TEXT-BOOK

ON

GEODESY AND LEAST SQUARES

PREPARED FOR THE USE OF

CIVIL ENGINEERING STUDENTS

BY

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PREFACE.

THIS book is an outgrowth of a course of lectures prepared for instruction at Cornell at a time when there was but little available in English on the subject of geodesy outside of the Coast Survey Reports.

Field practice in geodetic, hydrographic, and topographic surveying was introduced as a part of the course, the work advancing from year to year continuously over the territory as in actual practice. This furnished data for reduction, computation, and mapping, so that the course has been thoroughly practical.

The lecture notes have been revised from time to time and printed by one of the duplicating processes for the use of the students, as they have seemed to meet our needs better than the books which have been published on the subject.

In preparing the notes for publication they have been elaborated to give them the character of a text-book, and they have been thoroughly revised in the effort to bring the work up to the best standard practice.

Acknowledgment is due to many friends for data and suggestions, and especially to Mr. Ross M. Riegel, C.E., an instructor in the college, for intelligent and conscientious aid in carrying through the revision and proof-reading.

The material for the work has been gradually collected from various sources, some of which have been referred to in the text. The principal are: Clarke's Geodesy, Jordan's Vermessungskunde, Chauvenet's Method of Least Squares, Wright and Hayford's Adjustment of Observations, and the Reports and data furnished from the office of the Coast Survey.

For a bibliography of geodesy reference is made to Appendix No. 16 of the Coast Survey Report for 1887, prepared by Professor Gore.

ITHACA, N. Y., Nov., 1906

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GEODESY AND LEAST SQUARES.

PART I.

GEODESY.

CHAPTER I.

INTRODUCTION.

1. **Geodetic Survey.**—Geodesy is the science and art of making the measurements and reductions required in relatively locating with accuracy, on the earth's surface, points which may be widely separated. It hence supposes a knowledge of the figure of the earth, a knowledge of the various phenomena which affect physical measurements, and a knowledge of the construction and use of instruments, in addition to the accuracy of sight and touch so characteristic of the good observer.

A triangulation-net, or chain of triangles, is usually employed as giving the best results, both in quantity and accuracy, for the expenditure. Elevated points are chosen for the triangle vertices at distances apart, varying with the character of the survey, from a few miles up to a hundred; one or more level lines shorter than the others are selected for base lines, in such positions that they can be readily connected with the main net; signals are established which define the vertices accurately, yet are conspicuous enough to be seen by the aid of a telescope from the adjacent stations; the horizontal angles of the triangles, and usually the vertical also, or the inclinations of the sides, are then accurately measured with a theodolite, and the base lines with a base apparatus. All the triangle sides and the differences in elevation of the vertices can then be computed.

Usually the elevations above sea-level of one or more vertices are measured, while the latitudes of one or more vertices, and their longitudes with reference to some standard observatory or reference station, and the azimuths of one or more sides, are determined by astronomical observations. The actual positions on the earth's surface, both horizontally and vertically, can then be computed.

The objects of a geodetic survey are usually twofold:—

(a) The location or recovery of boundary and division lines or monuments, and the furnishing of a net with which to connect a topographic or hydrographic survey, so that the inaccuracies of the latter cannot accumulate over large areas.

(b) The accurate determination of the figure of the earth. The distance between the parallels or meridians through any two stations or vertices can be found from the triangulation, and their difference in latitude and in longitude from the astronomical observations. Dividing the difference in latitude in linear units by the angle in degree-measure will give the length of a degree of the meridian, or by the angle in π-measure will give its radius of curvature. From these values in different latitudes the semi-axes a and b, the meridian quadrant Q and the eccentricity e, or compression c,* can be computed, assuming the section an ellipse, or the actual form can be approximated. Similarly, the parallels, assumed to be circles, can be computed,—giving an ellipsoid of revolution,—or their actual shape can be approximated.

2. Historic Outline.—(a) In glancing at the development of the science of Geodesy we may note as of special interest:—

The first authenticated hypothesis of the spherical form of the earth by Pythagoras, who is supposed to have been born about 582 B.C.

The first determination of the circumference by Eratosthenes, 230 B.C. He originated the method of deducing the size of the earth from a measured meridional arc, for he found that while the sun's rays were vertical at noon during the summer solstice at

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* $c = \frac{a-b}{a}$.

Syene in southern Egypt, they made an angle $2\pi/50$ with the vertical at Alexandria in northern Egypt, and reasoned from this that the earth's circumference must be 50 times the distance between the points. The distance, according to the statements of travelers, was 5 000 stadia, giving 250 000 stadia for the circumference by assuming both points to be on the same meridian (Syene is nearly 3° east of Alexandria). Jordan (*Vermessungskunde*, Stuttgart, 1890, vol. 3, p. 2) estimates this value to be about 16% in excess by taking 1 stadium = 185^{m}, the exact value of a stadium being unknown.

Omitting the arc of 2 degrees of the meridian, which was directly measured with wooden rods under the direction of an Arabian caliph in 827, and the measurement made by Fernel in 1525 by counting the number of revolutions made by a carriage wheel in going from Paris to Amiens, and reducing the broken line to the meridian,—giving, by an unusual compensation of errors, a computed circumference only 0.1% in excess,—we come to

The arc measured by Snellius of Holland in 1615, it being the first in which the principle of triangulation was employed. He used 33 triangles; measured his base line with a chain, his angles with a sector having sights attached; and found a meridional arc of about 1° 11′. His computed circumference was 3.4% too small.

The introduction of cross hairs in the telescope, and its adaptation to angle instruments, by Picard in 1669. He extended a triangulation over an arc of about 1° 23′ from a base line nearly seven miles long, and derived the most accurate degree length thus far given. His angles were carefully measured with a sector of 10 feet radius, to which a telescope was attached.

The facts reported by Richer, on his return from an astronomical expedition in 1672, namely, that his clock, which beat seconds at Paris before starting, lost about two minutes per day while at the island of Cayenne, South America, and could only be corrected by shortening the pendulum 1¼ Paris lines.

The announcement by Newton in the *Principia*, 1687, of the theory of universal gravitation, and of the corollary of the oblate spheroidal form of the earth. The first was confirmed by Picard's more accurate degree length; for, with the diameter of the earth thus given, the force of gravity at the surface and the force required to hold the moon in its orbit were to each other inversely as the squares of the distances from the earth's center. The second was confirmed by the behavior of Richer's pendulum; for, since *

$$t=\pi\sqrt{\frac{l}{g}\left(1+\frac{h}{8l}\right)}$$

(where t is the time of oscillation in seconds in vacuo, l the length, g the acceleration of gravity, and h the versed sine of the semi-arc of oscillation, supposed small), an increase in t, for a given value of l in the lower latitude, indicated a decrease in g, or an increase in distance from the earth's center in approaching the equator.

The extension of Picard's triangulation each way from the vicinity of Paris to include a meridional arc of 8° 31′ between 1683 and 1716 by J. and D. Cassini, from which the length of a degree of the meridian was found to be less at the northern end than at the southern. The earth would thus be a prolate spheroid and not an oblate, as advocated by Newton, Huygens, and others.

Huygens had published in 1691 the results of experiments whereby he found that a flexible hoop when rotated about one of its diameters would become flattened at the poles if unrestrained. The controversy which arose finally induced the French Academy—as the French at this time took the lead in Geodesy—to send out two expeditions, one to Peru under the equator in 1735, the other to Lapland under the Arctic Circle in 1736, to definitely settle the question. The degree length in Lapland, when made known in 1737, was found to be greater than at Paris; Cassini's arc, when revised in 1744, gave a greater length for a degree of the meridian at the northern end than at the southern; so that, when the result from Peru was received about a year later, all agreed in confirming the oblate hypothesis. The details of the measures of these arcs are extremely interesting. The first is described by Maupertuis in *La figure de la Terre*, Paris, 1738, and by Outhier in *Journal d'un Voyage au Nord en 1736–37*, while its remeasure by Svanberg, 1801–3, is described in *Exposition des Opérations faites en Lapponie*, by J. Svanberg, Stockholm, 1805; the second, by Cassini de Thury, in *La méridienne de l'Observatorie de Paris, verifée*, Paris, 1744; and the third in *La figure de la Terre*, by M. Bouguer, Paris,

* Church's Mechanics, § 78.

1749, and *Mesure des trois premiers Degrés du Méridien* by M. de la Condamine, Paris, 1751. Clarke, *Geodesy*, Oxford, 1880, pp. 3–13, gives an excellent résumé of the work in Lapland and Peru.

The triangulation to connect the observatories of Paris and Greenwich proposed in 1783, and that to determine the earth's meridian quadrant in 1791 from the measure of an arc of about 9° 40′, extending south from the extreme northern end of France; one ten-millionth part of this meridian quadrant was to be used as a standard unit of length to be called a meter.

The French introduced the repeating circle (see § 31) on the first, and the Borda base apparatus (see § 60) on the second. With the one, the angle to be measured between two signals is added on the circle as many times as desired, or, as there are repetitions—as may be done with an ordinary railroad transit. Then, subtracting the initial reading from the final (with 360° added for each full circumference passed) and dividing by the number of repetitions, the value of the angle is found with the errors of graduation and of reading divided by the number of repetitions, or by as great a number as desired. With the other, the change in length of the measuring rod, due to a change in temperature, is inferred from the actual change with reference to a companion rod, having a different rate of expansion, forming a metallic, or Borda, thermometer. While the theoretical advantages have never been fully realized in either case, the importance of the principles developed may be inferred from the fact that both have held an important place in geodetic work from that time to the present. For descriptions of the French portion of the work see *Exposé des Opérations faites en France en 1787 pour la jonction des Observatoires de Paris et Greenwich*, by MM. Cassini, Mechain, and Legendre, and the three volumes entitled *Base du système métrique décimale*, by Delambre, Paris, 1806–10.

On the part of the triangulation which fell to the English, a Ramsden theodolite was introduced, of such excellent quality that the repeating circle and the corresponding method of repeating angles have never crossed the Channel. This instrument has remained in use on primary triangulation in England and in India to the present time; and Col. Clarke in 1880 (*Geodesy*, p. 14) says that, with the exception of some very trifling repairs, it is as good as when first used. The circle, 36 inches in diameter, was graduated with a dividing engine by dots into spaces of 15′; it is read by three micrometer microscopes to single seconds. The telescope has a focal length of 36 inches, and is supported by an axis two feet long. For a description of the work see *Account of the Observations and Calculations of the Principal Triangulation . . .* , by Capt. A. R. Clarke, R. E., London, 1858.

3. **Historic Outline.**—(*b*) The increased accuracy introduced by the French and English on the survey to connect Paris and Greenwich, and on the survey to determine the length of the meter, mark the close of the eighteenth century as the beginning of the era of modern geodesy.

General interest in the subject became awakened, and geodetic surveys began to extend over Europe, while the degree of accuracy attained, in some respects at least, compares not unfavorably with that of the present time, e.g., large triangles were easily closed within 3″ with the 36-inch Ramsden theodolite, a maximum limit which has long been prescribed by the U. S. Coast Survey for primary triangles, although the average closure is very much less.

In England the Ordnance Survey developed from the triangulation connecting Paris and Greenwich; it has extended over the entire kingdom, with a triangulation and detailed topography, under Gen. Roy, Capt. Mudge, Col. Colby, and Gen. James respectively, as directors. See *Account of the Trigonometrical Survey of England and Wales*, 1799, also *Account of the Observations and Calculations of the Principal Triangulation . . .* , by Capt. A. R. Clarke, London, 1858.

In India work was commenced in 1802 under Col. Lambton,—a short arc was measured in 1790 by Burrow (*Monatliche Correspondenz*, XII, 488);—it has been continued under Col. Everest, Sir A. Waugh, Lieut.-Gen. Walker, Col. Thuillier, and Col. Gore. The objects have been mainly topographic, but, in order to properly check the work over such large areas, chains of primary triangles, with an occasional tie-chain at right angles, have been carried along meridian lines, at such distances apart that the intervening country can readily be covered by secondary triangles. A meridional arc of about 23° 49′ has resulted, and an arc of the parallel of some 30°; the first is of value in degree-determinations, but the difference in longitude has not been determined with sufficient accuracy to warrant the use of the second. See *An*

Account of the Measurements of an Arc of the Meridian between the Parallels of 18° 03′ and 24° 07′ . . . , by Col. Everest, London, 1830, also *An Account of the Measurement of Two Sections of the Meridional Arc of India,* by Lieut.-Col. Everest, 1847, and *Account of the Great Trigonometric Survey of India,* by Lieut.-Gen. Walker, Col. Thuillier, and Col. Gore respectively, for the different volumes.

On the Continent geodetic work was begun in Prussia in 1802 by von Zach. In Switzerland and Italy work was begun in 1811, the object being to join the French triangulation and secure an arc of the parallel from the Atlantic Ocean to the Adriatic Sea; when completed in 1832 it was not found very satisfactory and has never received much credit.

In Russia the first work of value was begun in 1817, under Tenner and Struve; in 1855 a meridional arc of about 25° 20′, extending from the Danube to the North Sea, had been completed. The report of the work in the two volumes, *Arc du Méridien, de 25° 20′, entre le Danube et la mer glaciale mesuré depuis 1816, jusqu'en 1855 . . . ouvrage composé sur les différents matériaux et rédigé,* by F. G. W. Struve, St. Petersburg, 1860, is considered the greatest contribution yet made to the subject of the figure of the earth, and should be studied by all who are interested in Geodesy.

In Hanover Gauss measured a meridional arc for a degree-measure, 1821–23, and extended the triangulation over the country, 1824–44. His work is classic; to it is due the first application of the method of least squares to the adjustment of a triangulation-net; the theory of conical coordinates; the general theory of geodetic lines on curved surfaces; and the invention and use of the heliotrope.

In 1831 Bessel and Bayer began a triangulation to connect the chains of France, Hanover, Denmark, Prussia, and Bavaria, with that of Russia, and to serve for degree measurements. This work is also classic; the publication of the report, *Gradmessung in Ostpreussen und ihre Verbindung . . .* , by F. W. Bessel, Berlin, 1838, is thought by Col. Clarke to mark an era in the science of Geodesy, on account of the precision of the book, and of the work of which it treats, many of the methods which are there for the first time described being still in use.

The Russian and Austrian chains were connected between 1847 and 1851, and the Swiss and Lombardian chains at about the same time. The English and Belgian were joined in 1861.

But little work was done in Italy until the formation of the Italian Commission, 1865. Work was begun in Spain in 1858, and excellent results have been obtained under Col. Ibañez. A remeasure of the French arc of Delambre and Mechain was begun in 1870 under the direction of M. Perrier, and this was followed by an extension of the French and Spanish chain across the Mediterranean to Algiers in 1879, giving a meridional arc of 27°, extending from the Shetland Islands to the desert of Sahara.

The chains of Russia and England have just been connected through central Prussia, with small discrepancies between the ten base lines joined. Accurate topographic surveys and lines of geodetic levels have also been extended over the greater part of Europe.

The development of *least squares* has added much to the precision of geodetic work. The theory was first stated by Legendre in 1805, and added to by Adrian in 1808; but its full development was due to Gauss in 1809, and its first application to the adjustment of a triangulation was made by him in adjusting the Hanover arc, as already noted.

The method as now extended and perfected is applied in the reduction of every important geodetic survey.

4. Geodetic Work in the United States.—The English astronomers, Mason and Dixon, in running out the celebrated line bearing their name, found the position of the division line between Maryland and Delaware, which coincides approximately with the meridian, to be on low and level ground, and hence well adapted to direct measurement for a degree determination. Accordingly, with the aid of the Royal Society of London, they made a direct measurement with wooden rods, starting at the southwest corner of Delaware and extending into Pennsylvania, of about 1° 29′, and determined the azimuths of the different portions of the line and the latitudes of its extremities. The work, described in *London Philosophical Transactions*, 1768, by Mason and Maskelyne, is not accepted with much confidence.

The U. S. Coast Survey was authorized by Congress in 1807; but, owing to lack of funds, work was not commenced until 1817, and but little was done except in detached surveys along the coast until 1832. The triangulation, which was commenced in the vicinity of New York harbor, has been gradually extended along the entire Atlantic coast, along the Gulf coast, and along the greater part of the Pacific coast, not including Alaska. In 1871 the project was authorized of connecting the Atlantic and Pacific systems and of furnishing trigonometric surveys to such states as should make the necessary provisions for carrying on the topographic and geologic portions of the work.

This connection has been made by a chain of triangulation along the 39th parallel. A chain has also been extended approximately along the 98th meridian from latitude 28° to latitude 46°. Considerable work has been done at the Hawaiian Islands, Porto Rico, and the Philippines.

The opportunity afforded for state surveys has been improved by quite a number of states, while the country will eventually be covered with a triangulation-net which will compare favorably with any in Europe.

Since the extension to include interior work, the survey has been known as the Coast and Geodetic Survey. It was under the Treasury Department until 1903, when it was transferred to the Department of Commerce and Labor.

The superintendents, and times of their appointments, have been: F. R. Hassler, 1807; A. D. Bache, 1843; Benjamin Pierce, 1867; C. P. Patterson, 1874; J. E. Hilgard, 1881; F. M. Thorn, 1886; T. C. Mendenhall, 1889; W. W. Duffield, 1894; H. S. Pritchett, 1897; O. H. Tittmann, 1900. The yearly reports contain much valuable material, especially in the appendices.

The survey of the Northern and Northwestern Lakes was commenced in 1841, under the War Department; better instruments and methods were introduced in 1851, and the character of the work was gradually improved to 1870, when the survey passed under the charge of Gen. C. B. Comstock of the Corps of Engineers. From that date to the close in 1882 a continuous chain of triangulation, depending upon eight carefully measured bases, was extended from St. Ignace Island, on the north shore of Lake Superior, to Parkersburg in southern Illinois, a distance in latitude of 10°, and from Duluth, Minn., via Chicago, to the east end of Lake Ontario, a distance along its axis of 1300 miles, or of 16° in longitude. Some very excellent base line work has been done, and the triangulation has been carefully executed. The work was reorganized in 1892, and resurveys and extensions are now in progress. See *Primary Triangulation U. S. Lake Survey*, 1882, by Gen. C. B. Comstock; or see the yearly reports of the Chief of Engineers.

The Missouri and Mississippi River Commissions, under the charge of the Corps of Engineers, have extended triangulation a distance of 2550 miles along the Missouri and 1600 miles along the Mississippi.

5. **Degree Measurements.**—At the suggestion of Gen. Baeyer, the Middle European Association for the measurement of arcs in middle Europe was formed in 1864. At this time only three arcs of the meridian had been measured in Europe. Nineteen states gave support to the project, but the work was largely of local importance.

In 1887 the association was merged into the International Geodetic Association, and its scope enlarged to include the principal nations. Its reports, at first published yearly but now less frequently, are of great scientific interest, taking up the investigations by the different nations of such subjects as the variation of latitude, the force of gravity, the aberration of light, etc., in connection with degree measurements.

Two sets of values for the semi-axes of the ellipsoid of revolution which best corresponds with the earth are in use in Geodesy, one set by Bessel, the other by Col. Clarke.

The first was determined in 1841 from a discussion of 10 meridional arcs, total amplitude 50.6°, giving *

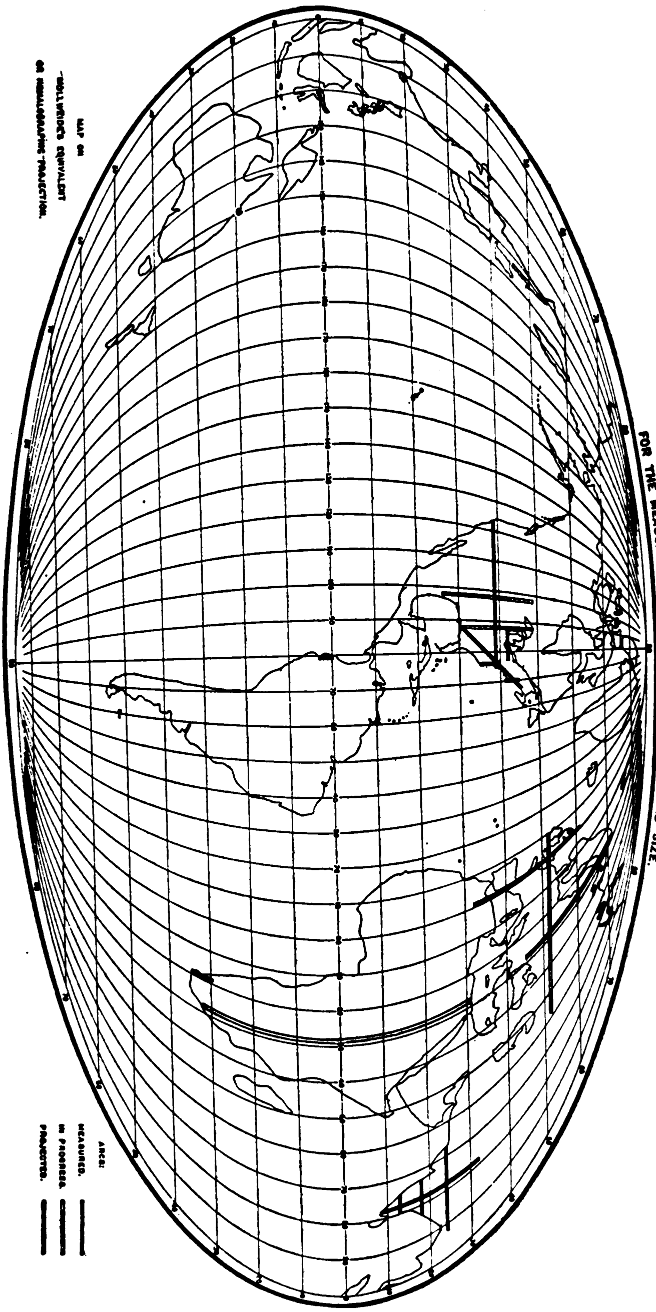
$$a=6\,377\,397^{\mathrm{m}};\quad b=6\,356\,079^{\mathrm{m}};\quad a-b=21\,318^{\mathrm{m}};\quad c=1/299.$$

The second was determined in 1866 from a discussion of 5 meridional arcs, the English, the Indian, the Russian, the Cape, and the Peruvian, total amplitude 76.6°, giving

$$a=6\,378\,206^{\mathrm{m}};\quad b=6\,356\,584^{\mathrm{m}};\quad a-b=21\,622^{\mathrm{m}};\quad c=1/295.$$

On account of the importance of the meridional arcs in Peru and Lapland, described in § 2, the French government began the remeasurement of the Peruvian arc in 1899, while Russia and Sweden began the measurement of a northern arc on the island of Spitzbergen. Their amplitudes are only 6° and 5° respectively,

* a = semi-major axis; b = semi-minor axis; c = compression = $(a-b)/a$.



PRINCIPAL ARCS—MERIDIONAL, PARALLEL, AND OBLIQUE.

FIG. 1.

but they are important on account of their positions, one near the equator, the other near the pole.

A large number of meridional arcs have been measured. The following have an amplitude of over 20°:—the French-English, from the north end of the Sahara desert to the Shetland Isles, 28°; the Russian, from the mouth of the Danube to the Arctic Ocean, 25°; and the East Indian, 24°.

Two important arcs of parallels have been measured:—an arc of 70° from Valentia, Ireland, to the Ural Mts., latitude 52°; and an arc of 48.8° from Cape May, N. J., to Point Arena, Calif., latitude 39°. Both of these arcs indicate sharper curvature than given by the Clarke ellipsoid.

The Eastern Oblique Arc of the Coast Survey, extending from the Bay of Fundy to the Gulf of Mexico, is important as being the first to utilize on a grand scale a measurement oblique to the meridian. The results were published in 1901.

Two long meridional arcs are now in progress, one of 65° from the Cape of Good Hope to Egypt, with the possibility of an extension through Asia Minor to join the Russian work. This would give an amplitude of 106°. The other of 50° along the 98th meridian from the southern coast of Mexico to the Arctic Ocean. The portion in the United States is nearly completed, while Canada and Mexico have both agreed to extend the work across their territory.

Former Supt. Pritchett states that the completion of this arc would, with the 39th parallel already completed, furnish data which could probably never be improved upon for this continent.

The relative location of these arcs is shown on Fig. 1.

CHAPTER II.

TRIANGULATION, RECONNOISSANCE, SIGNALS.

6. Primary, Secondary, Tertiary Triangulation. — When a triangulation is to be extended over a large tract of country, or between two or more distant points, a system of *primary* triangles is employed, which is characterized by the maximum development of which the topography will admit. This, in level or slightly undulating country, will allow of triangle sides of only 15 to 25 miles, on account of the height of signal and of observing-stand required to overcome the earth's curvature; while, in mountainous country, sides of from 40 to 60 miles are common, and those of from 100 to 150 miles are not unknown. Distances are determined with an accuracy or probable error of about 1/100 000, the range being from about 1/60 000 to 1/200 000.

If points are required nearer together than the primary stations, *secondary* ones are established. The triangles connecting them with the primary ones, or with each other, are called secondary triangles. Their sides usually vary from 5 to 25 or more miles, while an accuracy of from 1/20 000 to 1/50 000 is usually attained.

If an accurate topographic or hydrographic survey is to follow, points not more than from 1 to 3 miles apart will be required. The triangles connecting them with the secondary ones are called *tertiary* triangles. Their sides do not usually exceed about 5 miles, while an accuracy of from 1/5 000 to 1/20 000, or an average of 1/10 000 is usually attained.

