to the horizontal semidiameter.	Altitudes of the Sun or Moon.												
	0	5°	10°	15°	20°	25°	30°	40°	50°	60°	70°	80°	90°
	Decrements of the oblique semidiameters of the Sun or Moon.												
°	′ ″	″	″	″	″	″	″	″	″	″	″	″	
0	0 0	0	0	0	0	0	0	0	0	0	0	0	0
15	0 11.8	1.7	0.5	0.2	0.16	0.1	0.07	0.04	0.03	0.02	0.02	0.02	0
30	0 44.1	6.2	2.	0.9	0.6	0.3	0.28	0.16	0.11	0.09	0.07	0.06	0
45	1 28.2	12.5	4.	1.8	1.2	0.7	0.56	0.32	0.23	0.18	0.15	0.13	0
60	2 11.6	18.5	5.9	2.7	1.8	1.1	0.8	0.5	0.34	0.27	0.23	0.2	0
75	2 44.7	23.3	7.5	3.4	2.2	1.3	1.04	0.6	0.43	0.34	0.29	0.25	0
90	2 56.5	25.	8.	3.7	2.4	1.45	1.12	0.64	0.46	0.37	0.31	0 27	0

150. The latitude of an observatory or station is an element of practical astronomy which can be determined without any knowledge of the movements of the celestial bodies beyond the fact of the diurnal rotation, and, except the effects of refraction, without any data from astronomical tables: it is merely necessary to be provided with a mural, or any circle which can be placed in the plane of the meridian; or, for the ordinary purposes of geography, a sextant with an artificial horizon.

The observations required for determining this element, are the altitudes or zenith distances of any circumpolar star at the times when it culminates, or comes to the meridian of the station above or below the pole: and, if a circle which is capable of being reversed in azimuth be used, the zenith distance in each position may be taken; that is, with the graduated face of the circle towards the east, and again with that face towards the west, by which double process the error of collimation may be eliminated. The pole star is one which may be advantageously employed in the northern hemisphere, for this purpose, as the slowness of its motion will allow it to be observed on the same night, by direct view and also by reflexion.

Now, let C be the centre of the earth, A the place of an observer on its surface, and HPQ the circumference of the



observer's meridian in the celestial sphere: also let h A, a tangent to the earth's surface at A, in the plane of the meridian, represent the position of the observer's horizon; and let S and S′ be the apparent places of a fixed star at its lower and upper culmination. The distance of the star being so great that the effect of parallax is insensible, so that h A may be conceived to coincide with HC, drawn parallel to it through the centre of the earth, the apparent altitudes will be HS and HS′; and the effects of refraction at the lower and upper culmination being represented by Ss and S′s', s and s' become the true places of the star, and Hs, Hs' the true altitudes: then Ps being equal to Ps', half the sum of the true altitudes is equal to HP.

151. If the earth were a sphere, a plumb line suspended at A would take the direction AC, passing through C, the centre of the sphere, and if produced upwards it would meet the heavens in Z: this line ZAC would be perpendicular to hA or HC, and the angle HCP would be equal to ZCQ, which expresses the latitude of A. But, if it be assumed that the earth is a spheroid, and if eAq be an elliptical meridian, a plumb line at A would, on account of the equality of the attractions on all sides of a normal line AN, take the direction of that line, and NA being produced would meet the celestial sphere in Z′: the latitude obtained from the observation will in this case be expressed by the angle Z′Nq, or the arc Z′Q. This is called the nautical, or geographical latitude; while, Z being the geocentric zenith, the angle ZCQ, or the arc ZQ, is called the geocentric latitude.

152. To investigate a formula for reducing one of these kinds of latitude to the other, the following process may be used. Draw the ordinate AR; then by conic sections, we have $\quad$ Cq^2 : Cp^2 :: CR : NR;

and in the right-angled triangles ACR, ANR,

CR : NR :: tang. CAR : tang. NAR,

or CR : NR :: cotan. ACR : cotan. ANR (tan. ANR : tan. ACR).

Hence $\quad$ Cq^2 : Cp^2 :: tang. ANR : tang. ACR.

Let the equatorial and polar semiaxes, Cq and Cp, be to one another as 305 to 304, which from geodetical determinations (arts. 415. 418.) is the ratio adopted by Mr. Woolhouse in the appendix to the Nautical Almanac for 1836 (p. 58.): then we shall have $\frac{\mathrm{C}p^2}{\mathrm{C}q^2} = .9934$; therefore if the geographical latitude be represented by L, and the geocentric latitude by l, we have .9934 tang. L = tang. l.

The second line in the following table shows, for every tenth degree of geographical latitude, the number of minutes, &c., which should be subtracted from that latitude in order to obtain the corresponding geocentric latitude.

10°	20°	30°	40°	45°	50°	60°	70°	80°
′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″
3 45.5	7 21	9 54	11 16.2	11 27.3	11 17	9 56	7 23	3 56

153. The word parallax, is used to express the angle at any celestial body between two lines drawn from its centre to the points from whence it may be supposed to be viewed: or it is the arc of the celestial sphere between the two places which a body would, at the same instant, appear to occupy if it were observed at two different stations. The celestial arc between the sun and moon, or between the moon and a star, at any instant, will evidently subtend different angles at the eyes of two observers at different stations on the surface of the earth; and in order that a common angle may measure the same arc, it is necessary that each observed angle should be reduced to that which would be subtended by the same arc at the centre of the earth. When the arc measures the altitude of a celestial body above the visible horizon, the correction which must be applied in order to convert the observed angle of elevation to that which would have been obtained if the angular point had been at the centre of the earth, is called the diurnal parallax. This name has been given to it because it goes through all its variations between the times at which the body rises and sets, being greatest when the latter is in the horizon, and least when in the plane of the meridian; and since every parallax is necessarily in a plane passing through the two points of observation and the object, it is evident that the diurnal parallax will be in a plane passing through the spectator and the centre of the earth; that is, in a vertical plane.

154. The angular distance between two places to which, in the heavens, a celestial body is referred when it is supposed to be viewed from the sun and from the earth, or from the earth at two different points in its orbit, is called the annual

or, if z' express the true zenith distance,

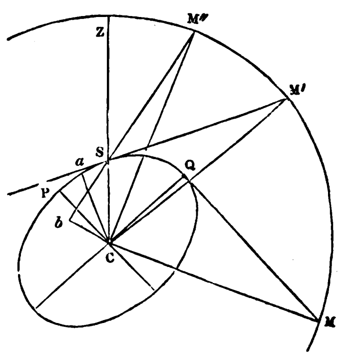
$$\sin. p = \sin. \text{P} \sin. (z' + p).$$

It may be observed that the greatest parallax of a celestial body takes place when the body is in the horizon; and, both above and below the horizon, the parallax diminishes with the sine of the distance from the zenith. The apparent place of a celestial body is always given by observation; but, in seeking the effect of parallax when the true place of the body is obtained by computation from astronomical tables, it will be convenient first to find the parallax approximatively from the formula $p' = \text{P} \sin. z'$ (where z' is the true zenith distance), and then to substitute that approximate value in the formula $\sin. p = \sin. \text{P} \sin. (z' + p')$; the result will in general be sufficiently near the truth. Or a second approximative value may be found from the formula $p'' = \text{P} \sin. (z' + p')$; and the value of p'' being substituted in the formula $\sin. p = \sin. \text{P} \sin. (z' + p'')$ will give a still more correct value of the parallax in altitude. The following series for the parallax in altitude in terms of the horizontal parallax and the true zenith distance is given in several treatises on astronomy:

$$p \text{ (in seconds)} = \frac{\sin. \text{P}}{\sin. 1''} \sin. z' + \frac{\sin.^2 \text{P}}{\sin. 2''} \sin. 2z' + \frac{\sin.^3 \text{P}}{\sin. 3''} \sin. 3z' + \&\text{c.}$$

and the three first terms are sufficient.

156. If it be assumed that the earth is of a spheroidal figure, and such as would be produced by the revolution of an ellipse about its minor axis, it must follow that the horizontal parallaxes, which are the angles formed at a celestial body by a semi-diameter passing through the place of the observer, will be the greatest at the equator and the least at the poles; and that they will vary with the distance of the spectator from the centre of the earth. The value of the geocentric horizontal parallax at any station whose latitude is given may be investigated in the following manner. Let PSQ be one quarter of the terrestrial meridian passing through any station S; also let P be the pole, and Q the point in which the meridian cuts the equator: again, let M and M′ be places of any celestial body,



sin. ZAM + sin. Z′BM : rad. :: AMB (in arc, or in seconds) : hor. par. (in arc or seconds) :

and thus the horizontal parallax of the moon or planet at the time of observation is found. It should be observed that this method is not applicable to planets beyond the orbit of Mars; and if the observers be not situated precisely on the same terrestrial meridian, it would be necessary to correct the observed zenith distance of the moon or planet, at one of the stations, on account of the variation in its declination during the time in which it is passing from one meridian to the other.

Since the horizontal parallax of a celestial body may be represented by the angle SMC (arts. 154. 156.), in which the angle at S is, or may be, considered as a right angle, and that SC = MS tan. SMC; it follows that, for different celestial bodies, the tangents of the horizontal parallaxes vary inversely with the distances of the bodies from the earth.

161. The parallax of a celestial body in altitude being obtained, the deviation of the apparent from the true place of the body in any other direction (as far as it depends upon the place of the observer) may be readily found by Plane Trigonometry, when the angle at the celestial body between a vertical circle passing through it and a circle of the sphere, also passing through it, in the direction for which the deviation is required, is known; since that deviation may be considered as one side of a right-angled triangle, of which the hypotenuse is the parallax in altitude: but this is not always the most convenient method of determining the deviation in a direction oblique to the vertical circle; and the following are investigations of formulæ, by which the parallaxes of a celestial body in right ascension and declination (that is, in the directions which are particularly required for the solution of problems relating to practical astronomy), are generally obtained.

Let ♈HQ (fig. to art. 147.) be part of the celestial equator, stereographically projected, P its pole, and let Z be the geocentric zenith of the spectator's station. Let S′ be the true place of the celestial body, suppose the moon, and ♈ the equinoctial point, so that at a given time ♈H′ is the moon's true right ascension (in arc) and PS′ her true north polar distance. At the same instant let S be the apparent place of the moon, so that ♈H is the moon's apparent right-ascension (in arc), and PS her apparent polar distance. Then PQ being the meridian of the station, QPS′ is the moon's true, and QPS her apparent, horary angle at the given time. Again, ZS′ is the true, and ZS is the apparent, zenith distance of the

Now, in order to eliminate ZS and ZS′, we have (art. 61.), in the spherical triangles, ZPS, ZPS′,

$$\sin. \text{ZS} : \sin. \text{PS} :: \sin. \text{ZPS} : \sin. \text{Z} \left(= \frac{\sin. \text{PS} \sin. \text{ZPS}}{\sin. \text{ZS}}\right),$$

$$\text{and} \sin. \text{ZS}' : \sin. \text{PS}' :: \sin. \text{ZPS}' : \sin. \text{Z} \left(= \frac{\sin. \text{PS}' \sin. \text{ZPS}'}{\sin. \text{ZS}'}\right).$$

Equating these values of sin. Z, we obtain

$$\frac{\sin. \text{ZS}'}{\sin. \text{ZS}} = \frac{\sin. \text{PS}' \sin. \text{ZPS}'}{\sin. \text{PS} \sin. \text{ZPS}}:$$

therefore, by substitution,

$$(\cos. \text{PS} - \cos. \text{PZ} \sin. \text{P}') \frac{\sin. \text{PS}' \sin. \text{ZPS}'}{\sin. \text{PS} \sin. \text{ZPS}} = \cos. \text{PS}';$$

$$\text{or } \left(\text{cotan PS} - \frac{\cos. \text{PZ} \sin. \text{P}'}{\sin. \text{PS}}\right) \frac{\sin. \text{ZPS}'}{\sin. \text{ZPS}} = \text{cotan. PS}';$$

that is,

$$\left(\tan. \text{D} - \frac{\sin. l \sin. \text{P}'}{\cos. \text{D}}\right) \frac{\sin. (\tau + a)}{\sin. \tau} = \tan. (\text{D} - \delta).$$

Thus, D being known, we might obtain δ, the parallax in declination; but, as in the process, logarithms with seven decimals must be employed, it would be advantageous to have a formula for δ alone.

The equation for cotan. PS′ may be put in the form

$$\text{cotan. PS} \sin. \text{ZPS}' - \text{cotan. PS}' \sin. \text{ZPS} = \frac{\cos. \text{PZ} \sin. \text{P}' \sin. \text{ZPS}'}{\sin. \text{PS}};$$

and dividing the first member by cotan. PS − cotan. PS′, it may be put in the form

$$\left\{\sin. \text{ZPS}' + \frac{\text{cot. PS}' (\sin. \text{ZPS}' - \sin. \text{ZPS})}{\text{cotan. PS} - \text{cotan. PS}'}\right\} (\text{cot. PS} - \text{cot. PS}') =$$
$$\frac{\cos. \text{PZ} \sin. \text{P}' \sin. \text{ZPS}'}{\sin. \text{PS}} \dots (\text{A}).$$

But (Pl. Trigon., art. 41.)

$$\sin. \text{ZPS}' - \sin. \text{ZPS} = 2 \sin. \tfrac{1}{2} (\text{ZPS}' - \text{ZPS}) \cos. \tfrac{1}{2} (\text{ZPS}' + \text{ZPS}),$$

$$\text{or} \qquad = 2 \sin. \tfrac{1}{2} a \cos. (\tau + \tfrac{1}{2} a),$$

$$\text{and cotan. PS} - \text{cotan. PS}' = \frac{\sin. (\text{PS}' - \text{PS})}{\sin. \text{PS} \sin. \text{PS}'};$$

therefore, by substitution, the equation (A) becomes

$$\frac{\sin. (\text{PS}' - \text{PS})}{\sin. \text{PS} \sin. \text{PS}'} \sin. \text{ZPS}' + \text{cotan. PS}' \, 2 \sin. \tfrac{1}{2} a \cos. (\tau + \tfrac{1}{2} a) =$$
$$\frac{\cos. \text{PZ} \sin. \text{P}' \sin. \text{ZPS}'}{\sin. \text{PS}};$$

Astron. Soc., vol. iv. part 2.), that the apparent dip or the angle h A L is expressed by the formula $\frac{1}{\sin. 1''}\left(\frac{2h}{r}\cdot\frac{n-2}{n}\right)^{\frac{1}{2}}$, where $\frac{1}{n}$ is the value of the refraction in terms of the angle A C K (in general $n=12$ as above stated). If n were equal to, or less than 2, it would follow from the formula that the dip then becomes zero or imaginary: and in these cases the surface of the sea is visible as far as objects on it can be distinguished, or the water and sky appear to be blended together so that no sea-line appears to separate them.

CHAP. VI.

DETERMINATION OF THE EQUINOCTIAL POINTS AND THE OBLIQUITY OF THE ECLIPTIC BY OBSERVATION.

167. WHEN the observer has succeeded in obtaining the latitude of his station, he is prepared with an astronomical circle, to ascertain the apparent declinations of fixed stars, of the sun, the moon, and the planets. For, supposing the observer to be in the northern hemisphere, it will be evident from an inspection of the figure in art. 150., in which HZO represents half the meridian, Z the zenith, P the pole, and Q the place where the equator cuts the meridian, that if any celestial body culminate south of the zenith, as at M, its observed distance ZM from the zenith at the time of culmination being subtracted from ZQ, the geocentric latitude of the station will give MQ the required declination; and this will be either north or south of the equator, according as the remainder is positive or negative. If it culminate north of the zenith and above the pole, as at M′, the declination will be equal to the sum of the observed zenith distance and the latitude of the station; and lastly, if it culminate north of the zenith and below the pole, as at M″, the declination will be equal to the supplement of such sum.

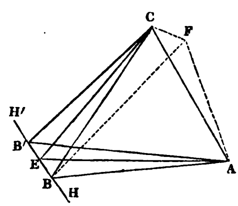
168. The declinations of those which are called fixed stars are not always the same; for independent of certain proper motions in the stars themselves, the plane of the equator, from which the declinations are reckoned, changes its position in the celestial sphere by the effects of planetary attraction on the terrestrial equator; but the declinations of the sun, moon, and planets are subject to considerable variations.

169. If the celestial body be the sun, the observations of the declinations may be considered as the first step in the determination of the elements of its apparent motions; for let it be imagined that, by means of the zenith distances observed during a whole year, beginning, for example, at midwinter, and continuing to the next succeeding midwinter, the declinations of the sun are obtained every time that the luminary arrives at the meridian of the station; the corrections on account of the errors of the instrument, and the effects of refraction and parallax being applied, on comparing such declinations one with another it will be found, that at midwinter

the more convenient to employ them as they are, or ought to be always present to the mind of the calculator. But since they involve operations which are frequently laborious, and which require a minute attention to accuracy at every step, mathematicians have found it necessary to investigate formulæ in which the differences only between the observed and reduced angles may enter; and thus the reductions are effected by the employment of logarithms extending to a few places of decimals, while the errors of the calculation have small influence on the result. The practice of using such formulæ, when possible, in preference to the direct processes, prevails, indeed, throughout the whole of the calculations connected with geodetical, as well as astronomical operations.

395. The following is an investigation of a formula for reducing any observed angle to the centre of a station.

Let A, B, C be the centres of three stations, and let E (near B) be the place of observation. Here AB and the angle BAC are supposed to be accurately known by previous operations; EB is supposed to be measured, and the angles CEB, CEA to be observed; and it is required to deter- the angle CBA.



Produce EB indefinitely towards H; then the angle CBH = CEB + ECB, and angle ABH = AEB + EAB: whence by subtraction, we have

$$\text{the angle CBA} = \text{CEA} + \text{ECB} - \text{EAB} \dots . (\text{A}).$$

Now (Pl. Trigo., art. 57.)

$$\text{AB : BE :: sin. AEB : sin. EAB} \left(= \frac{\text{BE}}{\text{AB}} \text{ sin. AEB}\right), \text{ and}$$

$$\text{CB : BE :: sin. CEB : sin. ECB} \left(= \frac{\text{BE}}{\text{CB}} \text{ sin. CEB}\right); \text{ also}$$

$$\text{sin. ACB : sin. CAB :: AB : CB} \left(= \text{AB}\frac{\text{sin. CAB}}{\text{sin. ACB}}\right).$$

Substituting this value of CB, we obtain

$$\text{sin. ECB} = \frac{\text{BE sin. CEB sin. ACB}}{\text{AB sin. CAB}};$$

and hence, sin. ECB − sin. EAB =

$$\frac{\text{BE}}{\text{AB}} \cdot \frac{\text{sin. CEB sin. ACB} - \text{sin. AEB sin. CAB}}{\text{sin. CAB}}.$$

the angular point and the two distant stations, it must be considered as appertaining to a spheroidal triangle: consequently the sum of the three must exceed two right angles by a certain quantity which, when the sides are several miles in length, becomes sensible; and should the observed excess be equal to that which is determined by computation, the correctness of the observations would thereby be proved. In the tract on Spherical Geometry (prop. 20.) it is shown that the area of a spherical triangle is equal to the rectangle contained by half the diameter of the sphere and a line equal to the difference between half the circumference and the sum of the arcs which measure the angles of the triangle, and this may be expressed by the formula $\text{A} = \text{R}\,(a+b+c-\pi)$; where A is the area of the triangle, R the radius of the earth supposed to be a sphere, $\pi = 3.14159$; and a, b, c are the several arcs (expressed in terms of a radius = 1) which measure the angles of the triangle: hence

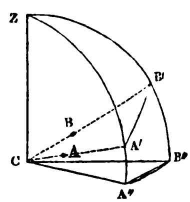
$$a+b+c-\pi \text{ (the required excess) } = \frac{\text{A}}{\text{R}}.$$

In employing the theorem for the purpose of verifying the observed angles of any triangle on the surface of the earth, the radius of the latter may be considered as equal to 20886992 feet. And the area of the triangle being generally expressed in square feet, the value of the spherical excess found by the above formula would be expressed in feet (linear measure): therefore in order to obtain the excess in seconds, the term $\frac{\text{A}}{\text{R}}$ should be divided by R sin. 1″ (equal to an arc, in feet, subtending an angle of one second at the centre of the earth), and thus the excess in seconds will be equal to $\frac{\text{A}}{\text{R}^2 \sin. 1''}$. It may be observed that the area A is supposed to be computed in square feet from the ascertained lengths of the sides, by the rules of mensuration, as if the surface of the triangle were a plane.

397. In the French surveys, instead of a theodolite, the repeating circle was used for taking the angles between the stations; and when all these were not equally elevated above the general surface of the earth, considered as a sphere or spheroid, it became necessary to reduce the observed, to horizontal angles. For this purpose it was merely required to find the angles, at the zenith of each station, in a spherical triangle of which the three sides are given: viz. the angular distance between the other stations and the angular distances

of these last from the zenith; and a rule for determining such angle at the zenith has been given in art. 69. But the following investigation leads to a formula by which the difference between the observed angular distance of two stations and the corresponding horizontal angle may with ease and accuracy be obtained.

Let A and B be the two stations between which the observed angle is taken, and C the place of the observer: then Z being the zenith of C in the celestial sphere, let ZCA″, ZCB″ be parts of two vertical circles passing through C and the two stations. Let CA″B″ be the plane of the observer's horizon, and imagine the lines CA, CB to be produced to A′ and B′ in the heavens: then A′CB′ is the observed angle and A″CB″, or the spherical angle at Z, is the required horizontal angle.



Now let the observed angle A′CB′ be represented by C, and the observed altitudes A′A″, B′B″ by a and b respectively; then, by either of the formulæ (a), (b), or (c), art. 60.,

$$\cos. Z = \frac{\cos. C - \sin. a \sin. b}{\cos. a \cos. b};$$

or since the arcs a and b are small, substituting a and b for their sines, and $1-\frac{a^2}{2}$ and $1-\frac{b^2}{2}$ for their cosines (Pl. Trigo., art. 46.), we have, rejecting powers of a and b higher than the second, $\cos. Z = \dfrac{\cos. C - ab}{1-\frac{a^2}{2}-\frac{b^2}{2}}$, or again, by division,

$$\cos. Z = \cos. C - ab + \tfrac{1}{2}(a^2+b^2)\cos. C.$$

Next, let Z be represented by C$+h$; then (Pl. Trigo., art. 32.) $\cos. Z = \cos. C \cos. h - \sin. C \sin. h$: but because h is small, the arc may be substituted for its sine and its cosine may be considered as equal to unity; therefore

$$\cos. Z = \cos. C - h \sin. C:$$

equating these values of cos. Z, we get

$$h \sin. C = ab - \tfrac{1}{2}(a^2+b^2)\cos. C,$$

or
$$h = \frac{ab}{\sin. C} - \tfrac{1}{2}(a^2+b^2)\ \text{cotan.}\ C.$$

In this equation, a, b, and h are supposed to be expressed in arcs; therefore, in order to have the value of h in seconds when a and b are expressed in seconds, each of the two terms in the second member of the equation must be multiplied by sin. 1″. Hence the value of Z may be found.

If one of the stations as B were in the horizontal line CB″, we should have $b=0$, and in this case $h = -\frac{a^2}{2}$ cotan. C.

398. In computing the sides of triangles formed by the principal stations on the earth's surface, when all the angles have been observed and one side measured or previously determined, three methods have been adopted. The first consists in treating the triangles as if they were on the surface of the sphere and employing the rules of spherical trigonometry. In the second method the three points of each triangle are imagined to be joined by lines so as to form a plane triangle; then, the given side being reduced to its chord and the spherical angles to those which would be contained by such chords, the other chord lines are computed by plane trigonometry, and subsequently converted into the corresponding arcs of the terrestrial sphere or spheroid. The third method consists in subtracting from each spherical angle one third of the *spherical excess*, and thus reducing the sum of the three angles of each triangle to two right angles; then, with the given terrestrial arc as one side of a plane triangle, computing the remaining sides by plane trigonometry, and considering the sides thus computed as the lengths of the terrestrial arcs between the stations. The first and third methods present the greatest facilities in practice, and all may be considered as possessing equal accuracy.

399. The computations of the sides of the triangles by the first of the above methods might be performed by making the sines of the sides proportional to the sines of the opposite angles: but in so doing a difficulty is felt on account of the imperfection of the logarithmic tables; for the sides of the triangles being small, the increments of the logarithmic sines are so great as to render it necessary that second differences should be used in forming the correct logarithms. This labour may be avoided by using a particular formula, which is thus investigated:

Let a be any terrestrial arc expressed in terms of radius $(=1)$; then (Pl. Trigo., art. 46.)

sin. $a = a - \frac{1}{6}a^3$ (rejecting higher powers of a)

$$\text{or } \frac{\sin. a}{a} = 1 - \tfrac{1}{6}a^2.$$

Now $\cos. a = (1 - \sin.^2 a)^{\frac{1}{2}}$; hence $\cos.^{\frac{1}{3}} a = (1 - \sin.^2 a)^{\frac{1}{6}}$:

or, by the binomial theorem, $\cos.^{\frac{1}{3}} a = 1 - \frac{1}{6} a^2$ (rejecting as before).

Therefore $\frac{\sin. a}{a} = \cos.^{\frac{1}{3}} a$ very nearly: and if a were expressed in seconds,

$$\frac{\sin. a}{a \sin. 1''} = \cos.^{\frac{1}{3}} a; \text{ or } \sin. a = a \sin. 1'' \cos.^{\frac{1}{3}} a;$$

or, in logarithms,

$$\log. \sin. a = \log. a + \log. \sin. 1'' + \tfrac{1}{3} \log .\cos. a \;.\;.\;.\; (\text{A}).$$

From this formula, while a (in seconds) is less than 4 degrees we may obtain the value of log. sin. a with great precision as far as seven places of decimals. Therefore, in any terrestrial triangle whose sides (in seconds) are represented by a, b, and c, and whose angles, opposite to those sides, are A, B, and C, if a were a given side, and all the angles were given by observation or otherwise, we should have (in order to find one of the other sides, as b, from the theorem sin. A : sin. a :: sin. B : sin. b (art. 61.)), by substituting the above value of log. sin. a,

$$\log. a + \log. \sin. 1'' + \tfrac{1}{3} \log. \cos. a + \log. \sin. \text{B} - \log. \sin. \text{A} = \log. \sin. b;$$

in which, since a is very small, its cosines vary by very small differences, and log. cos. a may be taken by inspection from the common tables. From the value of log. sin. b so found, that of b (in seconds) may be obtained by an equation corresponding to (A) above: thus

$$\log. b = \log. \sin. b - \log. \sin. 1'' - \tfrac{1}{3} \log. \cos. b;$$

in which, since b is very small, log. cos. b may be found in the common tables, that number being taken in the column of cosines, which is opposite to the nearest value of log. sin. b in the column of sines.

If a were expressed in feet it might be converted into the corresponding arc in terms of radius $(=1)$ on dividing it by R, the earth's mean radius in feet: and into seconds by the formula

$$\frac{a \text{ (in feet)}}{\text{R} \sin. 1''} = a \text{ (in seconds)}.$$

400. A formula for the reduction of any small arc of the terrestrial sphere to its chord may be investigated in the following manner: Let a be a small arc expressed in terms of radius $(=1)$; then (Pl. Trigon., art. 46.) we have

$$\sin. \tfrac{1}{2} a = \tfrac{1}{2} a - \tfrac{1}{8} \frac{a^3}{6} \text{ (rejecting powers of } a \text{ above the third)};$$

hence

$2 \sin. \frac{1}{2} a$, or chord . a, $= a - \frac{1}{24} a^3$, and $a -$ chord . $a = \frac{1}{24} a^3$: an equation which holds good whether a and chord . a be expressed in terms of radius ($= 1$) or in feet.

If a were given in seconds we should have $\frac{1}{24} (a \sin. 1'')^3$ equal to the difference between the arc and its chord in terms of radius ($= 1$); or in logarithms,

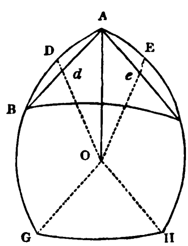
$3 (\log. a + \log. \sin. 1'') - \log. 24 = \log. (a -$ chord . $a)$ in terms of rad. ($=1$);

or if a were given in feet we should have $\frac{1}{24} \left(\frac{a}{R}\right)^3$ for the value of the same difference; or in logarithms,

$3 (\log. a - \log. R) - \log. 24 = \log. (a -$ chord . $a)$ in terms of rad. ($= 1$).

401. The reduction of an angle of a spherical triangle to the corresponding angle between the chords of the sides which contain it, may be thus effected:

Let the curves AB, AC be two terrestrial arcs constituting sides of the triangle ABC, and let their chords be the right lines AB, AC: let O be the centre of the sphere, and draw OG, OH parallel to AB, AC: draw also OD and OE to bisect the arcs in D and E, cutting the chords AB and AC in d and e. Then, since the angles AOB, AOC are bisected by Od and Oe, the angles AdO and AeO are right angles; therefore the alternate angles DOG, EOH will also be right angles, and DG, EH will each be a quadrant; also the arc GH or the angle GOH will be equal to the plane angle BAC between the chord lines. The sides AG, AH and the included spherical angle at A being known, the arc GH which measures the reduced or plane angle BAC may be computed by the rules of spherical trigonometry (as in art. 64.); and in like manner the angles between the chords at B and C might be computed. The excess of each spherical angle above the corresponding angle of the plane triangle formed by the chords of the terrestrial arcs is thus separately found; and it is evident that the sum of the three reduced angles will be equal to two right angles if the spherical angles have been correctly observed.



402. The third method of computing the sides of terrestrial triangles is the application of a formula which was inves-

tigated by Legendre, who, in seeking what must be the angles of a plane rectilineal triangle having its sides equal to those of a triangle on the terrestrial sphere, arrived at the conclusion that, neglecting powers of the sides higher than the third, each of the angles of the former triangle should be equal to the corresponding angle of the spherical triangle diminished by one third of the spherical excess found as above shown (art. 396.). This proposition may be demonstrated in the following manner. (See Woodhouse, "Trigonometry.") Let A, B, and C, as in the above figure, be the angles of a spherical triangle, and a, b, c expressed in terms of radius ($=1$) be the sides opposite to those angles; also let p represent half the sum of those sides. Then, (art. 66. (II)),

$$\cos.^2 \tfrac{1}{2} \text{A}, \text{ or } \tfrac{1}{2}(1+\cos. \text{A}) = \frac{\sin. p \sin. (p-a)}{\sin. b \sin. c}:$$

developing (Pl. Trigon., art. 46.) the second member as far as the third powers of p, a, b, c, we get

$$\tfrac{1}{2}(1+\cos. \text{A}) = \frac{(p-\frac{1}{6}p^3)\{(p-a)-\frac{1}{6}(p-a)^3\}}{bc(1-\frac{1}{6}(b^2+c^2))} =$$

$$\frac{p(p-a)}{bc}\left\{1-\tfrac{1}{6}(p^2+(p-a)^2\right\}.\frac{1}{1-\frac{1}{6}(b^2+c^2)};$$

$$\text{or } = \frac{p(p-a)}{bc}\left\{1-\tfrac{1}{6}(p^2+(p-a)^2)\right\}(1+\tfrac{1}{6}(b^2+c^2)),$$

$$\text{or again, } = \frac{p(p-a)}{bc}\left\{1-\tfrac{1}{6}(p^2+(p-a)^2-b^2-c^2)\right\}.$$

But $p = \frac{1}{2}(a+b+c)$ and $p-a = \frac{1}{2}(b+c-a)$:

substituting these values in the second co-efficient of $\frac{p(p-a)}{bc}$,

the numerator of that co-efficient will be found to be equal to twice the product of $\frac{1}{2}(a+c-b)$ and $\frac{1}{2}(a+b-c)$, that is to $2(p-b)(p-c)$: therefore

$$\tfrac{1}{2}(1+\cos. \text{A}) = \frac{p(p-a)}{bc} - \frac{p(p-a)(p-b)(p-c)}{3bc} \dots \text{A})$$

Now, in a plane triangle whose sides are expressed by a, b, c, and of which the angles opposite to those sides are represented by A′, B′, C′, we have (Pl. Trigon., art. 57.)

$$\cos.^2 \tfrac{1}{2} \text{A}', \text{ or } \tfrac{1}{2}(1+\cos. \text{A}') = \frac{p(p-a)}{bc};$$

also, by the rules of mensuration, the square of its area is equal to $\frac{1}{4}b^2c^2\sin.^2 A'$, and to $p(p-a)(p-b)(p-c)$; therefore

$$\tfrac{1}{4}b^2c^2\sin.^2 A' = p(p-a)(p-b)(p-c), \text{ or } \tfrac{1}{12}bc\sin.^2 A' = \frac{p(p-a)(p-b)(p-c)}{3bc}.$$

Substituting the equivalents in equation (A) we have

$$\tfrac{1}{2}(1+\cos. A) = \tfrac{1}{2}(1+\cos. A') - \tfrac{1}{12}bc\sin.^2 A',$$

or $$\cos. A = \cos. A' - \tfrac{1}{6}bc\sin.^2 A'.$$

Again, assuming $A = A' + h$, we have (Pl. Trigon., art. 32.)

$$\cos. A = \cos. A' \cos. h - \sin. A' \sin. h,$$

or, since h is very small, $\cos. A = \cos. A' - h \sin. A'$:

hence $h \sin. A' = \frac{1}{6}bc\sin.^2 A'$, or $h = \frac{1}{6}bc\sin. A'$.

But, as above, $\frac{1}{2}bc\sin. A'$ represents the area of a plane triangle of which the curve lines a, b, c, considered as straight are the sides: consequently

$$h = \tfrac{1}{3}(\text{area}), \text{ and } A = A' + \tfrac{1}{3}(\text{area}) \text{ or } A' = A - \tfrac{1}{3}(\text{area}).$$

In like manner $B' = B - \frac{1}{3}$(area), and $C' = C - \frac{1}{3}$(area).

Now the area of such triangle considered as on the surface of the sphere, and the radius of the latter being unity, has been shown to be equal to the excess of the arcs which measure the spherical angles, above half the circumference of a circle (the arcs being expressed in terms of radius $= 1$): hence, in employing the third method above mentioned, each angle of the spherical triangle on the surface of the earth must be diminished by one third of the spherical excess, in order to obtain the corresponding angle of the plane triangle, in which the lengths of the straight sides are equal to the terrestrial arcs whether expressed in seconds or in feet.

403. When a base line has been measured, or when any side of a triangle has been computed, it becomes necessary to reduce it to an arc of the meridian passing through one extremity of the base or side; and therefore the angle which the base or side makes with such meridian must be observed. For this purpose, as well as with the view of obtaining the latitude of a station by the meridian altitude, or zenith distance, of a celestial body, or of making any other of the observations which depend on the meridian of the station, the position of that meridian should be determined with as much accuracy as possible. Since the Nautical Almanacs now give at once the apparent polar distances and right ascensions of the principal stars, it is easy (art. 312.) to compute the moment at which any one of these will culminate; and a first

approximation to the position of a meridian line on the earth's surface may be made in the following manner.

A well-adjusted theodolite having a horizontal and a vertical wire in the focus of the object glass may have its telescope directed to the star a little before the time so computed; then, causing the vertical wire to bisect the star, keep the latter so bisected by a slow motion of the instrument in azimuth till the moment of culmination. At that instant the telescope is in the plane of the meridian; and the direction of the latter on the ground may be immediately indicated by two pickets planted vertically in the direction of the telescope when the latter is brought to a horizontal position.

But, employing a transit telescope or the great theodolite with which the terrestrial angles for geodetical surveys are taken, a more correct method of obtaining the position of the meridian is that of causing the central vertical wire of the telescope to bisect the star α Polaris at the time of its greatest eastern or western elongation from the pole; that is, about six hours before or after the computed time of culminating; and then, having calculated the azimuthal deviation of the star from the meridian, that deviation will be the angle between the plane of the meridian and the vertical plane in which the telescope moves. Consequently, by the azimuthal circle of the theodolite, the telescope can be moved into the plane of the meridian; whose position may then, if necessary, be correctly fixed by permanent marks.

404. To obtain the star's azimuth at the time of its greatest elongation from the pole, let N P Z be the direction of the meridian in the heavens; Z S that of the vertical circle in which the telescope moves at the time of such elongation: let P be the pole, S the star, and Z the zenith of the station. Then, in the spherical triangle Z P S, right angled at S, we have (art. 60. (*e*)),

$$\text{Rad. sin. PS} = \text{sin. PZ sin. PZS},$$

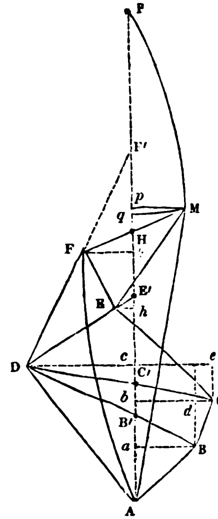
and P Z S is the required azimuth. The above method possesses the advantage of being free from any inaccuracy which, in the former method, might arise on account of the error of the clock; since the star at S appears for a moment stationary in the telescope, and consequently it can be bisected with precision at that moment by the meridional wire. The method of bringing a transit telescope, or that of a great theodolite, accurately to the meridian, at any time, or of determining its deviation from thence, has been explained in arts. 94, 95, 96.

405. When a meridian line had not been previously de-

termined, the method employed in the English surveys for obtaining the angle which the measured base, or any side of a triangle, made with the meridian, was to compute from the elements in the Nautical Almanac the moment when the star α Polaris was on the meridian, or when it was at the greatest eastern or western elongation; then, at the place where the angle was to be observed, having first directed the telescope of the theodolite to the star, at the moment, the telescope was subsequently turned till the intersection of the wires fell on the object which marked the other extremity of the base or side. With a telescope capable of showing the star during the day, should it come in the plane of the meridian while the sun is above the horizon, this angle may be thus taken; or it may be obtained at night, when the star culminates after sun-set, if a luminous disk be used to indicate the place of the station whose bearing from the meridian is required. When the observation was made at the time of either elongation, the azimuthal deviation of the star, computed as above, was either added to, or subtracted from the observed azimuth according as the star was on the same, or on the opposite side of the meridian with respect to the station; and the sum, or difference, was of course the required azimuth of the latter: but if the bearings of the station were observed at both elongations of the star, half their sum expressed immediately the required bearing. Another method of obtaining the azimuth of a terrestrial object is given in art. 341. By some of the continental geodists a well-defined mark or, by night, a fire-signal was set up very near the meridian of the station whose azimuth was to be obtained; then, by means of circumpolar stars or otherwise, they obtained the correct azimuthal deviation of that mark from the meridian; and the sum or difference of the deviation, and the observed angle between the mark and the station, was consequently, equal to the required azimuth.

406. Different processes have been employed for reducing the sides of the triangles to the direction of a meridian passing through a station at one extremity of the series. Among the most simple is that which was adopted by M. Struve in measuring an arc of the meridian from Jacobstadt on the Dwina to Hochland in the Gulf of Finland. It consisted in computing all the sides of the triangles as if they were arcs of great circles of the sphere, either by the rules of spherical trigonometry or by the theorem of Legendre above mentioned (arts. 398, 402.), one of them as AB being the measured base. Then, C, D, E, F, &c. being stations, imagining A and F, A and M, &c. to be joined by great circles; in the

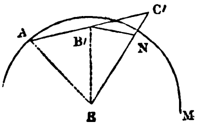
triangles ADF, AFM, &c. each of the arcs AF, AM, &c. were found by means of the two sides previously determined and the included angles ADF, AFM. Finally, the azimuthal angle PAD having been obtained by observations, the angle PAM was computed: and imagining a great circle M*p* to be let fall perpendicularly on the meridian AP, the meridional arc A*p* was calculated in the right-angled spherical triangle AM*p*.



A second process is that of computing by spherical trigonometry, or by the method of Legendre (art. 402.), the sides of the triangles; and then, in like manner, the lengths of the meridional arcs AB′, B′C′, &c. between the points where the sides of the triangles, produced if necessary, would cut the meridian. For the latter purpose an azimuthal angle, as PAB, must be observed: then, in the spherical triangle ABB′, there would be given AB and the two angles at A and B; to find AB′, BB′ and the angle B′. Again, in the triangle B′DC′, there would be given DB′ (the difference between the computed sides BD and BB′) the observed angle at D and the computed angle at B′; to find C′B′, DC′ and the angle C′. In like manner, in the triangle DC′E′, may be found DE′, C′E′ and the angle E′. In the triangle E′F′D, formed by producing DF till it cuts the meridian in F′; with DE′ and the angles at D and E′ may be computed E′F′, DF′ and the angle F′. Lastly, letting fall F*h* perpendicularly on the meridian, in the right angled triangle F*h*F′, with FF′ (=DF′—DF) and the angles at F and F′, the arc *h*F′ may be determined; and thus with E′*h* (=E′F′—F′*h*) and AB′, B′C′, C′E′, before computed, the length of the meridional arc from A to *h* is obtained.

407. When the sides of the triangles are the chords of the spherical arcs, and are computed by the rules of plane trigonometry, the following process is used for reducing those sides to the meridian. After calculating the chords AB, AD, BD, &c., the distance AB′ and the angle AB′B are determined by means of the angle PAB. Subsequently, with the angle DB′C′ (=AB′B), the side B′C′ is computed in the triangle B′DC′, and this is to be considered as a continuation of the former line AB′, which is supposed to have nearly the position of a chord within the earth's surface; but the plane of the triangle ABD being considered as horizontal, that of the next triangle BCD will, on account of the curvature of the

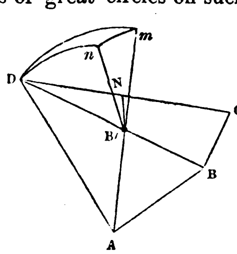
earth, be inclined in a small angle to the former plane: therefore, in order to reduce the computed line B′C′ to the plane of BCD, and allow AB′ and the reduced line to retain the character of being two small portions of the geodetical meridian, that line B′C′ may be supposed to turn on the point B′ as if it moved on the surface of a cone of which B′ is the vertex and DB′ or B′B the axis (that is, so as not to change the angle which it makes with DB′) till it falls into the plane BCD. Thus let AM be part of the periphery of the terrestrial meridian passing through A in the former figure, and let AB′ be the position of the chord AB′; then the first computed value of B′C′ may, in the annexed figure, be represented by B′C′, which is in the plane of the meridian and of the triangle ABD. And when, by the conical movement above mentioned, the line B′C′ comes into the plane of the triangle BCD in the former figure, it will have nearly the position of a chord line, and may be represented by B′N which terminates at N in a line imagined to be drawn from C′ to the centre of the earth.* Since the inclination of the plane ABD to that



* It must be remembered that (agreeably to what is stated in art. 387.) a geodetical meridian is a curve line the plane of which is every where perpendicular to the tangent plane or the horizon, at every point on the earth's surface through which it passes; and unless the earth be considered as a solid of revolution, the geodetical meridian is a curve of double curvature: the error which arises from considering it as a plane curve is, however, not sensible.