For surveys of less extent, the primary triangulation, and the secondary also, is sometimes omitted. Greater care and accuracy will then be required in the tertiary triangulation, as it must check its own work. In primary work the base lines are usually from

4 to 12 miles long, and they are placed from 150 to 600 miles apart, measured along the triangulation. In secondary work, which does not start from primary work or check upon it at sufficiently small intervals, they are about 2 to 3 miles long, and are placed at distances apart of from 50 to 150 miles. In tertiary work, which is not sufficiently checked by secondary, they are from ½ to 2 miles long, and are placed at intervals of from 10 to 40 miles. These distances vary with the character of the work and of the country, as well as with the individuality of the person conducting the survey.

7. Triangulation Systems.—In connecting two distant points, or in following a line as a coast or boundary, a principal chain of triangulation should be laid out, along which distances and azimuths or directions can be carried with the greatest accuracy and directness. At the end of the chain, and at as many intermediate points as may be thought necessary, a check is had by measuring a base and comparing the measured length with that computed through the chain.

In covering a large area with a network of triangulation, the method often employed is to extend around the area a main chain, which is checked by closing upon itself, and which serves as a framework with which to connect longitudinal chains. These in turn serve for transverse chains, which complete the gridiron of primary triangulation and allow the intervening areas to be reached by secondary and tertiary triangles. The discrepancies due to imperfect measurement are adjusted for each series, in order, and each is then considered perfect in fitting the next lower to it. The adjustment is thus comparatively simple, while if the whole area were covered with a series of continuous triangles, all measured with the same accuracy, the labor would increase so rapidly with the number of triangles as soon to become prohibitive, except by subdividing into more or less arbitrary sections.

The above methods should be flexible enough to allow of taking advantage of routes most favorable for the triangulation, even though they are some distance from the boundary, or do not give cross chains at right angles or at uniform distances apart.

The composition of the chain also deserves attention. In order to make a comparison of strings of practically the same length, Mr. C. A. Schott (*Coast Survey Report*, 1876, App. 20) takes a string of 10 * equilateral triangles with sides of unity; 3 regular hexagons with sides of unity, each divided into 6 equilateral triangles by joining a central point with the vertices; and 7 quadrilaterals with diagonals of unity (Fig. 2), and finds that:

The actual lengths of the strings will be 5, 5.2, and 4.95 respectively.

The numbers of stations will be 12, 17, and 16.

The numbers of the sides to be sighted will be 21, 34, and 36.

The total lengths of the sides will be 21, 34, and 29.6.

The areas covered will be 5, 9, and 4.04.

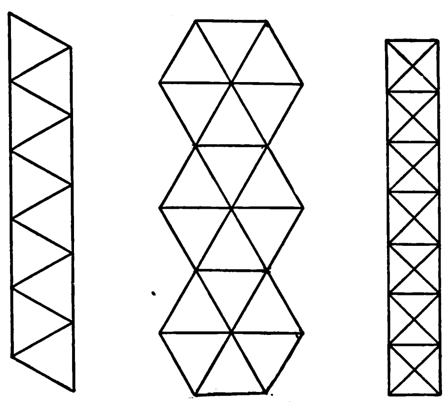


FIG. 2.

The number of checks upon the observed angles, due to geometric conditions, will be 10, 21, and 28. While these regular figures and separate systems usually are not feasible, the above comparison indicates that the single string of triangles is the most favorable for rapidity and economy, the difference being more apparent in level or prairie country, as only about two-thirds as many expensive elevated signals will be required; while, if the level ground be wooded, the additional saving in clearing only about two-thirds the length of lines will usually compel its adoption even for the best grade of work. The string of hexa-

* The 9 which Mr. Schott used is here changed to 10, to give the same actual advance of triangulation rather than that of extreme points. A quadrilateral in geodesy is a four-pointed polygon having all the vertices joined.

gons, or other polygons having their vertices joined to an interior point, commands attention when greater width and accuracy are desirable; while the string of quadrilaterals affords greater accuracy with less stations and less labor, and is the system usually adopted by the Coast Survey, except for densely wooded level country.

8. Coast Survey, Character of Figures.—In January, 1905, definite instructions were furnished the field parties of the Survey as to the degree of accuracy desired, the methods to be used, and the field computations required. They are the outcome of a study to secure greater effectiveness and economy in the triangulation.*

For *primary triangulation* the chain between base nets or base figures is made up of completed quadrilaterals and of central point figures, with all stations occupied. It is not to be allowed to degenerate, even for a single figure, to simple triangles. There must be two ways of computing the length through each figure, but with no overlapping of figures and no excess of observed lines beyond those necessary to secure the double determination of every length, except that in a four sided central point figure one of the diagonals may be observed.

For *secondary* the same rules apply, except that certain stations may be left unoccupied, as indicated in § 10. If supplementary stations are necessary in order to fix certain required positions, these occupied stations should be connected with the main scheme by the simplest figures possible in which there is a check, preferably by single triangles with all angles measured.

For *tertiary* the same rules apply as for secondary, including supplementary stations, while the main scheme is allowed to drop to simple triangles at any point where it is extremely difficult to be avoided.

9. Coast Survey, Uncertainty of Length.—In a given figure the accuracy of a side computed from the base increases with the number of checks upon the observed angles, and also with the size of the angles used, the sines varying much more rapidly for small angles than for large ones.

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* Condensed from a mimeograph print kindly loaned by the Superintendent.

In *Part II* it is shown that this relation is expressed by the following:—

$$p^2=\frac{4}{3}d^2\frac{N_d-N_c}{N_d}\Sigma(\delta_A{}^2+\delta_A\delta_B+\delta_B{}^2), \quad . \quad . \quad . \quad . \quad (1)$$

where p=probable error of the computed log side; d=p.e. of an observed direction; N_d=number of directions observed; N_c= number of conditions or checks to be satisfied; δ_A and δ_B=log differences for 1″ for the distance angles A and B of a triangle (A, opposite computed side, B, opposite given side); Σ=the summation for the triangles used in computing the side in question from the side supposed to be absolutely known.

In (1) the two terms $\frac{N_d-N_c}{N_d}$ and $\Sigma(\delta_A{}^2+\delta_A\delta_B+\delta_B{}^2)$ depend entirely upon the figures chosen, and are independent of the accuracy with which the angles are measured. Their product is therefore a measure of the strength of the figure with respect to length, in so far as the strength depends upon the selection of stations and of lines to be observed over. The second term depends upon the distance angles. The value for each triangle can be taken from Table XII, and these values summed as the total for the figure.

To compare two alternative figures (e.g., either quadrilaterals or central point figures) for the strength with which length is carried, proceed as follows:

1. For each figure take out the distance angles to the nearest degree, if possible, for the best and second best chains of triangles through the figure. These chains are to be selected at first by estimation, the estimate to be checked later by the results of comparison.

2. For each triangle in each chain enter Table XII with the distance angles as the two arguments and take out the tabular value.

3. For each chain (the best and second best) through each figure take the sum of the tabular values.

4. Multiply each sum by the factor $\frac{N_d-N_c}{N_d}$ for that figure. The product, viz. $\frac{N_d-N_c}{N_d}\Sigma(\delta_A{}^2+\delta_A\delta_B+\delta_B{}^2)$, will be called R_1 and R_2 for the best and second best chains respectively.

5. The strength of a figure is dependent mainly upon R_1, hence the smaller R_1 the greater the strength. R_2 contributes somewhat to the total strength, and the other weaker and progressively less independent chains contribute still smaller amounts. In deciding between figures they should be classed according to their best chains, unless these best chains are very nearly of equal strength and the second best differ greatly in the two cases.

N_d, the number of observed directions, is readily found by inspection of the figure, omitting the direction at each end of the starting line which is supposed to be completely fixed. For N_c, the number of checks in the figure, see *Part II*. For simple figures add the number of independent triangles (those in which the angles cannot be found by taking sums and differences of those used in other triangles) to the number of distance checks in the figure. Thus in a completed quadrilateral there are three observed directions at each station, a total of twelve; subtracting the one at each end of the fixed line, leaves ten. There are three independent triangles and one distance check.

$$\therefore \frac{N_d-N_c}{N_d}=\frac{10-4}{10}=0.60.$$

If a station at one end of the fixed line is unoccupied, two triangle closures and two directions drop out, giving a result $=0.75$.

If a station not on the fixed line is unoccupied, two triangle closures and three directions drop out, giving a result $=0.71$.

10. Coast Survey, Strength of Figure.—In a chain of *primary triangulation* between base nets the extreme limits for R_1 and R_2 are 25 and 80 respectively, in units of the sixth place of logs. These limits are reduced to 15 and 50 respectively, if it does not increase the estimated total cost over 25%.

For *secondary* between base nets the extreme limits are 50 and 150, and the 25% extra cost limits 25 and 80 respectively. One station in each figure may be left unoccupied, or, if all are occupied and considerable time could be saved by so doing, lines to the limit of three may be left observed in one direction only, subject to the above limitations for R_1 and R_2. In the latter case R_1 and R_2 are computed as if one outside station of the figure were omitted. The advantage of keeping R small, and thus avoiding the necessity of measuring bases at short intervals, is indicated in §11. For a triangle connecting a supplementary station with the main scheme $R < 50$.

For *tertiary* the same conditions are imposed as for secondary, except that the extreme limits of 50 and 150 may be exceeded when it would be extremely difficult to keep within them.

11. Examples of Strength of Figure.—The following are some of the examples given by the Coast Survey as guides on reconnoissance in selecting stations which will give small computed values for R. The heavy lines in each figure are the given and required sides.

The *completed quadrilateral.* Fig. 3. If square, $R_1=5$, $R_2=5$; R_1'*$=6$, $R_2'=6$. Shortening the length in the direction of the chain to one-half the width, forming a rectangle, strengthens the figure, but reduces the advance for given length sights: thus $R_1=1$, $R_2=1$; $R_1'=2$, $R_2'=2$. Increasing the length to double the width reduces the strength: thus $R_1=22$, $R_2=22$; $R_1''=27$, $R_2''=27$. Changing the quadrilateral to a rhombus with angles of 60° and 120° gives $R_1=5$, $R_2=12$; $R_1''=6$, $R_2''=15$. Lengthening the required side of the square to double the given side by extending

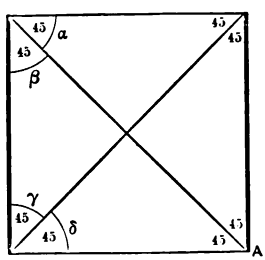


FIG. 3.

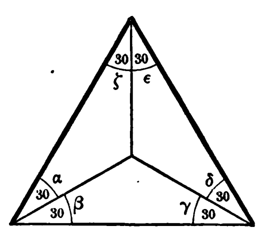


FIG. 4.

the end at A gives a figure of good strength for rapid expansion or contraction: thus $R_1=5$, $R_2=22$; $R_1''=6$, $R_2''=27$. If this extension is made one-half at each end, $R_1=7$, $R_2=7$; $R_1'=9$, $R_2'=9$.

The *triangle with interior point.* Fig. 4. If equilateral, with the point in the center, $R_1=2$, $R_2=12$; $R_1''=3$, $R_2''=15$. The

---

* R', any one station not occupied. R'', one station on fixed line not occupied. R'''. one station not on fixed line not occupied. R^{IV}, one outside station not occupied. R^{V}, center station not occupied.

effect of moving the interior point towards the unused side until $\beta=10°$ is to reduce R_1 to 2 and R_2 to 4, while $R_1''=3$ and $R_2''=5$. If moved the same amount towards the given side, $R_1=2$, $R_2=71$; $R_1''=3$, $R_2''=89$; $R_1'''=3$, $R_2'''=84$, thus weakening the figure for R_2.

The *quadrilateral with interior point.* Fig. 5. If square, with the interior point in the center, $R_1=13$, $R_2=13$; $R_1^{iv}=16$, $R_2^{iv}=16$; $R_1^{v}=17$, $R_2^{v}=17$. Increasing the length to twice the width gives $R_1=40$, $R_2=40$; $R_1^{iv}=47$, $R_2^{iv}=47$; $R_1^{v}=50$, $R_2^{v}=50$. Reducing the length to one-half the width gives $R_1=19$, $R_2=19$; $R_1^{iv}=22$, $R_2^{iv}=22$; $R_1^{v}=24$, $R_2^{v}=24$. These figures are all weaker than the corresponding shapes for the completed quadrilateral.

The *five-point figure with interior point.* Fig. 6. For the regular polygon, with interior point in the center, $R_1=10$, $R_2=15$; $R_1^{iv}=11$, $R_2^{iv}=16$; $R_1^{v}=13$, $R_2^{v}=19$.

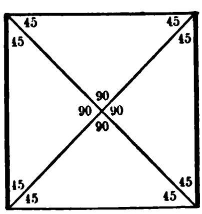


FIG. 5.

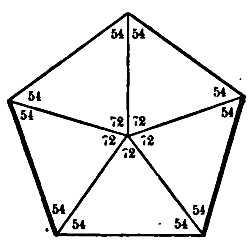


FIG. 6.

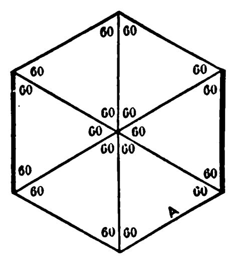


FIG. 7.

The *six-point figure.* Fig. 7. For the regular polygon, with interior point in the center, $R_1=11$, $R_2=11$; $R_1^{iv}=12$, $R_2^{iv}=12$; $R_1^{v}=14$, $R_2^{v}=14$. Changing the required side to A, $R_1=8$, $R_2=14$; $R_1^{iv}=9$, $R_2^{iv}=15$; $R_1^{v}=11$, $R_2^{v}=18$.

Base net. Fig. 8. This gives an expansion of two to one. $R_1=5$, $R_2=5$, $\frac{N_d-N_c}{N_d}=\frac{28-16}{28}=0.43$. R_1 can be reduced to 4, while R_2 would be increased to 7, by moving the base parallel to itself until the left-hand triangle has angles of 5°, 5°, and 170°.

Fig. 9 shows how to make an advance by a completed quadrilateral with one station unoccupied and obtain a small value for R. $R_1=1$, $R_2=1$.

Buffalo base-net, U. S. Lake Survey. Fig. 10. This base, about 4.2 miles long, was measured in 1875. It shows an enlargement of nearly 3/1 with $R_1 = 11$ and $R_2 = 16.5$.

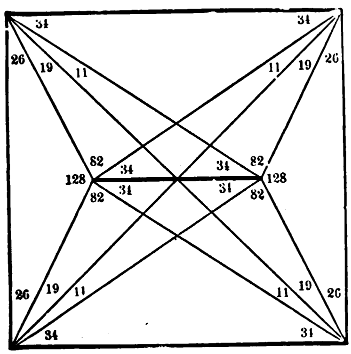


FIG. 8.

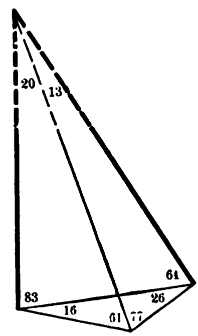


FIG. 9.

Fig. 11 shows a portion of the primary and secondary triangulation near the Edisto base of the Coast Survey, from *Report*, 1885, App. 10. The country is flat and wooded, no elevations of 20

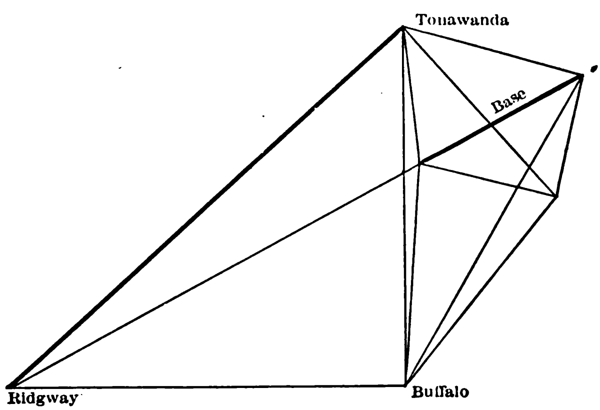


FIG. 10.

feet being available. The use for secondary sides of lines cleared for primary ones may be noted. In the same appendix may be found a sketch of the secondary triangulation of Boston Bay, an open country with suitable elevations.

Ex. 1. In Fig. 3 compare the strength of the figures *A* and *B*.

	α	β	γ	δ
For *A*.	40°	55°	50°	50°
For *B*.	68	20	15	80

Ex. 2. In Fig. 4 compare the strength of the figures *A* and *B*.

	α	β	γ	δ	ε	ζ
For *A*. . . .	33°	33°	28°	28°	29°	29°
For *B*	08	58	28	28	50	08

12. Coast Survey, Length of Line.—For *primary triangulation*, outside the base nets no line should be less than six kilometers. There is little, if any, gain in accuracy by making the lengths much

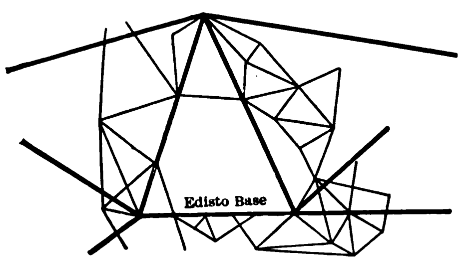


FIG. 11.

greater; therefore an effort is made to use the economic lengths of line, i.e., such lengths for each region as will make the cost of reconnoissance, building, and triangulation a minimum per mile of progress, subject to the limitations stated in the instructions.

For *secondary* and *tertiary* the economic lengths should govern as stated above, with the lower limit considerably below 6^{km}. On very short lines it is difficult to obtain the accuracy necessary to close the triangles within the required limit. On long lines there is danger of delays due to non-visibility of signals, while supplementary stations to reach required points are more apt to be needed.

13. Coast Survey, Frequency of Bases.—For *primary triangulation*, if the character of the country is such that a base site can be found near any desired location, R_1 between bases should equal about 130. This will give a chain of from 15 to 35 triangles, according to the strength of figures secured. Strong figures thus reduce the cost for base lines. If it is difficult to secure a base site at the desired location, R_1 may approach, but not exceed, 200. When this is done, there will be danger that an intervening base may be necessary; for, if the discrepancy between adjacent bases should exceed one part in 25 000, an intervening base must be measured.

For *secondary* and *tertiary* the same limits are used for R_1, while the discrepancy between adjacent bases (either measured ones or lines of primary or secondary triangulation used as bases) is extended to one part in 10 000 before requiring an intervening base for secondary, and one part in 5 000 for tertiary. For tertiary the intervening triangulation may be strengthened instead.

14. Coast Survey, Base Sites and Base Nets.—For *primary triangulation* grades of 10% for any 50-meter tape length are allowed, and ravines not over a tape length wide in the direction of the line are regarded as no obstacle to measurement. The length of base is limited between 4 and 12 kilometers.

For the base net as good geometrical conditions as possible are secured, even if the base is forced upon rough ground. It should not be longer than two ordinary figures of the main chain between bases. It may be strengthened by observing over as many lines of the net as can be obtained without excessive cost for building or cutting, provided the figure is not made so complicated as to be excessively difficult and costly to adjust.

For *secondary*, grades up to 20% are allowed if the geometrical conditions can be improved by so doing. The length is chosen primarily to give small values for R, and this will usually tend towards a long base, which in turn will lengthen the distance between bases. The economic length will be from 2^{km} to 8^{km} as a rule. The base net will be of the same character as to figures and strength as the chain between base nets. If made up of two or more figures, they may overlap, but should not be tied together.

For *tertiary*, the same conditions are imposed as for secondary, except that the economic length is not given or called for.

15. Coast Survey, Intersection Stations.—In selecting these it should be kept in mind that the geographic value of a triangulation depends upon the number of points determined, the area over which they are distributed, and the permanence with which they are marked, and that for a given area this value is lost when the points cannot be recovered. The chance of permanency is increased by increasing the number of points as well as by thorough marking.

These considerations should lead to the determination as intersection stations of many artificial objects of a permanent character, as lighthouses, church spires, cupolas, towers, and large chimneys; should lead occasionally to the determination of specially marked stations established for this particular purpose; and should frequently lead to the permanent marking upon the ground of topographic or hydrographic stations and their determination as intersection stations. The practice of permanently marking such hydrographic points as are in commanding positions, on promontories for example, will frequently obviate the necessity for a new triangulation when a later hydrographic survey is made. It is especially desirable to increase the area effectively covered for geographic purposes by selecting intersection stations which are outside the area covered by the main scheme.

When feasible without undue expense, the section corners and other survey marks should be connected with the triangulation, either by distance and direction from a station or by using them as intersection stations.

16. Elevation of Signal.—Usually the question of intervisibility of stations is best settled by actual observation; but, when the station points are not intervisible and signals can only be rendered so by elevation, the required heights may be difficult to determine by observation, unless there is a tree or other elevated object near, from the top of which the desired view may be had. In such cases, if the heights of the stations are known, and that of the intervening ground which obstructs the view can readily be determined, as would be the case for level ground or for a line passing over water, the required heights can be readily computed. In the vertical section through the two stations C and C', Fig. 12, let AA' be a straight line tangent at D; BDB' the line of sight between the two intervisible points B and B', concave downwards on account of refraction. Denote the distances AD, $A'D$, in miles, by k, k'; the required heights BC, $B'C'$, in feet, by h, h'; the radius, in miles, by R (log $R = 3.597317$); and the coefficient of refraction, with mean value 0.07, by m, or the refraction angle ADB by $m \times AOD$. Then, in the right triangle AOD,

$$(AC+R)^2=k^2+R^2, \text{ or}$$

$$AC=\frac{k^2}{2R}, \text{ nearly.}$$

$$\because ADB=mAOD=2mADC,$$

and the angles are small,

$$\therefore AB=2mAC, \text{ and } BC, \text{ in miles, or}$$

$$\frac{h}{5280}=AC-AB=\frac{k^2}{2R}(1-2m);$$

$$h=\frac{k^2}{2R}\times 0.86\times 5280,$$

$$k^2=1.743h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

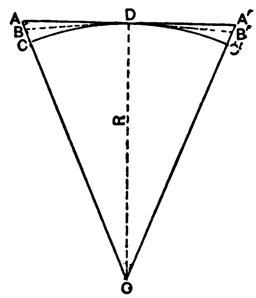


FIG. 12.

where k is in miles and h is in feet;

i.e., *the square of the distance in miles is about* $1\frac{3}{4}$ *times the required elevation in feet*—a convenient rule easily remembered.

For k in kilometers and h in meters, (2) reduces to

$$k^2=14.807h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The line of sight should not pass nearer the surface than 10 feet at the tangent point, on account of the lack of transparency and danger of lateral refraction, due to the disturbed lower air.

Ex. Two stations of the U. S. Lake Survey, Buchanan, on the north side of Lake Superior, and Brulé River on the south, are 10 and 19 feet above lake level respectively, and 16 miles apart. A signal 35 feet high was used at Brulé. How high should the instrument and observing-stand be elevated at Buchanan, in order to see the upper 20 feet of the signal at Brulé?

Solution. $19+35-20=34$; $34-10=24$, the available height at Brulé.

Placing $h=24$ in (2), $k=\sqrt{1.743\times 24}=6.5$ miles, the distance from Brulé to the tangent point. $16-6.5=9.5$ miles, the distance from the tangent point to Buchanan.

$$h'=\frac{(9.5)^2}{1.743}=52 \text{ feet.}$$

$52+10=62$, the required height above lake level, or 52 feet above the ground.

17. Hints in Selecting Stations.—For the main chains choose the highest elevations, even if at greater first cost on account of inaccessibility. The stations will then better command new ground, if at any time it becomes necessary to extend the work beyond its original limits; while high lines meet less atmospheric disturbance.

Avoid low lines and lines passing over cities, furnaces, etc.

In finally locating stations make certain that those intended to be intervisible really are so, even at the expense of time and patience in waiting for clearing weather; otherwise the observing party may suffer vexatious and expensive delays.

Select stations so that permanent station marks can be placed and protected, or so that accurate references can be had to permanent objects.

The base line site should be selected with reference to securing suitable ground for measurement and a convenient expansion through the base net. If the base is to be measured with bars, the line should be free from obstructions and quite smooth for a width of at least 12 feet; longitudinal slopes up to from 3° to 5°, depending upon the apparatus used, do not offer serious inconvenience or affect the accuracy of the measurement. The ends need not be intervisible from the ground, if they can be made intervisible by signals and observing-stands of moderate elevation. The measurements can be made along two straight segments, not differing widely in direction, if better ground will thus be secured. Narrow ravines can be crossed by bridges or trestlework with complete success, while a wide one, or a bog or similar obstruction to direct measurement, can be passed by triangulation without very serious decrease of accuracy.

If the measurement is to be made with a tape, as is now frequently done, rougher ground may be used, as indicated in §14.

18. Reconnoissance.—A general reconnoissance should precede the selection of stations, in order to become sufficiently familiar with the topography to be able to recognize the most prominent features and elevations, as seen from different points of view, and in order to determine the general scheme of triangulation and the general routes best suited to the ground, for aid in conducting the detailed reconnoissance.

Unless the surface is level and unbroken, points will be found which from their position or elevation will offer such advantages that they probably must be used for stations. Starting from these, others must lie within prescribed areas, in order to fulfill the required geometric and other conditions, and to use the economic lengths for sides.

From each of these probable station points, sights should be taken to the others, if visible, and also to such points in the prescribed areas as will possibly serve for stations.

Other available points can be occupied, and the process repeated, if necessary. Should a point be occupied which has not been cut from at least two other stations, sights must be taken upon at least three known points, when its position can be determined by § 19.

Magnetic bearings often aid in orientation on arriving at a new station, and in identifying objects already located, by giving approximate directions, while they sometimes aid in plotting, when insufficient angles have been taken.