Now, if AB′, B′C′ (in the above figure, and in the fig. to art. 406.) be considered as two small portions of the geodetical meridian, the vertical planes passing through these lines should be, respectively, perpendicular to the horizons at the middle points of the triangles ABD, BCD. Let it be granted that the vertical plane passing through AB′ is perpendicular to the plane of the triangle ABD; and let it be required to prove that while B′C′ (fig. to art. 406.) in the plane BCD makes the angle DB′C′ equal to the angle AB′B, in the plane ABD, the vertical plane passing through AB′ and B′C′ may be considered as perpendicular to the plane BCD.

Imagine, in the annexed figure, a sphere to exist having its centre at B′ and any radius as B′D; and let Dm, Dn be arcs of great circles on such sphere, the former in the plane ABD and the latter as much below the plane BCD as Dm is above it: let also B′m be the prolongation of AB′ in the plane ABD produced. Then, by the manner in which B′m was supposed to revolve to the position B′n (keeping the angle DB′m or AB′B equal to DB′n) the spherical triangle Dmn is evidently isosceles, and a great circle passing through D, bisecting mn, will cut mn at right angles. Let B′N be in the plane of this circle; it will also be in the plane BCD,



and the latter will be cut perpendicularly by the plane passing through of BCD is very small, the triangle B′NC′ may be considered as right angled at N, and the angle C′B′N as that between a tangent at B′ and a chord line drawn from the same point; consequently (Euc. 32. 3.) as equal to half the angle B′EN, or half the estimated difference of latitude between the points B′ and N. Therefore B′N (the reduced value of B′C) can be found; and in a similar manner the reduced values of C′E′, &c. may be computed. The first station A is on the surface of the earth; but the points B′, C′, &c., after the above reductions, are evidently below the surface: therefore the meridional arcs appertaining to the chords AB′, B′N′, &c. should be increased by quantities which are due to the distances of the points B′, N, &c. from the said surface.

When the sides of the triangles have been computed by Legendre's method (art. 402.), and the azimuthal angle between a station line and the plane of the meridian passing through one of its extremities has been observed; if perpendiculars be let fall from the stations to the meridian (the stations not being very remote from thence on the eastern or western side), the lengths of the perpendiculars and of the meridional arc intercepted between any station as A, and the foot of each perpendicular, may also be computed by the rules of plane trigonometry. For since the computed lengths of the station lines are equal to the real values of those lines on the surface of the earth, though the lines be considered as straight, the lengths of the arcs A*a*, A*b*, B*a*, &c. (fig. to art. 406.), computed from them (one third of the spherical excess for each triangle being subtracted from each angle in the triangle), will be the true values of those arcs. Consequently the whole length of the meridional arc A*p* will be correct. The following is an outline of the steps to be taken for the determination of the length of a meridional arc, as A*p*, by perpendicular arcs let fall upon it from the principal stations and by arcs coinciding with the meridian, or let fall perpendicularly on the others from the several stations.

Let AB (fig. to art. 406.) be the measured base; PAB the azimuthal angle observed at A; and let B*a*, C*b*, &c. be the perpendiculars let fall from the stations B, C, &c. on the meridian AP: then, in the triangle A*a*B, we have AB (supposed to be expressed in feet), the angle *a*AB and the right angle at

A, B′, *m*, *n*: or the plane B′*mn*, which is the plane of the geodetical meridian passing through B′C′, may be considered as at right angles to the plane BCD. In like manner the plane passing through the next portion C′E′ of the geodetical curve, in the plane of the triangle CDE, may be considered as at right angles to the plane of that triangle; and so on.

a; to find Aa and Ba. In like manner in the triangle DAc, we have AD, the right angle at c and the angle DAc (equal to the difference between the angles DAB and PAB); to find Dc and Ac. Again, imagining Bd to be drawn parallel to AP, in the triangle CdB we have BC, the right angle at d and the angle CBd (=ABC−ABa−aBd, the last being a right angle); to find Cd and Bd; thus we obtain Ab (=Aa+Bd) and Cb (Cd+aB). In the triangle DCe, we have DC, the right angle at e and the angle DCe (the complement of DCb); to find De and Ce: hence we obtain Ac and Dc a second time. The values may be compared with those which were determined before; and if any difference should exist, a mean may be taken. In the like manner the computation may be carried on to the end of the survey; and the whole extent of the meridional arc from A to p as well as the lengths of the several perpendiculars may be found.

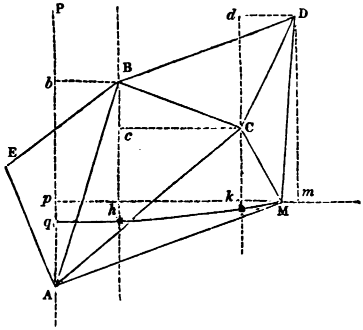
But at intervals in the course of the survey other azimuthal angles as PMF must be obtained by observation: then, since the angle FMp will have been found from the preceding computations, and the angle PMp by the solution of the right angled spherical triangle PMp; the sum of these two may be compared with the observed azimuth, and the accuracy of the preceding observations may thus be proved. The angle PHM being computed in the spherical triangle PHM, that azimuthal angle may be employed to obtain the meridional arcs and the perpendiculars beyond the point H.

The process above described is particularly advantageous when it is intended to make a trigonometrical survey of a country as well as to determine the length of an extensive meridional arc; for the spherical latitudes and longitudes of the stations A, B, C, &c. might be found from the above computations, and thus the situations of the principal objects in the country might be fixed. For this purpose it is convenient to imagine several meridian lines to be traced at intervals from each other of 30 or 40 miles; and to refer to each, by perpendiculars, the several stations in the neighbourhood. The lengths of these perpendiculars will not, then, be so great as to render of any importance the errors arising from a neglect of the spherical excess in employing the rules of plane trigonometry for the purpose of making the reductions to the several meridian lines.

408. If a chain of triangles be carried out nearly in the direction of an arc perpendicular to any meridian, the situations of the stations may, in like manner, be referred to that arc by perpendiculars imagined to be let fall on the latter; and the lengths of the arcs and of the perpendiculars may be

computed as before. The difficulty of obtaining the longitudes of places with precision is an objection to the employment of this method in the survey of a country; and the same objection exists to the measurement of an arc on a parallel of terrestrial latitude. The measured length of an arc on a perpendicular to a meridian, and on a parallel of latitude, have, however, been used in conjunction with the measured arc of the meridian at the same place, as means of determining the figure of the earth.

In a triangulation carried out from east to west, or in the contrary direction, the sides of the triangles may be computed as arcs of great circles of the sphere: then with these sides and the included angles, the distances AB, AC, AD, &c. of the several stations may be obtained by spherical trigonometry; and from the last of these as M, letting fall Mp perpendicularly on the meridian of A, the arcs Ap, Mp may be computed in the right angled spherical triangle ApM. There must subsequently be obtained the distance from M to q on the arc of a parallel circle, as Mq, drawn through M, and the distance from p to q on the meridian.



It has been shown in art. 71. that pq in seconds is approximately equal to $\frac{1}{4}$ P^2 sin. 2 PM sin. 1″ (fig. to that article), the radius of the sphere being unity. Now if the arc Mp, computed as above mentioned, were in feet, and MC the semidiameter of the earth be also expressed in feet; since $\frac{\text{M}p}{\text{MC}}$ is equal to the measure of the angle MCp at the centre, and that MC sin. MCp = MC′ sin. MC′q, each member being equal to MN; also, since MC′ = MC sin. PM, we have, considering Mp as an arc of small extent, and putting Mp in feet for MC sin. MCp, also for sin. MC′q putting its equivalent sin. P or P sin. 1″,

$$\frac{\text{M}p}{\text{MC sin. PM}} = \text{P sin. } 1''.$$

Therefore

$$pq \text{ sin. } 1'' \text{ (or } pq \text{ in arc, rad.} = 1) = \frac{\text{M}p^2 \text{ sin. 2 PM}}{4 \text{ MC}^2 \text{ sin.}^2 \text{ PM}}; \text{ or}$$

putting 2 sin. PM cos. PM for sin. 2 PM (Pl. Trigon., art. 35.), and l for the latitude of M, pq (in arc, rad. $=1$) $= \frac{\text{M}p^2}{2\ \text{MC}^2} \tan l$; and, in feet,

$$pq = \frac{\text{M}p^2}{2\ \text{MC}} \tan.\ l.$$

In the same article it has been shown that the difference between Mq and Mp (in arc, rad. $=1$) is approximatively equal to $\frac{1}{6}$ Mp^3 sin.^{3} 1″ tan.^{2} l. Now if the arc Mp were in feet, and MC the semidiameter of the earth be also expressed in feet, Mp sin. 1″ in the last expression, in which Mp is supposed to be in seconds, would be equivalent to $\frac{\text{M}p}{\text{MC}}$, and that expression would become $\frac{1}{6} \frac{\text{M}p^3}{\text{MC}^3}$ tan.^{2} l; therefore the difference between Mp and Mq in feet is, when Mp and MC are in feet, equal to $\frac{1}{6} \frac{\text{M}p^3}{\text{MC}^2}$ tan.^{2} l, by which quantity Mq exceeds Mp.

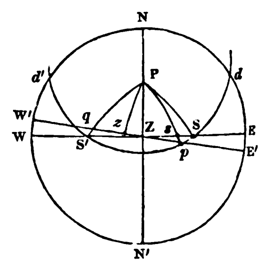
If, after the several distances AB, BC, CD, &c. have been computed in the triangulation, the latitudes of the stations and the bearings of the station lines from the terrestrial meridian passing through one extremity of each be observed or computed; the lengths of the several arcs of parallel circles, as Bb, Cc, Dd, &c., drawn from each station to the meridian passing through the next may be calculated and subsequently reduced to the corresponding arcs qh, hk, &c. on the parallel of terrestrial latitude Mq, which passes through any one, as M, of the stations. The sum of all such arcs will be the value of that whose length it may have been proposed to obtain.

Whether the chain of triangles extend in length eastward and westward, or in the direction of the meridian, the value of pq must be subtracted from the computed value of Ap in order to obtain the length of the meridional arc comprehended between A and the parallel of latitude passing through M: then the latitudes of A and M being determined by computation or found by celestial observations, the difference between the latitudes of A and M will become known, and such difference compared with the measured length of Aq will, by proportion, give the length of a degree of latitude at or near A. In like manner the difference between the longitudes of A and M, obtained by celestial observations, by chronometers or otherwise, if compared with the measured lengths of Mp and Mq, will, by proportion, give the lengths near A of an arc of one degree on a great circle perpendicular to the meridian and on a parallel of terrestrial latitude.

409. The usual method of finding the latitude of a station as A or M for geodetical purposes is similar to that which has been described in the chapter on Nautical Astronomy (art. 334.), some fixed star which culminates very near the zenith being employed, in order to avoid as much as possible the error arising from refraction; and the altitude or zenith distance being observed with a zenith sector (art. 107.). On the continent, however, lately, the latitudes of stations have been obtained from observed transits of stars at the prime vertical on the eastern and western sides of the meridian; and the following is an explanation of the process which may be used.

The observer should be provided with a transit telescope which is capable of being moved in azimuth, or with an altitude and azimuth circle: that which is called the horizontal axis should be accurately levelled, and the telescope should be brought as nearly as possible at right angles to the meridian. This position may be obtained by first bringing the telescope correctly to the meridian by the methods explained in arts. 94, 95.; and then turning it 90 degrees in azimuth by the divisions on the horizontal circle.

Let WNE represent the horizon of the observer, Z his zenith, and P the pole of the equator; also let NZN′ represent the meridian, WZE the prime vertical, dSS′d' part of the star's parallel of declination, and let S and S′ be the places of the star at the times of observation. Imagine hour circles to be drawn through P and S, P and S′; then PS, PS′, each of which is the star's polar distance, are known from the Nautical Almanac, and if the times of the transits be taken from a clock showing mean solar time, the interval must be converted into sidereal time by the table of time equivalents, or by applying the "acceleration:" the sidereal interval being multiplied by 15 gives the angle SPS′. From the equality of the polar distances this angle is bisected by the meridian, and the angles at Z are right angles; therefore, the effects of refraction being disregarded, we have in the right angled triangle PZS (art. 62. (f'))



$$\text{rad. cos. ZPS} = \text{cotan. PS tan. PZ},$$

and PZ is the required colatitude of the station.

But the true value of the angle ZPS is diminished by a small quantity depending on the change produced by refraction in the star's zenith distance, and a formula for determining the amount of the diminution may be thus investigated. In the right-angled triangle ZPS we have (art. 60. (e))

$$\sin. \text{ZS} = \sin. \text{ZPS} \sin. \text{PS}:$$

Differentiating this equation, considering PS as constant,

$$\cos. \text{ZS}\, d\text{ZS} = \cos. \text{ZPS}\, d\text{ZPS} \sin. \text{PS}.$$

But, from the equation, $\sin. \text{PS} = \frac{\sin. \text{ZS}}{\sin. \text{ZPS}}$, and this value of sin. PS being substituted in the differential equation, there is obtained

$$\frac{\cos. \text{ZS}}{\sin. \text{ZS}}\, d.\text{ZS} = \frac{\cos. \text{ZPS}}{\sin. \text{ZPS}}\, d\text{ZPS};$$

or $$\text{cotan. ZS}\, d\text{ZS} = \text{cotan. ZPS}\, d\text{ZPS};$$

whence $$d\text{ZPS} = \frac{\text{cotan. ZS}}{\text{cotan. ZPS}}\, d\text{ZS}, \text{ or } = \frac{\text{tan. ZPS}}{\text{tan. ZS}}\, d\text{ZS}.$$

And if for dZS we put the value of refraction corresponding to the star's altitude, the resulting value of dZPS may be added to that of the angle ZPS which was determined from the half interval between the observations in order to obtain the value of ZPS which should be employed in the above formula for PZ.

Ex. At Sandhurst, November 23. 1843, the interval between the transits of α Persei at the prime vertical was found to be, in sidereal time, 2 ho. 52′ 22″: consequently half the hour angle = 21° 32′ 45″. The star's north polar distance from the Nautical Almanac was 40° 41′ 50″.5, its apparent zenith distance by observation was 13° 51′ (nearly); and consequently its refraction = 14″.2 (= dZS). Let this be represented in the figure by sS, and the corresponding variation of the hour angle by sPs: then, by the formula above,

log. dZS (14″.2)	- -	1.152288
-log. tan. ZPs (21° 32′ 45″)		9.596415
-log. tan. ZS (13° 51′) co. ar.		0.608097
		1.356800
log. dZPS=		22″.74 (= SPs)
ZPs=21° 32′ 45″		
ZPS= 21 33 7.74		

In the triangle ZPS.		
log. cos. ZPS	- -	9.9685218
— log cotan. PS	- -	0.0654716
= log. tan. PZ	- -	9.9030502
= 38° 39′ 28″, the colatitude required.		

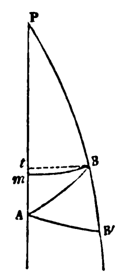
If the time at the station should be well known, the sidereal time corresponding to the middle of the interval between the times of observation may be computed; and this should agree with the sidereal time at which the star is on the meridian, that is, with the star's right ascension. If such agreement be not found to subsist, the error must arise from the transit telescope not being precisely in the prime vertical: let it be supposed that the telescope is in the plane of the vertical circle W′E′; then the places of the star at the times of observation will be p and q, and Pz let fall perpendicularly on W′E′ will denote the colatitude obtained from the above formula. The true colatitude PZ may then be found in the triangle PZz; for Pz has been obtained as above, the angle at z is a right angle, and the angle ZPz is equal to the difference between the star's right ascension and the sidereal time at the middle between the observations: therefore (art. 62. (f'))

$$\cos. \text{ZP}z = \tan. \text{P}z \ \text{cotan. PZ};$$

hence cotan. PZ $= \dfrac{\cos. \text{ZP}z}{\tan. \text{P}z}$; or, in this last expression, putting for tan. Pz its value $\dfrac{\cos. \text{ZPS}}{\text{cotan. PS}}$ before found, we have cotan. PZ$=\dfrac{\cos. \text{ZP}z \ \text{cotan. PS}}{\cos. \text{ZPS}}$, or tan. PZ $= \dfrac{\cos. \text{ZPS} \tan. \text{PS}}{\cos. \text{ZP}z}$.

The result immediately deduced from the observation must be reduced (art. 152.) to the geocentric latitude.

410. In the progress of a geodetical survey, it becomes necessary, frequently, to determine by computation the difference between the latitudes and the longitudes of two stations as A and B, when there are given the computed or measured arc AB, the latitude and longitude of one station as A, and the azimuthal angle PAB: and if the earth be considered as a sphere, the following processes may be employed for the purpose.



Let P be the pole of the earth, and let fall Bt perpendicularly on the meridional arc PA: then, by the usual rules of spherical trigonometry, we have

I. (art. 62. (f')) rad. cos. PAB = cotan. AB tan. At;
whence At and consequently Pt are found.

II. (art. 60. 3 Cor.) cos. AB cos. Pt = cos. At cos. PB;
whence PB the colatitude of B is found.

III. (art. 61.) sin. PB : sin. A :: sin. AB : sin. P;
and the angle APB is equal to the difference between the longitudes of A and B; and

IV. sin. PB : sin. A :: sin. AP : sin. ABP (= the azimuthal angle at B).

But avoiding the direct processes of spherical trigonometry, Delambre has investigated formulæ by which, the geodetical arc between the stations being given with the observed latitude and the azimuth, at one station, the differences of latitude and longitude between the stations, and the azimuth at the other station may be conveniently found.

Thus, as above, for the difference of latitude, let P be the pole of the earth, A a station whose latitude has been observed, and PAB the observed azimuth of B; it is required to find the side PB, and subsequently the angles at P and B. In the triangle PAB we have (art. 60. (*a*), (*b*), or (*c*))

$$\cos.\ \text{PB} = \cos.\ \text{A} \sin.\ \text{PA} \sin.\ \text{AB} + \cos.\ \text{PA} \cos.\ \text{AB}.$$

Now, let l be the known latitude of A; dl the difference between the latitudes of A and B ($=$ Am if Bm be part of a parallel of terrestrial latitude passing through B); then $l+dl$ is the latitude of B, and the equation becomes

$$\sin.\ (l+dl) = \cos.\ \text{A} \cos.\ l \sin.\ \text{AB} + \sin.\ l \cos.\ \text{AB},$$

or (Pl. Trigo., art. 32.)

$$\sin.\ l \cos.\ dl + \cos.\ l \sin.\ dl = \cos.\ \text{A} \cos.\ l \sin.\ \text{AB} + \sin.\ l \cos.\ \text{AB},$$

or again (Pl. Trigo., arts. 36, 35.)

$$\sin.\ l\ (1-2 \sin.^2 \tfrac{1}{2} dl) + 2 \cos.\ l \cos.\ \tfrac{1}{2} dl \sin.\ \tfrac{1}{2} dl = \cos.\ \text{A} \cos.\ l \sin.\ \text{AB} + \sin.\ l\ (1-2 \sin.^2 \tfrac{1}{2}\ \text{AB});$$

whence

$$-2 \sin.\ l \sin.^2 \tfrac{1}{2} dl + 2 \cos.\ l \cos.\ \tfrac{1}{2} dl \sin.\ \tfrac{1}{2} dl = \cos.\ \text{A} \cos.\ l \sin.\ \text{AB} - 2 \sin.\ l \sin.^2 \tfrac{1}{2} \text{AB},$$

or dividing by 2 cos. l,

$$-\tan.\ l \sin.^2 \tfrac{1}{2} dl + \cos.\ \tfrac{1}{2} dl \sin.\ \tfrac{1}{2} dl = \tfrac{1}{2} (\cos.\ \text{A} \sin.\ \text{AB} - 2 \tan.\ l \sin.^2 \tfrac{1}{2} \text{AB}).$$

Let the second member be represented by p; also for $-\tan.\ l$ put q, and divide all the terms by $\cos.^2 \frac{1}{2} dl$; then

$$q \tan.^2 \tfrac{1}{2} dl + \tan.\ \tfrac{1}{2} dl \left(= \frac{p}{\cos.^2 \frac{1}{2} dl}\right) = p\ (1 + \tan.^2 \tfrac{1}{2} dl),$$

or $$(q-p) \tan.^2 \tfrac{1}{2} dl + \tan.\ \tfrac{1}{2} dl = p.$$

Treating this as a quadratic equation, we obtain

$$\tan.\ \tfrac{1}{2} dl = -\frac{1}{2\,(q-p)} + \frac{1}{2\,(q-p)} (1 + 4p\ (q-p))^{\frac{1}{2}},$$

or developing the radical term by the binomial theorem,

$$\tan.\ \tfrac{1}{2} dl = \frac{1}{2\,(q-p)} \{2\ (q-p)\,p - 2\ (q-p)^2 p^2 + 4\ (q-p)^3 p^3 - \&c.\};$$

or $\tan.\ \frac{1}{2} dl = p - qp^2 + (1 + 2q^2)\,p^3$, rejecting the powers of p above the third.

Next, developing the arc $\frac{1}{2} dl$ in tangents (Pl. Trigo., art. 47.) as far as two terms, we have

$\frac{1}{2}dl = \tan.\ \frac{1}{2}dl - \frac{1}{3}\tan.^3\ \frac{1}{2}dl;$

$\quad = p - qp^2 + (1 + 2q^2)\,p^3 - \frac{1}{3}\,p^3$, rejecting as before,

or $\quad = p - qp^2 + 2p^3\,(\frac{1}{3} + q^2);$

whence $dl = 2p - 2qp^2 + \frac{4}{3}p^3\,(1 + 3q^2)$.

But $2p = \cos.\ \text{A} \sin.\ \text{AB} - 2 \tan.\ l \sin.^2 \frac{1}{2}\text{AB}$;

$2\,qp^2 = -\frac{1}{2}\cos.^2\ \text{A}\ \sin.^2\ \text{AB}\ \tan.\ l + 2\cos.\ \text{A}\ \sin.\ \text{AB} \sin.^2 \frac{1}{2}\text{AB} \tan.^2 l$,

and $\frac{4}{3}\,p^3 = \frac{1}{6}\cos.^3 \text{A} \sin.^3 \text{AB}$, rejecting the remaining terms, which contain powers of AB higher than the third.

Now (Pl. Trigo., art. 46.) $\sin.\ \text{AB} = \text{AB} - \frac{1}{6}\,\text{AB}^3$; therefore the first term in the value of $2p$ becomes $\text{AB} \cos.\ \text{A} - \frac{1}{6}\,\text{AB}^3 \cos.\ \text{A}$; also the equivalent of $\frac{4}{3}p^3$ may be put in the form $\frac{1}{6}\,\text{AB}^3 \cos.^2\ \text{A} \cos.\ \text{A}$. This being added to the second term just mentioned, the three terms are equivalent to

$$\text{AB} \cos.\ \text{A} - \frac{1}{6}\ \text{AB}^3 \sin.^2\ \text{A} \cos.\ \text{A}.$$

Again, putting $\frac{1}{2}$ AB for its sine, the second term in the value of $2p$ becomes $-\frac{1}{2}\ \text{AB}^2 \tan.\ l$; and the first term in the equivalent of $-2qp^2$ becomes $+\frac{1}{2}\,\text{AB}^2 \cos.^2 \text{A} \tan.\ l$. These terms being added together, produce the term

$$-\frac{1}{2}\ \text{AB}^2 \sin.^2 \text{A} \tan.\ l.$$

Thus we obtain for dl or Am (the required difference of latitude)

$$\text{AB} \cos.\ \text{A} - \frac{1}{2}\,\text{AB}^2 \sin.^2 \text{A} \tan.\ l - \frac{1}{6}\,\text{AB}^3 \sin.^2 \text{A} \cos.\ \text{A}\ (1 + 3\tan.^2 l) - \&c.$$

Here AB and dl are supposed to be expressed in terms of radius $(= 1)$; if AB be given in seconds, we shall have, after dividing by sin. 1″,

$$dl \text{ (in seconds)} = \text{AB} \cos.\ \text{A} - \frac{1}{2}\,\text{AB}^2 \sin.\ 1'' \sin.^2 \text{A} \tan.\ l - \frac{1}{6}\,\text{AB}^3 \sin.^2 1'' \sin.^2 \text{A} \cos.\ \text{A}\ (1 + 3\tan.^2 l) - \&c.$$

This formula is to be used when the angle PAB is acute: if that angle be obtuse, as the angle PAB′, its cosine being negative, the formula would become

$$dl \text{ (in seconds)} = -\,\text{AB} \cos.\ \text{A} - \frac{1}{2}\,\text{AB}^2 \sin.\ 1'' \sin.^2 \text{A} \tan.\ l + \frac{1}{6}\,\text{AB}^3 \sin.^2 1'' \sin.^2 \text{A} \cos.\ \text{A}\ (1 + 3\tan.^2 l)\ \&c.$$

If in either of these expressions we write for l the term $l' - dl$, in which case l' represents the latitude of B; then since, as a near approximation,

$$\tan.\ (l' - dl) = \tan.\ l' - \tan.\ dl,$$

the term $-\frac{1}{2}\ \text{AB}^2\ \sin.\ 1'' \sin.^2 \text{A} \tan.\ l$ becomes

$$-\frac{1}{2}\ \text{AB}^2 \sin.\ 1'' \sin.^2 \text{A} \tan.\ l' + \frac{1}{2}\ \text{AB}^2 \sin.\ 1'' \sin.^2 \text{A} \tan.\ dl;$$

and for dl putting AB sin. $1''$ cos. A, its approximate value, we get

$$-\tfrac{1}{2}\,\text{AB}^2 \sin. 1'' \sin.^2 \text{A} \tan. l' + \tfrac{1}{2}\,\text{AB}^3 \sin.^2 1'' \sin.^2 \text{A} \cos. \text{A}.$$

If to the last of these terms we add the term $-\tfrac{1}{6}\,\text{AB}^3 \sin.^2 1''$ $\sin.^2$ A cos. A, the first expression for dl above becomes, neglecting the terms containing $\tan.^2 l$,

$$\text{AB} \cos. \text{A} - \tfrac{1}{2}\,\text{AB}^2 \sin. 1'' \sin.^2 \text{A} \tan. l' + \tfrac{1}{3}\,\text{AB}^3 \sin.^2 1'' \sin.^2 \text{A} \cos. \text{A};$$

in which all the terms are negative when the angle at A is obtuse.

This formula corresponds to that which denotes the value of Mp in art. 360.; and from it we should have

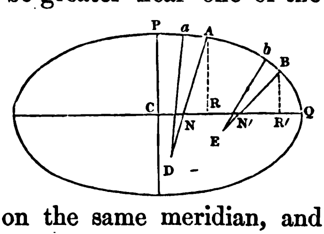
$$\text{PA} = \text{PB} + \text{AB} \cos. \text{A} - \tfrac{1}{2}\,\text{AB}^2 \sin. 1'' \sin.^2 \text{A} \tan. l' + \&c.$$

conformably to the value of PZ in art. 352.

The angle at P, or the difference of longitude between the stations A and B, and also the azimuthal angle at B, will be most conveniently found by the direct formulæ III. and IV. above given.

Delambre observes that the meridional arcs computed on the hypothesis that the earth is a sphere do not differ sensibly from those which would be obtained on the spheroidal hypothesis; because the sums of the spheroidal angles of the triangles are, without any appreciable error, equal to those of the spherical angles, and the chords of the sides are the same. Thus we obtain correctly the sides of the triangles and the arcs between the parallels of latitude whatever be the figure of the earth (provided it do not differ much from a sphere). He adds that the consideration of the spheroidal figure of the earth is of little importance in the calculations, except for the purpose of changing into seconds the terrestrial arcs which have been measured or computed in feet. This may be done by dividing such arc by the radius of curvature at the place, instead of dividing it by what may be assumed as the mean radius of the earth.

411. The length of a degree of latitude being found, from the results of admeasurement, to be greater near one of the poles of the earth than near the equator, it follows that the earth is compressed in the former region, or that the polar semi-axis is shorter than a semidiameter of the equator. For, let A a and B b represent the lengths of two such terrestrial arcs, supposed to be on the same meridian, and



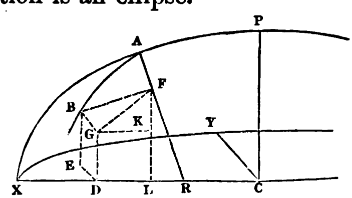
imagine AD, aD, BE, bE to be normals drawn to the earth's surface; the angles at D and E will each be equal to one degree: and since Aa, Bb may be considered without sensible error as portions of circles, the sectors ADa, BEb will be similar, and the lines AD, BE, which may be considered as radii of curvature, will be to one another in the same ratio as the arcs. Therefore Aa, which is the nearest to the pole P, being longer than Bb, AD will be longer than BE, or the surface of the earth about A will be less convex than the surface about B, which is the nearest to the equator.

412. Adopting now the hypothesis that the earth is a spheroid of revolution, such that all the terrestrial meridians are ellipses whose transverse axes are in the plane of the equator, and the minor axes coincident with that of the earth's rotation; the principal investigations relating to the values of terrestrial arcs will be contained in the following propositions.

PROP. I.

Every section of a spheroid of revolution, when made by a plane oblique to the equator, or to the axis of rotation, is an ellipse.

Let XYC, XPC and PCY be three rectangular co-ordinate planes formed by cutting a spheroid in the plane of the equator (XYC) and in those of two meridians at right angles to one another; and let a plane cut the spheroidal surface in AB and the plane XPC in AR, which for the present purpose may be supposed to be a normal to the spheroid at A; it is required to prove that the section is an ellipse.



Let CD ($= x$), DE ($= y$) and EB ($= z$) be rectangular co-ordinates of any point B in the curve line AB; and let RF ($= x'$), FB ($=y'$) be rectangular co-ordinates of B with respect to AR: also let l represent the angle ARX, and θ the inclination of the normal or vertical plane ABR to the co-ordinate plane XPC. Imagine a perpendicular to be let fall from B on the plane XPC, it will meet the plane in G; and join G, F: draw also FL in the plane XPC perpendicular, and GK parallel to XC. Then we shall have, BF and FG being perpendicular to AR so that the angle BFG is the inclination of the plane ABR to the plane XPC,

$$y \; (= \text{BG or ED, or} = \text{BF sin. BFG}) = y' \sin. \theta;$$

also, DL or GK being equal to GF sin. GFK or GF sin. l, and GF to BF cos. BFG or BF cos. θ,

x (= CD or CR + RL + DL) = CR + x' cos. l + y' cos. θ sin. l;
z (= GD or FL − FK) = x' sin. l − y' cos. θ cos. l.

Substituting these values of x, y and z in the general equation $\text{A}z^2 + \text{B}(x^2 + y^2) = \text{C}$ of a spheroid, the latter will manifestly become of the form

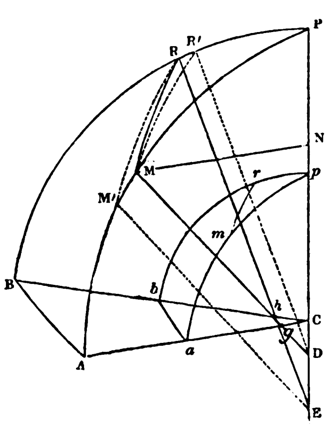
$$mx'^2 + ny'^2 + px'y' + qx' + ry' = s,$$

which is the equation to a line of the second order: the variables in it being the assumed co-ordinates of B, it follows that the equation appertains to the curve line AB; and since the curve, being a plane section of a solid, returns into itself, it is an ellipse.

PROP. II.

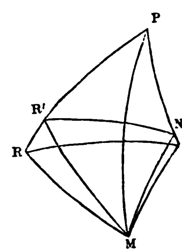
413. When the sides of a terrestrial triangle do not much exceed in extent any of those which are formed in a geodetical survey, the excess of the three angles above two right angles is, without sensible error, the same whether the earth be considered as a spheroid or a sphere.

This may be proved by comparing the angles of a spheroidal, with those of a spherical triangle, when the angular points of both have the same latitudes and equal differences of longitude, agreeably to the method pursued by Mr. Dalby in the "Philosophical Transactions" for 1790.



Let PRM be a spheroidal triangle of which one point as P is the pole of the earth, and let prm be a corresponding triangle on the surface of a sphere: let also ABC be the plane of the equator. Imagine the normals MD and RE to be drawn cutting AC and BC in g and h; then the angles MgA RhB, which express the spheroidal latitudes of M and R, will be equal to the angles mCa, rCb respectively; and we shall have MD parallel to mC and RE parallel to rC. Imagine also the planes DMR′, ERM′ to pass through MD and RE parallel to the plane rmC; then R′D will be parallel to rC, and consequently to RE; and M′E will be parallel to *m*C, consequently to MD: also the angles PR′M and PM′R are, respectively equal to the spherical angles *prm* and *pmr*. But since, within the limits of the terrestrial triangles, the deviation of the produced normal ER from the plane MDR conceived to pass through the normal MD, is imperceptible from the station M, we may consider the planes MDR and MER as coincident; and the supposed vertical plane passing through MR to cut the parallel planes MR′D, RM′E under equal angles of inclination, so that the angle RMR′ may without sensible error be considered equal to MRM′: then the spheroidal azimuthal angle PRM will be as much less than the spherical angle PR′M ($=prm$) as the spheroidal azimuthal angle PMR exceeds the spherical angle PM′R ($=pmr$). Consequently the sum of the spheroidal angles at R and M will, without sensible error, be equal to the sum of the spherical angles *prm*, *pmr*.

The like will be true for another triangle as PNM, in which we shall have the sum of the angles PMN, PNM equal to the sum of the angles PMN′, PN′M; therefore, describing the arcs RN, R′N′, we have the angles



$$\text{PRM} + \text{PMR} = \text{PR}'\text{M} + \text{PMR}', \text{ and}$$
$$\text{PNM} + \text{PMN} = \text{PN}'\text{M} + \text{PMN}':$$

then by addition,

$$\text{PRM} + \text{PNM} + \text{RMN} = \text{PR}'\text{M} + \text{PN}'\text{M} + \text{R}'\text{MN}'.$$

Again in the triangles PRN, PR′N′, we have as before

$$\text{PRN} + \text{PNR} = \text{PR}'\text{N}' + \text{PN}'\text{R}';$$

and subtracting this equation from the preceding, we have

$$\text{NRM} + \text{RNM} + \text{RMN} = \text{N}'\text{R}'\text{M} + \text{R}'\text{N}'\text{M} + \text{R}'\text{MN}'.$$

That is, the sum of the angles in the spheroidal triangle RMN is without sensible error equal to that of the angles in the triangle R′MN′, which are those of a spherical triangle *rmn*, whose angular points correspond, in latitude and longitude, with the points R′, M, N′.

By a different investigation, Legendre has ascertained that the difference between the spherical and the spheroidal excess of the angles of a triangle, above two right angles, in the greatest triangle ever formed on the surface of the earth, does not amount to $\frac{1}{20}$ of a second; therefore, in computing the sides of the terrestrial triangles, the latter may always be considered as appertaining to the surface of a sphere.

PROP. III.

414. To find the radius of curvature at a certain point A (fig. to art. 412.) in the periphery of a vertical section of a spheroid.

It has been already proved that the general equation for the curve line produced by such a section is

$$mx'^2 + ny'^2 - px'y' + qx' + ry' = s \quad . \quad . \quad . \quad . \quad (\text{I}),$$

and comparing the co-efficients of the variables x' and y' with those of the same variables in the equation $az^2 + bx^2 + by^2 = a^2b^2$ of a spheroid, when for x, y, z, are substituted their equivalents (art. 412.) it will be seen that

$$\begin{array}{ll|ll} m & = a^2 \sin.^2 l + b^2 \cos.^2 l & q & = 2\,\text{RC}\, b^2 \cos.\, l \\ n & = b^2 + (a^2 - b^2) \cos.^2 l \cos.^2 \theta & r & = 2\,\text{RC}\, b^2 \sin.\, l \cos.\, \theta \\ p & = 2\,(a^2 - b^2) \sin.\, l \cos.\, l \cos.\, \theta & s & = (a^2 - \text{RC}^2)\, b^2. \end{array}$$

also a, representing half the equatorial diameter, b half the polar diameter of the spheroid, and $a^2 - b^2$ or a^2e^2 representing the square of the excentricity of the ellipse XAP, we have, by conic sections,

$$\text{RC} = \frac{ae^2 \cos.\, l}{(1 - e^2 \sin.^2 l)^{\frac{1}{2}}}, \text{ and AR} = \frac{b^2}{a\,(1 - e^2 \sin.^2 l)^{\frac{1}{2}}}.$$

Now x' and y' being the co-ordinates of any point in a curve, the usual formula for the radius of curvature is, when dy' is considered as constant,

$$\text{R} = \frac{\left\{1 + \left(\frac{dx'}{dy'}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x'}{dy'^2}};$$

and differentiating the equation (I) twice successively, considering dy' as constant; also, since the radius is required for the point A where $y' = 0$, making y' equal to zero after each differentiation, we have

$$\frac{dx'}{dy'} = -\frac{r - px'}{q + 2mx'}$$

$$\text{and } \frac{d^2x'}{dy'^2} = \frac{-2m\left(\frac{dx'}{dy'}\right)^2 + 2p\left(\frac{dx'}{dy'}\right) - 2n}{q + 2mx'}.$$

But, from the preceding values of p and r, and since, at the point A, $x' = \text{AR}$, it will be found that $r = px'$, or $r - px' = 0$; therefore the radius R of curvature at the point A becomes equal to $\frac{q + 2mx'}{2n}$; in which replacing m, n, q and x' by their values above, we obtain

$$\text{R} = \frac{ab^2}{(1 - e^2 \sin.^2 l)^{\frac{1}{2}} \; b^2 + (a^2 - b^2) \cos.^2 l \cos.^2 \theta\}}.$$

When the ellipse AB coincides with the plane of a terrestrial meridian as XAP, we have $\theta = 0$; and designating the radius of curvature in this case by R′ (which is then in the direction of AR) we have

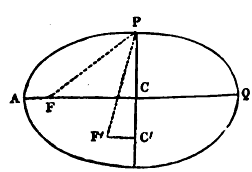
$$\text{R}' = \frac{b^2}{a\,(1 - e^2 \sin.^2 l)^{\frac{3}{2}}} \text{ or } = \frac{a\,(1 - e^2)}{(1 - e^2 \sin..^2 l)^{\frac{3}{2}}}:$$

again, if the said ellipse be perpendicular to the plane of a meridian XAP, we have $\theta = 90$ degrees; and designating the radius of curvature in that case by R″ (which is then also in the direction of AR) we have

$$\text{R}'' = \frac{a}{(1 - e^2 \sin.^2 l)^{\frac{1}{2}}}.$$

The above formulæ may be transformed into others, equivalent to them, in which ε, the compression $\left(= \frac{a - b}{a}\right)$ is employed instead of e, the excentricity. Thus, by the nature of the ellipse we have $\frac{a^2 - b^2}{a^2} = e^2$; now since the difference between a and b is very small, if we put $2a$ for $a + b$ and multiply the above value of the compression by $\frac{a + b}{2a}$ $(= 1)$ we shall have $\frac{a^2 - b^2}{2a^2} = \varepsilon$; then comparing this equation with that for e^2, we obtain $\frac{1}{2}e^2 = \varepsilon$, or $e^2 = 2\varepsilon$.

Again, let APQ represent a meridian of the terrestrial spheroid, AQ a diameter of the equator, P one of its poles, and F the focus. Imagine PF′ to be equal to the semi-transverse axis, and F′C′ drawn perpendicularly to PC, to be equal to the difference between the semi-transverse and semi-conjugate axes: also let the angle F′PC′ be represented by I. Then, when PF′, or PF, or AC, is



equal to unity, F′C′ ($= \varepsilon$) becomes equal to sin. I: and because the difference between the semi-axes is small, PC ($= 1 - \varepsilon$) is sometimes put for PC′, and is represented by cos. I; or

$1 - \cos. \text{I}$ is put for ε. But $1 - \cos. \text{I}$ is (Pl. Trigo., art. 36.) equal to $2 \sin.^2 \frac{1}{2}\text{I}$; therefore ε is equivalent to $2 \sin.^2 \frac{1}{2}\text{I}$: or putting $\frac{1}{2} \sin. \text{I}$ for $\sin. \frac{1}{2}\text{I}$ because the angle I is very small, $\frac{1}{4} \sin.^2 \text{I} = \sin.^2 \frac{1}{2}\text{I}$; whence ε may be represented by $\frac{1}{2} \sin.^2 \text{I}$; or 2ε, that is, e^2 by $\sin.^2 \text{I}$.

PROP. IV.

415. The lengths of two meridional arcs as Aa, Bb (fig. to art. 411.), one near the pole and the other near the equator, and each subtending one degree, being given by admeasurement; to find the ratio which the earth's axes have to each other, the meridian being an ellipse.

Imagine the normals aD, AD, bE, BE to be drawn from the extremities of the arcs till they meet in D and E respectively; and let AD, BE intersect CQ, the equatorial semidiameter, in N and N′: also let the angles ANQ, BNQ (the geographical latitudes of the points A and B) be represented by l and l' respectively.