A hasty outline profile sketch of the ground in the vicinity of each object sighted will aid very materially in identification from surrounding stations, while if the estimated distance in miles is written on the sketch near the object, and the circle reading or bearing is written above on a vertical through it (see Fig. 13),

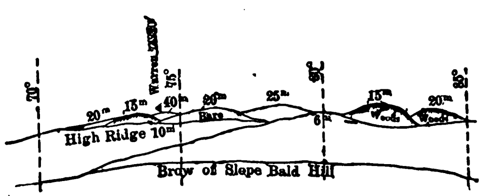


FIG. 13.

very clear and concise notes will result. The obstructed arcs at a station should be noted, also their cause, and whether they can be removed by cutting or by signal elevation. Should the location of stations be likely to prove difficult, vertical angles should be taken to aid in deciding upon the intervisibility of signals by giving differences of elevation.

A plat of this preliminary triangulation should be kept up by angles, starting from a known or assumed side; or by computed triangle sides, if greater accuracy is desired. Then, working from probable station points or from stations already located, the possible point in a given area is picked out which will best fulfill the conditions imposed as to length of line, intervisibility, etc. In the same manner as many new ones are chosen from the plot as desired.

On reaching an elevated point it is quite difficult, without experience, to orient one's self and be able to identify signals and topographic features at distances of 40 to 50 miles, even under the most favorable conditions. When, as is often the case, the features are not prominent, and the air is thick with haze and smoke for days at a time, the skill and patience of the experienced are fully taxed. With wooded elevations the observations must be usually taken from the top of a tree, or, if none can be found of sufficient height, from the top of a ladder formed by splicing several ladders together and supporting them by guys.

High elevations with the summits free from timber afford the best station sites. Wooded summits require sight lines to be cut through. These for pole signals on long lines should be about 100 feet wide, and they should be extended so far back of the station that the signal will not be seen against near woods.

As the summits broaden or the timber becomes valuable, elevated signals and observing-stands should be considered before clearing the lines, although they generally should not be adopted unless a considerable saving will result.

Parallel wooded ridges may present much difficulty if so near together that the triangle sides must reach over an intermediate ridge, instead of spanning an intermediate valley. The direction across the ridge to an invisible station can be found from the plat or from § 21 when the required signal elevation can be found from vertical angles or from carefully taken aneroid barometer readings; but, if two or more ridges intervene, actual tests from ladder-tops, or an examination of the entire line, will be necessary.

In level country an elevation of 70 feet for signal and observing-stand will allow of 20-mile sides. If wooded, these had best be used in a chain of nearly equilateral triangles having all the lines cut through; but if clear, as on prairie, quadrilaterals with diagonals of 21 miles and sides of about 15 will add only one more station in 30 miles of progress, which will be more than compensated for by the increased precision attained.

If the level ground be cultivated and contain patches of valuable timber, the difficulties will be so much increased, even if the ground be rolling, that the greatest care and skill will be required to avoid insuperable obstacles. Sometimes chains of secondary triangles along the watercourses have proved effective.

Full notes and sketches should be taken of the points most important for the subsequent work. Among these are the means of access, the timber which can be found at the site for the signal, the roads which have to be opened by the angle party in occupying the station, the places nearby where board can be had, etc.

The efficiency and economy of the survey will depend very materially upon the skill, good judgment, and experience of the person who conducts the reconnoissance.

If a topographic or hydrographic survey is to follow, the intersection stations should include as many prominent objects, usually from 1 to 3 miles apart, as may be needed by the topographer in tying up his work or by the hydrographer in taking angles to locate soundings, etc., such as church spires, cupolas, chimneys, flags in prominent trees, large white crosses or triangles painted upon rocky cliffs, etc. Well shaped triangles are not so important as the securing of a sufficient number of convenient points for the topographer, since the errors introduced do not accumulate over large areas.

19. N-point Problem.—To determine the position of a point when only angles at the point have been observed between known stations:

Lay off the angles on tracing cloth in order around a point, place the cloth on the plat and move it until each line passes through the station to which it belongs, when the vertex can be pricked through. Two angles will locate a point, giving the 3-point problem, except when the point lies on or near a circle passing through the three stations on which the sights are taken; 3 or more angles are better, forming a check. A 3-armed protractor is often used in place of the tracing cloth; also a sheet of paper, by cutting out a narrow strip along each line near the portion to be used.

When a more accurate solution is desired than can be had from a careful plat on a large scale, a numerical one is used. In Fig. 14 let S be the required point at which the angles $P, P', P'', \ldots$ have been observed upon the known stations, F, G, $H, \ldots$ B is also known, it being an angle between known stations.

In the triangles, SFG, SGH, by formula 18]*

$$n' = \frac{a \sin A}{\sin P} = \frac{b \sin C}{\sin P'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

$$\therefore \ a \sin A \sin P' - b \sin C \sin P = 0. \quad . \quad . \quad . \quad . \qquad (b)$$

In the polygon $SFGH$, $P + P' + A + B + C = 360°$,

whence $C = Q - A$, where $Q = 360° - (P + P' + B)$. . . . (4)

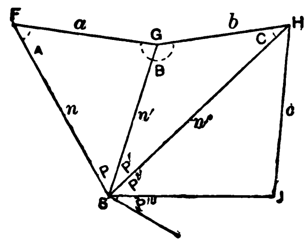


FIG. 14

Substituting in (b), with the expansion of sin (Q–A) from formula 3],

$$a \sin A \sin P' - b \sin P \ (\sin Q \cos A - \cos Q \sin A) = 0, \text{ or}$$

$$a \sin P' - b \sin P \sin Q \cot A + b \sin P \cos Q = 0$$

$$\cot A = \cot Q + \frac{a \sin P'}{b \sin P \sin Q}, \quad \text{or}$$

$$\cot A = \cot Q \left(1 + \frac{a \sin P'}{b \sin P \cos Q}\right). \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Having A, all the angles of the triangles become known, whence

$$n = \frac{a \sin (P + A)}{\sin P}; \quad n'' = \frac{b \sin (P' + C)}{\sin P'}; \text{ etc.} \quad . \quad . \qquad (6)$$

Ex. At Sheldrake Point, Cayuga Lake, N. Y., the following angles were observed upon three known stations, Willets, King's Ferry, and Kidders.

Required the position of Sheldrake.

Observed.			Given.		
Wil.-Shel.-King's	$= P =$	118° 15′.2	Wil.-King's	$= a =$	7150.2 ft.
King's-Shel.-Kid.	$= P' =$	58 31 .2	King's-Kid.	$= b =$	3050.7
			Kid.-King's-Wil.	$= B =$	101° 08′.0

* Formulas marked thus, ...]. are to be found in Table I at back of book.

Solution.

By (4)	$Q=82°\ 05'.6.$		
" (5)	$a=7150.2$, log		3.85432
	$P'=58°\ 31.'2$, sin		9.93086
			3.78518
	$b=3050.7$, log	3.48440	
	$P=118°\ 15'.2$, sin	9.94490	
	$Q=82°\ 05'.6$, cos	9.13849	2.56779
	$+16.497$		1.21739
	1.		
	17.497		1.24297
	$Q=82°$ 05.6, cot.		9.14264
	$A=22$ 22.1, cot.		0.38561

$C=59°\,43.5=Q-A$ by (4).

By (6) $a=7150.2$	3.85432	By (a) $b=3050.7$	3.48440
$P+A=140°\ 37'.3$, sin.	9.80239	$C=59°\ 43'.5$, sin.	9.93632
	3.65671		3.42072
$P=118°\ 15'.2$, sin.	9.94491	$P'=58°\ 31'.2$, sin.	9.93086
$n=5149.9$ ft.	3.71180	$n'=3089.3$ ft.	3.48986

Computing n' by the first equation of (a) the same value is found as above.

20. Two-point Problem.—If two unknown stations, C and D, Fig. 15, see each other, and also two known stations, A and B, their positions can be determined by measuring the angles ACB, BCD, CDA, ADB, as follows:

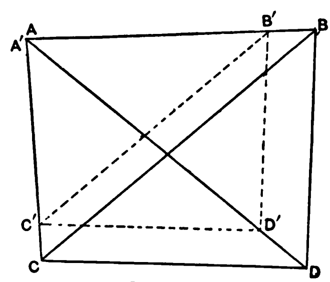


FIG. 15.

Draw the line $C'D'$ of convenient length on tracing-cloth and at C' and D' lay off the measured angles; the intersection of the two lines which pass through A will determine its position on the cloth, and similarly for B; join A' and B'; place the cloth on the plat so that A' will coincide with station A and B' will fall on the line AB of the map, produced if necessary; prick through the points B', C', and D'. Then through B draw parallels to $B'C'$ and $B'D'$; their intersections with AC' and AD' produced will determine C and D on the map.

If more accuracy is desired, assume CD as unity and compute

AC and AD in the triangle ACD, and BC and BD in the triangle BCD. Having two sides and the included angle in ACB, AB can be found, formula 20]. The ratio of the true value to the computed one will be the ratio which the other sides bear to their computed values.*

Ex. The following angles were observed at Giles and Elm of the C. U. Skaneateles Lake Survey in 1892 upon the known stations Haight and Olmstead.

Haight-Giles-Elm = 50° 02′ 17″
Olm.-Giles-Elm = 35° 05′ 03″
Giles-Elm-Olm. = 88° 01′ 27″
Giles-Elm-Haight = 50° 04′ 29″
Haight-Olmstead = 12,944 feet.

Required the positions of Giles and Elm.

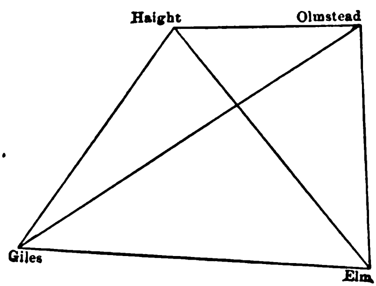


FIG. 16.

21. Direction of Invisible Stations.—If enough angles have been taken so that the stations can be platted by methods already given, the direction of the line joining any two can be taken directly from the plat with a protractor. Or, starting from some known side, the sides of the preliminary triangles can be computed from the observed angles when, by assuming a meridian, the distance in latitude and longitude of each point from an initial one can be computed as in an ordinary land survey. The tangent of the azimuth of the line joining any two points can then be found by dividing the difference in longitude by that in latitude. The line can then be cleared from either end if obstructed by timber, or the height of signal for intervisibility can be determined if the obstruction is an intervening ridge.

* For fuller treatment of the N- and two-point problems see *Zeitschrift für Vermessungswesen*, 1888, p. 140.

For an example in difficult country in northern Alabama see *Coast Survey Report*, 1885, App. 10.

If two stations, C and D, each see two points, A and B, Fig. 15, the direction to trim from one to the other can then be found as follows: At A measure BAD and DAC, and at B, DBC and CBA. Compute AD in the triangle ABD and AC in the triangle ABC, calling AB unity; then in ACD two sides and the included angle are known, from which the angles at C and D can be found by formula 20]. Or the directions can be found by platting.

22. Outfit for Reconnoissance.—When accurate angles are required, a light transit with good telescope is most convenient. The needle will give bearings, while, by adding a level to the telescope tube and a gradienter screw or good vertical circle, elevation angles can be measured with sufficient accuracy for determining intervisibility. An aneroid barometer is also convenient for determining differences of elevation. For distances over 25 miles a reconnoitering glass with stand will be found desirable on account of the larger telescope. If care is taken in setting up to place the tripod head level, the small horizontal circle will give angles quite accurately.

In a wooded country where angles have to be measured from tree tops, a sextant will be necessary; also a telescope or field-glass for identifying the stations, and a set of spurs or creepers for climbing. An azimuth or pocket compass is convenient; also the best available map of the region.

To these should be added some 100 feet of about $\frac{3}{8}$-inch manila rope, a ball of twine, an ax, and material for different colored flags to be spread out upon trees or other objects for temporary signals. An assistant who is quick and handy at all kinds of work and who is used to climbing, and a horse and covered wagon, will complete the outfit. Much of the traveling will necessarily be on foot, or possibly on horseback if the country is hilly or wooded.

If away from all supplies, a cook and the usual camp outfit will be necessary; while for primary triangulation, in rough country with good railroad facilities, like much of New England, it may be more convenient to travel the long distances between stations by rail, hiring a horse when use can be made of one.

23. Signals.—After the exact station points have been located, the signals which are to be erected over them to give definite points for sighting in measuring the angles should fulfill the following conditions:—

They should be conspicuous, so as to be readily seen and distinguished from surrounding objects; they should have a well-defined central line or point upon which to fix the cross-hairs; they should have little or no *phase*, i.e., this line or point should not change in apparent position with the direction of the illumination by direct sunlight; they should be firm in position, unless of the class which require an attendant; they should be cheap, or light and portable; while often it is convenient if, when in place, they will allow an instrument to be set up over the station point.

With these general requirements in mind, the relative advantages offered by the different signals to be described will be more readily appreciated.

24. **Pole Signals.**—When height is not required for intervisibility, one of the most common forms of signal consists of a vertical pole set in or on the ground and supported by braces or wire guys, or of a pyramid or tripod surmounted by a pole.* On sharp mountain peaks, where only small, stunted timber can be found, the rectangular pyramid, Fig. 17, is convenient. A signal with height of apex of from 12 to 18 feet, and legs from 3 to 5 inches in diameter at the top, can be erected, and a center pole 8 to 12 feet long inserted by 3 men, without tackle. By inclosing the top with boards, cloth, or slats made from small poles, visibility can be given, while the apex and pole remain for accurate bisection. The pole can be increased to any desired diameter by nailing on slats or poles after erection, while the signal can be anchored to the rock by wiring the legs to anchor-bolts, or by wire guys extending from the top of the pole.

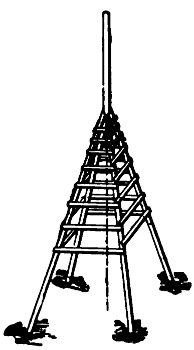


FIG. 17.

On flatter peaks more height must be given for visibility, rendering the tripod signal, Fig. 19, more convenient. By bolting all four pieces together on the ground, with a 1- to 1¼-inch bolt, as shown in Fig. 18, or, better, with the head raised 6 feet on a bent or staging, 5 men can raise a 25 to 35 foot signal of round timber, each piece being 5 or 6 inches in diameter at the top, with no special outfit except about 30 feet of rope. Pits are dug, or stones piled up to prevent two of the feet from slipping; the head is then lifted and pushed to position by the third leg, when the pole is made vertical by pulling down the large end with a rope; it is secured by spiking braces to the tripod legs.

If the angles at the station are to be measured with the signal in place, the legs should be so placed as not to obstruct the lines of sight to the other stations. They should extend a couple of feet into the ground; or, if on rock, be securely tied to anchor bolts by wire rope, or notched and horizontal cross-pieces attached and

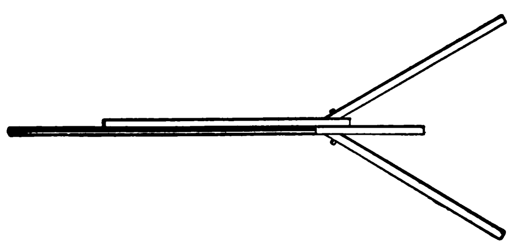


FIG. 18.

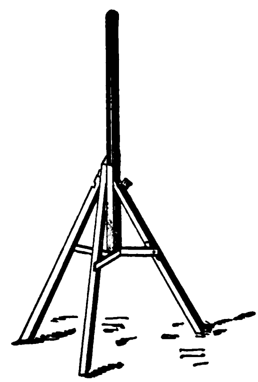


FIG. 19.

loaded with stones. Wire guys from the top of the pole may also be desirable.

A tin cone or barrel of larger diameter than the pole is often placed at the top, especially when the tripod head will not be seen against the sky.

The pole should not be more than 6 to 8 inches at the tripod head, even for a large signal, on account of the weight in erection. It can afterwards be increased, or the pole straightened, by nailing on light slats; or, when lumber is available, a square box of 2-inch plank in place of the pole will give diameter without increased weight. One or more slats along the center of each side will make it more nearly cylindrical.

A very convenient and portable signal for tertiary work can be made by supporting a pole on a tripod having a light, cast iron head and about 10-foot legs. By holding the pole in position with wire guys, a signal 15 to 20 feet high can be made very stable, while there is room enough underneath to set up an instrument. Any portion of the pole can be enlarged to any desired diameter by light slats.

25. Diameter and Height.—The diameter of pole for short lines may be large enough to subtend an angle, as seen by the observer, of 4 or 5 seconds; but, as the distance and the power of the telescope increases, the angle should diminish, according to Coast Survey practice, down to one second for about 15 miles, and not fall below this value for greater distances (see also § 26).

Diameter to subtend one second at

1 mile	=0.307 inch	40 miles	=12.3 inches
10 miles	=3.100 inches	60 "	=18.4 "
20 "	=6.100 "	80 "	=24.6 "

Increased diameter beyond that necessary for visibility gives increased range to the cross-hairs in bisection, and introduces the uncertain element of *phase* with cylindrical signals which do not show against the sky.

The height of signal in feet should be about one-half the distance in miles, plus 10. Less height may answer for long lines, or for signals on sharp peaks with a sky background, but height adds to visibility without diminishing accuracy, and with only the increased cost of construction.

A signal to be seen against the sky should be painted black or wound with black cloth, one to be seen against the ground should be painted white or wound with white cloth; unless two colors are needed on the same signal for ready identification from surrounding objects, when the pole, or pole and tripod, can be painted in alternate rings of black and white or red and white, each ring being several feet wide.

26. Signals without Phase.—Various signals have been devised to avoid phase, or the effect produced by the unequal illumination by direct sunlight of the portion of the signal facing the observer, whereby the apparent and real centers do not coincide. One devised by Bessel for the Prussian triangulation in 1831, and used on the U. S. Lake Survey, consists of a board in place, or in front of, the pole, with its face perpendicular to the line of sight. On the latter survey a width was given of about 4 seconds, as seen by the observer, yet good results were obtained. The station must be visited and the board changed each time the observing party moves to a new station.

Another, designed in 1881, and used on the Mississippi River Survey for distances of from 5 to 12 miles, gave excellent results. It consists of a horizontal board 6 inches in diameter, to the circumference of which are attached 4 stiff vertical wires, 90° apart, each 5 feet long. These wires are held in position by a wire ring at the top, and another one-third the distance from the top, each joint being well soldered. Two opposite wires are connected for the upper and lower thirds by a white cloth, and the other two for the central third by a black cloth; 4 guy wires are attached at the central ring, and the board rests on a tripod or other support.

27. Elevated Signals and Observing-stands.—When the signal and the instrument at the station require elevating, and no existing structure can be utilized, a suitable one must be erected.* In order that the instrument may not be disturbed by the movements of the observer, its support must be independent of the observer's. The standard tripod for the instrument and scaffold for the observer, adopted by the Coast Survey for heights from 32 to 96 feet, are described in the *Report* for 1882, App. 10. See also App. 9, p. 158, and *Pri. Tri. U. S. Lake Sur.*, p. 318.

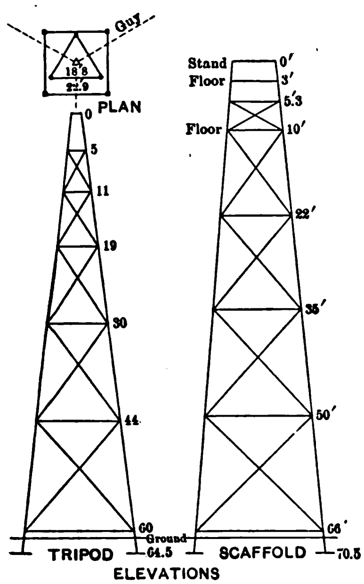


Fig. 20.

The more recent standard used on the 98th meridian work is described in App. 4, *Report* for 1903. Fig. 20 shows the dimensions for a 60-foot tripod and a 66-foot scaffold. The tripod legs are in three lengths, scarf spliced with a 30-inch lap, and spiked, viz.

* An ingenious adaptation of existing natural features to the purposes of a signal is given in *Transcontinental Triang'n*, 1900, p. 601. The observer's platform was supported by small scaffolds upon the top of a tree trunk, and the instrument stand was built around the trunk, but not touching it.

6″ by 6″ by 24′ at bottom, 5″ by 5″ by 22′, 5″ by 5″ by 24′ at top. The bottom horizontal and diagonal braces are 3″ by 4″, the others 2″ by 4″. The scaffold legs are scarf spliced as above. The pieces are 6″ by 6″ by 24′ at bottom, 5″ by 5″ by 26′, 5″ by 5″ by 26′ at top. The two bottom horizontal and diagonal braces are 3″ by 4″, all the others 2″ by 4″.

The tripod legs are brought together at the top, while the scaffold legs are about 7 feet apart 3 feet from the top. The floor for the observer is 10 feet below the top of the scaffold, and the top of the tripod is cut off so as to support the instrument at the exact height required by the observer. A top floor is placed 3 feet from the top for the light tender, and a 3′ by 3′ table at the top in the center, for supporting the lamps and heliotropes. Two trap-doors are placed in the lower floor, one where the ladder comes up and the other in another corner, for hoisting the instrument.

In putting on the bracing, the legs for both structures are sprung out somewhat between the ends, for increased stiffness. Holes are dug for the legs and foot planks inserted. Instead of placing these all on the same level, the legs can be cut to fit.

For erection, two legs of the tripod are placed on the ground in relative position, and the bracing spiked on. A bottom piece of a scaffold leg is used as a gin-pole, and the tripod legs brought to position and guyed; the third leg is then drawn to place and the bracing spiked on the other two sides. Two opposite sides of the scaffold are then put together and raised, using the tripod in place of the gin-pole. The bracing for the other two sides is then attached.

The cost for these signals for the season of 1902, with lumber at about $25 per 1000 feet at the yard, was about $3.20 per vertical foot, measured to the light table.

These slender signals are cheaper than the broader ones of the old type, yet the experience of the season of 1902 indicates that there is less twist due to sun and less vibration due to wind, even without the protecting cotton sheet frequently used on the sunny or windward side of the scaffold.

For different heights the upper 10 feet of the signal remains the same, while the lower horizontal tie changes length as follows, the scaffold leg in each case being 6 feet longer than the tripod leg:—

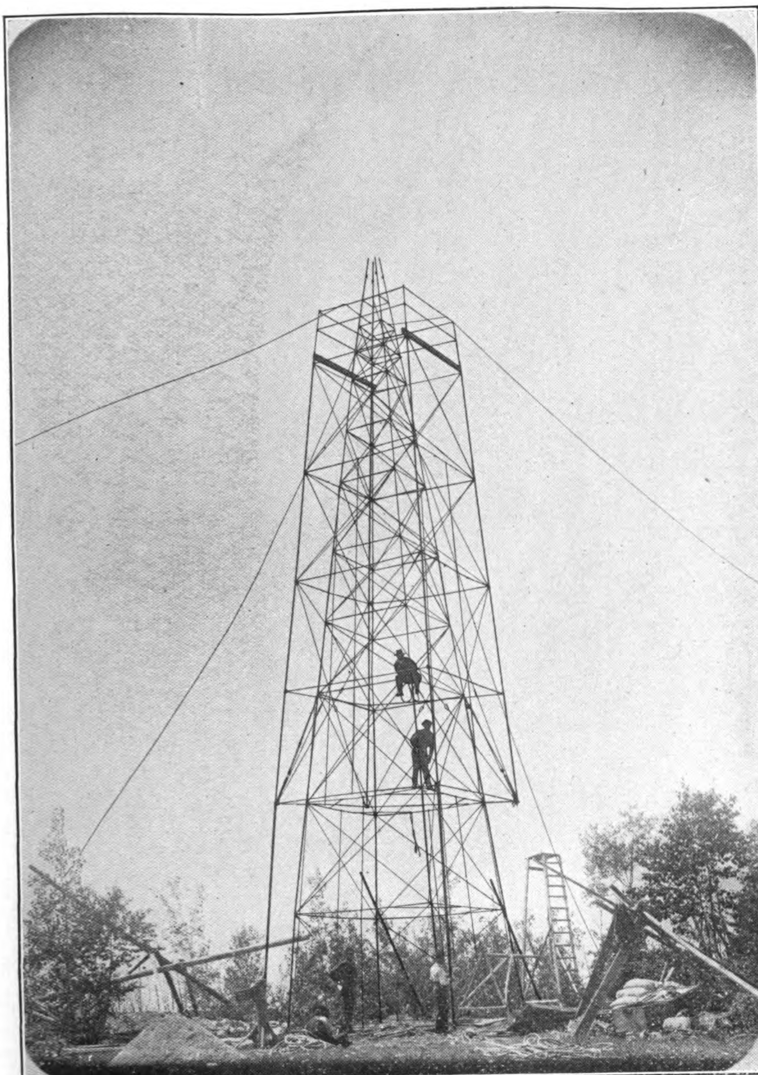


Fig. 21.

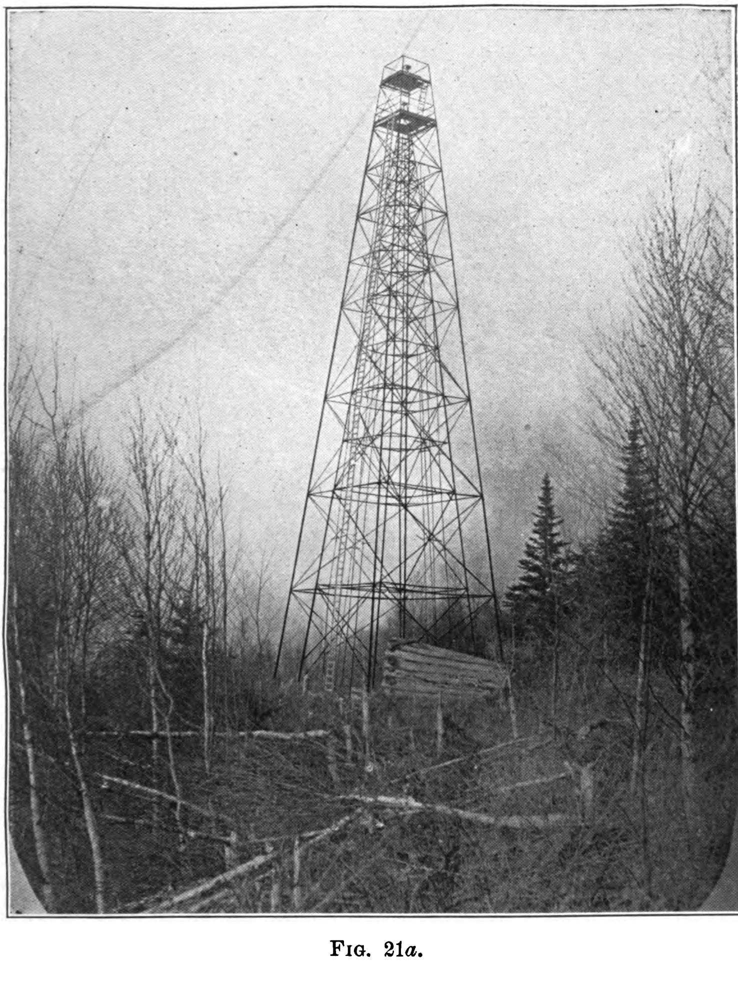


FIG. 21*a*.