Then, by the similarity of the sectors ADa and BEb, representing AD and BE, the radii of curvature, by R and R′, we have

$$\text{R} : \text{R}' :: \text{A}a : \text{B}b, \text{ or } \text{R} . \text{arc B} = \text{R}' . \text{arc A} \quad . \; . \; . \; . \; . \quad (\text{A}).$$

But (art. 414.) $\text{R} = \dfrac{a(1-e^2)}{(1-e^2 \sin.^2 l)^{\frac{3}{2}}}$; or, developing $\dfrac{1}{(1-e^2 \sin.^2 l)^{\frac{3}{2}}}$ or $(1-e^2 \sin.^2 l)^{-\frac{3}{2}}$ by the binomial theorem and neglecting powers of e above the second,

$$\text{R} = a\,(1-e^2)\,(1 + \tfrac{3}{2}e^2 \sin.^2 l) = a\,(1 - e^2 + \tfrac{3}{2}e^2 \sin.^2 l).$$

In like manner $\text{R}' = a\,(1 - e^2 + \tfrac{3}{2}e^2 \sin.^2 l')$:

therefore the above equation (A) becomes

$$\text{B}\,(1 - e^2 + \tfrac{3}{2}e^2 \sin.^2 l) = \text{A}\,(1 - e^2 + \tfrac{3}{2}e^2 \sin.^2 l'),$$

and from this by transposition we obtain

$$e^2 = \frac{\text{A} - \text{B}}{\text{A} - \text{B} - \frac{3}{2}(\text{A} \sin.^2 l' - \text{B} \sin.^2 l)}.$$

Thus e^2 is found:

but $e^2 = \dfrac{a^2 - b^2}{a^2}$; whence $a^2(1-e^2) = b^2$, and $a(1-e^2)^{\frac{1}{2}} = b$:

developing the radical as far as two terms, we get $1 - \frac{1}{2}e^2 = \dfrac{b}{a}$, in which substituting the numerical value of e^2 above, we have the required ratio of the semi-axes a and b.

The numerical value of e being substituted in the above equation R . arc B $=$ R$'$. arc A, there might from thence be obtained the value of a; and from the values of e and a, that of b might be found. Thus the equatorial and polar semi-diameters of the earth would be completely determined.

PROP. V.

416. To investigate the law according to which the lengths of the degrees of terrestrial latitude increase, from the equator towards either pole.

It has been shown in the preceding article, that, in the plane of the meridian,

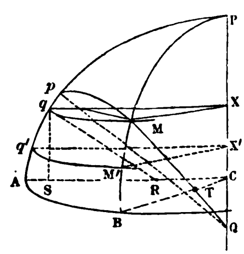
$$R = a\,(1 - e^2 + \tfrac{3}{2} e^2 \sin.^2 l).$$

But, at the equator, where $l = 0$, the radius of curvature becomes equal to $a\,(1 - e^2)$; therefore, by subtraction, the increment of the radius, for any latitude l, above its value at the equator, is equal to $\frac{3}{2} a e^2 \sin.^2 l$: or the increments of the radii vary with the square of the sine of the geographical latitude of the station.

Now the lengths of the degrees of latitude have been shown (art. 415. (A)) to vary with the radii of curvature in the direction of the meridian of any station; therefore, by proportion, the increments of the degrees of latitude vary with the increments of the radius; that is, with the square of the sine of the latitude.

PROP. VI.

417. To determine the radius of any parallel of terrestrial latitude, and find the length of an arc of its circumference between two points whose difference of longitude is given.



Let PqA be a quarter of the elliptical meridian passing through q; and let qR be the direction of a normal at q. Let AC ($=a$) be a radius of the equator, and PC ($=b$) be the polar semi-axis: let also qX ($=$ SC or x) be a radius of the parallel Mq of latitude; and let $a e$ be the excentricity of the elliptical meridian. Then, by the nature of the ellipse, we have

$$Sq^2 = \frac{b^2}{a^2}(a^2 - x^2) \text{ and } SR^2 = \frac{b^4}{a^4} x^2:$$

now let l be the latitude of q, or the value of the angle A R q; then

$$\text{S}\,q^2 = \text{R}\,q^2 \sin.^2 l \text{ and } \text{S}\,\text{R}^2 = \text{R}\,q^2 \cos.^2 l.$$

From these equations we obtain

$$\text{R}\,q^2 = \frac{b^2\,(a^2 - x^2)}{a^2 \sin.^2 l}, \text{ and also} = \frac{b^4\,x^2}{a^4 \cos.^2 l}:$$

equating these values of Rq^2, we obtain

$$(a^4 - a^2 x^2) \cos.^2 l = b^2 x^2 \sin.^2 l;$$

or, substituting $a^2 - a^2 e^2$ for its equal b^2 (by conic sections) we get after dividing by a^2,

$$(a^2 - x^2) \cos.^2 l = (1 - e^2)\,x^2 \sin.^2 l,$$

or
$$a^2 \cos.^2 l - x^2 = -e^2 x^2 \sin.^2 l;$$

whence $x^2 = \dfrac{a^2 \cos.^2 l}{1 - e^2 \sin.^2 l}$, or x $(=$ X $q) = \dfrac{a \cos. l}{(1 - e^2 \sin.^2 l)^{\frac{1}{2}}}$.

Now the angle q X M (= A C B) is the measure of the spherical angle A P B which is the difference between the longitudes of M and q; and if this angle, in seconds, be denoted by P we shall have evidently, for the value of the arc M q,

$$\text{X}\,q \,.\, \text{P} \sin. 1'', \text{ or } \frac{a \cos. l}{(1 - e^2 \sin.^2 l)^{\frac{1}{2}}} \,.\, \text{P} \sin. 1''.$$

PROP. VII.

418. To find the ratio between the earth's axes from the length of a degree on the meridian combined with the length of a degree of longitude on any circle parallel to the equator.

The radius of curvature in the direction of the meridian being represented by R, we have R 3600 sin. 1'', or (art. 414.) $\dfrac{a\,(1 - e^2)\,3600 \sin. 1''}{(1 - e^2 \sin.^2 l)^{\frac{3}{2}}}$; or again, developing the denominator by the binomial theorem, $a\,(1 - e^2 + \frac{3}{2} e^2 \sin.^2 l)\,3600 \sin. 1''$, for the length of a degree of the meridian: and supposing one extremity (more properly the middle point) of this measured degree to have the same latitude as the parallel on which the other degree is measured, the length of the latter will (art. 417.) be expressed by $\dfrac{a \cos. l\; 3600 \sin. 1''}{(1 - e^2 \sin.^2 l)^{\frac{1}{2}}}$, or after development, by $a \cos. l\;(1 + \frac{1}{2} e^2 \sin.^2 l)\;3600 \sin. 1''$. Let the first of these arcs be represented by A and the second by A′; then, if the latter be subtracted from the former, we shall have

$$\text{A} - \text{A}' = \{a - a \cos. l - a e^2 + \tfrac{3}{2} a e^2 \sin.^2 l - \tfrac{1}{2} a e^2 \cos. l \sin^2 l\}\; 3600 \sin. 1'';$$

whence

$$\frac{\text{A}-\text{A}'-a\,(1-\cos.\,l)\,3600\sin.\,1''}{(\frac{3}{2}a\sin.^2 l-a-\frac{1}{2}a\cos.\,l\sin.^2 l)\,3600\sin.\,1''}=e^2,$$

or

$$\frac{\text{A}-\text{A}'-a\,(1-\cos.\,l)\,3600\sin.\,1''}{(a\sin.^2 l+\frac{1}{2}a\sin.^2 l-a-\frac{1}{2}a\cos.\,l\sin.^2 l)\,3600\sin.\,1''}=e^2;$$

or again,

$$\frac{\text{A}-\text{A}'-2\,a\sin.^2\frac{1}{2}l\;3600\sin.\,1''}{(a\sin.^2 l\sin.^2\frac{1}{2}l-a\cos.^2 l)\,3600\sin.\,1''}=e^2:$$

neglecting e^2 in the above value of A we may, as an approximation sufficiently near the truth, put $\frac{\text{A}}{3600\sin.\,1''}$ for a; when we shall have

$$e^2=\frac{\text{A}-\text{A}'-2\,\text{A}\sin.^2\frac{1}{2}l}{\text{A}\sin.^2 l\sin.^2\frac{1}{2}l-\text{A}\cos.^2 l}=\frac{\text{A}\cos.\,l-\text{A}'}{\text{A}\sin.^2 l\sin.^2\frac{1}{2}l-\text{A}\cos.^2 l}.$$

The value of e being thus found, the ratio of the axes may be obtained as in Prop. IV. (art. 415.): and in a similar manner, the length of a degree perpendicular to the meridian being found by multiplying the value of R″ (art. 414.) into 3600 sin. 1″, may the ratio be obtained from the measured length of a degree of latitude combined with that of a degree on an ellipse perpendicular to the meridian.

PROP. VIII.

419. To find the distance in feet, on an elliptical meridian, between the foot of a vertical arc, let fall perpendicularly on the meridian from a station near it, and the intersection of a parallel of latitude passing through the station. Also, to find the difference in feet between the lengths of the vertical arc and the corresponding portion of the parallel circle.

Let P (fig. to Prop. VI.) be the pole of the world, C its centre, M the station; and let Mp, Mq be the two arcs, as in art. 410. Since normals to the elliptical meridians PA, PB, at the points M and q, may, without sensible error, be considered as meeting in one point Q on the axis PC; it is evident that the values of cos. P and of 1 − cos. P, or 2 sin.$^2\frac{1}{2}$P will be the same whether the earth be a spheroid or a sphere, and consequently that they will coincide with the values given in art. 71. Now MQ is equal to $\frac{\text{MX}}{\cos.\,\text{XMQ}}$ (Pl. Trigon., art. 56.), and XMQ is equal to MTB, the geographical latitude of M; therefore, putting l for the latitude of M, and substituting for MX or qX its value $\frac{a\cos.\,l}{(1-e^2\sin.^2 l)^{\frac{1}{2}}}$ (art. 417.), we have MQ equal to $\frac{a}{(1-e^2 \sin.^2 l)^{\frac{1}{2}}}$: consequently, in art. 408., putting for the semidiameter MC the length of the normal MQ, there will be obtained pq (in feet) = $\frac{\text{M}p^2}{2a}$ tan. l $(1-e^2 \sin.^2 l)^{\frac{1}{2}}$; or, extracting the root as far as two terms, pq (in feet) $= \frac{\text{M}p^2}{2a}$ tan. l $(1-\frac{1}{2}e^2 \sin.^2 l)$.

Again, considering Mp as a circular arc of which MQ is, without sensible error, equal to the radius of curvature, it is evident that the difference between Mq and Mp may be obtained from its value in art. 408., on substituting for MC the above value of MQ; and it follows that on a spheroid,

$$\text{M}q - \text{M}p \text{ (in feet)} = \tfrac{1}{6}\,\frac{\text{M}p^3}{a^2} \tan.^2 l\,(1 - e^2 \sin.^2 l).$$

The value of Mq being obtained for any given parallel of spheroidal latitude, it may be reduced to a corresponding arc, as M$'q'$, having equal angular extent in longitude, by the proportion MX : M$'$X$'$ :: Mq : M$'q'$.

PROP. IX.

420. To investigate an expression for the length of a meridional arc on the terrestrial spheroid; having, by observation, the latitudes of the extreme points, with assumed values of the equatorial radius and the excentricity of the meridian.

Let l and l' represent the observed latitudes of the two extreme stations, a the radius of the equator, and ae the excentricity; then (arts. 414, 415.) R, the radius of curvature at a point whose latitude is l, $= a\,(1 - e^2 + \frac{3}{2}e^2 \sin.^2 l)$, or putting for $\sin.^2 l$ its equivalent $\frac{1}{2}\,(1 - \cos.\ 2\,l)$,

$$\text{R} = a\,(1 - \frac{e^2}{4} - \tfrac{3}{4}e^2 \cos.\ 2\,l).$$

Let m (in measures of length) express the length of a meridional arc from the equator to the point whose latitude is l; then dm may represent an increment of that length, and if dl (in arc. rad. $=1$) represent the corresponding increment of latitude, we shall have

$$\text{R}\,dl = dm;\ \text{whence}\ a\,(1 - \frac{e}{4} - \tfrac{3}{4}e^2 \cos.\ 2\,l)\,dl = dm.$$

Integrating this equation between l and l', corresponding to m and m',

$$a\left\{l'-l-\frac{e^2}{4}(l'-l)-\tfrac{3}{8}e^2(\sin. 2\,l'-\sin. 2\,l)\right\}=m'-m;$$

or putting for e^2 its equivalent $2\,\varepsilon$ (art. 414.) and for $\sin. 2\,l'-\sin. 2\,l$ its equivalent (Pl. Trigon., art. 41.) we have

$$a\left\{1-\tfrac{1}{2}\varepsilon-\tfrac{3}{2}\varepsilon\,\frac{\cos.(l'-l)\sin.(l'-l)}{l'-l}\right\}(l'-l)=m'-m.$$

If $l'-l$ be expressed in seconds, the factor and the denominator must, each, be multiplied by $\sin. 1''$.

If $m'-m$ be considered as representing the length of the meridional arc, obtained in measures of length from the triangulation, this equation may be used as a test of the correctness of the assumed values of a and e.

421. In the following table are contained a few determinations of the lengths of a degree of latitude in different regions of the earth, from which the fact of a gradual but irregular increase of such lengths in proceeding from the equator towards either pole is manifest.

Country.	Lat. of the Middle Point in the Arc.	Length of a Degree of Lat. in English Feet.
Peru - -	1° 31′ 0″ S.	362804
India - -	12 32 21 N.	362988
Cape of Good Hope -	33 18 30 S.	364716
France - -	44 51 2 N.	364540
England - -	52 35 45 N.	364968
Russia - -	58 17 37 N.	365350

The value of the earth's ellipticity, deduced from the measured lengths of meridional arcs, is liable to some uncertainty within, however, very narrow limits; and the ratio of the equatorial to the polar diameter is supposed to be nearly as 301 to 300. In the appendix to the Nautical Almanac for 1836 it is assumed to be as 305 to 304; whence ε (the compression) would be equal to 0.00247.

422. The fundamental base line is unavoidably small when compared with the distances between the stations which form the angular points of most of the primary triangles in a geodetical survey: and since, when the three angles of each triangle are actually observed, the most favourable condition is that the triangles be as nearly as possible equilateral, it follows that the distances between the stations, or the sides of the triangles, should increase gradually as the stations are more remote from the base line, till those sides become of any

magnitude which may be consistent with the features of the country or the power of distinguishing the objects which serve as marks.

423. The stations whose positions have been determined by the means already described, are so many fixed points from which must commence the operations for interpolating the other remarkable objects within the region or tract of country; and, for the measurement of the angles in the secondary triangles, there may be employed such an azimuth circle as has been described in art. 104., while, for triangles of the smallest class, a good theodolite of the kind employed in common surveying will suffice.

A process similar to that which has been already described, may be followed in fixing the positions of the secondary stations; and where such a process is practicable, no other should be employed, the most accurate method of surveying being that which consists in observing all the angles of the triangles formed by every three objects. Should circumstances, however, prevent this method from being followed, or permit it to be only partially put in practice, the verification obtained by observing the third angle of each triangle must be omitted; and it must suffice to obtain, with the theodolite, the angles, at two stations already determined, between the line joining those stations, and others drawn from them to the station whose position it is required to find. Frequently, also, it will be convenient, in a triangle, to make use of two sides already computed, and the angle between them, obtained by observation, to compute the third side and the two angles adjacent to it.

When a side of some primary triangle, or any line determined with the requisite precision, is used as the base of a secondary triangle, or as a base common to two such triangles, and the angles contained between the sides of the triangles have been observed; the rules of plane trigonometry will, in general, suffice for the computation of the lengths of the sides. The stations may also be laid down on paper, if necessary, by a graphical construction, a proper scale being chosen from which the given length of the base line is to be taken. If at any two stations already determined, there be taken, by means of a surveying compass, or the compass of the theodolite, the bearings of any object whose position is required, from the true or magnetic meridian of those stations; the intersection of lines drawn from the stations on the plan, and making, with lines representing those meridians, angles equal to the observed bearings, will give the position of the object with more or less accuracy, according to the delicacy of the needle, and the precision with which the directions of the meridians have been previously determined.

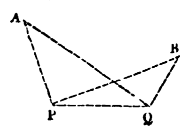
It would be proper to observe at three or more known stations the angles contained between the lines imagined to join them, and other lines supposed to be drawn from them to each of the objects whose positions are required; in order that, by the concurrence, in one point, of the intersections of all the lines tending towards each object, the correctness of the operations might be proved. Such concurrence is, however, scarcely to be expected, and the mean point among the intersections must be assumed as the true place of the object.

424. It occurs frequently, in the secondary operations of a survey, particularly in those which take place on a sea-coast, that the position of an object, or the position and length of a line joining two remarkable objects, are to be determined when, from local impediments, it would be inconvenient to convey the instrument to any known stations from whence the objects might be visible; and, in order to meet such cases, the following propositions are introduced. As a graphical construction alone may sometimes suffice, there are given, with the formulæ for the computations, the processes of determining the positions and distances by a scale.

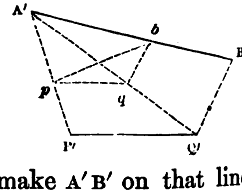
PROB. I.

To determine the positions of two objects, and the distance between them, when there have been observed at those objects, the angles contained between the line joining them, and lines imagined to be drawn from them to two stations whose distance from each other is known.

Let P and Q be the two objects whose positions are required, A and B the stations whose distance AB from each other is known; then QPB, QPA, PQA, PQB will be the observed angles.



On paper draw any line as pq, and at p and q lay down with a protractor, or otherwise, angles equal to those which were observed at P and Q; then the intersections of the lines containing the angles will determine the points A′ and b. Produce the line A′b if necessary, and, with any convenient scale, make A′B′ on that line equal to the given distance from A to B: from B′ draw B′Q′ parallel to bq till it meets A′q, produced, if necessary, in Q′;



and from Q′ draw Q′P′ parallel to *pq* till it meets A′*p*, produced, if necessary, in P′. The figures A′*pqb* and A′P′Q′B′ are (Euc. 18. 6.) similar to the figure APQB formed by lines imagined to join the objects on the ground: therefore P′Q′, A′P′, &c., being measured on the scale from whence A′B′ was taken, will give the values of the corresponding distances between the objects.

The processes to be employed in computing the distances are almost obvious. Thus, let any number, as 10 or 100, represent *pq*. Then, in the triangle A′*pq* there are known the angles A′*pq* (= APQ), A′*qp* (= AQP), with the side *pq*; from whence (Pl. Trigon., art. 57.) A′*p* may be found. In the triangle *pqb* there are known the angles *bpq* (= BPQ), *pqb* (= PQB), with *pq*; to find *pb*. In the triangle A′*pb* there are known the angle A′*pb* (= APB) and the sides A′*p*, *pb*; from which A′*b* may be found. Again, from the similarity of the figures,

$$\text{A}'b : \text{A}'\text{B}' \; (=\text{AB}) :: pq : \text{P}'\text{Q}' \; (=\text{PQ});$$

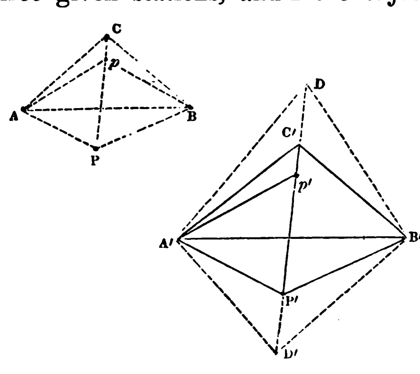
and by like proportions any other of the required distances may be found.

The construction and formulæ of computation will, manifestly, be similar to those which have been stated, whatever be the positions of the stations P and Q with respect to A and B.

PROB. II.

425. To determine the position of an object, when there have been observed the angles contained between lines imagined to be drawn from it to three stations whose distances from each other have been previously determined.

Let A, B, C be the three given stations, and P the object whose place is to be determined; then APC, CPB, or APB will represent the observed angles.



With the three given distances AB, AC, BC, lay on paper, by any convenient scale, the triangle A′B′C′, and on the side A′B′ make the angles B′A′D, A′B′D, respectively, equal to the observed angles CPB, APC; through D and C′ draw an indefinite line, and from A′ a line to meet it, suppose in P′, making the angle B′A′P′ equal to B′DC′: the point P′ will be the station. For if the circumference of a circle be imagined to pass through A′, D, and B′, since the angle B′A′P′ is by construction equal to B′DP′, those angles will be in the same segment (Euc. 21. 3.), and the circumference will also pass through P′; therefore the angle B′P′C′ will be equal to B′A′D, and the angle A′P′C′ to A′B′D. But the angles B′A′D, A′B′D are by construction equal to BPC, APC; therefore the latter angles are, respectively, equal to B′P′C′ and A′P′C′: thus, the angles at P′, on the paper, are equal to the corresponding angles which were observed at P; and P′ represents the object P: therefore the lines A′P′, C′P′, B′P′, measured on the scale, will give the required distances in numbers.