Length of tripod leg in feet..	10	15	25	45	60	75
" " bottom tripod tie. .	6	7.5	10	14	18	19
" " " scaffold tie.	10	11.5	13.5	17	22	23

For heights greater than 75 feet the bottom sections should be heavier than those given for 60 feet. Up to 90 feet they may be framed and erected without bolts, as described. If over 90 feet high, the top must be framed with bolts and then raised to position and erected on top of the lower part.

In exposed situations ⅜-inch wire rope guys with turn buckles may be used, although but few of the signals have been collapsed by storms.

The method of erection on the U. S. Lake Survey for heights up to 140 feet was to put together one side of the observing tower on the ground, attach radiating ropes at different points, all leading to the rope through the block; erect a derrick boom, and haul the side to position with teams. The side was then held by guys and the block shifted to it, and one side of the inner tripod hauled up and held in the same way. Then the third leg of the tripod was hauled up and the braces attached to the side already in position, and the opposite side of the tower was raised and the braces attached. Sills some 3 feet underground were used for the tower, but not for the tripod. The station mark was placed after the signal was up. The work was let by the vertical foot. The contractor, with 15 men and 2 teams, would frame, erect, and complete a signal in 2 days.

Recently the Lake Survey has adopted an elevated signal and observing-stand, made of gas pipe, which has proved very satisfactory, and more economical than timber, on account of its portability.

The method of erection is shown in Fig. 21. The upper portion of the instrument pyramid is put together on the ground and then erected. The upper part of the observing stand is then put together around the pyramid. Tackles are attached at the four corners as shown, and this upper portion of the stand is drawn up so that the next section can be attached underneath.

Afterwards either structure can be drawn up, if the other is high enough for attaching the blocks, and an additional section attached, the workmen remaining on or near the ground. Guys are used for steadying during erection.

Fig. 21*a* shows a completed signal 108 feet high erected on Great Duck Island, Lake Huron.

A portable tripod and scaffold, having a floor about 12 feet high, is shown in Fig. 22. The tripod legs are 6 by 8 inches, 18

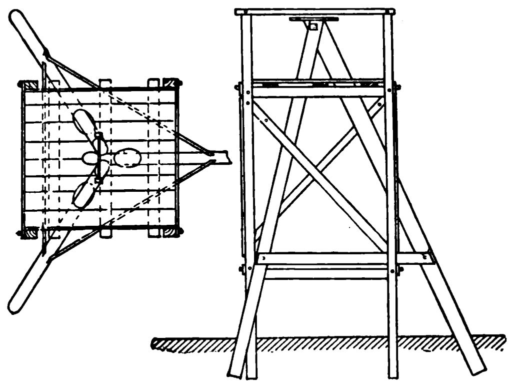


FIG. 22.

feet long. They are held by an inch bolt 16 inches long, and by three horizontal braces. The scaffold posts are 5 by 5 inches, 16½ feet long; the horizontal braces are 7 feet long, and the diagonal ones 10 feet. The posts are interchangeable, and the braces are

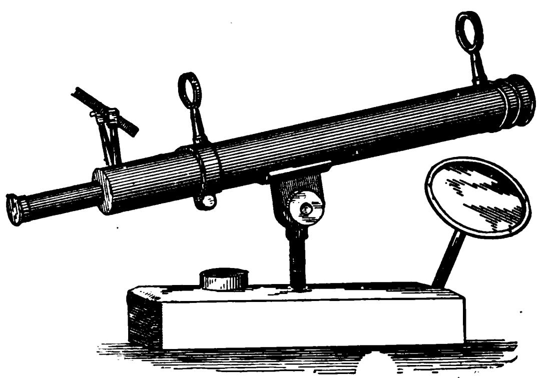


FIG. 23.

held by wood-screws. The posts all extend about 2 feet into the ground, and the floor is placed from 2 to 3 feet below the top. Only a few hours are required for erection after everything is in readiness.

In crossing the plains of India, hollow masonry towers, 50 feet or more in height, were extensively used for the support of the instrument; while in the early French surveys, church spires, large towers, etc., were often used with inaccurate results, due to phase.

28. Heliotropes.—One of the most common forms in use is called the gas pipe heliotrope, Fig. 23. A piece of 2-inch iron pipe

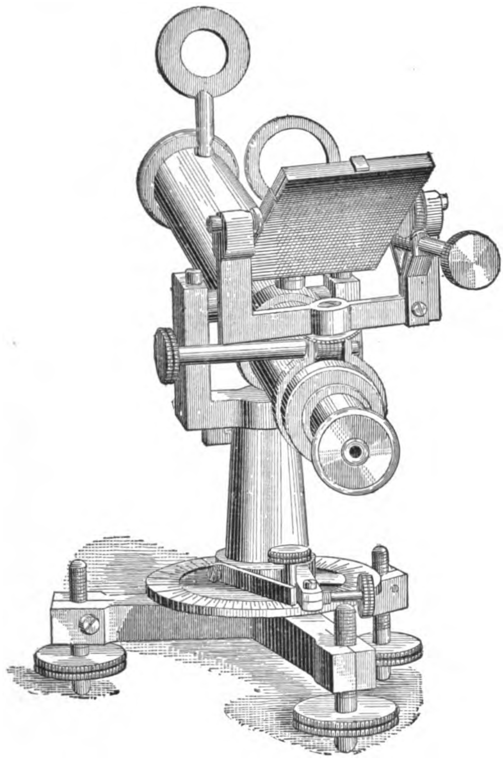


Fig. 23*a*.

serves as a telescope tube, while it carries 2 rings or diaphragms, each with about an inch opening, and a $2\frac{1}{4}$-inch plate glass mirror having motion about a horizontal and a vertical axis. The whole instrument is supported by a wood-screw, which can be screwed into a tripod head or other block. It is set up directly over the station mark, or on line and a few feet in front of it, and the cross-hairs of the telescope brought on the distant observing party.

The mirror is then turned so that the reflected sunlight will pass through the first or near diaphragm and give a concentric ring of light around the second, which is a little smaller. This ring is maintained by gently tapping the mirror at intervals of from ½ to 2 minutes.

The adjustment of the instrument should be tested by bringing the cross-hairs on an object within a few hundred feet, throwing the light as above, and noting if it falls as far above the object as the rings are above the cross-hairs.

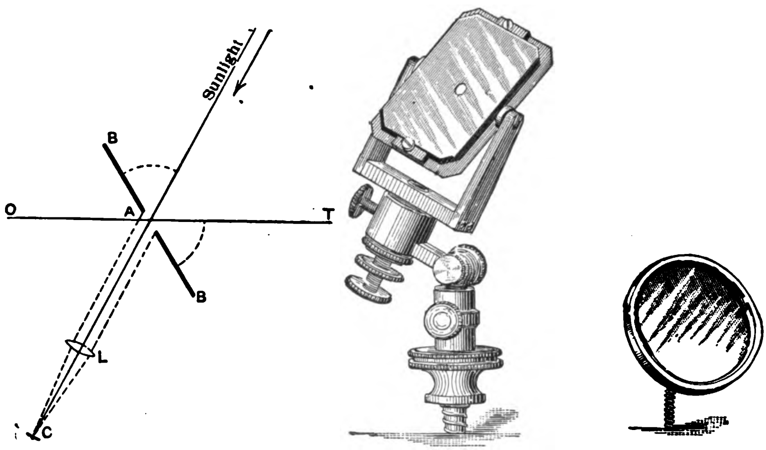


FIG. 24.

A more recent form, used on the Coast Survey, is shown in Fig. 23*a*. The mirror is rotated by tangent screws and the telescope can be more accurately pointed.

The Steinheil heliotrope differs from that already described in having only one mirror and no rings, making it very simple and convenient for reconnoissance work.

The axis of the frame is hollow, and it contains a small lens, *L*, Fig. 24, and a white reflecting surface, *C*, usually chalk, at the focus of the lens.

By turning this axis towards the sun a hole through the silvering of the mirror allows a beam of sunlight to reach the lens and be concentrated upon the white surface. It is reflected from the surface back to the lens and emerges in parallel rays which reach the back of the mirror in a direction just opposite to that of the incident rays. Enough of these rays will be reflected from the back to give an image of the bright spot, C, and in a direction, AO, directly opposite to the reflection of sunlight from the face of the mirror. Hence, if the eye be placed at O so as to see the observing party through the opening A in the direction AT, and the mirror be turned until the bright spot C is seen (the axis pointing towards the sun), the sunlight will be reflected in the direction OAT to the observing party.

The distance from the reflecting surface to the lens is adjustable for focus.

When the alignment has been once secured, if there is no natural landmark in range, a pole should be set up at a distance of 100 to 200 feet, so that its sharp top will be on or a little below the line. The light can then be shown, and will often be seen by the observing party, on days when haze and smoke will prevent the heliotroper from seeing even the outline of the hill or mountain at the observing station.

A second mirror is usually supplied, which can be screwed up and light reflected from it to the first, if at any time the first falls in shadow or its angle of incidence becomes so great that the reflected beam will not fill the diaphragm.

Extreme accuracy in pointing is not essential, the range being about the diameter of the sun, or 32 minutes.

About a 2-inch mirror is used for lines from about 20 to 60 miles, and usually in connection with pole or other signals. For shorter lines a pasteboard or other screen with a smaller opening should be attached to the second ring. For longer lines larger mirrors are used. Thus, on the U. S. Lake Survey, for the longest lines a common mirror, 9 by 12 inches, was set up, and light thrown through a circular hole in a wooden screen some 20 feet distant in the direction of the observing station, this having a diameter of from 8 to 10 inches on sides of 90 to 100 miles. On the longest line ever observed, Mt. Shasta-Mt. Lola in northern California, 192 miles, a helio 12 inches square was used.

Wilson, *Topographic Surveying*, gives

$$X = .046d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for the length of the side of the mirror in inches, where the distance d is in miles and $d > 10$.

Too much light gives, by irradiation, a diameter too large for accurate bisection, and increases the unsteadiness. An opening suited to the distance, or one which will subtend from one-fourth to one-fifth of a second, will give in quiet air a small, bright disk easy to bisect.

An intelligent and very faithful person should be picked out for the heliotroper; otherwise delay and vexation will result. If he is to occupy the station a long time, he can usually be picked up with economy in the locality; if for only a short time, it may be more economical to have one who is familiar enough with the work and with instruments to go to new stations and establish himself without assistance, when directed by the observing party.

29. Night Signals.—Lamps with 10-inch reflectors for short lines and the Drummond light for long ones were used on the English Ordnance Survey in the last century, while night signals have been extensively used in the recent prolongations of the *Nouvelle méridienne de France* by M. Perrier, and Argand lamps and heliotropes are exclusively used in India.

The electric light, in the focus of a reflector 20 inches in diameter and 24 inches focal length, proved very successful recently on a line of 168 miles across the Mediterranean, where, on account of fog and mist, a 12-inch heliotrope had failed to once show during a three months' trial.

On the Coast Survey experiments were made in 1879 with night signals (*Report*, 1880, App. 8), and the conclusion reached that night observations were a little more accurate than day observations; that the average time of observing in clear weather could be more than doubled; and that reflector lamps, or optical collimators, burning coal-oil could be used to advantage up to some 40 miles, while the magnesium light would be better and cheaper for longer lines.

The optical collimators used were M. Perrier's French lamps, Fig. 25. The large lens has a focal length of 24 inches, and a diameter of 8 inches. The emergent rays subtend an angle of about 1°. The intensity of light, as compared with the magnesium, was about as 2 to 5. It made a very pretty mark to point upon on clear nights, but, at a distance of 43 miles, it would often be scarcely visible in the telescope, and would not allow of illuminating the cross-hairs, when the magnesium light was clearly visible.

The magnesium lamp was made up of a small lamp, an 8-inch reflector, a clockwork, and a reel of magnesium tape, which was fed into the lamp by the clockwork and burned intermittently in the focus of the reflector. For accurate bisection a pasteboard screen was used to reduce the diameter on all but hazy nights on a line of 60 miles. The apparatus weighed but 5 pounds, but the tape was expensive.

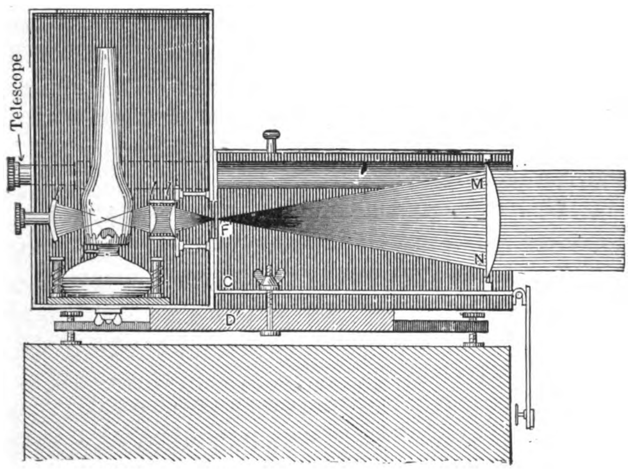


FIG. 25.

The acetylene lamp is now used on the Survey (*Report*, 1903, App. 4). It is made from an ordinary bicycle lamp by substituting for the lens a cylindrical tube containing two 5-inch lenses. It is mounted upon a metallic stand which is provided with adjusting-screws for pointing. The weight when packed with 5 pounds of carbide is but 22 pounds. It has been used on lines from 3 to 31 miles long with satisfactory results, the light under unfavorable conditions sometimes becoming a small, bright point, finally growing too faint to be observed, rather than a faint blur, as in the case of the oil lamp.

Collimators and reflectors, with kerosene lamps, were both successfully used on the N. Y. State Survey for distances up to about 50 miles. The field however was left dark, and the cross-hairs illuminated from behind, giving light lines in the dark field.

30. Station Reference.—In referencing a station, the object should be to render the recovery of the locality and of the exact station point as easy and certain as possible, at any time, and by any one unfamiliar with the country but familiar with the kind of work. The station point is usually marked by an underground and by a surface mark. The underground mark should be placed below frost and plow, or some 3 to 4 feet below the surface. It may consist of any material which is durable, foreign to the locality, and capable of receiving and retaining an exact center mark.

Jugs and bottles, cut stone blocks, and hollow cones of stoneware are among the most common. The stone block, holding a copper bolt and surrounded by masonry, is much used at the ends of base lines, where a very accurate mark is essential, on account of working up from so short a side.

The surface mark should not be in contact with the underground mark, while it should project enough above the surface to be readily found. A stone post, with the top dressed some 4 to 6 inches square and the center marked by a cross or hole, is much used; often the number of the station or the initials of the survey are cut near the top.

Should the station be on firm rock, a hole is drilled some 12 to 15 inches deep and filled with lead or sulphur, or a copper bolt is inserted with a wedge at the bottom which tightens as the bolt is driven down. Along coasts and rivers, where stations are forced out within reach of the action of the water, and on soft, yielding, and shifting soil, much difficulty may be met in securing proper station marks without undue expense. Screw-piles protected by masonry or riprap, etc., are among the expedients resorted to when reference cannot be had to near, permanent objects, or to reference marks set for the purpose. A stake driven down in soft, wet soil, a hole made with a bar and filled with quicklime or charcoal in impervious soil, mounds, references to trees, etc., are among the marks often used for the less important stations.

A topographic sketch of the station and its surroundings should be given, on which are shown the features likely to aid in identification, and especially those objects which can be used for reference points. This should be accompanied by the distances to these points, taken with steel tape if near enough, or by including them in a sweep of angles which includes one or more distant objects and a magnetic bearing. If to these are added the kind of a signal, with the heights above the station mark of the points most convenient for sighting in measuring vertical angles, as tripod head, top of pole, etc.; the name of the landowner or person who has been requested to look after the station, or of those who would know most of its position; the name of the nearest railroad station, and the best method of approach, the description will be reasonably complete.

The various tertiary points sighted upon should be described, to aid the topographer in identifying stations with ease and certainty, and to aid in securing the stations for use in future topographic and hydrographic work.

A station should be named from the popular name of the hill or locality, from some well known peculiarity of the ground, from the owner of the land, or in such a way as to best call attention to the special locality. Numbers are sometimes used in the computations and records, as being more concise.

On the Coast Survey at least one reference mark of a permanent character is called for, not nearer than 10 meters for each station of the main scheme, to be accurately connected by distance and direction. These reference marks are preferably on fence or property lines, and should be always in a locality chosen to avoid disturbance by cultivation, erosion, and building. Such reference marks are desired at all marked stations.

Descriptions are required for all marked stations. For each station which is in itself a mark, as a cupola, sharp peak, etc., a description is required, or the angle records must show the exact point used. Each land section corner connected with the triangulation must be fully described. Nothing is to be put into the description which does not aid in recovery or identification. A sketch should not be used as a substitute for words, but all essential facts which can be stated in words should be so stated, even if shown on the accompanying sketch.

CHAPTER III.

INSTRUMENTS AND OBSERVING.

31. Development of Angle Instruments.—When Snellius of Holland introduced the principle of triangulation in 1615, angles were measured with quadrants, rectangles, or semicircles graduated on their peripheries, and having alidades with sights attached. Defects in graduation were early detected, and efforts made to remedy them by using large radii; 6 to 7 feet was the smallest radius for a sector, while those of 180 feet were not uncommon with the Arabian astronomers.

A means of measuring parts of a division was devised by Nuñez in 1542; the present form of the vernier was first used by Vernierus in 1631; the entire circle was first used by Roemer in 1672; and the first micrometer and cross-hairs in the telescope were used by Picard, although constructed by Auzout in 1666.

The great advance in instrument construction dates from 1783, when the survey to connect the observatories of Paris and Greenwich was begun.

The *French* brought out the *repeating circle* constructed upon the principle pointed out by Tobias Mayer in 1752. Fig. 26 shows the general construction. The horizontal circle just above the leveling screws is an auxiliary not essential in the measurement of angles. The long vertical axis is forked at the upper end to carry the short horizontal axis which supports the repeating circle on one side and a counterweight on the other. The plane of the circle will thus rotate with the horizontal axis, while the circle itself can be rotated in its plane which is parallel with the axis.

The horizontal and vertical axes allow of placing the circle in any plane from horizontal to vertical, while its motion in that plane about an axis perpendicular to the horizontal axis allows of measuring angles on any part of the circle.

The circle carries two telescopes, one above, the other below, both eccentric, each capable of rotation about the axis, with independent clamps and tangent screws; the lines of collimation are parallel to the circle, and the position of the upper telescope can be read by means of verniers.

To measure an angle the following steps are necessary: Bring the plane of the circle into the plane of the objects; clamp the upper telescope at zero; rotate the circle until the upper telescope bisects the right hand object and clamp the circle (the old French circles were graduated counter-clockwise); bring the lower telescope to the left hand object and clamp; unclamp circle and rotate until lower telescope bisects right hand object and clamp; loosen upper telescope, and bring onto left hand object. The reading will now be twice the angle, for in rotating the circle so that the lower telescope changes from the left hand to the right hand object, the zero rotates through the same angle to the right of the right hand object, and the upper telescope must be brought back over the angle to reach the right hand object and once again from the right to the left, giving a reading of twice the angle. The above steps are continued until a sufficient number of repetitions have been taken, when the last reading (increased by the proper number of 360°'s) is divided by the number of repetitions for the value of the angle. It should be noted that this angle may be in an oblique plane. In such a

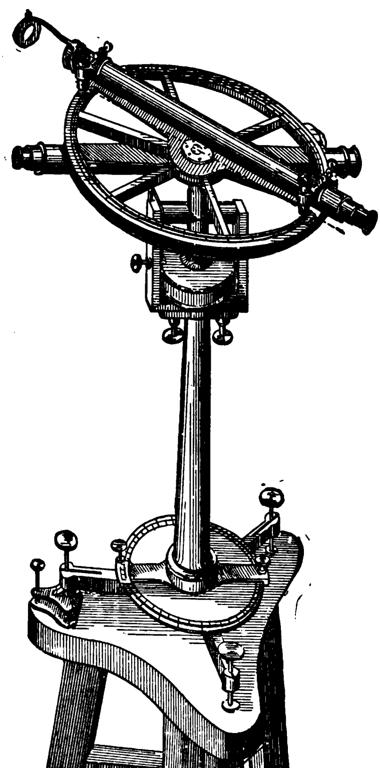


FIG. 26.

case the vertical angles upon the two points must be measured before the horizontal angle between them can be determined.

In measuring vertical angles a level on the side of the lower telescope comes up in position, not shown in Fig. 26, to serve for the reference horizon when the circle is vertical.

At the same time (in 1783) the *English* brought forward the celebrated *Ramsden theodolite*, partially described in § 2, which in its essential principles is the same as the modern theodolite and does not need separate description.

The different parts of an instrument will be taken up in detail, beginning with the telescope.

32. Normal Vision.—The eye is an optical instrument, consisting essentially of a series of transparent refracting media bounded by curved surfaces, forming a lens, and a delicate network of nerve fibers, spreading out from the optic nerve, forming the retina. A pencil of light entering the eye is refracted by the lens and brought to a focus upon the retina, and the impression is carried to the brain along the optic nerve.

The normal eye at rest is supposed to be adjusted for parallel rays; the curvature of the lens and its distance from the retina will increase with the nearness of the object up to the limit of distinct vision, which is some 8 to 10 inches; the pupil or aperture for the admission of light is also adjustable. The distance from the center of the lens to the retina is about 0.6 inch.

With this ratio of distances of retina and object from lens (0.6 and 8) the image will be only 0.6/8 = .075 times as large as the projected object.

For the projected object at the distance of 8 inches from the eye,

$$\text{Angular magnitude for } 1^{\mu} = \frac{0.000\ 039\ 4}{8 \sin 1''} = 1'',$$

where μ = the millionth part of a meter, = 0.000 039 4 inches.

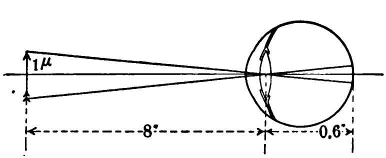


Fig. 27.

The minimum angle between two bright points or lines upon a dark ground, or the reverse, which the eye can distinguish without running them together is found to be about

60″. This would give for the distance on the retina between the images

$$60 \times .075 = 4.5^{\mu}.$$

The surface is made up of minute papillæ or nerve elements called *rods* and *cones* from 2^{μ} to 6^{μ} in diameter, with an average of 4.5^{μ}; the retina thus shows no power to distinguish impressions on parts of a papilla.

A single dark line upon a bright ground can be distinguished, it is said, when the visual angle is only one-fiftieth as large as the above (image 0.09^{μ}).

According to Prof. Forster's investigations as given in Jordan's *Handbuch der Vermess.*, Vol. II, p. 147, the minimum distance b, between a hair and a scratch, which can be distinguished in bisecting a division mark upon a bright scale, as with the cross-hairs of a micrometer microscope, Fig. 28, is 2.5^{μ} measured upon the retina. With this width of line the probable error of the bisection, with a magnifying power of 25, was found to be 0.25^{μ} measured upon the retina. This width 2.5^{μ}, referred to the object and unaided vision, would correspond to $b = 2.5/.075 = 33^{\mu}$, or a visual angle of 33″; while the probable error of bisection would be one-tenth as great. A power of 33 would thus give a probable error of 0.1^{μ} in bisecting a division, and b would $= 1^{\mu}$.

FIG. 28.

It is found that if b be increased 16-fold, or so as to cover 8 papillæ or nerve elements, a power of 85 is necessary for a probable error of 0.1^{μ} in bisection; and if widened to cover 1^{5}, a power of 150 is necessary.

33. The Astronomical Telescope.—This in its simplest form consists of two biconvex lenses fixed in a tube: the eyepiece and the object glass. Its advantages over the unaided eye in accurately sighting an instrument upon a point are: (*a*) increased light; (*b*) magnifying power; and (*c*) the use of cross-hairs.

The following are from Geometric Optics:

A lens is a portion of a refracting medium bounded by two surfaces of revolution having a common axis; this axis is called the *axis* of the lens. The surfaces of revolution are usually spherical or plane; if they do not intersect, the lens is supposed to be bounded by a cylinder in addition having the same axis. The *thickness* is the distance between the bounding surfaces measured on the axis. The *optical center* is a point of the axis, usually within the lens; if any ray of light pass through this point, its direction after passing through the lens will be parallel to its direction before, a slight offset taking place for oblique rays on account of the refraction towards the normal on entering the lens.

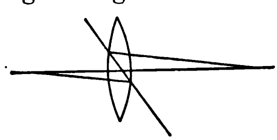


FIG. 29.