The like construction might be used if A, C, and B were in one straight line. If the point D should coincide with C′ the case would evidently fail; and the determination of P′ will be less accurate as D falls nearer to C′.

The formulæ for computation may be briefly stated thus: In the triangle A′B′C′ the three sides are given; therefore one of the angles, as B′A′C′, may be found (Pl. Trigon., art. 57.). In the triangle B′A′D all the angles are known, and the side A′B′; therefore the side A′D may be computed. In the triangle DA′C′, the sides A′D, A′C′, and the angle C′A′D are known; therefore the angles A′DC′, A′C′D may be found. In the triangle A′C′P′ all the angles and the side A′C′ are known; therefore the sides A′P′, C′P′ may be computed. Lastly, in the triangle A′B′P′ there are known the side A′B′ and all the angles (for A′B′P′ is equal to the computed angle A′DC′); therefore P′B′ may be obtained.

By this proposition the Observatory of the Royal Military College at Sandhurst was connected with three stations whose positions are given in the account of the Trigonometrical Survey of England. The stations are Norris's Obelisk, which may be represented by A; Yately Church Steeple, represented by B, and the middle of a Tumulus near Hertford Bridge represented by C; P representing the centre of the dome in the Observatory.

With an altitude and azimuth instrument whose circles are, each, twenty inches in diameter, the following angles were taken: —

$$\text{APC} = 134° \; 32' \; 4'', \; \text{APB} = 171° \; 37' \; 16''; \text{ whence}$$
$$\text{CPB} = 37° \; 5' \; 12'';$$

and from data furnished by the Trigonometrical Survey, there were obtained AB = 20252 feet, AC = 18086 feet, and BC = 8243.66 feet: with these data and the observed angles there were found by computation, as above, AP = 6985.2 feet, CP = 12488 feet, and BP = 13315.3 feet. There was, at the same time, observed the bearing of Norris's Obelisk from the meridian of the Observatory, which was found to be S. 86° 11′ 6″ E., with which, and the computed distance AP, it was found by the formulæ in art. 410. that the difference between the latitudes of the Obelisk and the centre of the dome is 8″.19, and the difference, in time, between the longitudes is 7″.28; the Observatory being northward and westward of the Obelisk. Hence, from the latitude and longitude of the latter in the Trigonometrical Survey, it is ascertained that the latitude of the Observatory is 51° 20′ 32″.99, and its longitude, in time, is 3′ 3″.78 westward of Greenwich.

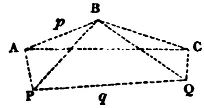
If the object whose position is required were, as at p, within the triangle formed by the three given stations A, B and C, the observed angles being then ApC, CpB or ApB, the construction would be similar to that which has been given, except that the angles B′A′D′, A′B′D′ must in that case be made, respectively, equal to the supplements of BpC and ApC, and the angle B′A′p' equal to B′D′C′; for then, as in the former, the circumference of a circle supposed to pass through A′, B′ and D′ would also pass through p', and this point would represent the object. The formulæ of computation would be precisely the same as in the other case.

When many points are to be determined in circumstances corresponding to those which are stated in this proposition, it is found convenient to obtain their positions mechanically by means of an instrument called a *station pointer*. This consists of a graduated circle about the centre of which turn three arms extending beyond the circumference, and having a chamfered edge of each in the direction of a line drawn through the centre: by means of the graduations these arms can be set so as to make with one another angles equal to those which have been observed, as APC and CPB; and then, moving the whole instrument on the paper till the chamfered edges of the arms pass through the three points A′, B′ and C′, the centre of the instrument will coincide with P′, and consequently will indicate on the paper the required position of the object.

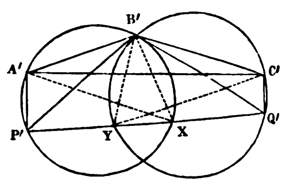
The position of an object, as P, with respect to two given stations as A and B, may be found by the method of cross-bearings in art. 364.

PROB. III.

426. To determine the positions of two objects with respect to three stations whose mutual distances are known, when there have been observed, at the place of each object, the angles contained between a line supposed to join the objects and other lines imagined to be drawn from them to two of the three given stations, some one of these being invisible from each object.



Let P and Q be the two objects whose positions are required; A, B and C the three stations, and let APB, APQ or BPQ, PQB, PQC or BQC be the four angles which have been observed.



With any scale lay on paper a triangle A′B′C′ having sides equal to the given distances AB, BC, AC, then on A′B′ describe (Euc. 33. 3.) a segment A′P′B′ of a circle, which may contain an angle equal to APB, and on B′C′ describe a segment B′Q′C′ which may contain an angle equal to BQC: it is manifest that the representations of P and Q will be somewhere on the circumferences of those segments.

Now, in order to discover readily what should be the next step in the process, imagine P′ and Q′ to be the places of the two objects, and imagine lines to be drawn as in the figure: then wherever P′ and Q′ be situated, the angles A′P′B′ and B′Q′C′ will be equal to the corresponding angles APB and BQC; also the angle B′A′X will be equal to B′P′Q′, and B′C′Y to B′Q′P′. Therefore, if the angles B′A′X, B′C′Y be made respectively equal to BPQ and BQP, a line drawn through X and Y, and produced if necessary (as in the figure), will cut the circumferences of the circles in P′ and Q′; and these points will represent the required places of the two objects, since it is manifest that the angles at P′ and Q′ will be equal to the observed angles at P and Q. The lines P′Q′, A′P′, &c., being measured on the scale, will give the required distances in numbers.

The points P and Q may be on opposite sides of the triangle ABC, as at p and q, or one of them may be within and the other on the exterior of the triangle, but the construction will be similar to that which has been given. It is evident that the case would fail if the points X and Y should coincide with one another.

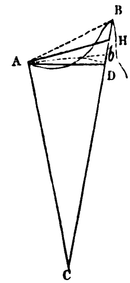
The formulæ for the computation may be briefly stated thus. In the triangle A′B′C′ the three sides are known, therefore the angle A′B′C′ may be found (Pl. Trigo., art. 57.). In the triangle A′B′X there are given A′B′, and the angles B′A′X (= BPQ) and A′XB′ (= APB); therefore the side B′X may be found. In like manner, in the triangle C′B′Y, there are given the side B′C′ with the angles B′C′Y (= BQP) and B′YC′ (= BQC); therefore the side B′Y can be found. By subtracting (in the figure) the angle A′B′C′ from the sum of the angles A′B′X and C′B′Y, the angle XB′Y will be obtained; and then, in the triangle XB′Y, the sides B′X, B′Y and the contained angle will be known, and consequently the angles B′XP′ and B′YQ′ may be computed. In the triangle B′P′X there are known all the angles and the side B′X; whence B′P′ may be found; in like manner may B′Q be found in the triangle B′Q′Y. In the triangle B′P′Q′ all the angles and the side B′P′ are known; therefore P′Q′ may be found; and lastly, the distances A′P′, C′Q′ may be found if required in the triangles A′B′P′ and C′B′Q′.*

427. A geodetical survey of a country is always accompanied by determinations of the heights of its principal mountains; for which purpose, in general, the angular elevations of their summits are observed by means of the vertical circle belonging to the azimuth instrument or theodolite; with which, after due allowance for the effects of refraction, and the curvature of the earth, the required altitudes may be computed by the rules of plane trigonometry.

The allowance for terrestrial refraction is to be determined by the method explained in art. 165.: and the correction of an angle of elevation on account of the deflection of the earth's surface, supposed to be spherical, from a plane touching the sphere at the place of the observer may be investigated in the following manner. Let C be the centre of the earth, A the place of the observer's eye, and B the summit of a mountain: let CA and CB be radii of the earth, and imagine CD to be equal to CA, so that A and D may be level points; also let AH be perpendicular to AC, so as to represent a horizontal line at A: then the angle BAH, after allowance

* Much useful information respecting the processes to be employed for "filling-in" a geodetical triangulation will be found in the works on trigonometrical surveying by Capt. Frome, Royal Engineers (Weale, 1840); Major Jackson, Hon. E.I.C., Seminary, Addiscombe (Allen and Co., 1838); and Mr. G.D. Burr, Royal Military College, Sandhurst (Murray, 1829).

for refraction, is the elevation of B above the horizon of A.



Draw the straight line AD; then the triangle ACD will be isosceles, and the angle CAD together with the half of ACD is equal to a right angle: but the angle CAH is a right angle; therefore HAD is equal to half the angle at C. This angle is known, since the radius of the earth is known, and the horizontal distance of A from B, which may be considered as equal to the arc AD, is supposed to be given by former operations. When B is, as in the figure, elevated above AH, the angle HAD must evidently be added to BAH in order to have the corrected elevation DAB: but if B were, as at *b*, below AH, the angle HA*b*, of depression, should be subtracted from HAD or from half the angle at C. Then, with AD as a base, the angle ADB as a right angle, and the correct elevation DAB or DA*b*, the height BD or *b*D might be computed.

The mean length of a degree on the earth's surface being 364547 feet, when AD is equal to 1 mile, or 5280 feet, the angle ACD $= 52''.14$; and then the deflexion DH is equal to 8.0076 inches. Now $(2\text{CD}+\text{DH}).\text{DH}=\text{AH}^2$ (Euc. 36.3); but DH being very small compared with 2CD, the factor 2CD+DH may be considered as constant; therefore DH, the other factor, may be said to vary with the square of AH, or very nearly with the square of AD. Thus, for any given distance AD, not exceeding a few miles, we have

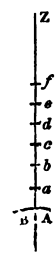
$$(5280 \text{ feet})^2 : (\text{AD, in feet})^2 :: 8.0076 \text{ in.} : \text{DH},$$

and the last term is the deflexion, in inches, corresponding to the given distance. Hence, if BH were computed in the triangle AHB, the angle at H being by supposition a right angle and, AH being considered as the given distance, the value of DH found from the proportion might be added to BH in order to have the correct height BD.

428. The determination of the heights of mountains by trigonometrical processes, is, with good instruments, susceptible of a high degree of accuracy; but the operations being laborious the relative heights of the remarkable summits in a mountainous country are more generally ascertained by means of the observed heights of the mercurial column in a barometer at the several stations. When Colonel Mudge and Mr. Featherstonehaugh determined, in North America, the levels along the axis of elevation, from the head of the Penobscot and St. John's rivers to the Bay of Chaleurs, they

were provided with several barometers, one of which was kept constantly at the Great Falls of St. John, in a building to which the air had free access: the man who had it in charge made the observations at 8 A. M., at noon, and at 4 P. M. daily, and the travelling party always endeavoured to make their observations as nearly as possible simultaneously with those at the Falls. The instruments employed for such purposes are, in general, similar to the ordinary barometer but more delicate; occasionally, however, a *Siphon Barometer*, of a kind invented by M. Gay Lussac, and now frequently made in this country, has been used. A formula for determining the relative heights of stations, from the heights of the mercurial columns supported by the atmosphere at the stations, may be thus investigated.

429. The particles of the atmosphere which surrounds the earth gravitate towards the surface of the latter, but, being of an elastic nature, they exert an expansive force in every direction; and, when the atmosphere is tranquil, that part of the force which, at any point, acts from below upwards is in equilibrio with the gravity or weight of the column of air which is vertically above that point. Now, in any small volume of air within which the density may be considered as uniform, that density is proportional to the external force which, acting against it, tends to compress it: and this force of compression on every part of the surface of the volume is equal to the weight of the column of air which is incumbent on that part; it follows therefore that, since the weight of a volume of air of uniform density is proportional to the density, the weights of small portions of air, equal in volume, in every part of the atmosphere (gravity being supposed constant), will be the same fractional parts of the weights of the atmospherical columns above them. This law being admitted, let A B be part of the surface of the earth, A Z the indefinite height of a slender cylindrical column of the atmosphere, and let this be divided into strata of equal thickness A*a*, *ab*, *bc*, &c. Then, taking W to represent the weight of the column Z*f*, $\frac{1}{n}$W may represent the weight of the small column *ef* and $W+\frac{1}{n}W$, or $\frac{n+1}{n}W$, will express the weight of the column Z*e*. Again, by the law above mentioned, $\frac{1}{n}\left(\frac{n+1}{n}W\right)$ will represent the weight of the small column *de*, and



$\frac{n+1}{n}\text{w} + \frac{1}{n}\left(\frac{n+1}{n}\text{w}\right)$, or $\left(\frac{n+1}{n}\right)^2 \text{w}$, will be the weight of $\text{z}d$. Continuing in this manner it will be found that the weights of the columns $\text{z}f$, $\text{z}e$, $\text{z}d$, &c., and also of the columns ef, ed, dc, &c., will form a geometrical progression whose common ratio is $\frac{n+1}{n}$: that is, the weights or pressures at f, e, d, &c. will form a geometrical progression, while the depths fe, fd, fc, &c., form an arithmetical progression.

430. But, by the nature of logarithms, if a series of natural numbers be in a geometrical progression, any series of numbers in an arithmetical progression will be logarithms of those natural numbers; therefore, if there were a kind of logarithms adapted to the relation between the densities of the air and the depths of the strata, on finding the densities of the air at any two places as between a and b and between c and d, such logarithms of those densities would express the depths fa and fd, and the difference between the logarithms would be equal to ad, which is the height of d above a. Thus $ad =$ log. dens. at a − log. dens. at d, or

$$ad = \log.\ \frac{\text{dens. at } a}{\text{dens. at } d}.$$

There are no such logarithms, but, from the general properties of logarithms, the formula may be adapted to those of the ordinary kind. Thus, the weight of an atmospherical column, as $\text{A}f$, is equal to the sum of the increasing weights of the series of strata from f down to A, and may be represented by the product of the greatest term (the weight of the air in $\text{A}a$) by some constant number M, which is therefore the modulus * of the system of logarithms whose terms are fe, fd, fc, &c., to $f\text{A}$. Now M may represent the height of a homogeneous atmosphere whose uniform density is equal to that of the stratum $\text{A}a$, and whose weight or pressure on A, is equal to that of the real atmosphere; and since such column, at a temperature expressed by 31° (Fahrenheit's

* From the series 1, 2 (art. 213. Elem. of Algebra), we have $n\text{M}w = \log. (1+w)^n$ where $n\text{M}w$ may represent $f\text{A}$, and $(1+w)^n$ the weight of the stratum in $\text{A}a$: let the first member be represented by x and $(1+w)^n$ by y. Then, by subtraction, in the series 2, the increment dx of $n\text{M}w$ is $\text{M}w$; and after developing $(1+w)^n, (1+w)^{n+1}$ by the binomial theorem, we have, by subtraction, the value of dy, the increment of y: from the values of these increments it will be found that $y\,dx = \text{M}\,dy$. But the integral of $y\,dx$ expresses the sum of the weights of all the strata; therefore the integral of $\text{M}\,dy$, that is, $\text{M}y$, or the product of the greatest term (the weight of $\text{A}a$) by the modulus M of the system, is equal to that sum or to the weight of the column $\text{A}f$.

thermometer), would hold in equilibrio at the level of the sea a column of mercury equal in height to 30 inches, it follows (the heights of two columns of homogeneous fluids, equal in weight, being inversely proportional to their densities) that the height of the column of homogeneous atmosphere at that temperature would be 4343 fathoms, and this may be considered as the value of M. The modulus of the common logarithms is 0.4343; and since, in different systems, the logarithms of the same natural number are to one another in the same proportion as their moduli, we have

$$0.4343 : 4343 :: \text{com. log.} \frac{\text{dens. at } a}{\text{dens. at } d} : 10000 \text{ com. log.} \frac{\text{dens. at } a}{\text{dens. at } d},$$

and the last term is equivalent to atmospheric log. $\frac{\text{dens. at } a}{\text{dens. at } d}$.

But the heights of the columns of mercury in a barometer, at any two stations, a and d, are to one another in the same ratio as the densities of the atmosphere at those stations, or as the weights of the columns of air above them; therefore the height of d above a may be expressed, in fathoms, by the formula

$$10000 \text{ common log.} \frac{\text{barom. at } a}{\text{barom. at } d},$$

or by its equivalent, 10000 (log. barom. at a – log. barom. at d).

Thus the height of the column of mercury in a barometer being observed at any two stations, as a and d, as nearly as possible at the same time, there may be obtained the relative heights of the two stations.

In using the formula it is evident that the heights of the mercurial columns at the two stations ought to be reduced to those which would have been observed if the temperature of the air and of the mercury were 31°. In order to make such reduction, since the mean expansion of a column of mercury is a part expressed by .000111 of the length of the column for an increment of temperature equal to one degree of Fahrenheit's thermometer, if this number be multiplied by the difference between the temperature at each station and 31° (that temperature being expressed by the thermometer attached to the instrument), the product will be the expansion of the column (in parts of its length) for that difference. Therefore, multiplying this product into the observed height of the column, in inches, at each station, we have the expansion in inches; and this last product being subtracted from the observed height, if the temperature at the station be greater than 31°, or added if less, there remains the corrected height of the column. The logarithms of these corrected heights, at the two stations, being used in the last formula above, the resulting value of ad will be a first approximation to the required height. By experiments it has been found that the relative height thus obtained varies by $\frac{1}{435}$ of its value for each degree of the thermometer in the difference between 31° and the mean of the temperature of the air at the two stations: consequently, if d be the difference between 31° and the mean of two detached thermometers, one at each station, the correction on account of the temperature of the air will be expressed by the formula $\frac{ad}{435}\,d$, where ad is the first approximate value of the height. This correction is to be added to that approximate height when the mean of the detached thermometers is greater than 31°, and subtracted when less.

The result is very near the truth when the height of one station above the other does not exceed 5000 or 6000 feet and when the difference of temperature does not exceed 15 or 20 degrees: in other cases, more accurate formulæ must be employed, and that which is given by Poisson in his "Traité de Mécanique" (second ed. No. 628.), when the measures are reduced to English yards, and the temperatures to those which would be indicated by Fahrenheit's thermometer, is

$$\text{H} = \text{A}\,\{\log.\ \text{barom. at } a - \log.\ k\};$$

in which H is, in yards, the required height of one station above the other, as d above a.

$$\text{A} = \frac{20053.95}{1 - .002588 \cos.\ 2\lambda}\left(1 + \frac{t + t' - 64^\circ}{900}\right).$$

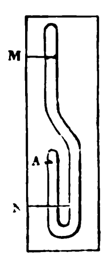
$$k = \text{height of barom. at } d, \times \left(1 + \frac{\text{T} - \text{T}'}{9990}\right).$$

t and t' are the temperatures of the air, by detached thermometers at the two stations.

T and T′ are the temperatures of the mercury, by attached thermometers.

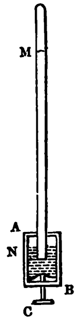
λ is the common latitude of the stations, or a mean of the latitudes of both if the stations be distant from each other in latitude.

431. The siphon barometer is a glass tube formed nearly as in the annexed figure, and containing mercury; it is hermetically sealed at both extremities, and has at A a very fine perforation, which allows a communication with the external air without suffering the mercury to escape. The atmosphere pressing on the mercury at N, balances the weight of a column of that fluid whose upper extremity may be at M. There is



a sliding vernier at each extremity of the column; and, the zero of the scale of inches being below N, the difference between the readings at M and N on the scale is the required height of the column of mercury.

In other mountain barometers the tube is straight, and its lower extremity, which is open, enters into a cistern AB containing mercury: the bottom of the cistern is of leather, and by means of a screw at C, that mercury can be raised or lowered till its upper surface passes through an imaginary line, on which, as at N, is the zero of the scale of inches; M being the upper extremity of the column of mercury in the tube, the height MN is read by means of a vernier at M. The external air presses on the flexible bottom of the cistern, and this causing the surface of the mercury at N to rise, or allowing it to fall, the corresponding variations in the elasticity of the air in the part AN of the cistern, produce the same effect on the height of the mercurial column MN, as would be produced by the external air if it acted directly on the surface at N.



The barometer invented by Sir H. Englefield has no screw for regulating the surface of the mercury in the cistern with respect to the zero of the scale of inches; and the atmosphere, entering through the pores of the box-wood of which the cistern is formed, presses directly at N on the surface of the mercury; there can, consequently, be only one state of the atmosphere in which the surface is coincident with that zero. The exact number of inches and decimals, on the scale, at which the extremity M of the column of mercury stands when the surface at N coincides with the zero is found by the artist, and engraven on the instrument; and, when the top of the column is at that height (or at the neutral point, as it is called) no correction is necessary on account of the level of the mercury in the cistern. In other cases such correction is determined in the following manner: — The ratio between the interior area of a horizontal section through the cistern, and the area of a like section through the bore of the tube, is ascertained by the artist and engraven on the instrument: let this ratio be as 60 to 1: then the lengths of cylindrical columns, containing equal volumes, being inversely proportional to the areas of the transverse sections, the required correction will be $\frac{1}{60}$ of the difference between the height of the neutral point, and that at which the top of the column stands in the tube. This correction must be added to, or subtracted from, the height read on the scale according as the top of the column is above or below the neutral point.