For spherical surfaces this point is found by drawing any two parallel radii, joining the points where each cuts its own surface, and noting the intersection of this line with the axis. The ratio of the distances of the centers of curvature from the optical center equals the ratio of the radii. When one surface is plane, the optical center is found at the other surface. The principal focal length of the lens, f, is found from

$$\frac{1}{f}=(n-1)\left(\frac{1}{r}+\frac{1}{r'}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where r and r' are the radii of the surfaces, and n the index of refraction.

The fundamental equation connecting conjugate foci is

$$\frac{1}{f}=\frac{1}{f'}+\frac{1}{f''}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where f'' is the distance of the object and f' that of the image (negative if the image is virtual*).

34. **Magnifying Power.**—In Fig. 30, let O be the object glass and O' the eyepiece. The rays of light from the arrow head, A, will be brought to a focus at A' where the ray through the optical center O meets the focal plane, and those from C at C', these two central rays preserving their direction beyond the lens, but suffering a slight offset as indicated in § 33. Join A' and C' with the optical center of the eyepiece. All the rays of light coming from A and C which pass through the telescope will emerge in pencils

* For further discussion of lenses, telescopes, etc., see W. Watson's *Physics*, Book IV, Chap. III.

parallel with, or slightly diverging from, these two directions $A'O'$, $C'O'$, if adjusted for distinct vision for a normal eye. The lines $A'O'$, $C'O'$, give directions only for the rays passing through A' and C'.

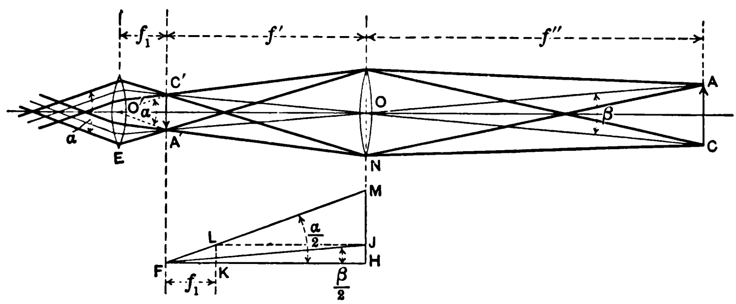


FIG. 30.

Without the telescope, the angular magnitude of the object with the eye at O would be β. With the telescope, the angular magnitude is α.

To find the magnifying power draw $FH=f'$, the focal length of the objective; erect the perpendicular $HJ=\frac{1}{2}A'C'$; take $FK=f_1$, the focal length of the eyepiece; erect the perpendicular $KL=HJ$, join J and L with F, giving $HFJ=\frac{1}{2}\beta$, and $HFL=\frac{1}{2}\alpha$.

Extending FL to M to refer both images to the same distance, the ratio of the apparent magnitudes or the magnifying power will equal the ratio of HM to HJ.

But $$HM:HJ=HM:KL=FH:FK=f':f_1,$$

or, $$\text{magnifying power, } G=\frac{f'}{f_1}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

i.e., *the magnifying power equals the focal length of the object glass divided by the focal length of the eyepiece.*

Also,

$$HM:HJ=\tan\tfrac{1}{2}\alpha : \tan\tfrac{1}{2}\beta,$$

or, $$G=\tan\tfrac{1}{2}\alpha : \tan\tfrac{1}{2}\beta, =\frac{\alpha}{\beta}, \text{ nearly}, \quad . \quad . \quad . \quad . \quad (11)$$

i.e., *the magnifying power equals the angular magnitude as seen through the telescope divided by the angular magnitude as seen with the naked eye, nearly.*

Since, by (9), f' increases with the nearness of the object, G will be greater for a near than for a distant object; f' for parallel rays is taken as the standard.

For normal eyes the eyepiece would be focused for a virtual image at the distance of most distinct vision, or from about 8 inches to infinity; myopic (near-sighted) eyes, unless corrected by glasses, would require that the eyepiece be pushed in, and hypermetropic (far-sighted) eyes that it be pulled out, thus changing f_1 and G.

In Fig. 31 it may be noted that the extreme rays from a point A striking the object glass at the distance apart D will intersect at a' in the focal plane and may be focused to emerge in parallel lines ($\alpha=0$, $\beta=0$) at the distance apart d'.

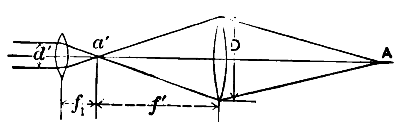


FIG. 31.

From similar triangles, neglecting the thickness of the lenses,

$$\frac{D}{d'}=\frac{f'}{f_1},$$

or from (10)

$$G=\frac{D}{d'}, \quad \dots \dots \dots \quad (12)$$

i.e., *the magnifying power equals the diameter of the clear aperture of the object glass divided by that of the emergent cylinder of rays from a point.*

For the magnifying glass, or simple microscope, with the eye placed close to the lens as with the eyepiece, take $FH=8$ inches, the distance for normal vision; $FK=f_1$; $HJ=\frac{1}{2}AC$. Then, from (10),

$$G=\frac{8}{f_1}. \qquad (13)$$

If an objective is added, making a compound microscope,* it will magnify the image AC in the ratio f'/f'' (AC being now regarded as replacing $A'C'$ of Fig. 30).

Hence $$G=\frac{8}{f_1}\frac{f'}{f''}. \qquad (14)$$

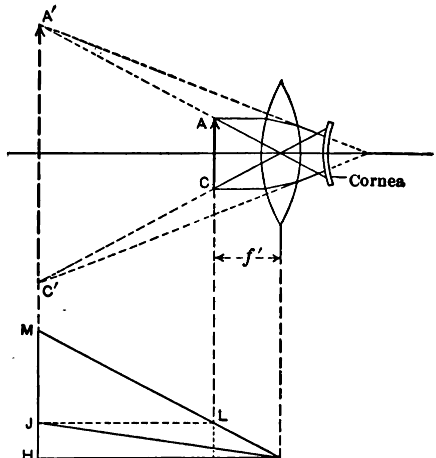


FIG. 32.

Ex. 1. Find the power of a magnifying glass having a focal length of 1″ and placed so that the virtual image will appear 8″ from the eye.

Ex. 2. Show that the change in focal length of an object glass with distance, has the effect of placing the initial point for stadia work at the focal length in front of the object glass.

Ex. 3. Find the range in motion of the object glass slide to focus at distances from 10 feet to infinity, for $f=9''$, $G=25$.

35. Measurement of Magnifying Power.—(*a*) Set up the telescope where two prominent well defined objects can be seen symmetrically with reference to the center of the field, on looking through the object end, and focus for parallel rays. Set up a transit back of the telescope, and measure the angle A subtended by the objects as seen through the telescope. Remove the telescope; set the center of the transit in the position occupied by the eyepiece and measure the angle A' between the same objects as seen directly.

* With the magnifying glass or compound microscope the eye cannot replace the lens or objective and focus on the object as with the telescope, the distance being less than 8 inches.

Then, by (11),

$$G=\frac{\tan\frac{1}{2}A'}{\tan\frac{1}{2}A}=\frac{A'}{A}\text{ (nearly)}. \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

(*b*) Focus the telescope for parallel rays; point it towards the sun, or a bright sky, and measure the diameter d' of the emergent cylinder at the eyepiece as thrown upon a paper screen; measure the clear diameter D of the objective by pushing a pencil in from the edge until it will just cast a shadow on the screen, and noting the reduction from the apparent diameter. Square pieces of paper of different sizes, moistened and placed around the circumference, will show the clear diameter more accurately than the pencil point.

By (12), $$\frac{D}{d'}=G\text{ (approx.)}.$$

(*c*) Sight to a speaking rod, a clapboarded house, or other object which will answer for a scale of equal parts. While looking through the telescope at a scale unit with one eye count the number of units which it covers as seen by the other or free eye; this number will be the power G' for the given distance.

To find G, the power for parallel rays: Measure the distance f''' from the center of the objective to the back of the cross-hair diaphragm when focused for the above scale reading, and the distance f' when focused on a distant object. From (10),

$$G=\frac{f'}{f'''}G'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

The method (*a*) is the most accurate; (*b*) will give fair results except for high powers for which it is difficult to measure d' with sufficient accuracy; (*c*) is the most convenient for low powers.

Ex. The angle subtended by two objects when seen looking into the object end of the telescope focused for parallel rays was $A=2'\,11''$.

The angle subtended, as seen directly, was $A'=1°\,18'\,06''$. Required G.

Solution.

By (15), approx. $$G=\frac{1°\,18'\,06''}{2'\,11''}=\frac{4686}{131}=36.$$

By (15), $$G=\frac{\tan\,39'\,03''}{\tan\,1'\,05\frac{1}{2}''}=36.$$

36. Intensity and Brightness.—Let D = diameter of the object glass; d = that of the eyepiece; d' = that of the emergent cylinder of rays; d_1 = that of the pupil of the eye, assumed = 0.2 inch by Chauvenet for astronomical work, and 2.24 mm. or 0.09 in. by Jordan for geodetic work, the actual size varying with the individual and with the brightness over a greater range than indicated by the above values; m = percentage of light striking the object glass, from a given point in the optical axis, which passes through the lenses, = 85% for the best telescopes, and often falling to 60%.*

With the unaided eye, the cone of rays which can enter it from a given object has a diameter d_1. With the telescope, the diameter of the cone which may be condensed to enter it is D. The quantities of light, for the same distance from the object, will vary as the squares of these diameters. Hence, allowing for the loss due to absorption and reflection, the intensity or ratio of the light received by the eye through the telescope, to that received by the unaided eye, is

$$I = m\frac{D^2}{d_1^2}.$$

For all this light to enter the eye, $d_1 \geqq d'$, or, substituting the value of d' from (12),

$$d_1 \geqq \frac{D}{G}.$$

If $d_1 \leqq \frac{D}{G}$, as may be the case with telescopes designed for special purposes, the effective diameter of the object glass will be reduced to d_1G. This value substituted in the value of I will give mG^2. Collecting results,

$$\left.\begin{aligned} I &= m\frac{D^2}{d_1^2} \quad \text{when} \quad d_1 \geqq \frac{D}{G}, \\ I &= mG^2 \quad \text{when} \quad d_1 \leqq \frac{D}{G}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (17)$$

* It is stated by Nolan in *The Telescope* that about 7% is lost by each lens, one-half of this being reflected back from the outer surface and the other half from the inner surface as it passes through.

Experiments at the University give m about 60% for the older telescopes with terrestrial eyepieces.

Owing to the magnifying power, this light appears to come from an area G^2 times as large as without the telescope.

Hence the brightness, or light per unit area as compared with the naked eye,

$$\left.\begin{aligned} B&=\frac{I}{G^2}=m\frac{D^2}{d_1{}^2G^2} \quad \text{when} \quad d_1 \geqq \frac{D}{G}, \\ B&=m \quad \text{when} \quad d_1 \leqq \frac{D}{G}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (18)$$

Tabulating (18) for different values of D and G, we have the following

TABLE FOR BRIGHTNESS.

G	m	d_1 inches.	*D*, or Aperture in Inches.						
			1	1½	2	2½	3	3½	4
10	0.85	0.09	0.85	0.85	0.85	0.85	0.85	0.85	0.85
		0.20	0.21	0.48	0.85	0.85	0.85	0.85	0.85
20	0.85	0.09	0.26	0.59	0.85	0.85	0.85	0.85	0.85
		0.20	0.05	0.12	0.21	0.33	0.48	0.65	0.85
30	0.85	0.09	0.12	0.26	0.47	0.73	0.85	0.85	0.85
		0.20	0.02	0.05	0.09	0.15	0.21	0.29	0.38
40	0.85	0.09	0.07	0.15	0.26	0.41	0.59	0.80	0.85
		0.20	0.01	0.03	0.05	0.08	0.12	0.18	0.21
60	0.85	0.09	0.03	0.07	0.12	0.18	0.26	0.36	0.47
		0.20		0.01	0.02	0.04	0.05	0.07	0.09

A glance at the table will show that with the powers in common use, viz., about 20 for a 1-inch aperture, 25 for a 1¼, 30 to 40 for a 1½, 60 for a 2-inch, the brightness is from 10% to 25% for Jordan's value of d_1, which is full large for sunny weather; while it is only from 2% to 5% for Chauvenet's value, which is none too large for work in thick woods near nightfall or on dark November days. This serious loss of brightness at times when most needed, due to the failure of the aperture of the telescope to respond, like that of the eye, to variations in illumination, can be met by using an eyepiece of lower power in dull weather.

It should be noted that the ratio of the brightness of the sky and all other objects seen in the telescope remains constant whatever the loss. For this reason the loss is not very noticeable until quite large.

In looking at a fixed star, the more perfect the telescope, the more nearly will the image appear as a bright point, regardless of the power; the brightness will therefore increase directly with the intensity, there being no magnification. The brightness of the field will however reduce as G^2, as the area from which the light appears to come is increased in that ratio. This is why fixed stars can be seen in the daytime with telescopes of small apertures and large powers when they are invisible to the naked eye, the darkened field allowing them to show through as at night; also why faint stars can be seen at night which would be invisible with the same telescope and a lower power. On the other hand, faint nebulæ, tails of comets, etc., which have nearly the same degree of brightness as the sky, become invisible under high powers, because, although the ratio remains constant, the difference in brightness soon becomes too small to be distinguished by the eye.

Ex. 1. Find the intensity and brightness of a stadia rod at 1000 feet with a telescope having a power of 30 and a $1\frac{1}{4}$-inch object glass, as compared with the naked eye at 100 feet.

Solution.

$$I = m\frac{D^2}{d_1^2} = 0.85\frac{(1.25)^2}{(0.09)^2} = 164.$$

$B = I/G^2 = 18\%$ as compared with the eye at 1000 ft.

The light radiating from any point of the rod in cones, its intensity would diminish as the square of the distance, so that moving the telescope from 100 feet to 1000 feet from the rod would reduce the intensity $(100/1000)^2$ fold. The linear magnification, G, would be reduced 100/1000 fold.

$$\therefore\ I' = \frac{I}{100} = 1.6. \quad G'^2 = \left(\frac{30}{10}\right)^2 = 9. \quad B' = \frac{I'}{G'^2} = 18\%.$$

Ex. 2. An engineer's level telescope has a power of 35 and a $1\frac{3}{8}$-inch object glass. Find the intensity and brightness.

Ex. 3. Find the distance to which a leveling rod can be removed from the level of Ex. 2 for the divisions to be of the same size on the retina as when the rod is seen at a distance of 10 feet with the naked eye.

37. Field of View.—It is customary to limit the focal plane by a circular diaphragm to about $0.5f_1$ on account of the difficulty of securing good images with an eyepiece of larger field.

From Fig. 33, since the image of each object is on the line joining the object with the optical center, we have, by (10),

$$\tan \gamma^\circ = \gamma \tan 1^\circ = \frac{0.5f_1}{f'} = \frac{0.5}{G}.$$

But tan 1°=0.017,

$$\therefore \gamma = \frac{30}{G} \text{ (approx.)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

E.g., Mag. power,	$G=10$	20	30	40	60
Angular field of view,	$\gamma=3°$	1° 30′	1° 00′	0° 45′	0° 30′

As the field becomes small, the eyepiece is often made movable in order to include a greater range in one direction, either altitude or azimuth, by moving it with a tangent screw, the simul-

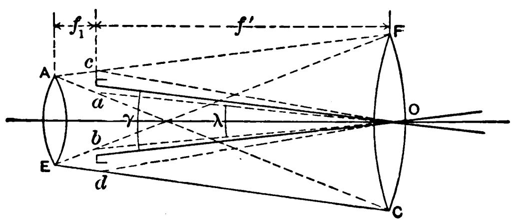


FIG. 33.

taneous field being as above. In order to find λ, the angle of the bright field, or field from which all the light which passes through the object glass from a given point will reach the eyepiece, draw the diagonal lines AC and EF, and join their intersections with the focal plane, a and b, with the optical center O. All the rays coming through the object glass from any point on aO will pass through the focus a, and reach the eyepiece, those from C passing just to the limit at A. Similarly for bO. Hence aOb, $=\lambda$, is the angle of the *bright field*, or field for total light.

From this field out the intensity and brightness both diminish, and they would reach zero at cOd were the field not restricted to γ by the diaphragm. Since γ is about equal (not much larger than λ) objects should retain their brightness nearly or quite to the edge of the field.

In order to take in the whole extent of this field the eye must be placed at the point in which the axes of the extreme pencils, diverging from the center of the object glass, meet the axis of the telescope after emergence. The position of the eye is therefore at the focus of the eyepiece which is conjugate to the center of the object glass. The telescope tube is prolonged to this point and furnished with an eye stop.

38. Spherical and Chromatic Aberration.—The simple telescope described above would be satisfactory only for very low powers. For, with spherical surfaces, the only ones which can be conveniently ground, the rays from near the border of the lens are brought to a focus nearer than those passing through the central portion; the distance along the axis between these foci is an indication of the *spherical aberration*. Spherical aberration is reduced for a given diameter by increasing the focal length of the lens, as a less portion of the sphere is used. Again, the different colors have different indices of refraction as seen from the spectrum, the violet coming to a focus nearest the lens, and the red the farthest; the distance along the axis between these foci indicates the *chromatic aberration*. To obviate these difficulties, the object glass is usually composed of two simple lenses (see Fig. 34), an outer double convex one of crown glass having a low dispersive or spectrum-forming power, and an inner double concave one of flint glass having a high dispersive power but with flatter curvature. The dispersive powers can thus be made equal for any two colors of the spectrum by a proper relation between the focal lengths, rendering the combination nearly *achromatic*, while the sharper curvature of the convex lens leaves a residual of converging refractive power. The combination can be rendered nearly aplanatic, or free from spherical aberration, by giving proper radii of curvature to the four surfaces. The two adjoining surfaces usually have the same curvature; they are sometimes united by Canada balsam to prevent the loss of light by reflection from the inner surfaces, sometimes cemented around the outside to prevent the entrance of moisture, and sometimes they are held in place by the cell simply.

The grinding of the lenses and the first polishings are extremely simple. The finishing of a fine object glass requires great skill and patience on the part of the optician, as the effect of every flaw in the glass and defect in the grinding must be counteracted by polishing here and there each of the four surfaces, with the finger alone or using a little of the finest rouge and water, until after repeated tests the desired degree of perfection is attained.

With two lenses thus adjusted to each other, it is evident that their relative positions in the cell cannot be disturbed without injury.

39. Eyepieces.—The correction for the eyepiece is usually made by using two separate lenses of the same kind of glass placed at such a distance apart that the colored rays produced by the first lens shall fall at different angles of incidence upon the second and become recombined. The two lenses may be treated like a single one with the equivalent focal length, as found from optics.

$$f_1 = \frac{f_1' \cdot f_1''}{f_1' + f_1'' - a}, \quad \ldots \ldots \ldots \quad (20)$$

where f_1', f_1'', are the focal lengths of the separate lenses, and a is the distance between them.

The *Huygenian*, or negative eyepiece, is one of the best when cross-hairs are not required. It consists of two plano-convex lenses, Fig. 34, with the plane sides towards the eye, the focal length of the farther or field glass being three times that of the

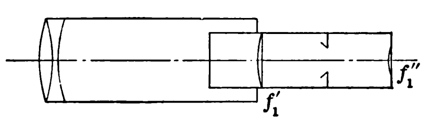


FIG. 34.

nearer or eye glass. They are placed about half the sum of the focal lengths apart. The field glass receives the converging rays from the object glass before they have reached the focus, and brings them to a focus between the lenses. Cross-hairs are often placed at the focus to define certain portions of the field, as in the sextant telescope, but not for accurate measurements, since the cross-hairs will be distorted, seen through the eye glass only, while the object will not be, seen through the corrected combination.

Airy replaces the plano-convex field glass by a concavo-convex, increasing the flatness of the field.

The *Ramsden*, Fig. 35, is the form most commonly used when accurate measurements with cross-hairs or micrometer are required. It is a positive eyepiece, i.e., it receives the diverging rays from the object glass after they have passed the focus. The two plano-convex lenses have their convex sides turned towards each other; they have the same focal length, and are placed two-thirds the focal length apart, giving by (20) an equivalent focus of three-fourths that of one of the lenses.

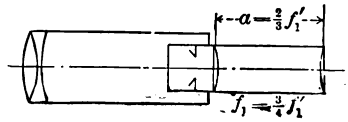
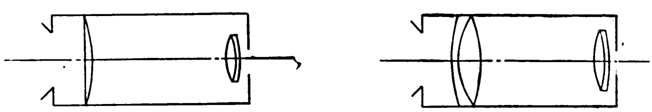


FIG. 35.

The *Kellner* and the *Steinheil* are modifications of the Ramsden which are coming into favor on account of the greater flatness of

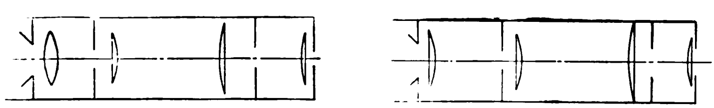


Kellner Eyepiece. Steinheil Eyepiece.

FIG. 36.

the field or freedom from spherical aberration. In the former the eye glass is an achromatic combination, and in the latter both are achromatic (Fig. 36). The former has the larger field. None of these eyepieces invert the image, and as the object glass inverts, the objects all appear inverted.

The *terrestrial eyepiece* consists of four lenses, the object being to invert the image so that objects seen through the telescope will appear erect. Quite an appreciable loss of light results from



Airy Eyepiece. Fraunhofer Eyepiece.

FIG. 37.

the two extra lenses (14% on the basis of Mr. Nolan's estimate of 7% per lens), and a serious shortening of the focal length of the object glass for a given length of telescope, which increases the difficulty of securing a flat field. Two combinations are shown, the Airy and the Fraunhofer.

Diagonal eyepieces.—For convenience in looking at very high objects, a mirror of polished speculum metal is placed between the two lenses of the eyepiece, at an angle of 45°, so that the light emerges perpendicular to the telescope tube. This erects the object (reverses the image) in altitude but not in azimuth. For objects near the zenith, the perpendicular portion of the tube must frequently be lengthened for accessibility. This is secured by placing the mirror between the central lenses of the terrestrial eyepiece, which then inverts the object in altitude and leaves it erect in azimuth.

Instead of the speculum mirror, a glass isosceles right-angled triangular prism can be used with less loss of light.

40. Cross-hairs.—Since with the telescope the image of any point is at the intersection of the focal plane with a line through the point and optical center of the object glass, this optical center may be taken as a fixed point for all lines of sight. The intersection of a horizontal and a vertical hair placed in the focal plane (it should be in the optical axis) will give a second fixed point. The line joining them, called the *line of collimation,* is taken for the direction of the telescope; its greater precision is due to the magnifying power and increased light of the instrument.

In pointing, the eyepiece is first focused upon the cross-hairs and then the object glass upon the object; the focal plane of the object glass is thus brought to coincide with that of the cross-hairs, so that the latter will remain fixed upon the object as the eye is moved from side to side behind the eyepiece. The first is for the eye of the observer, and this focus does not need to be disturbed when once properly made; the second is for the distance to the object, which requires change with each new distance.

Spider lines are usually used for cross-hairs. Some prefer to have them spun directly by a spider as needed, others to take them from cocoons. They should be opaque, cylindrical, free from dust, and as small as is compatible with distinct visibility. Platinum wires are used by some instrument makers as being more opaque and less liable to stretch with age. The requisite fineness is obtained by coating with silver, drawing down the wire, and afterwards removing the silver by nitric acid.

A glass diaphragm with etched lines is sometimes used in place of cross-hairs, with perhaps some advantage as to permanence of position but with the disadvantage of loss of light, and the magnification of all dust on the glass unless thick and the cross-hair side inclosed in a sealed cell.

The *reticule* of wires consists of one horizontal and one vertical for the ordinary surveying instruments. Sometimes stadia wires are added. For triangulation work the vertical wire should be replaced by an X for pole signals, while on the Coast Survey two vertical wires some 15″ apart and two horizontal wires much farther apart are used for heliotropes and night signals. The horizontal wires limit the portion of the vertical wires to be used for bisection. For astronomical work several horizontal and vertical hairs are used, either equidistant or arranged in groups symmetrically with reference to the center. The linear distance required between the wires can be computed and laid off with a micrometer from the focal length of the object glass as measured on the outside of the tube to the cross-hair diaphragm. Or, better and more accurately, by using a micrometer microscope as an eyepiece and measuring the distance subtended by the divisions of a rod at a measured distance; from this distance the required distance between wires is readily computed and laid off by the micrometer. Allowance must of course be made for the change in focal length for parallel rays. The angular distance can be determined from astronomical observation, or directly from circle readings.

41. Tests of Telescope.—To test for *spherical aberration*, reduce the effective area of the object glass about one-half by a ring of black paper and focus upon a well defined point. Then remove the ring of paper and cover the other half of the object glass; the distance the latter must be moved in or out for distinct vision, which should be small if any, is an index of the spherical aberration.

To test for *definition*, focus upon small clear print at a distance of 20 to 100 feet, depending upon the magnifying power, and note if the print is as sharp and well defined as when viewed with the naked eye at a distance of 8 to 10 inches. Poor definition may be due to spherical aberration, or to inaccurate curvature, or to variable density or non-centering of the lenses.

To test for *centering*, or for the coincidence of the optical axes of the different lenses, fix a white paper disk, about one-eighth inch in diameter with sharp outline, in the center of a black surface, and look at it when placed in a good light at a distance of 30 to 40 feet. If the image of the disk, when a little out of focus, is surrounded on all sides by a uniform haze, the centering is good.

Astronomical objects are sometimes preferred for testing as follows: The correction for *spherical aberration* is well made when the image of a star under favorable atmospheric conditions appears as a small well defined point or round disk. Having this in the best focus, the slightest motion of the object glass out or in should enlarge the image, it remaining circular if the lens is symmetrical throughout; while in the most perfect telescopes the image will enlarge to several concentric circular rings of light before disappearing. An imperfect, unsymmetrical lens will give distorted rings, or only a confused mass of irregularly colored light. If the glass is not homogeneous, bright stars will show "wings" which it is impossible to remove by perfection of figure or adjustment. The defective portion can be found by covering up different portions of the object glass and testing.