The difficulty of transporting the usual mountain barometer overland has induced travellers to use, for the purpose of determining the relative heights of stations, the instrument invented by Dr. Wollaston, and called by him a *thermometrical barometer*. This consists of a thermometer, which may be of the usual kind, but very delicate, whose bulb is placed in the steam arising from distilled water in a cylindrical vessel about five inches long, the water being made to boil by an oil or spirit lamp. Now it is known that water boils when the elastic power of the steam produced from it is equal to the incumbent pressure of the atmosphere; and thus, the temperature at which, in the open air, the water boils, will depend upon the weight of the atmospherical column above it. Therefore, since this weight becomes less as the station is more elevated, it is evident that water will boil at a lower temperature on a mountain than on a plain at its foot; and, for the purpose of determining the height of the mountain, it is only necessary to find an expression for the elastic power of steam, at a given temperature under the pressure of the atmosphere, in terms of the height of an equivalent column of mercury in a barometer.

From the experiments of De Luc, M. Dubuat has obtained a formula equivalent to

$$h = \left(\frac{t}{75.18}\right)^{3.84},$$

$$\text{or log. } h = 3.84\,(\text{log. } t - 1.876105), \text{ or log. } h = 3.84 \text{ log. } t - 7.204,$$

where h is the height of the ordinary barometrical column expressed in English inches; t is the temperature at which the water boils in the open air, expressed by degrees of Fahrenheit's thermometer, and reckoned from the freezing point (32°) as zero. The value of h being thus obtained for each of the two stations, the height of one station above the other may be found by the formula for the usual mountain barometer.

432. The processes of a trigonometrical survey for determining the figure of the earth are too extensive to be frequently put in practice; they also require the combined efforts of many persons, and they involve expenses which are beyond the resources of private individuals. On the other hand, the variations of terrestrial gravity at different places on the earth's surface are capable of being determined with great precision by comparing together the observed times of the vibrations of pendulums at the places: and as these variations, though in part due to a want of homogeneity in the

mass of the earth, depend chiefly on the deviation of its figure from that of an exact sphere; a series of well-conducted experiments on the vibrations of pendulums at stations remote from each other, which may be made by two or three persons at a comparatively small expense, afford the most convenient means of ascertaining the form of the earth.

433. The pendulums which are employed for this purpose are generally of the kind called invariable; that is, they are not provided with a screw by which their lengths may be increased or diminished like the pendulums applied to ordinary clocks; so that their lengths can only vary by the expansion or contraction of the metal in consequence of the variations of temperature to which they may be subject. The effects arising from the variations of temperature and from all the other circumstances which interfere with the action of gravity upon them, are determined by theory and applied as corrections to the observed times of the vibrations.

These pendulums are employed in two ways: they may be attached to the machinery of a clock for the purpose of continuing the oscillations and registering their number, or they may be unconnected with any maintaining power, and left to vibrate by the action of gravity till the resistance of the air and the friction on the point of suspension bring them to a state of rest. The late Captain Kater, availing himself of that property of vibrating bodies by which the centres of suspension and oscillation are convertible, constructed pendulums which admit of being made to vibrate upon either of those centres at pleasure; by this construction the effective length of the pendulum (which is the distance between those centres) is easily found, by measurement, when the places of the centres have been determined by the experimental number of vibrations made upon each being equal, in equal times; and such invariable pendulums are now generally employed by the English philosophers for the purposes of experiment. They are furnished with steel pivots or axles (called knife edges) at the two places which are to be made alternately the centres of suspension and oscillation; and these rest upon the upper edge of a prism of agate or wootz, so that the pendulums may vibrate with as little friction as possible.

434. In making the experiments with a detached pendulum, the latter is placed in front of a clock regulated by mean solar, or sidereal time, but quite unconnected with its motion. On the pendulum of the clock is placed a disk of white paper opposite to a vertical wire crossing an opening in the rod of the detached pendulum. A telescope is fixed a few feet in front of the pendulums, so that, when these are put in

motion, the disk of paper may be seen to pass over the field of view. Now, since the detached pendulum and that of the clock have not exactly the same velocity, if we suppose the vertical axis of the former to have been originally in coincidence with the centre of the disk, these will separate from each other by the excess of the velocity of one above that of the other: but after a certain number of oscillations, they will again coincide, moving in opposite directions; and then the detached pendulum may be said to have gained or lost one oscillation. For example, if 30 vibrations of the clock pendulum had been observed in the interval between two consecutive coincidences, it is evident that the invariable pendulum must have made either 29 or 31 vibrations. After being in coincidence the pendulums separate as before, and again, subsequently, they coincide; and so on. The number of oscillations made by the pendulum of the clock between two, three, or more coincidences is counted, and the times of the several coincidences are shown by the clock, when they take place: then the number of oscillations made by the clock pendulum in the time of any number of coincidences (suppose n) is to the number of oscillations of the detached pendulum in the same time (which number in this case will exceed or fall short of the former number by n), as 86400 seconds (the number of oscillations made by the clock in a mean solar, or sidereal day, according as the clock is regulated by solar or sidereal time) are to the number of oscillations made by the detached pendulum in the same time, at the station.

With respect to the invariable and attached pendulum, the number of vibrations performed by it in a mean solar, or sidereal, day is ascertained by observing the times indicated on the dial of the clock to which it is attached, at the end of equal intervals of time, as 12 or 24 hours. And in both cases it is most convenient to determine the measure of time by the transits of stars.

Captain (now Colonel) Sabine's experiments in his two voyages during the years 1822, 1823, were made with pendulums whose lengths were invariable except in respect of temperature. In the first voyage the pendulum was detached from any clock-work, so that after a certain number of oscillations it rested: but in the second voyage the pendulum was attached to a clock, and its oscillations were continued by the maintaining power of the clock. The uniformity of the maintaining power was inferred from that of the extent of the arc of vibration.

435. The following propositions contain the principal subjects relating to the corrections which are required for the

purpose of determining the intensities of gravity in different places, and the ellipticity of the earth, by the vibrations of pendulums.

PROP. I.

To reduce the number of vibrations made on a given arc, which is supposed to be constant, to the number which in an equal time would be made in an arc of infinitely small extent.

If t be the time of performing a vibration in an infinitely small arc, and t' the time of vibration in an arc of given extent ($= 2a$), we have by Mechanics, as a near approximation,

$$t' = t\left(1 + \frac{\sin.^2 a}{16}\right).$$

Now, let N be the number of vibrations made in an infinitely small arc by a simple pendulum in the time T, and let N′ be the number of vibrations made in the same time T by a pendulum of equal length, and vibrating in an arc equal to $2a$; then t and t' being the times of making one vibration in each case, we have

$$t = \frac{\mathrm{T}}{\mathrm{N}}, \text{ and } t' = \frac{\mathrm{T}}{\mathrm{N}'}:$$

these values of t and t' being substituted in the preceding equation, we get

$$\mathrm{N} = \mathrm{N}'\left(1 + \frac{\sin.^2 a}{16}\right).$$

Therefore, if N′, the number of oscillations made in an arc equal to $2a$ during a given time, as a sidereal day, be given, we may obtain the number which the same pendulum would have made in an infinitely small arc, in an equal time.

Note.—If, as usual, the arcs continually diminish, the mean of the extents observed at the commencement and end of the time may be considered as equal to $2a$, half of which is the arc a in the formula.

PROP. II.

436. To find the correction of the length of a pendulum, or of the number of vibrations made in a given time, on account of the buoyancy of the air.

The loss of weight to which a pendulum is subject when it vibrates in air being equal to the weight of an equal volume of air; we have (since the loss of weight is equivalent to a diminution of the action of gravity), putting S for the specific gravity of the pendulum, s that of air, and f for the force of gravity on a pendulum vibrating in air,

$$S : s :: f : df,$$

where df represents an increment of the force of gravity; and this increment must be added to the force of gravity deduced from the experimental vibrations of the pendulum in air, in order to have the force of gravity on the pendulum in vacuo. But the force of gravity is proportional to the length of the pendulum, when the times of making one vibration are equal; therefore the above proportion becomes, l being the length of a pendulum vibrating seconds in air,

$$S : s :: l : dl,$$

and $l + dl$ becomes the length of a pendulum vibrating seconds in vacuo.

Now if it were required to find the excess of the number of vibrations performed in vacuo over the number performed in air, in an equal time, by the same pendulum; imagine the pendulum, whose length in vacuo is $l + dl$ (as above determined) to be reduced to the length l of the pendulum in air. Then, since, by Mechanics, the lengths are inversely as the squares of the number of vibrations performed in equal times,

$$l : l + dl :: N'^2 : N^2;$$

where N' is the number of vibrations performed in a given time by the pendulum $l + dl$ in vacuo (equal to the observed number of vibrations described by l in air), and N the number of vibrations performed in vacuo by the pendulum l in the same time: and by division,

$$l : dl :: N'^2 : N^2 - N'^2;$$

where $N^2 - N'^2$ is an augmentation $(= a)$ of N'^2, or of the square of the number of vibrations described in air by an equal pendulum in an equal time, and $N'^2 + a = N^2$: or again, $S : s :: N'^2 : a$, and thus a may be found. From the equation $N'^2 + a = N^2$, we have $(N'^2 + a)^{\frac{1}{2}} = N$, and developing the first member by the binomial theorem as far as two terms, we have $N' + \frac{1}{2}N'^{-1}a = N$; or $N' + \dfrac{a}{2N'} = N$; consequently $\dfrac{a}{2N'}$ is the increment, which being added to N', gives the value of N.

The value of s, the specific gravity of the air, should be determined for the state of the barometer and thermometer at the time of the experiment. And it may be observed that no notice has been taken of the retardation caused by the air which is moved with the pendulum as the latter vibrates.

PROP. III.

437. To find the correction of the length of a pendulum on account of temperature. Or to reduce the number of oscillations made in a given time at an observed temperature to the number which would be made in an equal time at a standard temperature.

N. B. The variation in the length of the pendulum corresponding to an increment of temperature expressed by one degree of the thermometer is supposed to be known from the tables of the expansions of metals by heat, or from experiments made on the pendulum itself.

Let e express the expansion, in terms of the length of the pendulum, for one degree of Fahrenheit's thermometer; let t be the standard temperature, and t' the observed temperature: then $t-t'$ $(=\pm b)$ is the difference between the observed and standard temperatures, and $\pm be$ is the expansion or contraction of the pendulum in terms of its length; consequently, l being the experimented length of the pendulum (in inches) from the point of suspension to the centre of oscillation at the standard temperature, $(1 \pm be)\,l$ is the length of the pendulum in inches at the observed temperature; let this be represented by l'. Then, by Mechanics,

$$l : l' :: n'^2 : n^2;$$

where n' is the observed number of vibrations in a sidereal day, for example, and n is the number of vibrations which would be performed in an equal time if the pendulum were corrected for expansion, or reduced to the length which it would have at the standard temperature.

It may be observed, that the expansion of a pendulum is to be ascertained by immersing it successively in fluids of different temperatures, and measuring its lengths by a microscopical apparatus; the difference between the lengths so measured will be the amount of expansion corresponding to the number of degrees in the difference between the temperatures of the fluids. Or the correction for expansion may be found by observing the number of vibrations performed in a sidereal day in apartments brought by artificial means to different temperatures: thus, the corrections being first applied for the magnitude of the arc of vibration and for the buoyancy of the air, the difference between the numbers of vibrations in a day, divided by the difference of the temperatures in degrees of the thermometer, gives the number of vibrations, or the parts of a vibration, which, for one day, are due to a difference of temperature expressed by one degree.

PROP. IV.

438. To reduce the length of a pendulum at any station to that which it should have in order that it may perform the same number of vibrations in an equal time at the level of the sea.

Let R be the semidiameter of the earth at the level of the sea in the latitude of the station; h the height of the station above the level of the sea; l the observed length of the seconds' pendulum, and l' the required length at the level of the sea. Then, the lengths of pendulums vibrating in equal times varying directly as the force of gravity, and the force of gravity varying inversely as the square of the distance of the station from the centre of the earth, we have

$$l' : l :: (\text{R} + h)^2 : \text{R}^2; \text{ whence } l' = l\frac{(\text{R} + h)^2}{\text{R}^2};$$

on developing $(\text{R} + h)^2$ and rejecting powers of h above the first, we get

$$l' = l\left(1 + \frac{2h}{\text{R}}\right).$$

Note.—Captain Kater's allowance for the height of a station is rather less than that which would be given by this rule, because he used a co-efficient depending on the attraction of the matter between the level of the sea and the place of observation.

439. In Captain, now Colonel, Sabine's "Account of Experiments to determine the Figure of the Earth" (1825), there is given (page 351.) from his own experiments, with those of Biot, Kater, and others, a table of the lengths of pendulums at several stations, extending from the equator to Spitzbergen; and from that table the following values have been taken.

Stations.	Latitudes.	Lengths of Pend. in Inches.
Sierra Leone -	8° 29′ 28″	39.01997
Jamaica - -	17 56 7	39.03510
New York - -	40 42 43	39.10168
London - -	51 31 8	39.13929
Unst - -	60 45 26.5	39.17164
Hammerfest - -	70 4 5	39.19519
Spitzbergen - -	79 49 58	39.21469

Now it is proved in Mechanics that the length of a seconds pendulum varies directly with the force of gravity, and (Airy's Tracts, Fig. of Earth, art. 63.) that in any place whose latitude is λ, the force of gravity is expressed by $1 + n \sin.^2\lambda$, where n is the excess of the force of gravity at the pole above the force at the equator, the latter force being represented by unity, and the earth being a spheroid; therefore the force of gravity at a station whose latitude is λ, is to the force at the equator, as $1 + n \sin.^2\lambda$ is to 1: hence, substituting l, the length of a seconds pendulum at the station for the force of gravity there, and A for the length of the like pendulum at the equator, we have

$$l = A + An \sin.^2\lambda.$$

In like manner, for a place whose latitude is λ', representing the length of the seconds pendulum there by l', we have

$$l' = A + An \sin.^2\lambda'.$$

Subtracting the former equation from the latter, we get

$$l' - l = An\,(\sin.^2 \lambda' - \sin.^2 \lambda):$$

whence An may be found, and, by substitution in either equation, the separate values of A and of n are obtained.

The conclusions drawn by Colonel Sabine from all his pendulum experiments, are that A is equal to 39.0152 inches; whence $n = 0.005189$: from this last, by Clairaut's theorem $\frac{5m}{2} - n = e$ (Airy's Tracts, Fig. of Earth, art. 62.), the value of e, the ellipticity of the earth, may be found. In the formula, m represents $\frac{1}{289}$, the value at the equator of the centrifugal force, that of gravity there being unity; and from the above data it will be found that e is equal to $\frac{1}{289}$, or very nearly equal to m: by the measured lengths of degrees of latitude, e is found to be about $\frac{1}{303}$.

From the above equation for l we have $l - A = An \sin.^2\lambda$; it follows, therefore, that the increase in the length of a seconds pendulum from the equator towards either pole varies with the square of the sine of the latitude of the station: also, the intensities of gravity at different stations being proportional to the length of the seconds pendulums at the stations, the increase of the force of gravity from the equator towards either pole varies with the square of the sine of the latitude.

440. Experiments for determining the elements of terrestrial magnetism being now usually carried on in connection with geodetical operations, it will be advantageous to notice

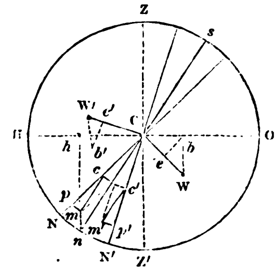
in this place their nature and the manner of performing them.

Those elements are of three kinds:—the *Declination* or, as it is generally called, the variation of the magnetized needle; the *Inclination* or dip of the needle, and the intensity of the earth's magnetic power.

441. The ordinary variation, or azimuth compass, is well known: the needle is supported near its centre of gravity, and is allowed to traverse horizontally on the point of a vertical pivot made of steel, and the box is furnished with plane sights, which may be placed in the direction of the meridian, or may be turned towards a terrestrial object, as the case may require. Compasses of a superior kind, like that of Colonel Beaufoy, differ from the others in having, instead of plane sights, a small transit telescope, by which the middle line of the box may, by the observed transits of stars, be placed accurately in the direction of the geographical meridian.

The usual dipping needle is a bar of steel which, before the magnetic quality is communicated to it, is balanced accurately upon its centre of gravity, where, by a horizontal axis of steel, terminating above and below in what is called a knife-edge, the needle rests, on each side, on the edge of an agate plate. With good needles there is an apparatus consisting of screws by which the centre of gravity and centre of motion are rendered coincident; this adjustment is made before the needle is magnetized by causing it to vibrate on the points of support, and observing that it comes to rest in a horizontal position; then, after reversing it on its axis, so that the uppermost edge of the needle becomes the lower, the needle is again made to vibrate, and alterations, if necessary, are made till the needle is found to settle horizontally in both situations. The needle thus prepared, and being duly magnetized, is placed in a plane coinciding with what is called the magnetic meridian (a vertical plane passing through a well-balanced compass needle); when the position which it assumes indicates, by means of the graduated circle on whose centre it turns, the absolute inclination or dip of the needle, or the line of direction of the resultant of all the magnetic forces in the earth. On being made to vibrate like a pendulum, the number of oscillations performed in a given time affords one of the means of determining the intensity of terrestrial magnetism in the direction of that resultant.

The intensity of terrestrial magnetism in horizontal and vertical directions are resolved parts of the intensity in the direction of the resultant, that is, in the direction assumed by the magnetic axis of the dipping needle. Thus H O being a horizontal line passing through the centre C of the needle's motion, in the plane of the magnetic meridian, and *ns* being the direction of the needle when subject to the force of terrestrial magnetism, so that the angle H C *n* is the inclination or dip; on drawing *nh* perpendicularly to H O, the three lines *n*C, C*h*, *nh* will, respectively, represent the absolute intensity and its horizontal and vertical components: hence if *f* represent the absolute intensity, and *d* the dip, we shall have *f* cos. *d* for the horizontal intensity, and *f* sin. *d* for the vertical intensity.



442. The dipping needle is not always so simple as that which has been described: in order to increase the facility of its vibrations, Mayer of Gottingen attached at the centre of gravity of the needle a wire perpendicular to its length, and in the plane of its vibration; this wire, which carries at its extremity a brass ball, may, by inverting the needle, be either above or below the latter, and the number of vibrations made in a given time may be observed in both positions of the needle. The intention, in separating the centre of motion from that of gravity, is to give the needle a power, resulting from the weight, to overcome the friction of the axis, and allow it, after having vibrated, to return with greater certainty to the same point on the graduated circle than if the centres of gravity and motion were coincident. In using the needle, the dip or inclination should be observed with the axis in one position, and again with the axis reversed, so that the upper edge may become the lower: and a mean of the two readings should be taken. Also, should the situation of the brass ball be such that its centre does not, when the needle is in a horizontal position, lie vertically above or below the centre of gravity of the latter, four such observations should be made, two with the poles of the needle in their existing state, and two others with the poles reversed. The reversion of the poles is effected by the usual method of magnetizing needles.

443. In order to obtain the correct dip from two observations, when the centre of gravity is alternately below and above the point of support; let Z O N H be the vertical circle

in the plane of the magnetic meridian: let CN be the actual direction of the needle when a weight W is applied at the end of the wire CW perpendicularly to NC, and let Cn be the direction which the needle ought to assume by the influence of terrestrial magnetism when the centre of gravity of the whole needle coincides with the point of support. Then, HO being a horizontal line, the angle HCn is the true inclination or dip ($=d$), and HCN ($=$CWb) the false dip ($=d'$), Wb being a vertical line drawn through W to represent the weight of the ball W.

Now, an equilibrium must be supposed to exist between the weight at W and the force of magnetism acting at c (the centre of gravity of the arm NC); the former force causing the needle to turn about C towards CH, and the latter causing it to turn about C towards CZ′. Let the magnetic force acting at c be represented in magnitude and direction by cm parallel to Cn, and let fall mp perpendicularly on CN; also from b let fall be perpendicularly on CW. Then, by Mechanics, the force be being supposed to act at W in a direction parallel to be, and pm to act at c perpendicularly to CN,

$$be \times \text{CW} = mp \times \text{C}c \;\ldots\ldots\; (\text{A}):$$

now $$be\;(= \text{W}b \text{ sin. CW}b) = \text{W}b \text{ sin. } d',$$

and $mp\;(= cm \text{ sin. } mcp = cm \text{ sin. NC}n) = cm \text{ sin. } (d-d')$.

When the needle is inverted, so that the weight W may be above it, as at W′, let its position be CN′: then HCn, being the true dip as before, the angle HCN′ ($=d''$) will be the false dip; and the resolution of the forces being similar in both cases, we have

$$b'e' \times \text{CW}' = m'p' \times \text{C}c' \;\ldots\ldots\; (\text{B}):$$

also $b'e' = \text{W}'b' \text{ sin. } d''$ and $m'p' = c'm' \text{ sin. } (d''-d)$.

Substituting the values in the equations (A) and (B), and cancelling the terms which by their equality destroy one another in division, there results

$$\frac{\text{sin } (d-d')}{\text{sin. } d'} = \frac{\text{sin. } (d''-d)}{\text{sin. } d''}:$$

Developing the numerators by trigonometry, the equation becomes

$$\frac{\text{sin. } d \text{ cos. } d' - \text{cos. } d \text{ sin. } d'}{\text{sin. } d'} = \frac{\text{sin. } d'' \text{ cos. } d - \text{cos. } d'' \text{ sin. } d}{\text{sin. } d''},$$

or sin. d cotan. $d' -$ cos. $d =$ cos. $d -$ cotan. d'' sin. d; or again, dividing by sin. d, and transposing,

$$\text{cotan } d' + \text{cotan. } d'' = 2 \text{ cotan. } d:$$

thus, from the observed values of d' and d'', that of d, the true dip, may be found.

In the "Transactions" of the Royal Society of Sciences at Gottingen for 1814, the following formula for the true dip d is investigated in the case of four observations being made with the needle and its poles in direct and reversed positions, as above mentioned.

Let d' d'' be the values of the dip in a direct and a reversed position of the needle, as in the last case; and, after the poles are reversed, let d''', d^{iv} be values of the dip in a direct and reversed position of the needle: then, putting

A for cotan. $d' +$ cotan. d'',

B for cotan. $d' -$ cotan. d'',

C for cotan. $d''' +$ cotan. d^{iv}, and

D for cotan. $d''' -$ cotan. d^{iv},

there is obtained

$$2 \text{ cotan. } d = \frac{\text{A.D}}{\text{B} + \text{D}} + \frac{\text{B.C}}{\text{B} + \text{D}}.$$

444. The place of the centre of magnetism in each arm of a balanced magnetized needle may be found precisely as the centre of gravity in any solid body would be found; and, if the needle be cylindrical or prismatical, that centre would be in the middle of the length of each arm: the centre of oscillation might also be found for a needle as it would be found for a common pendulum; and by the theory of pendulums, it may be shown that the intensities of magnetical attractions in different parts of the earth are inversely as the squares of the times in which a given number of vibrations are made, or directly as the squares of the number of vibrations made in a given time.

In making experiments with a needle, the number of vibrations made in a given time, and in an arc of a certain extent, must be reduced to the number which would be made in the same time in an arc of infinitely small extent: corrections should also be made for the buoyancy of the air; and the formulæ to be used for these purposes are similar to those which have been already given for the vibrations of pendulums by gravity.

A variation of temperature affects the magnetism of a needle as well as its length; and a formula, depending on the length, similar to that which may be employed for a common pendulum, would not be sufficiently precise for the correction of the error arising from that cause. The method employed to find experimentally the co-efficient for reducing the time of making a given number of vibrations at a certain temperature to the time in which an equal number would be made at a standard temperature, consists in counting the number of vibrations performed in a given time by the needle when placed in a vessel within which the temperature of the air may be varied at pleasure. For this purpose the apparatus is placed in a vessel of earth, or a trough of wood, with a glass top; and this vessel is placed within another: between the two, cold water first, and hot water afterwards, is poured, in order to bring the temperatures to any convenient states, which may be indicated by a thermometer; care is, however, taken not to raise the temperature by the hot water higher than about 120° (Fahrenheit), lest the magnetism of the needle should thereby be permanently changed.

Now, let T, in seconds, be the time in which a needle makes any number (suppose 100) of vibrations in the vessel when the temperature is t, and T' the time of making an equal number of vibrations when the temperature is raised to t'; then

$$t'-t : T'-T :: 1^\circ : \frac{T'-T}{t'-t},$$

and this last term is the increase, in seconds, in the time of making that number of vibrations in consequence of an increase of temperature expressed by one degree of the thermometer. If the value of this fraction be represented by m, then $m(\tau - \tau')$, in which τ' denotes a standard temperature, suppose 60° (Fahr.), and τ the temperature at which an observation was made, will express the increase or diminution due to the observed time of any number of vibrations, with respect to the time of making an equal number at the standard temperature. Consequently, if T be the observed time of making any number of vibrations, and T' the required time of making an equal number of vibrations at the standard temperature, we should have

$$T + m(\tau-\tau') = T'.$$

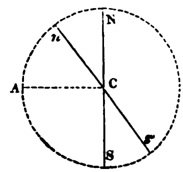
The magnetic condition of a needle ought to be ascertained at certain intervals of time, and allowance must be made for any variations which may be detected when observations at considerable intervals of time are compared together. In the "Voyage of the Beagle," it is stated that the time in which the needle performed 300 vibrations had increased in 5 years from 734″.45 to 775″.8. The state of a needle should not, however, be interfered with during the progress of any course of experiments for determining the intensity of terrestrial magnetism.

445. For the more delicate researches relating to terrestrial magnetism it is now found convenient, instead of the usual variation compass, to employ what is called a Declination Magnetometer; and, instead of ascertaining the intensity directly by a dipping needle, to employ horizontal force and vertical force magnetometers, by which the intensities in those directions may be determined; and, from them, the direct intensity may be deduced.

446. The Declination Magnetometer is a bar magnet from twelve to fifteen inches long, nearly one inch broad, and a quarter of an inch in thickness; this is made to rest horizontally in a stirrup of gun-metal, which is suspended, from a fixed point above, by fibres of untwisted silk nearly three feet long, and the whole is inclosed in a box to protect it from the agitations of the air. The bar is furnished with two sliding pieces, one of which carries a glass lens, and the other a scale finely graduated on glass, the scale being placed at the focus of the lens. By means of a telescope at a certain distance, the divisions on the scale may be seen through the lens (in the manner of a collimating instrument), and minute changes in the position of the axis of the magnet may therefore be detected by the divisions of the scale, which may be observed in coincidence with a fixed wire in the telescope.

In order to have the suspending threads free from torsion at the commencement of the observations, a bar of gun-metal is previously placed in the stirrup, and allowed to remain there till the threads are at rest, when a button, carrying the point of suspension, is turned horizontally so as to bring the bar into the direction of the magnetic meridian; after which the bar is removed, and the magnet is introduced in its place.

It is obvious, however, that though the threads may be free from torsion when the needle lies in the plane of the magnetic meridian, yet, as soon as by changes in the declination, the needle turns from that position, the threads will become twisted, and the apparent deviation from the mean position of the needle will be less than the true deviation. In order to ascertain the correction due to this cause of error, the ratio of the force of torsion to the magnetic force must be found by experiment. For this purpose, N S representing the position of the suspended needle when in the magnetic meridian, let the button of the torsion circle be turned upon its centre, vertically above C, till its index has described any angle N C A; (suppose a right angle), then the needle taking a position as ns and resting between the force of torsion acting horizontally from n towards A, and the force of magnetism acting horizontally from n towards N, if H represent the former force, and F the latter, we shall have by Mechanics,



$$\text{F} : \text{H} :: \text{AC}n : n\text{CN},$$

and by proportion,

$$\text{F} + \text{H} : \text{F} :: \text{ACN} : \text{AC}n; \text{ whence } \frac{\text{F}+\text{H}}{\text{F}} = \frac{\text{ACN}}{\text{AC}n}.$$

The terms in the second member of this equation being given by the experiment, the ratio of F to F + H, or of 1 to $1 + \frac{\text{H}}{\text{F}}$, is found. Any change in the declination of the needle, which may be observed with the instrument, is to the corresponding change, free from the error produced by torsion, as 1 is to $1 + \frac{\text{H}}{\text{F}}$; and hence the observed change multiplied by $1 + \frac{\text{H}}{\text{F}}$ gives the corrected change.

447. In order to obtain the value of the horizontal intensity of terrestrial magnetism by the instrument, a magnetized bar, called a *Deflector*, is placed in a horizontal position at right angles to the magnetic meridian, and in a line imagined to be drawn through the centre of the suspended magnet. Its centre is to be placed at two different distances from the latter on this line; and, in each position, the observer is to take notice of the deflexions produced by its attraction on the declination magnet when the north end of the deflecting bar is turned successively towards the east and towards the west; half the difference of the deflexions with the north end towards the east, and afterwards towards the west, being considered as the required deflexion at each distance. The experiments are then to be repeated on the other side of the suspended magnet at equal distances from its centre, and a mean of the four deflexions, two on each side at equal distances, is to be taken

for the deflexion at each of the two distances. Now it is demonstrated by M. Gauss (Intensitas Vis Magneticæ Terrestris, 1833) that if m represent the momentum of magnetism in the suspended bar, X the horizontal intensity of the earth's magnetism, R and R′ the distances of the centre of the deflecting bar from that of the suspended magnet, and ϕ, ϕ', the means of the observed angles of deflexion at those distances,

$$\frac{m}{\text{X}} = \frac{\text{R}'^5 \tan.\phi' - \text{R}^5 \tan.\phi}{2\,(\text{R}'^2 - \text{R}^2)}\left(1 + \frac{\text{H}}{\text{F}}\right);$$

therefore the ratio between the horizontal force of terrestrial magnetism, and the magnetism of the bar or needle, can be found from the experiment.

Next, the declination magnet being removed, the deflecting bar is to be attached to the suspension threads, and allowed to vibrate horizontally, and the time T of one vibration is to be determined from at least 100 oscillations: then, the momentum K of the bar's inertia being calculated, we have

$$m\text{X} = \frac{\pi^2\,\text{K}}{\text{T}^2\left(1 + \frac{\text{H}}{\text{F}}\right)}.\ *$$

From these two equations m may be eliminated and the value of X may be found.

If the value of the inclination or dip (d) be found by observation with the inclination instrument or dip circle in the plane of the magnetic meridian, the most correct method of ascertaining the vertical intensity will be to deduce it

* Here mX expresses the momentum of the horizontal force of terrestrial magnetism by which the bar is made to vibrate; and, K being the momentum of the bar's inertia, $\frac{m\text{X}}{\text{K}}$ denotes the angular velocity of the bar; but, by the nature of the centre of oscillation, in Mechanics, l being the distance of that centre from the point of suspension, and g the force of gravity, $\frac{mg}{ml}$, or $\frac{g}{l}$, or its equivalent $\frac{\pi^2}{\text{T}^2}$ ($\pi = 3.1416$), denotes the like angular velocity; therefore $m\text{X} = \frac{\pi^2\text{K}}{\text{T}^2}$. The factor $\left(1 + \frac{\text{H}}{\text{F}}\right)$ is the correction of the square of the time of vibration on account of the torsion of the suspension thread, as above shown. The bar being rectangular, the momentum K of its inertia with respect to a vertical axis passing through its centre of gravity may be shown, by Mechanics, to be equal to $\frac{1}{12}$ M (a^2+b^2); in which expression a is the length, b the breath, and M the mass of the bar.

from the value X, of the horizontal intensity, found as above. It is manifest that, if Ch (fig. to art. 441.) represent X, hn will represent the vertical intensity which, consequently, is equal to X tan. d. Let this be represented by Y; then the total intensity, represented by R and acting in the direction Cn, may be obtained from the formula $R = \sqrt{(X^2 + Y^2)}$.

448. That which is called the horizontal-force-magnetometer, and which is also employed for determining the horizontal intensity of terrestrial magnetism, is a bifilar instrument consisting of a magnetized bar or needle, suspended by a slender and very flexible wire passing under a pulley (from the axle of which a stirrup carrying the magnet is suspended) and through two perforations in a small bar above the pulley; the wire is attached at its extremities to a plate at the top of the apparatus, so that the two halves of its length are parallel to one another when the needle is in the plane of the magnetic meridian.

Now, turning the plate at the top of the apparatus till the needle is made to take a position at right angles to the magnetic meridian, resting there in equilibrio between the horizontal component of magnetic attraction by which it is drawn towards that meridian, and the force of torsion by which it is turned from it; then, the force of torsion being computed, that of the magnetic component in the horizontal direction may be obtained. The same instrument is employed to determine, by the observed extent of the horizontal oscillations of the needle about its mean place, at right angles to the magnetic meridian, the ratio between the corresponding variations of the horizontal intensity, and the whole amount of that component. The formula for this purpose, as given in the "Report of the Committee of Physics," published by the Royal Society, 1840, is

$$\frac{\Delta F}{F} = -\text{cotan. } v \, \Delta \, u,$$

in which F represents the horizontal intensity, ΔF its variation, v is the angle formed by lines joining the upper extremities and the lower extremities of the suspending wires, and u, expressed in terms of radius, is the observed deviation of the magnet from its mean place.

The vertical force magnetometer is used for determining the variations in the vertical component of terrestrial magnetism. It consists of a magnetic bar or needle about 12 inches long, having a horizontal axle formed into a knife edge and resting upon two agate planes which are supported on pillars of copper. The needle is provided, on each arm, with a screw; one of these acts as a weight to keep the needle in a horizontal position, and the other, to render the centre of gravity nearly coincident with the centre of motion. The apparatus has an azimuthal motion by which the needle may be allowed to vibrate in the plane of any verticle circle; and it is usually placed at right angles to the magnetic meridian. From the observed extent of the vertical vibrations of the needle about its mean place, in a horizontal plane, at right angles to the magnetic meridian, the ratio between the corresponding variations of the vertical intensity and the whole amount of that component may be determined.

The formula for this purpose, in the "Report" above mentioned, is,

$$\frac{\Delta \text{F}}{\text{F}} = \frac{\text{T}'^2}{\text{T}^2} \text{cotan.}\, \theta \, \Delta \, \eta,$$

in which F represents the vertical intensity, Δ F its variation, θ the dip or inclination, T and T′ are the times in which the needle vibrates in vertical and horizontal planes respectively, and η, expressed in parts of radius, is the observed deviation of the magnet from its mean place.

449. For the sake of regularity, in the expressions for the numerical values of the intensities of terrestrial magnetism, all measures of length, as well the linear dimensions of the magnetic bars or needles as the distances of the deflecting bars from them, should be given in feet and decimals of a foot; the mass of a bar should be denoted by its weight in grains; the times in which vibrations are performed should be expressed in seconds, and angular velocities by the decimals of a foot described in one second on the arc of movement. In such terms, from the formulæ for $\frac{m}{\text{X}}$ and $m\text{X}$ above, Lieutenants Lefroy and Riddle, in 1842, found 3.72 as an approximation to the value of the horizontal intensity at Woolwich (Phil. Trans. 1843, p. II.), at the same time the inclination, or dip, was found to be 69° 3′; therefore X tan. d, the vertical intensity, would have been expressed by 9.72, and $\frac{\text{X}}{\text{cos.}\, d}$, the total intensity, by 10.404.

At present the unit of absolute intensity is taken to represent the state of that element on the peripheries of two curve lines surrounding the earth, and containing between them a band of irregular breadth crossing the geographical equator. The northern limit of this band was, at one time, supposed to be the line of least intensity; and, with reference to this unit, the intensity at London is expressed by the number 1.372.

450. When the results of several observations and experiments for the determination of an astronomical or geodetical element have been obtained, they are next to be combined together, so as to obtain from the whole that value of the element which has the greatest probability of being strictly or nearly true: the principles upon which such combinations are made have been already explained (arts. 328, 329, &c.) and it is intended now to show by a few examples in what manner the equations of condition are formed from the circumstances of a case.

One of the most simple is that in which, at any station, there may have been observed the angles subtended by the distances between two objects A and B, between B and another object C, and also between A and C, which may be either the sum or the difference of the other angles, suppose the sum; and it is required to find the most probable values of the two first angles when, the observations not being equally good, the weights due to the three observed angles are represented by 4, 7, and 9.

Let the observed angle between A and B be $42°14'6''.5$, between B and C be $34°18'4''.25$, and between A and C be $76°32'14''.5$. Now, in order to save trouble, since there is a presumption that the errors can only exist in the seconds, let the degrees and minutes be omitted till the operation is concluded, and let x, y, z be the most probable values of the seconds in the different angles: then $(6.5-x)\,2$, $(4.25-y)\sqrt{7}$, and $(14.5-x-y)\,3$ are the three errors multiplied by the square roots of their respective weights (art. 326.) in order to bring them to the same degree of precision; and they constitute the first members of three equations of condition, the other members being the presumed errors, which may be represented by E_1, E_2, and E_3.

Multiplying each equation by the coefficient of x in it, with its proper sign, and making the results equal to zero, agreeably to the method of *Least Squares* (art. 330.), we have

$$2(-6.5+x)=0$$

and $$3(-14.5+x+y)=0,$$

whose sum is $$-56.5+5x+3y=0.$$

Again, multiplying each by the coefficient of y in it, and making the results equal to zero, we get

$$\sqrt{7}.(-4.25+y)=0$$

and $$3(-14.5+x+y)=0,$$

whose sum is $-54.737+5.644y+3x=0$.

From these sums we obtain

$$x=8''.05,\ y=5''.42 \text{ and } x+y=13''.47.$$

Prefixing to these numbers the degrees and minutes, the results will be the most probable values of the three angles.

If a relatively correct determination of the instant at which a certain star is bisected by the middle wire of a transit telescope were required; it might be obtained by a combination of several equations of the form

$$\text{T}-t-x-ay+bz=\text{E},$$

in which T is the true right ascension of a star as given in the Nautical Almanac, t is the observed time of the transit of the same star at the middle wire, x is the error of the sidereal clock supposed to be too fast, y is the error of collimation, in time, on the star's parallel of declination (art. 88.) and its coefficient a is the secant of the star's declination; also z is the azimuthal deviation of the telescope, in time, and its coefficient b is that which is designated n in art. 95. If t in the equations were strictly correct the first member of each would be equivalent to zero, but, on account of the errors of observation, that member must be made equivalent to some small error which is represented by E.

Multiplying the left hand member of each equation first by the coefficient of x in it, then by that of y, and afterwards by that of z, and making the sums of the equations separately equal to zero; there may, from the three equations, be obtained the most probable values of x, y and z; and from these, by substitution, the most probable time t of the transit.

Equations of condition for the length of a meridional arc may be of the form given in art. 420.; in which representing a by x, ε by y, $l'-l$ (the coefficient of a) by p and $\left(\frac{1}{2}+\frac{3}{2}\frac{\cos.(l'-l)\sin.(l'-l)}{l'-l}\right)(l'-l)$ by q; also $m'-m$ by L, we shall have

$$px+qxy=\text{L}, \text{ or } \text{L}-px-qxy=\text{E}$$

(E representing the supposed error arising from incorrectness in the data). From any number of these equations there may be obtained the most probable values of x and y, that is of a and ε. (Fig. of Earth, Ency. Metrop., p. 218.)

Equations of condition for the lengths of pendulums may be of the kind $l = \text{A} + \text{A}\,n \sin.^2 \lambda$ (art. 439.), the values of l being obtained directly from the experiments, and A and n deduced by computation; and, putting the equations in the form $l - \text{A} - \text{A} \sin.^2 \lambda\, n = \text{E}$ (E representing the presumed error) the most probable values of A and n may be obtained. (Sabine's "Pendulum Experiments," p. 349, &c.)

In some circumstances the determination of the most probable values of elements depends upon several independent relations which it is necessary to satisfy simultaneously, and this subject has been treated by M. Gauss in his "Supplementum Theoriæ Combinationis, &c.," Gottingen 1828. An application of the method has been made by Mr. Galloway in the "Memoirs of the Astronomical Society for 1844," to a triangulation executed in England, when the trigonometrical survey of the country was commenced; and the following is an outline of the process.

Every observed angle is considered as subject to a small error, and the equations of condition are of four different kinds. Those of the first kind are founded on the fact that the three angles of each geodetical triangle are equal to two right angles together with the spherical excess; and its form is

$$a + x + b + y + c + z = 180° + \text{excess};$$

in which a, b, c denote the three observed angles, or rather their several weighted means, and x, y, z are the unknown errors which may exist in the angles respectively. Equations of the second kind are found by making the sum of all the horizontal angles at any one station, when they comprehend all the circuit of the horizon, equal to 360 degrees; and their form is

$$a + x + b + y + c + z = 360,$$

in which a, b, c represent the angles, and x, y, z the errors. Again, in each triangle let a side be taken which is common to that and the next triangle, in the latter let a side be taken which is common to it and to a third triangle, and so on in a circuit ending with a side which is common to the last and first triangles; then, the sines of these sides being to one another as the sines of the opposite angles, (art. 61.), if the observed angles which are opposite to the sides be represented by a, b, c, &c., and the errors by x, y, z, x', y', z', &c., there may be formed a series of the fractions

$$\frac{\text{sin.}\,(a+x)}{\text{sin.}\,(b+y)},\quad \frac{\text{sin.}\,(b+y)}{\text{sin.}\,(a+x)},\quad \frac{\text{sin.}\,(b+y)}{\text{sin.}\,(c+z)},\quad \frac{\text{sin.}\,(c+z)}{\text{sin.}\,(b+y)},\ \text{\&c.},$$

which being multiplied together have unity for the equivalent of their product: in this manner there may be formed equations of a third kind. Equations of a fourth kind may be formed when an observed angle is made up of two or more angles, each of which has been also observed, by making the whole angle equal to the sum of all its components.

The values of the errors or corrections in the several equations of condition are to be determined by the method of least squares, and the angles are at the same time to satisfy those equations: thus, not only will the most probable values of the angles be obtained; but, from one extremity to another of a triangulation, the sides computed with those angles will be the same in whatever order the series of triangles may be taken.

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[Vide pp. 2 and 4.

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