The correction for *chromatic aberration* is well made when, after focusing on a bright object as the moon or Jupiter, pushing in the eyepiece slowly will give a ring of purple and pulling it out one of pale green; thus showing that the extreme colors of the spectrum, red and violet, have been corrected.

The *flatness of the field* depends mainly upon the correction for the spherical aberration of the eyepiece. It can be tested by drawing a square some 6 to 8 inches on a side, with heavy black lines upon white paper, and looking at it when flat and at such a distance as to nearly fill the field of view. If the lines appear perfectly straight, the field is flat. A telescope may distort the image appreciably without introducing any error in ordinary work, but it is objectionable for stadia work and inadmissible when measurements are to be taken in the field with a micrometer eyepiece.

The object glass should be mounted so that its optical axis coincides with the axis of the telescope tube. The object glass slide should be parallel to this same line, and the vertical plane of collimation should contain it, when adjusted perpendicular to the telescope axis.

The rear end of the object glass slide is sometimes supported by an adjustable collar for ease in meeting the above requirements, but with first-class workmanship it is usually considered unnecessary, while it adds an element of instability. The accuracy of workmanship can be appreciated by remembering that 10 seconds of arc will subtend only 0.000049 of an inch for a focal length of 10 inches.

The *object glass slide* is tested by placing the vertical wire in adjustment for distant objects (slide drawn in) and then testing the adjustment for near ones (object glass slide pushed out). This test is of more importance for ordinary instruments than for geodetic or astronomical ones, for which the precaution is taken not to disturb the slide or focus of the object glass between sights which are combined on the supposition of a fixed line of collimation. This is possible for sights over 1½ miles, no matter what the inequality, while it is not for short sights unless they are nearly equal.

The horizontal line of collimation is not restricted so closely as the vertical; so that if it is adjusted parallel to the object glass slide, the deviation from the optical axis of the object glass, or from the axis of the telescope, will have but little effect. This adjustment is effected by the three peg method of adjusting a level.

42. Level Tubes.—These for accurate work are carefully ground with emery on a revolving arbor which has been turned so as to give the desired curvature. The tube is slowly rotated about its axis so as to distribute the grinding uniformly around the circumference. The surface is then polished, the tube filled and tested on a level tester for uniform curvature by noting if equal angular changes will give a uniform motion of the bubble. For delicate levels the defects found after this rough grinding must be corrected, requiring repeated trials and much skill and patience.

The upper inner surface, when completed, must be highly polished to render the friction of the bubble as small and uniform as possible.

The tube should be of uniform bore and thickness and of hard glass.

The liquid used for filling is usually alcohol for the more common levels, alcohol with a little ether added for fluidity for more sensitive ones, and sulphuric ether, with possibly a little chloroform for the most sensitive ones.

For delicate levels a chamber is added at one end so that the bubble can always be used at about its normal length for greater convenience and accuracy; a change of length with the temperature changes the zero, if the curvature or size at one end differs from that at the other, while a short bubble is more sluggish and its position of rest more affected by friction and by local defects of the tube than a long one. The best results will be obtained with the length used by the maker in testing the tube. The tube should not be directly held in rigid metallic supports on account of the danger of distortion from pressure due to changes of temperature. The support should be at two points only and with rings of cork or other yielding material which will give sufficient stability.

A very sensitive level should be inclosed in a glass box or tube so as to form a closed air space, to diminish local distortion from sudden changes of temperature.

The value of a division should be determined for different portions of the tube to test uniformity, and at different temperatures to determine the temperature coefficient if any. An appreciable coefficient will usually denote a cramping of the tube by the supports.

43. Graduated Circles.—The process of graduating a circle can be readily understood from the description of the Saegmuller dividing engine given in *Zeit. für Instrumentenkunde*, 1894, p. 84. (See Fig. 38.) The engine is built entirely of cast iron and steel with the wearing parts hardened. These metals have nearly the same coefficient of expansion so that slow temperature changes have no appreciable effect upon the working of the parts. For dividing large circles a glass case is provided, and the machine kept automatically at a constant temperature by a thermostat. The weight of the large one-meter horizontal circle and its axis (over 500 pounds) is almost wholly relieved from the bearings. The circle is rotated by means of two diametrically opposed worm gears driven by pinions at the left, as shown in the figure These pinions are driven through ratchet wheels by a long rack working back and forth on the bed plate. The teeth on each end of the rack are cut to match the corresponding teeth on the pinions so that it is claimed the worm gears move absolutely alike in driving the large circle.

One turn of the worm gear rotates the circle through five minutes of arc, so that there is always a whole number of turns

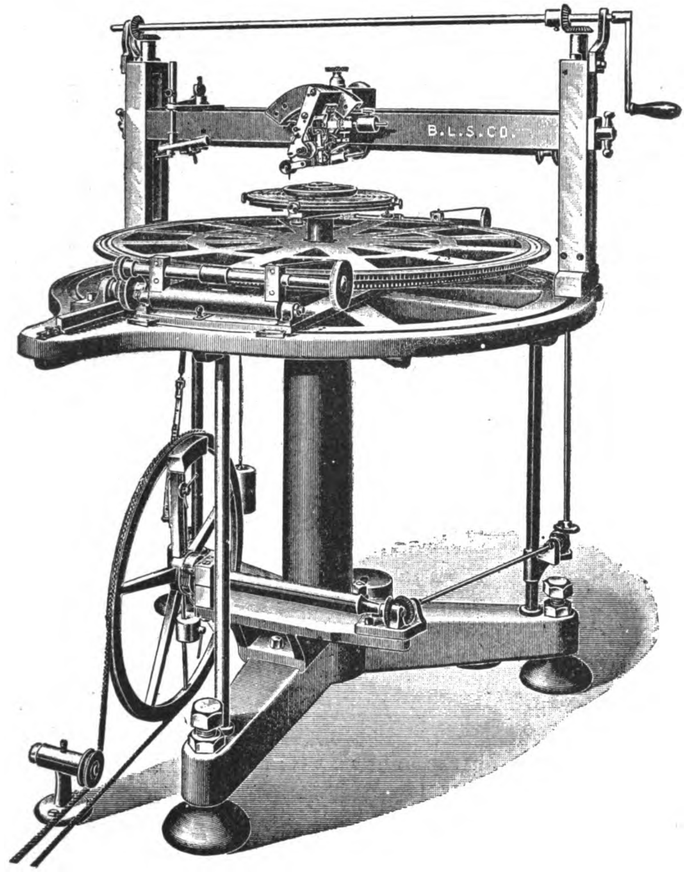


FIG. 38.

between division lines; this number is regulated by the motion of the rack between adjustable stops. The rack is driven forward by the belt wheel through the flexible pitman and adjustable crank pin, and pulled backwards by a weight, the ratchet pawls releasing promptly so that the worm screws are not disturbed. The belt wheel also drives the cutting tool at the top of the machine, acting through two cams, it being connected by bevel gearing in such a way as to allow of setting the cutting tool for any sized circle as shown. The cutting mechanism can be set to cut radially, vertically, or in an inclined direction.

The circle to be graduated is fastened to the upper plate and centered by the abutting screws shown, until a contact level attached to the cross-bar (pivoted at one end, with a stiff vertical leg pressed against the side of the hole for the axis of the circle by the unbalanced weight of the level) remains horizontal during rotation. The cutting tool, having a fine steel or diamond point, cuts during the return stroke of the rack; an elliptic gear giving a slow cutting and a quick return stroke.

The large circle has five-minute division lines, and it is assumed that the worm gear was adjusted to correspond with these as nearly as may be. It was found, however, that there were several irregularities in the division lines. To eliminate these errors the worm gears were fastened to a plate resting on the bed plate and capable of a slight rotation about the axis of the circle. A ring carrying 360 radial adjusting screws was attached to the under side of the circle and a horizontal lever attached to the bed plate so that the short arm engaged with the worm screw plate and the long arm rested against one or more of the screw heads. With these screw heads adjusted in the circumference of a circle concentric with the axis, no motion of the worm screw plate would take place on rotating the circle. With these adjusted for the errors in the gear, motion is given to the worm gears independent of their rotation so that the new circle is accurately divided.

For small circles the dividing is done automatically. For large ones, a micrometer microscope (see § 44) is attached to the frame and the main divisions copied from the engine circle, allowing for its graduation errors by moving the hair of the microscope for each scratch. The automatic principle is then applied to the short arcs between these lines, and checking on them.

The method of graduating the engine circle is well brought out in a description of the reconstruction of the Coast Survey dividing engine in App. 12, *Report* 1879. The process of continual bisection was employed until the circle was divided into 256 parts. The equality of the arcs after each set of bisections was tested by setting two micrometer microscopes the proper distance apart and measuring the discrepancies with the micrometer screw. After finding the corrections for these divisions the worm screw was used in changing to the degree system of graduation, applying the corrections step by step.

With a 10-inch circle, an error of 0.0001 of an inch in a division or in centering will give an error of $\frac{0.0001}{5 \sin 1''} = 4.1$ seconds; showing the extreme accuracy necessary in centering and in graduating a circle which is to be read to tenths of seconds.

Five-minute spaces are usually the finest cut upon large circles, and 10-, 20-, or 30-minute spaces are the smallest upon smaller circles. Intermediate readings are taken with verniers or micrometer microscopes. The vernier is too well known to need a description here.

44. **Micrometer Microscopes.**—These are usually used in place of verniers when readings closer than about 5″ are required. Cross-hairs are attached to a frame which is moved through a box perpendicular to the microscope tube by an accurate micrometer screw working against spiral springs, as shown in Fig. 39.

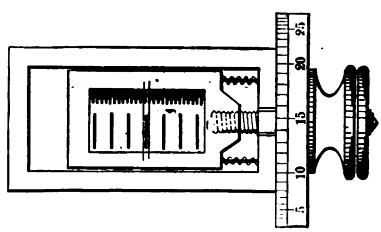


FIG. 39.

If the microscope has a flat field and the screw a uniform pitch the apparent motion of the cross-hairs across the limb will be proportional to the turns of the screw, giving an accurate means of subdividing the spaces on the circle. A common division of the limb is into 5-minute spaces, the objective being placed at such a distance that 5 turns of the screw will move the wires over one space; each turn will then give a minute, marked by a tooth on the comb in the edge of the field, as shown, while seconds can be read from the head of the screw by dividing it into 60 equal parts.

Two parallel hairs are usually used, placed far enough apart so that when brought over a division a bright line will show on each side between the hair and scratch; the equality in width of these light lines being judged more accurately than the bisection of a scratch by a single hair.

To take a reading, the micrometer screw is turned with the increasing numbers on the head, moving the hairs from the zero of the comb back to the first division of the limb to the right (apparent left), the number of teeth passed and the reading of the head giving the minutes and seconds from the division to the zero. Usually the motion of the screw is reversed, turning against the graduation on the head, until the hairs bisect the division to the left of the zero. Only the reading on the head is noted, and this should differ but slightly from the first if the microscope is adjusted so that 5 complete turns cover an average space.

The bisection is usually made with the positive motion of the screw to avoid lost motion. If the lost motion is appreciable, however, it indicates a poor condition of the micrometer which should receive attention.

The probable error of a single bisection should be about $0''.2$.

Ex. 1. Given a micrometer screw having 100 threads per inch, and an objective having a principal focal length of 1 inch. Find the distance to place the objective from a 10-inch circle so that one turn of the screw shall equal one minute, and find the focal length of the eyepiece for a magnifying power of 25.

Ex. 2. If, in Ex. 1, 5′ spaces on the circle read 5′ 04″ on the screw, find the distance to move the microscope towards the circle to put it in adjustment.

45. The Run of the Micrometer.—The micrometer is adjusted, as stated in § 44, so that the nominal number of turns, usually 5, will move the hairs over a 5-minute space. This can be only approximately realized owing to the imperfections of the micrometer and graduated circle, the inaccuracies of bisection and reading, and the disturbances due to changes in temperature.

The correction for run is made in several different ways by different observers, while many equally good observers regard it as an unnecessary refinement if the microscopes are kept in proper adjustment.

The method given in App. 9, *Coast Survey Report*, 1882, by Asst. R. D. Cutts, appears to be one of the most reasonable. In the angle record book (see § 57) the first and second readings are entered in separate columns and the means, for the two or three micrometer microscopes of the circle, set underneath for each reading of the circle. The mean of these two means is put in a third column and the difference, or the error of run, in a fourth. These errors of run are averaged for the station (or for as great a portion of the work at the station as they remain constant) for the error of runs. The error of runs as thus determined is freed from the inaccuracies of bisection and graduation of the lines on the circle and is due to non-adjustment of the microscope. The correction due to it is applied as a correction to the pitch of the screw, proportional to the length, while the inaccuracies of pointing and reading for the two scratches are averaged in the mean.

To find this correction, let a be the first reading; b, the second reading; r, the average run of the micrometer, positive when the first readings average greater than the second.

$$\text{Correction to } a=\frac{-r}{300''}a.$$

$$\text{Correction to } b=\frac{r}{300''}(300''-b).$$

$$\text{The mean, } m=\frac{a+b}{2}.$$

$$\text{Correction to } m=\frac{r}{300''}\,\frac{[300-(a+b)]}{2}.$$

$$\text{Correction to } m=\frac{r}{2}-m\frac{r}{300}. \qquad (21)$$

$$r=\frac{\Sigma(a-b)}{n}.$$

This correction has the same sign as r for $m<2'\ 30''$, and the opposite sign for $m>2'\ 30''$. Its value can be found from Table X with r and m or a as arguments.

See also the *Run of the Micrometer*, by George Davidson, *Coast Survey Report*, 1884, App. 8.

46. Errors of Graduated Circles.—These may be due to an eccentricity of the upper motion or inner axis with reference to the center of the graduation, or they may be due to errors in the division lines themselves. The error due to the plane of the circle not being horizontal when the axis of the upper motion is vertical, as indicated by the levels remaining in the center during rotation, is so small in an instrument in which the limb will remain flush with the vernier, or the micrometer microscopes in focus during rotation, that it can be neglected.

The error due to eccentricity is of more importance with instruments for ordinary surveying work than with those for geodetic or astronomical work, for with the latter all the microscopes or verniers are used in making a reading, and it can be readily shown that the mean of any number of equidistant verniers is free from eccentricity.

Let G be the center of the graduated circle; G', the center of the axis for the upper motion; GG', the eccentricity; $EE'=2R$, the line joining the centers; α', the angle AGE, made up of the index reading α and the micrometer readings A', B', C'; and e in degree measure

$$=\frac{GG'}{R \sin 1''}.$$

For two micrometers 180° apart:

From the 1st. $\alpha'=\alpha+A'-e\sin\alpha'$.
" " 2d. $\alpha'=\alpha+B'-e\sin(180°+\alpha')$
$=\alpha+B'+e\sin\alpha'$.

$\therefore$ Mean, $\alpha''=\alpha+\frac{1}{2}(A'+B')$, which is free from eccentricity.

For three micrometers 120° apart:

From the 1st. $\alpha'=\alpha+A'-e\sin\alpha'$.
" " 2d. $\alpha'=\alpha+B'-e\sin(120°+\alpha')$.
" " 3d. $\alpha'=\alpha+C'-e\sin(240°+\alpha')$.

By form. 8], $\sin(120°+\alpha')+\sin(240°+\alpha')=2\sin(180°+\alpha')\cos 60°$
$=-2\sin\alpha'\times\frac{1}{2}$
$=-\sin\alpha'$.

$\therefore$ Mean, $\alpha'=\alpha+\frac{1}{3}(A'+B'+C')$, which is free from eccentricity.

Similarly it can be shown that the mean of any number of equidistant micrometers will be free from eccentricity.

Some instrument makers put in radial abutting capstan-head

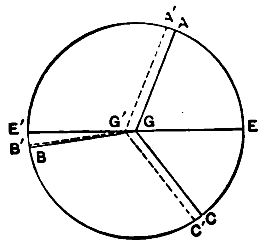


FIG. 40.

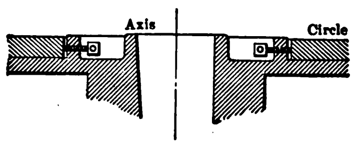


FIG. 41.

screws between the circle and hollow axis which supports the upper motion, so that the eccentricity can be adjusted out before the plate is screwed fast to the flange of the axis.

The graduation errors proper are divided into accidental and periodic. The former follow the law of errors of observation given in Least Squares, hence their effect is diminished as the square root of the number of lines used.

The latter occur at regular intervals according to some law, and may therefore be expressed as functions of the reading itself. The sum of all the corrections for periodic error, including those for eccentricity, must have the general form

$$\psi(\alpha) = \beta' \sin(\alpha + \gamma') + \beta'' \sin(2\alpha + \gamma'') + \beta''' \sin(3\alpha + \gamma''') + \text{etc.}, \quad (22)$$

where $\psi(\alpha)$ denotes the correction to the angle α, and β', γ', β'', γ'', etc., are constants. The shorter the period of any error, the higher is the multiple of α in the term representing it.

Chauvenet, *Astronomy*, Vol. II, p. 52, shows what terms are eliminated by taking the mean of a number of equidistant microscopes and how to determine the constants for a given circle by taking equidistant readings around the circumference. R. S. Woodward, *Report, Chief of Engrs. U. S. A.*, 1879, Part III, App. M.M., p. 1974, takes up the terms not eliminated by means of a number of equidistant microscopes and finds their effects upon a measured angle. He shows that if the distance between verniers is divided by the number of repetitions, and the circle is moved forward by this quotient each time, so that the initial readings are evenly distributed over the space between two microscopes, nearly all the terms will be eliminated from the mean. Also that the remaining terms tend to add up zero or eliminate as the number of observations increases, so that the effect may be neglected with a large number of observations.

In applying the formulas to some of the Lake Survey instruments (*Pri. Tri. U. S. L. Survey*, 1882) he finds periodic errors ranging from 1″.7 to 2″.

In Saegmuller's *Catalogue*, 1903, p. 7, comparisons of 10′ spaces 10° apart around the circle are given for six 8-inch circles made for the Geol. Survey. The greatest discrepancy for one instrument was found to be 1″.58, and for another 0″.55; those for the other four being between these limits. The discrepancies for these small arcs would indicate values for the accidental errors of graduation rather than for the periodic ones. On the basis of these Mr. Saegmuller makes the claim that his engine will graduate automatically to within 2″ to 3″, while if corrections are applied for the main divisions, and the automatic principle is used only for small arcs, each line will be correct within 1″.

47. Repeating and Direction Instruments.—The component parts have been quite fully described in the preceding paragraphs, and the French repeating circle in § 31. *For repeating instruments*, the circle is from 8 to 12 inches for primary triangulation, and from 5 to 9 inches for secondary and tertiary. The power of the telescope varies from about 60 to 20 with a diameter of object glass from about 2¼ to 1¼ inches. Two verniers or microscopes are common, and the upper and lower motions are the same as with the ordinary transit.

An 8-inch altazimuth, made by C. L. Berger & Sons, with repeating horizontal and non-repeating vertical circle, reading to 10″ and 20″ respectively, by verniers, is shown in Fig. 42.

To repeat an angle, the upper motion is set at the desired initial reading and the telescope pointed on the left hand object by the lower motion; it is then pointed on the right hand object by the upper motion, back to the left hand by the lower and to the right hand by the upper, etc., until the desired number of repetitions has been reached.

A 12-inch Coast Survey *direction instrument* is shown in Fig. 43, and a 50-centimeter one in Fig. 44, the former as made by the Bausch, Lomb, Saegmuller Co. Both have micrometer eyepieces.

The only essential difference between this and the repeating instrument is in the removal of the tangent screw for the lower motion, which prevents the use of the ordinary method of repeating angles; the object being to add to the stability of the circle Sometimes the lower motion is wholly removed, so that the circle

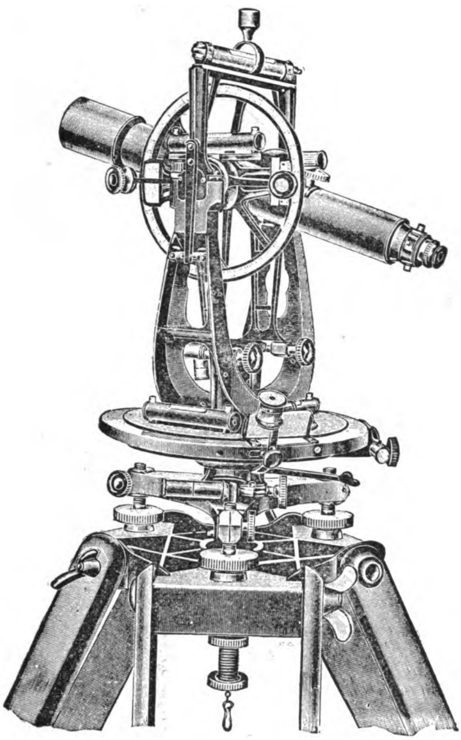


Fig. 42.

can only be rotated by motion below the leveling screws, but this arrangement is less convenient.

Rather larger circles are used than for repeating instruments for the same class of work, 15- to 18-inch circles being common, with about 8 inches as a minimum. Micrometer microscopes are used in place of verniers, 3 for the larger and 2 for the smaller circles.

The telescope can be made to transit, as shown in Fig. 42, in which case a vertical circle is added large enough to measure vertical angles, forming an altazimuth. Many observers, however, prefer short standards for greater stability, which requires that the telescope be taken out of the wyes for reversal, and often that vertical angles be measured with another instrument.

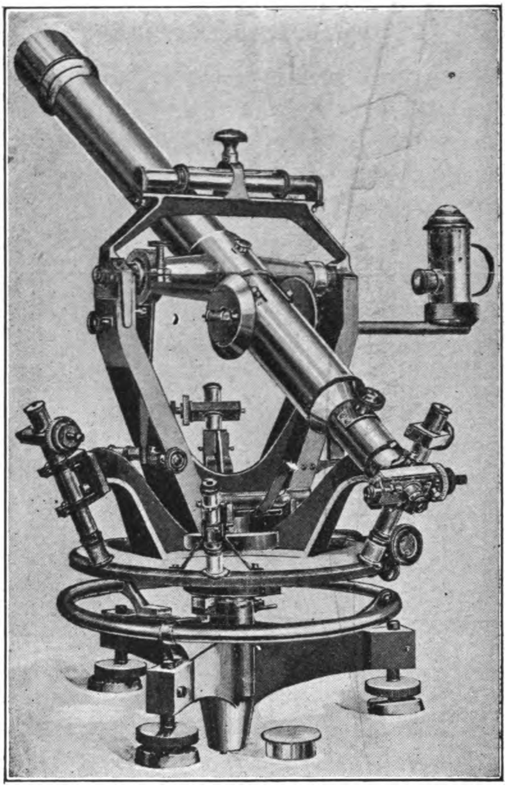


FIG. 43.—12-inch Coast Survey Theodolite.

48. Adjustments.—*Plate levels* perpendicular to vertical axis. These are adjusted by reversals as usual.

Line of collimation perpendicular to horizontal axis when the telescope is focused for parallel rays. Sight to a well defined distant point and clamp the horizontal motions. Reverse the telescope by carefully lifting it from the wyes and changing the ends of the axis. Adjust until the point is covered by the cross-hair, in both positions of the telescope.

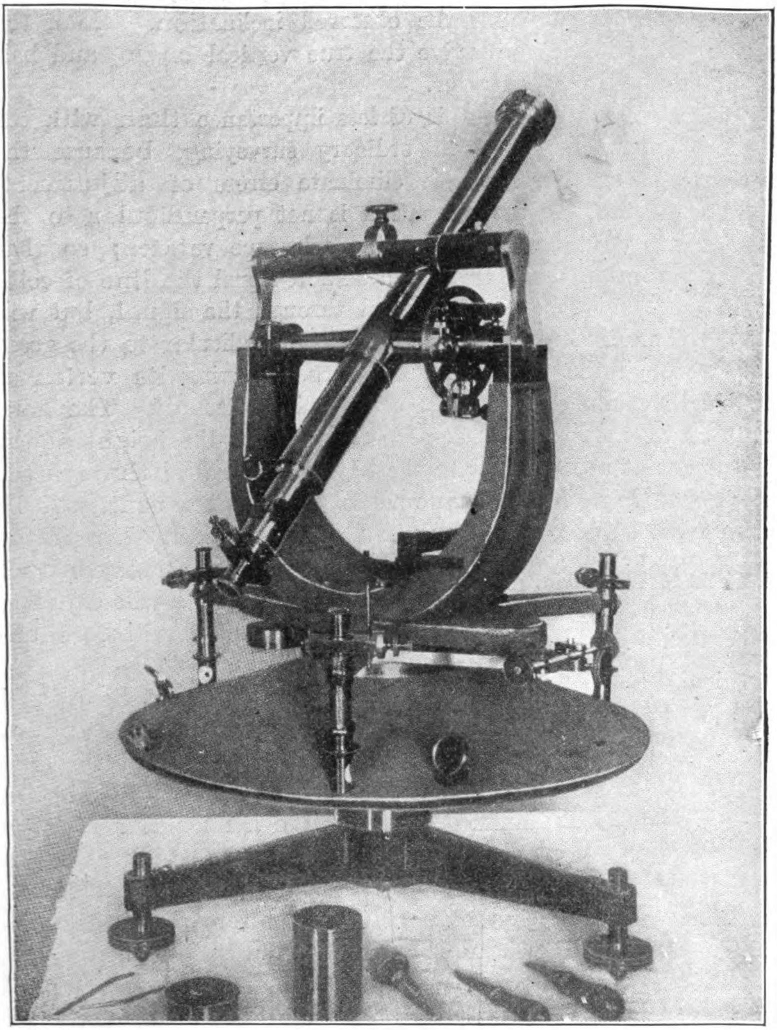


FIG. 44.—50-centimeter Coast Survey Theodolite.

Horizontality of Telescope Axis.—This can be adjusted by means of the striding level more accurately than by the method used for smaller instruments.

Index Error of Vertical Circle.—Take a reading with telescope direct and another with telescope reversed upon a well-defined point with bubble of reference level in the center, or the readings may be corrected for the observed inclination. Half the sum of the readings will give the true vertical angle, and half the difference the index error.

Accuracy of adjustment is of less importance than with the smaller instruments used in ordinary surveying, because the observations are arranged to eliminate errors of adjustment. Thus, if the line of collimation is not perpendicular to the axis, it will describe a cone as the telescope rotates; so that in plunging up or down through a distant signal the line of collimation will not follow the vertical through the signal, but will cut the plane through the vertical and perpendicular to the great circle through the points in an hyperbola having its vertex at the height of the instrument and its axis horizontal. The horizontal angle measured is then from a point at the height of the instrument at a distance x to the left of the vertical through the signal (see Fig. 45). Upon reversal the measurement will be taken from a point x' to the right of the vertical. But if the collimation error has remained constant and the axis is horizontal, x will equal x' and the error of collimation will be eliminated by taking the mean.

If the telescope axis is not horizontal when the plate levels are in the center, the plane of collimation through the distant signal will not be vertical but inclined, referring the horizontal angle to a point at a distance x to the left, as in Fig. 45. Upon reversal, the plate levels remaining in the center, the error will be the same but in the opposite direction. The mean will eliminate the error as before.

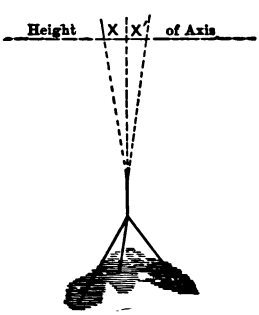


FIG. 45.

49. **Determination of Instrumental Constants.**—*Value of 1ᵈ of Level.*—Set up the instrument on a firm support where it will be protected from sudden changes of temperature, and place the level on the telescope with its axis in the vertical plane of the telescope. If the tube is chambered, take a bubble of about normal length. Move it by means of the vertical tangent screw

from one end of the tube to the other back and forth, setting at regular intervals in seconds and reading both ends of the bubble.

If the circle cannot be read closely enough, rod readings at a distance of 103.1 feet will give 2″ of arc per 0.001 foot on the rod.

If a level trier is available, it will be more convenient than either of the methods given above. Fig. 46 shows one with a 21-inch base having a micrometer screw graduated to 2″, as made by Berger & Sons. The wyes and grooves allow of supporting an unmounted level tube as shown, or a Y-level telescope or a transit without unmounting the level.

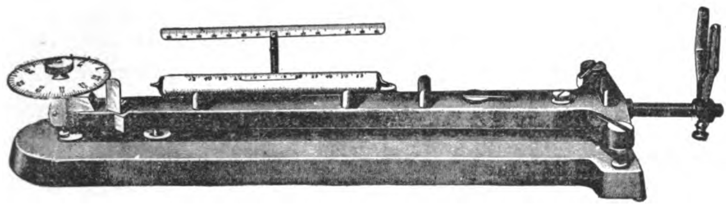


FIG. 46.

Value of 1^d of Micrometer Eyepiece.—If the screw is horizontal (which can be tested by noting that motion of the screw does not change the altitude of the horizontal hair), set the micrometer at a given reading and sight to a well defined point by the upper motion and read the circle; turn the micrometer, say 5 turns, and bring the hairs upon the same point by the upper motion, then read the circle; continue the process until the desired accuracy has been secured. The difference in the circle readings divided by the number of turns will give the value of one turn for the different parts of the screw. If the screw is vertical, the same method may be employed with the vertical circle if it is suitable.

A more accurate method involving more labor is by means of following a circumpolar star near upper culmination for the horizontal screw, or near elongation for the vertical screw with the circle clamped, depending upon the observed time intervals for the angles as described in Campbell's *Astronomy*, § 61.

Wire Intervals.—These may be determined by the methods given for 1^d of the micrometer.

The circle can be investigated by the methods referred to in § 46, while the methods for the telescope have already been given.

50. The Method of Direction Observations.—With the direction instrument it is customary to consider directions swung from a reference line, just as, with the compass, bearings are considered as swung from the meridian. Angles between the lines are not primarily considered in either case, although often used in the later computations. The method of direction observations is the one in most common use in this country. A reference line is taken, which may be the signal most easily seen under varying atmospheric conditions, or a mark set for the purpose at a sufficient distance to avoid changing focus (not less than 1½ miles).

The signals are sighted in order around the horizon in the direction of the graduation, beginning with the reference line, and the micrometers read for each; the telescope is then reversed, not changing the ends of the axis in the wyes if it has to be taken out for reversal, and the signals are sighted in the reverse order around the horizon, ending with the mark. This forms a set, and as many sets are taken as required.

The first signal each time should be approached with the telescope from the same direction as for the others in the half-set, so that the tendency of the circle to be dragged around by the friction of the upper motion will be taken up before the first reading. For each set the circle is shifted so that the readings for each signal are uniformly divided over the circle. In order to eliminate periodic error, as pointed out in § 46, the circle should be shifted each time approximately $\frac{360°}{mn}$, where n is the number of sets, and m the number of equidistant microscopes. Attention is called to the difference between two and three micrometer microscopes in reversing for a direction instrument. With two only two points of the circle are read, while with three six points are read, the reverse positions coming midway between the direct instead of coinciding with them. On this account it is better to shift the circle $\frac{360°}{2mn}$ for three microscopes rather than $\frac{360°}{mn}$. If the instrument has no lower motion, it is inconvenient to shift the circle after each set. The Coast Survey practice in such cases used to be to choose either 5 or 7 positions, equidistant 360°/5 or 360°/7, and to take an equal number of sets in each position, such that the total would give the required accuracy.

In setting upon the reference line, the zero of the micrometer should be advanced $1/n$ of the smallest division of the limb each time, in order to distribute the micrometer readings uniformly over the space. This will give a uniform division of the readings upon each of the other objects sighted, so that the average of the micrometer readings upon each object will be nearly the same, and the correction for error of runs for each angle will disappear.

The objections to this method of observing are thus stated in the *N. Y. S. Sur. Report* for 1887 by Mr. Wilson: "An objection to the method of directions is that it is very difficult, practically impossible indeed, to secure full sets upon primary points where the highest degree of precision is desirable and where broken sets are decidedly objectionable. In addition to this drawback to the method, another and very serious one arises from the length of time consumed in taking readings and bisections to several distant primary stations.

"When the theodolite is supported upon a high tower, as is frequently the case, the entire instrument is continually twisting in azimuth as the tower is subjected to the heat of the sun's rays. It is therefore of great importance that the intervals between sights should be as short as possible and that the two series in each set should be taken in about the same space of time. Frequently, however, one-half of a set may be taken in five minutes, while the other may require ten or fifteen." The broken sets are afterwards filled up by new sets, including the missing stations and the reference line.

51. The Method of Simple Angle Measurement.—In this the number of points in each series is reduced to the smallest possible number, or two. The angle between each signal and the reference line, or the angles between adjacent signals, can be measured independently. Or, the measurements can be so arranged that between n_1 stations $\frac{1}{2}n_1(n_1-1)$ angles will be measured. Thus starting with the first station as a reference line and swinging to the right to each of the others will give n_1-1 angles (Fig. 47); then from the second to each of the others to the right (not including the first) n_1-2 angles; then from the third, etc., to the n_1-1, from which only one angle is measured.

The sum of the series = first term plus last term, multiplied by one-half the number of terms, $=\frac{1}{2}[(n_1-1)+1](n_1-1)=\frac{1}{2}n_1(n_1-1)$, as stated above. This gives the same number of pointings, (n_1-1), upon each signal. Each angle is repeated the same number of times, and this number is taken large enough to give the required accuracy. To eliminate periodic error, the initial reading for each repetition of an angle is increased by $\frac{360°}{mn}$, as in § 50, m being the number of microscopes and n the number of repetitions of the angle. To reduce the effect of accidental circle errors, Schreiber, *Zeit. für Vermess.*, 1878, pp. 209–240, divides the arc between initial readings for the different repetitions of an angle $\left(\frac{360°}{mn}\right)$ by the number of angles, n_1-1, to be measured from the first reference station, and increases the initial reading for each new angle by this amount, starting from zero.

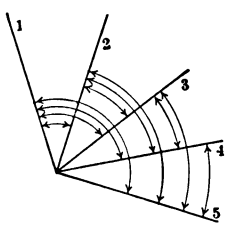


FIG. 47.

The initial readings for the angles measured from the other stations as initial lines are taken from the first set, using one each time which has not already been used with either of the lines forming the angle. An example of the settings at a station where six signals were sighted may be seen in *N. Y. S. Sur. Report*, 1887, p. 145.

This method requires the same number of pointings and readings as the preceding for two stations, $\frac{4}{3}$ as many for three stations, $\frac{6}{4}$ as many for four stations, etc., *provided* the visibility of the signals will allow of always taking full sets by the first method. For long lines, as in primary triangulation, these ratios will be less owing to imperfect sets by the first method, while if the delays in waiting for signals to show in order to complete sets are taken into account, the advantages will often be with this method.

Another advantage of this method is that angles can be measured whenever two signals are visible, provided atmospheric conditions are favorable, allowing more time to be utilized while in the field, and each signal to be sighted when under the most favorable conditions as to illumination and steadiness.

52. The Method of Repetitions.—The impression is quite general that this method will not give as good results as those with a direction instrument, described above, but the method has been a favorite one with many most excellent observers, and the results obtained have fully justified their preference. When the upper motion is always rotated in the same direction, errors due to twist of observing stand, drag of circle by friction of upper motion, travel of clamps, etc., are not eliminated by reversing the telescope, and the resulting angles will usually be too small, although sometimes too large. This is obviated by taking one-half the repetitions upon the angle, and the other half upon its explement, always swinging from left to right with the upper motion. Errors which tend to make the angle too small will thus also tend to make the explement too small, or the angle derived from it too large.

On the N. Y. S. Survey the practice was to take three repetitions of the same angle with telescope direct, reading the circle at beginning and end; then three repetitions of the explement with telescope reversed, still swinging the upper motion with the graduation, which is equivalent to "unwinding" the circle, i.e., the third repetition will bring the reading back nearly to the initial one. The explement thus only enters in the direction of the swing for the upper motion, and not in the figures recorded. They took six sets of six repetitions each for an angle, and the results with only an 8-inch circle were as satisfactory on primary work as with a direction instrument.

The angle from a reference line around to each signal can be measured; the angles between adjacent signals can be measured, closing the horizon and giving a sum of 360°; or the angles may be measured as shown in Fig. 47. In the first method (used on the N. Y. State Survey) the angles at a station are considered independent, in the second there is one check, while in the third a more complicated station adjustment is required. The initial readings for the different sets of an angle should differ by $\frac{360°}{mn}$ as usual, while if the angles are measured as in Fig. 47 the additional precaution can be taken of having no two readings alike upon the same signal.

53. Conditions Favorable for Observing. — To support the instrument tripod, or stand, three solid posts are set in the ground vertically some two feet with tops level, one for each tripod leg, and well tied together and braced by nailing on boards. The dirt is then tamped around the posts and the center often filled with stone. When an elevated observing stand is used (see § 27) the tripod or inner tower supports the instrument directly without its tripod, and the outer tower the observer. In all cases the height of the instrument should be such that the observer can look through the telescope when standing erect comfortably.

Some observers use a more or less portable observatory for the protection of the instrument from sun and air currents while observing, but the more common practice is to use a tent for primary and secondary work, and an umbrella or other simple shelter for tertiary. The tents used on the N. Y. S. Survey were octagonal for ground stations and square for elevated observing stands, both 8 feet in diameter, with walls 6 feet high, and made of 8-oz. duck. They were supported by 8 poles, one being in the center of each side for the square tent. The wall was in one piece, supported at the top by small pockets which slip over the tops of the poles, with a flap one foot wide at the bottom to tack to the floor to shut out the wind and dust, and a triangular-shaped door large enough to admit instrument boxes as well as the observers. The top was in one piece, held up in the center a foot above the eaves by a rope attached to a small thimble sewed on the outside, with flaps about a foot wide at the eaves which were strapped to the walls. Guy ropes extended from near the tops of the poles, to pegs if on the ground, or to the railings or other parts of the observing stand if elevated. Floor space is better economized by placing the tent eccentric over the station on account of storing instrument boxes, etc. Care should be taken not to obstruct lines of sight by tent poles.

The walls can be lowered a foot for observing, or a window one foot wide can be cut around the tent at the height of the eye or telescope and covered by a flap on each side when not in use.

Sheets of 8-oz. duck are sometimes used on two sides of an elevated observing tower to protect the inner or instrument stand from the wind to prevent vibration, or from the sun to prevent station twist, the exposed stand having a tendency to rotate in azimuth with the sun during a bright sunny day and to return at night.

The best time for observing is on a day when the sky is overcast; next to this is a calm, pleasant, late afternoon; evenings from about an hour after sunset until about midnight are also favorable.

The hours for observing on the Coast Survey in the summer season used to be from sunrise until 8 A.M. and from 4 P.M. until sundown. Vertical angles were measured from 12 M. to 1 P.M., and in the afternoon until within an hour of sundown. On the 98th meridian triangulation the observing hours were from 3 to 11.30 P.M., sighting on heliotropes and acetylene lamps for horizontal measurements, and from 11.30 A.M. to 4.30 P.M. for vertical measurements. No attempt apparently was made to distribute the observations over different days.

Lines of sight passing close to the surface are most disturbed by heat waves and other atmospheric disturbances, producing the appearance in the telescope often described as "boiling." Lines over furnaces and cities are objectionable, while those over bodies of water are not usually so clear as those over land; high lines are least affected by atmospheric disturbances.

For the best results, and especially with pole signals, the readings for an angle should be distributed over different days or divided between forenoon and afternoon, to equalize the effects of lateral refraction, side illumination of signals, etc. No readings should be taken under any improper conditions of the atmosphere, as shown chiefly by the appearance of the signals. The instrument should be handled with a light touch and with a certain degree of rapidity, yet in completing a pointing it should be done carefully and deliberately, without worry or bias as to the result, watching the signal long enough to be certain that it is really in the line of collimation and not temporarily there due to parallax or a sudden change of refraction either lateral or vertical.

54. Coast Survey, Instruments, and Observing.*—For *primary work* direction instruments are in general to be used. Repeating instruments are used when the station would be difficult of occupation with a direction instrument, or when the movements of the observer might disturb the instrument in azimuth, as on light-houses or buildings.

* From Instructions for Triangulation; see § 8.

The method of direction observations is used in observing with sixteen sets, the initial readings increasing by 15° 00′ 10″ in groups of three and by 18° 56′ 20″ between the groups, starting with about 0° 00′ 40″, and ending with about 237° 04′ 20″, except that the interval is increased 10″ whenever necessary to avoid whole minutes in a setting.

Broken sets are filled out by observing the missing signals in connection with the original reference line, or with a line to some other one station, and only one, of those already used in that set. Little time is spent in waiting for a doubtful signal to show, full sets being of less importance than economy of observing.

In observing at or upon supplementary stations, only four sets of readings are taken, using the first four positions of the circle. For intersection stations, one set is required and a second taken if it can be done under favorable conditions without much delay to observations in the main scheme. Each set on intersection stations is to contain one and only one of the main scheme or supplementary stations. It is considered important to have at least three lines to each intersection station for a check, but a station is not to be neglected simply because only two lines to it can be secured.

For a repeating instrument, six repetitions make a set; they are taken with telescope direct and followed by six repetitions of the explement with telescope reversed. Ten sets are used, five on the angle and five on the explement, for stations of the main scheme, measuring the single angles between adjacent lines of the primary scheme and closing the horizon.

For supplementary stations, four sets of six repetitions each are made, with no check angles except that required in closing the horizon.

For intersection stations, two sets of three repetitions each, one on the angle and one on the explement, are used, swinging from some line of the main scheme or from a supplementary station.

For *secondary work* an 8-inch direction instrument, used on its own tripod and protected from sun and wind by an umbrella is recommended. Five positions of the circle are used, the initial readings starting with 0° 01′ and increasing by 72° 02′. The minimum number of measurements is five, one in each position, and the maximum number ten, two in each position, unless in special cases more are necessary for the desired degree of accuracy. Only the additive readings of the micrometers (turning with the increasing numbers on the head) are taken for each microscope. At least once a month special readings are taken to determine the run of the micrometer, and an adjustment made if over 2″. Broken sets are filled out the same as for primary.

If a repeating instrument is to be used, a 10-inch Gambey is recommended, on its own tripod and with an umbrella. The minimum number of observations is two sets of six repetitions each, and the maximum number four sets, unless in special cases a larger number is necessary to secure the required accuracy. The single angles are measured and the horizon closed the same as for primary work. If the failure of signals to show, prevents the carrying out of this scheme, as near an approach as possible is made and the remaining signals are then observed in a second series with one only of those observed in the first series, measuring the single angles between adjacent signals and the angle necessary to close the horizon.

For intersection stations one set is to be used for a direction instrument and two sets of three repetitions each for a repeating instrument, including one station of the main scheme in the set for the first case, and swinging from a main station for each intersection station in the second case.

For indefinite and temporary objects for topographic or hydrographic signals, the telescope is set on a signal of the main scheme with the circle reading about zero, the objects are then sighted in order of azimuth and the series closed with a reading on the first station. A reading with telescope reversed may be taken as a check when thought necessary, although it is not required.

For *tertiary work* two sets are required for an 8-inch direction instrument and from two to four sets of six repetitions each for a 7-inch Berger repeating instrument. The horizon is closed as usual. For intersection stations the telescope is set on a main station with the circle at or near zero and the signals are observed in order of azimuth, closing with the starting line; the telescope is then reversed and the signals read in the reverse order, starting and closing with the reference line. For indefinite or temporary objects the method is the same as used in secondary woık.

55. Coast Survey, Vertical Measurements.—For *primary work* vertical measurements are made for all lines of the main scheme, and the readings are taken on as many days as possible during the occupation of the station, but the occupation is not prolonged in order to secure such measurements. Two measurements per day, each with telescope direct and with telescope reversed, are all that are required. These may be made between 11.30 A.M. and 4.30 P.M., but on account of the diurnal variation in refraction it is desired to have them measured at the same hour each day, as, e.g., between 3 and 4 P.M. If made by the micrometric method (by measuring differences of inclination to the different signals with the screw of the micrometer eyepiece, the telescope remaining fixed), double zenith distances are measured on at least two of the lines radiating from the station.

For supplementary and intersection stations, two measurements (each with telescope direct and reversed) over every line of which the horizontal direction is measured are all that are required, except when the observations upon such stations are for the purpose of connecting with precise level or tidal bench marks. In this case the rules for the main scheme are applied, and readings are taken at both ends of every line more than 5 km. long, even though horizontal measurements are necessary in one direction only.

For secondary and tertiary work, if the triangulation is inland or along a coast where prominent hills are determined by intersections and the elevations are desired, a continuous series of vertical angle measurements is carried through the main scheme and connected with accurately determined elevations as frequently as possible. For the main scheme, two measurements, each with telescope direct and reversed, on each day of occupation are all that are required. For intersection stations and indefinite objects, one measurement, telescope direct and reversed, is all that is necessary if taken from each station from which the horizontal measurement is taken.

56. Accuracy of Results.—The limit adopted by the Coast Survey in closing triangles is about 3″ for primary, 8″ for secondary, and 15″ for tertiary, the average closures being a little over 1″, from 2″ to 3″, and from 4″ to 5″, respectively. For primary work these values apply to the main scheme, while for secondary and tertiary they apply to the supplementary triangles as well. In selecting the conditions under which to observe, and the methods, instruments, and signals, speed and economy are to be considered rather than greater accuracy than the above. For the main scheme in secondary work observations are to be taken at any time of day when a moderately definite object to point upon can be seen. For the intersection stations and for tertiary work no delay is to be incurred to secure good seeing.

These standards of accuracy used in connection with the strength of figures and frequency of bases of § 10 and § 13 will in general insure a probable error of a base line as computed from an adjacent base of about 1/88 000 for primary, 1/35 000 for secondary, and 1/20 000 for tertiary triangulation; with actual discrepancies between bases less than 1/25 000, 1/10 000, and 1/5 000, respectively.

On the N. Y. S. Survey the observing party took the precaution to adjust the observations at a station while still in the field, in order that extra sets could be taken, or defective ones repeated, in case some of the directions did not show sufficient accuracy. The limit for the mean square error of a direction was placed at $0''.5$ for primary work and $1''.0$ for secondary and tertiary.

57. Forms for Record.—These for both repeating and direction instruments, as used on the Coast Survey, are given on p. 92.

With the repeating instrument, if the angles are measured from a reference-line, the pointing and reading in closing the set for the zero initial reading for the first angle can be used as the zero initial reading for the second angle, etc. If the angles between adjacent lines are measured, the reading can be saved by using the lower motion for the initial pointing for the next angle.

The form given for the direction instrument is for a limited number of positions, rather than for a change in the position of the circle after each set.

FORM OF RECORD FOR REPEATING INSTRUMENTS.

Station , Date , Observer , Recorder , Inst.

Station.	Time.	Tel.	Rep.	A.	B.	C.	Mean.	Angle.	Mean of D and R.	Remarks.
Mt. Equinox. .	7h 05m		0	30° 04′ 13′	04′ 35″	04′ 52″	04′ 33″.3			
			1	100 30 20				70° 26′ 15″.2		
Greylock.		D	3	241 23 10	23 35	23 12	23 19. 0		70° 26′ 16″.1	
								17. 0		
		R	3	30 04 20	04 24	04 40	04 28			

FORM OF RECORD FOR DIRECTION INSTRUMENTS.

Station , Date , Observer , Recorder , Inst. , Position IV.

Series and Number.	Object Observed.	Time.	Tel.	Mic.	Circle.			Dif. r.	Mean.	Cor. for Run.	Cor'd Mean.	Remarks.
					° ′	d	d					
IV.	Azimuth			A	271 31	49.0	50.2					
21	Mark	8h 43m	D	B		55.0	55.0					
				C		60.2	62.0					
						54.7	55.7	−1.0	55.2			
22	Manomet	47	D	A	35 26	48.4	48.2					
	Hill			B		50.5	52.8					
				C		56.0	58.0					
						51.6	53.0	−1.4	52.3			

58. Phase.—In bisecting a bright, reflecting cylindrical signal, seen against a dark ground in sunlight, the apparent center will usually be on one side of the true one, owing to *phase.*

Let r = radius of the cylinder; α = the angle between sun and signal (measured at the observing station at the time of the observations); D = the distance to the station, β = the correction to the angle in seconds.

(*a*) Pointing made upon the bright reflecting line.

$$\sin\beta=\frac{r\sin(90^\circ-\frac{1}{2}\alpha)}{D},$$

or

$$\beta=\frac{r\cos\frac{1}{2}\alpha}{D\sin 1''}\quad.\ .\ .\ .\ .\ .\ .\ .\ .\quad(23)$$

β being so small that $\sin\beta=\beta\sin 1''$.

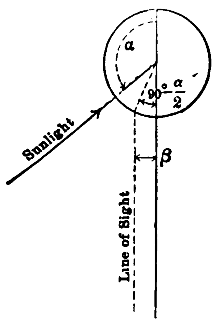


FIG. 48.

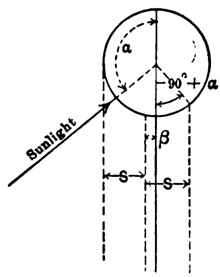


FIG. 48*a*.

(*b*) Pointing made by bisecting the illuminated portion.

Bisect the angle $2S$, subtending the illuminated portion as seen by the distant observer for the line of sight.

Then $\quad\sin(S+\beta)=(S+\beta)\sin 1''=\frac{r}{D},$

$$\sin(S-\beta)=(S-\beta)\sin 1''=\frac{r\sin(\alpha-90^\circ)}{D}$$

$$=\frac{-r\cos\alpha}{D}.$$

Subtracting,

$$2\beta \sin 1'' = \frac{(1+\cos\alpha)r}{D} = \frac{2r\cos^2\frac{1}{2}\alpha}{D}, \text{ by form. 11]}.$$

$$\therefore\ \beta = \frac{r\cos^2\frac{1}{2}\alpha}{D\sin 1''}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

59. Eccentricity.—The signals during the measurement of angles should be carefully watched, and if at any time found out of center the amount and direction with reference to one of the sides should be measured and the date noted. By plotting this data to a large scale and laying off the lines to the other stations with a protractor, any perpendicular can be scaled with sufficient accuracy.

If e = perpendicular distance from the signal to the line joining the stations,

Correction for eccentricity in seconds,

$$\gamma = \frac{e}{D\sin 1''}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

which will apply to each line whether the eccentricity be that of signal or instrument. A sufficiently accurate value of D can be found by solving the triangles with the approximate angles.

If a numerical solution is preferred for e, let d = distance of signal or instrument from the station; α = angle between the lines d and D.

$$e = d\sin\alpha.$$

Substituting in (25),
$$\gamma = \frac{d\sin\alpha}{D\sin 1''}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

Ex. At Pt. Figuras Lighthouse the instrument was set up 2.112 meters from the center of the station and the following readings taken to find the correction for eccentricity for each of the lines:

Pt. Figuras L. H. $d = 2^{m}.112$. $\text{Log}\left(\frac{d}{\sin 1''}\right) = 5.6391$.

Stations.	α.	log sin α.	log D.	$\log\frac{\sin\alpha}{D}$.	log γ.	γ.
Center.	0° 00′.0					
Curazon.	142 05 .3	9.7885	3.7976	5.9909	1.6300	+42″.66
Lebron.	235 23 .6	9.9154_n	4.0383	5.8771_n	1.5162_n	−32 .83
Mala Pasqua	252 12 .6	9.9787_n	4.1600	5.8187_n	1.4578_n	−28 .69

CHAPTER IV.

BASE-LINE MEASUREMENT.

60. Early Forms of Apparatus.*—The Lapland base in 1735 was measured with wooden rods laid directly upon the ice on the river Torneo. Four rods were used with end contacts, giving less danger of disturbance in moving a rod forward and placing it than with only two. Later, in measuring the Hounslow base of the English Ordnance Survey it was found that although the pine rods were thoroughly seasoned and soaked in boiling linseed oil there was an appreciable change of length with change of moisture and their use was abandoned. The base was then measured with a chain supported its entire length and stretched by a weight over a pulley, and with glass rods. Glass has a small coefficient of expansion, but its volume change lags behind the temperature change so that it is not suitable for base bars.

Borda apparatus (French), Fig. 49. This apparatus, referred

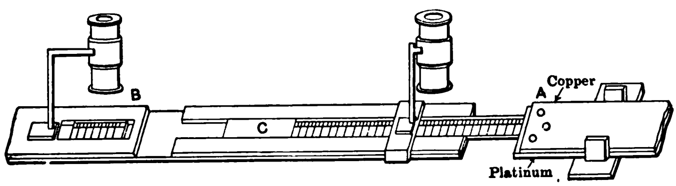


FIG. 49.

to in § 2, was designed to do away with the difficulty of finding the temperature and change in length of a metallic bar from mercurial thermometers during rapid changes by measuring the

* For more complete descriptions, see Clarke's *Geodesy* and Francoeur's *Géodésie*.

actual difference in expansion of two metallic bars. The components were platinum and copper, the platinum with a small coefficient of expansion for the measuring bar and the copper with a large coefficient placed above for temperature or differential expansion. They were fastened together at the rear end as at *A* and the difference in expansion measured at the front end by a graduated scale on the copper and a vernier on the platinum as shown at *B*. Contact was made by the slide *C*. The scales were read by microscopes. There were four base bars, each two toises ($=3.^{m}898$) long. In use they were sheltered by a flat board cover just above the metal bars and were supported by blocks at about the quarter points. These blocks had sharp pins below to hold in the ground and screws above to adjust for elevation.

Struve apparatus (Russian). In this an effort was made to obtain the temperature of the bar with mercurial thermometers by placing the bulbs in contact with the metal and slowing down the rapidity of the temperature changes. This was accomplished by wrapping the simple iron bar in cloth and raw cotton and letting the thermometer bulbs, one near each end, into the body of the bar. No correction is required for "contact" as with the Borda apparatus, and a constant pressure between the surfaces is secured without danger of disturbance of the rear bar, the spring yielding as the contact end of the contact lever presses against the other bar until the long arm reaches the zero of the scale shown in Fig. 50.

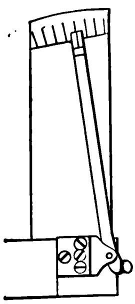


Fig. 50.

Offsets to the ground are made with a transit set up at right angles to the line and 25 feet distant, the position being held over night by a cube moving in a slide on the top of an iron pin driven 2 feet into the ground. The cube has a fine line cut on it for bisection.

Bessel apparatus (German), Fig. 51. In this apparatus a zinc bar *Z* and an iron one *E* are fastened together at the rear end of the measuring unit as in the Borda system. The contact is made and the differential expansion is measured by the insertion of glass wedges between the steel knife edges *St*. These wedges are graduated in ordinates increasing by intervals of 0.01 Paris line=0.00089 inch.

Colby apparatus (English), Fig. 52. The components, brass and iron, are used to compensate for temperature, and not to

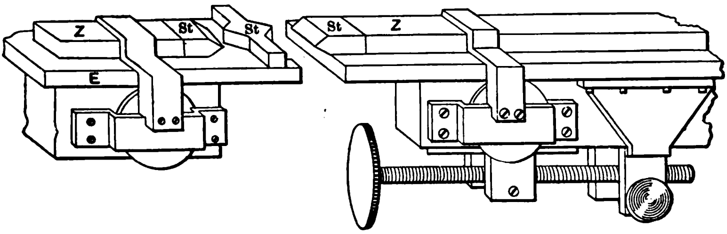


FIG. 51.

measure expansion as with the Borda and Bessel apparatus. The bars are placed side by side and fastened at the center as shown.

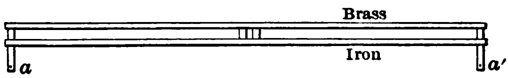


FIG. 52.

The microscopic dots, a, a' on the compensating levers remain fixed for equal changes of temperature in the two rods. These dots are on the side of the case so that the microscopes of Fig. 53

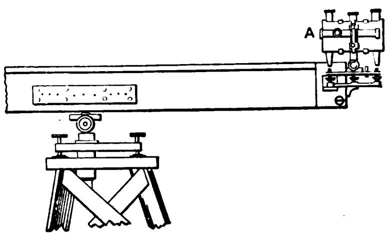


FIG. 53.

can be placed over them, one over its dot directly, the other over the dot of the other bar by pushing the bar back for "contact." The common axis serves as a telescope tube for transfers to the ground, its verticality being indicated by the attached level. The telescope shown at A serves to align the microscope case. The upper plate connecting the microscopes is brass, the lower iron, compensating the distance between the focal points. The bar is 10 feet long and the microscopes 6 inches apart.

Ex. 1. Find the units of the Borda scale, Fig. 49, such that an increase of one in differential expansion shall indicate an expansion of 1^{μ} per meter for the measuring component. Length for differential expansion assumed = 3.8^{m}. For coefficients of expansion see Table I.

Ex. 2. Find the error in the computed length of the Bessel (2 toise) base apparatus due to a difference of 1° in the temperature of the two components.

Ex. 3. Find the length of the compensating levers of Fig. 52 for a distance of 3 inches between the two components.

Ex. 4. Find the angle for the Bessel wedge of Fig. 51 for divisions 0.05 inch apart.

61. Bache-Würdemann Apparatus.—(*Coast Survey Rep.*, 1873, App. 12.) Fig. 54. Length 6^{m}. This, like the Colby, is a compensating apparatus. The two component bars, brass and

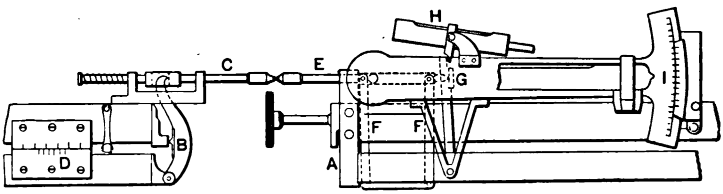


FIG. 54.

iron, are rigidly attached at the rear end to the block *A* and supported throughout their length by rollers, while the front ends are connected by a compensating lever *B*. The bars and rollers are inclosed in a trussed case.

The contact rod *C* projects through the end of the case, while the Borda scale *D* can be read through a window in the side. The contact rod *E* at the rear end is held in position by the parallel levers *F*, *F*, pivoted at the bottom of the brass bar. Its inner end knife edge rests against the cylindrical surface *G*. By bringing the base bar back through the case with a tangent screw (after the bubble of the level sector has been brought to the center), the contact rod, *E*, resting against the rear bar, the surface *G* is forced forward, bringing the bubble of the contact level *H* to the center for contact. When in this position the axis of the cylinder *G* is the axis of the level sector *I*, so that inclining the bar for slopes does not disturb the contact distances or level so long as the level sector tube remains horizontal.

The cross sections of the brass and iron components are so arranged that, while the two have equal absorbing surfaces, their masses are inversely as their specific heats, allowance being made for their different conducting powers. Both surfaces are varnished to give equal absorbing power, and the whole is protected by a double spar shaped tin case painted white to prevent rapid changes of temperature.

The heads of the supporting metallic tripods are adjustable vertically, laterally, and longitudinally, the motions for the rear one being controlled by rods running to the contact man at the rear of the bar. Each tripod leg is adjustable by rack and pinion and by foot screw. These foot screws rest in grooves in a cast-iron bearing plate placed on the ground.

The end of a bar is transferred by a transit at right angles.

Ex. If the 2-second divisions of the contact level of Fig. 54 are 0.1 inch apart, and the inclination can be read to 0.1 div., find the sensitiveness of contact for a 3-inch arm for the contact level and a 0.9 ratio for the lever *G*.

62. Porro Apparatus (French).—(Fig. 55.) In this a return is made to the method of measurement with chain and pins, the

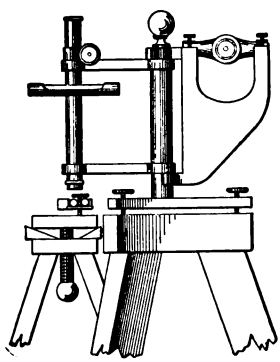


FIG. 55.

base bar taking the place of the chain, and four microscopes with very firm supports that of the pins. As originally designed, the rod was made of fir, varnished and incased in a copper tube; but as soon modified, the fir was replaced by two metals, forming a Borda thermometer.

The microscope (Fig. 55) has two objectives, one for plumbing over a point on the ground and the other for sighting at the bar, a cap with a central opening shutting off the light which does not pass through both when looking at the bar.

The telescope of the rear stand is used for alignment by sighting along the line at an offset target and then aligning the front stand, a scale taking the place of the front telescope axis.

63. Repsold Apparatus.—(*Pri. Tri. U. S. L. Survey*, p. 133.) This is of the Porro type. The components, steel and zinc, are placed side by side in a 4-inch iron tube; they are fastened at the center and are free to expand each way upon rollers; their ends are cut away to the neutral axes and graduated platinum plates are attached. In measuring, the micrometer microscope is set upon the zero of the steel bar for contact and a reading taken upon the nearest division of the zinc bar for expansion.

The tube stands or tripods are placed at the ends of the bar or tube, so that the front one for the first position becomes without disturbance the rear one for the second position, etc. The legs end in foot plates which rest on iron pins driven into the ground. The heads are adjustable vertically, laterally, and longitudinally. The tube is lengthened by a bracket at each end, the rear bracket resting on a knob in the center of the tube stand head, while the front bracket carries two rollers, one V-shaped, which rest on tracks on the tube stand head. The microscope stand is placed opposite the tube stand, a long bracket supporting the microscope over the end of the bar. A plank supported at the ends is used by the observer in reading the microscope, so as not to disturb the ground near the legs.

The bar is aligned by a telescope on the tube and its inclination measured with a level sector.

To set a microscope over the starting point, the tube stand head is removed and a telescopic tube called a cut-off tube placed over the rock crystal knob marking the end of the base. The tube is made vertical by an attached level and the microscope set on the zero of a horizontal scale at the top, a direct and reverse reading eliminating any index error of the scale. The tube is then removed and the end of the base bar brought under the microscope. At the end of a day's work the cut-off tube is used to transfer to a cast-iron knob, held by centering screws to a large cast plate supported by stakes.

64. Ibañez Apparatus.—(Spanish. *Engrg. News,* March, 1884, p. 133.) Fig. 56. This is an outgrowth of experience in Europe with the complicated forms due to the use of the Borda thermometer for temperature or compensation. The bar is a 4-meter 110-pound

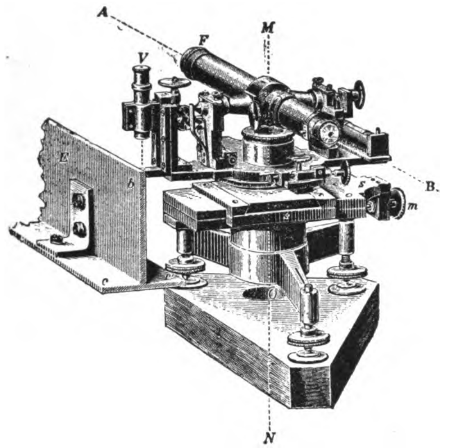


FIG. 56.

iron $\perp$ bar without case or cover. Marks are engraved on small platinum disks at points $0^m.5$ apart, while four mercurial thermometers with bulbs encased in iron filings are attached.

Underground monuments are set in advance, dividing the base line into daily stints, and no transfers to the ground are allowed at other points. Dependence is placed upon rapid continuous work (160^m per hour, Aarberger Base) between these points, and the use of a shelter tent for freedom from errors due to instability and to temperature changes.

In starting, the telescope F is replaced by one having its axis near the object end so that it can be made vertical and set over the monument at N; F is returned and sighted to a target on the line at A; the next microscope stand is set up 4^m ahead and a target at M, taking the place of the telescope axis, is brought into line by sighting through F; its aligning telescope is replaced

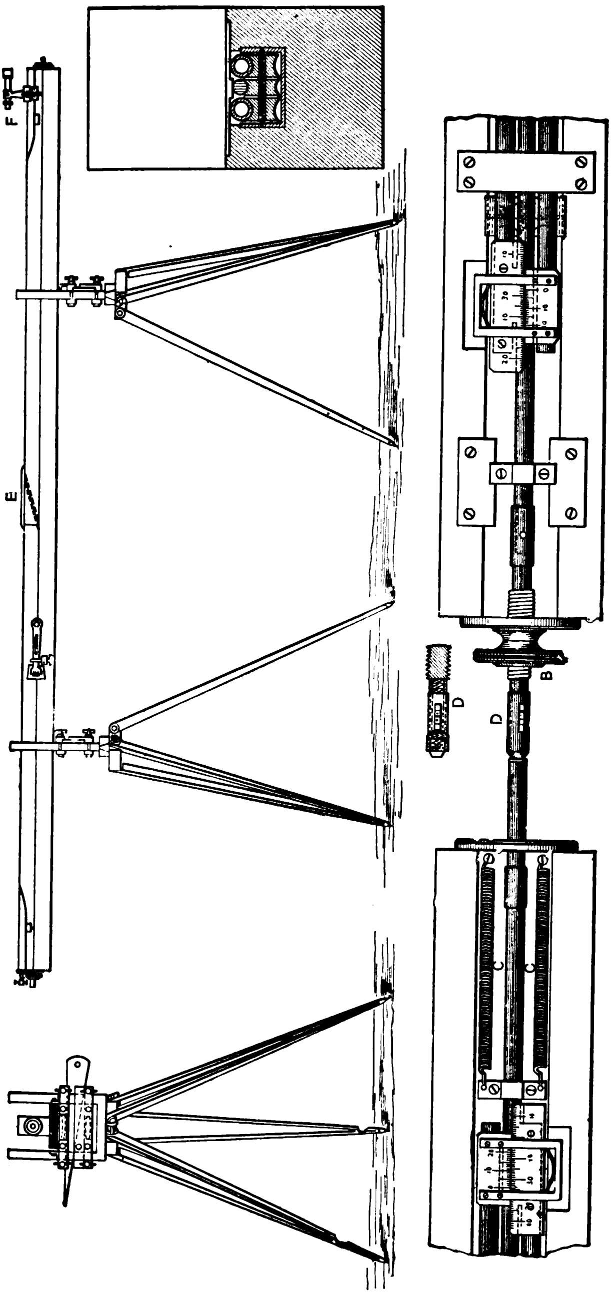


Fig. 57.

and sighted to the target ahead; the bar E is then brought under the microscope V, the dot b at the rear end being accurately bisected, while the front microscope is moved longitudinally on the slide S to bisect the dot at the front end; the third stand is set up like the second and the bar moved forward. When a monument is reached, a stand is set over it as in starting, the bar put in position and a $0^m.5$ scale used to measure the distance from the microscope to a dot on the bar.

65. Coast Survey Secondary Apparatus.—(*Rep.*, 1880, App. 17.) Fig. 57. The construction is clearly shown in Fig. 57 from Saegmuller's Catalogue. The measuring rod is steel 4^m or 5^m long. The outside tubes are zinc, one fastened to the steel at the rear with its Borda scale at the front, the other at the front with scale at the rear. Each scale is read by a magnifying glass at the top of the case. Contact is made by drawing the front rod back through the case by the tangent screw B, the spiral spring tending to push off the contact slide D, which yields to the pressure of the rear bar until the scratch on the slide coincides with that on the index attached to the bar. The tangent screw B works against the springs C at the front end of the bar. The mercurial thermometer E is attached to the case and its bulb is not in contact with the bar. The case is a pine joist about $3'' \times 8''$. The tripods are mainly of wood; the cross-bars can be clamped to the standards at any height.

The alignment telescope is at F.

With the College bars the Borda readings have been abandoned as unsatisfactory; the case has been covered with hair felt and canvas, and the thermometer has been replaced by two near the quarter points with their bulbs in close contact with the steel bar and surrounded by iron filings.

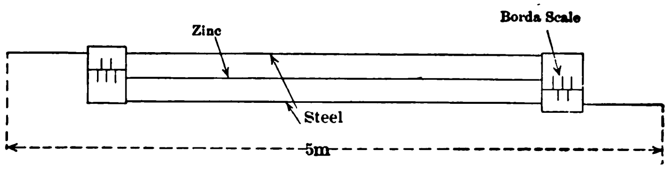


FIG. 58.

66. Coast Survey Gridiron Compensating Apparatus.—(*Rep.*, 1882, App. 7.) Fig. 58. For compensation the expansion of the steel must balance that of the zinc for equal temperature changes of the two components. The details of the secondary apparatus, § 65, are elaborated for the case, contacts, and tripods.

Ex. 1. Find the lengths of the components of Fig. 58 for a base bar 5^m long.

Ex. 2. Sketch the construction and find the lengths for a brass and steel combination 6^m long.

67. Coast Survey Duplex Apparatus.—(*Rep.*, 1897, App. 11.) Fig. 59. This apparatus is made up of two separate bars with contact slides, a steel tube and a brass one; the bars are placed

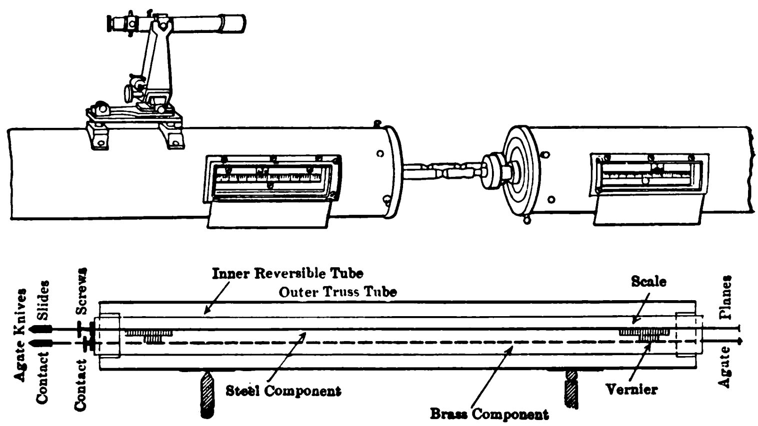


FIG. 59.

$1\frac{1}{8}''$ apart in the brass tube, which can be rotated 180° about its axis in the outside supporting steel tube. In use, double contacts are made, steel to steel and brass to brass, the accumulated differential expansion showing itself by the movement of one rod upon the other as noted by reading the vernier and scale at each end at the beginning and end of the measurement of a section.

About two reversals, or rotations of the tube, are required per day, arranged symmetrically as to rising and falling temperature and so as to have the same number of bars placed in each position.

The outer tube is covered with felt and canvas, and the bars are used under a portable tent drawn by a team as the work proceeds. In the later work (*Rep.*, 1901, App., 3) no shelter is used, a speed of fifty 5-meter bars per hour is claimed to be easily maintained with a well trained force, and a record of one hundred bars per hour has been made with this apparatus.

68. Coast Survey Iced Bar.—(*Rep.*, 1892.) Fig. 60. This apparatus was designed to avoid the temperature difficulty by surrounding the bar with melting ice when in use and thus securing a fixed temperature. The bar is of steel, 5.02^{m} long, with a section 8^{mm} by 32^{mm}, or $\frac{5}{16}$ by $1\frac{1}{4}$ inches. The upper half is cut away at the ends and platinum-iridium graduation plugs inserted with their upper surfaces in the neutral surface of the bar. One line transverse to the length of the bar and two longitudinal lines

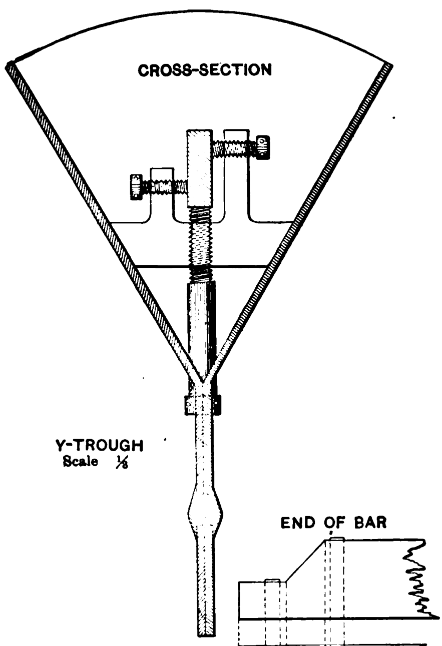


FIG. 60.

are ruled on each plug. These transverse lines establish the ends of the measure. Eleven German-silver plugs, projecting slightly, are inserted in the upper surface for alignment vertically by means of a striding level and horizontally by stretching a fine wire along the top.

The bar is supported on saddles 0.5^m apart. The lateral adjusting screws at the ends are opposite and at the height of the lower one shown in the figure. The vertical ones project through the trough, the holes around them serving for drainage. The alignment of the bar is made when the trough is loaded with ice. The ends of the trough are closed with wooden blocks and the whole is covered by a close fitting jacket of heavy white cotton felt. The weight is 82^{kg} (180 pounds) exclusive of the bar and ice. An ice crusher is considered an essential part of the outfit; the one used is a modification of the Creasy made in Philadelphia, Pa. The trough is mounted on two cars, with an adjustment laterally, longitudinally, and vertically for each for centering and focusing the ends of the bar under the microscopes. The cars have three wheels each and run on a portable track of 0.3^m gage. Three sections of track are provided, each 5^m long. Each track section is carried forward as the measurement proceeds instead of carrying the base bar.

The micrometer microscopes and the cut-off tube of the Repsold apparatus of § 63 were used for the Holton base where the iced bar was first used. The microscope stands had been destroyed by fire, so that wooden posts with cast-iron caps were used instead for the 100^m comparator and the standard kilometer which were measured with the apparatus. For the comparator measurements the apparatus was sheltered by a shed, and for the kilometer the microscopes were sheltered by umbrellas.

69. Standards of Length.—All measurements of the Coast Survey have been referred to one of the twelve original iron meter bars standardized in 1799 by the French Committee in terms of the toise which had served as a standard unit in measuring the meridional arcs of France and Peru. In November 1899 the Government received three platinum-iridium bars of the Prototype meter standardized by the International Bureau at Paris, and from early in 1900 these have referred the Coast Survey standard to the International.

The length of the iron bar is now taken

$$=1^m+0.2^\mu\pm0.6^\mu,$$

as the result of recent comparisons, instead of

$$=1^m-0.4^\mu, \text{ as given in 1799.}$$

In App. 6 of the *Report* for 1893 it is stated that no legal standard of weight or length was adopted by Congress until July 1866 (a Troughton 82-inch scale had been used by the Treasury Department as a standard in collecting duties, etc.), when the metric system was legalized and the weights and measures in common use were defined in terms of the metric units, giving

$$1 \text{ yard} = \frac{3600}{3937} \text{ meters}; \quad 1 \text{ pound} = \frac{1}{2.2046} \text{ kg.} \quad . \quad . \quad (27)$$

As a result, the Survey now uses $1^m = 3.2808\frac{1}{3}$ feet, instead of $1^m = 3.280869$ feet, as formerly.

Standards are divided into *end measures* and *line measures;* with the former the length is between the end surfaces, and end contacts are used; with the latter, the length is between lines or points near the ends and micrometer microscopes are usually used.

70. Comparators.—For *end measures* the contact level comparator is used by both the Coast Survey and Lake Survey (*Pri. Tri. U. S. L. Sur.*, p. 56). A diagrammatic sketch is shown in Fig. 61. The ends are in duplicate, each bed plate being fastened

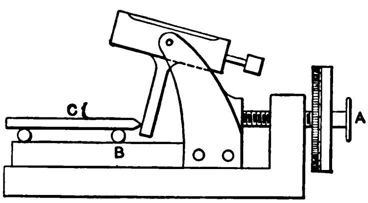


FIG. 61.

to a solid masonry pier for the best results, although an I beam or even a stick of timber is sometimes used to hold them in relative position.

The micrometer screw A moves the slide B on the horizontal bed plate. This slide carries the contact level and the grooved rollers for the piece C. The unbalanced weight of the outer end of the level forces the vertical arm against the piece C and insures a slight pressure against the end of the bar when in place.

To compare a bar with the standard of the same nominal length, the two bars are mounted side by side and placed first one then the other between the contact ends of the pieces C, each micrometer screw is turned until the bubble comes to, or very near, the center, when the screw and bubble are each read. The sum of the micrometer readings one at each end, each corrected for level, gives an expression for length of bar; the difference for the bar and standard gives the difference in length sought, and this difference applied to the length of the standard will give the length of the bar. Usually the bars would be compared in air after they had remained in place at a constant air temperature long enough to have reached that temperature. Thermometers in contact with one or both bars are used.

For the Würdemann comparator of the Lake Survey, 1^d of the level = about $0^{in}.00001$ and 1^d of the micrometer = about $0^{in}.0001$. The College has a field comparator constructed from Brown and Sharpe micrometer calipers reading to $0^{in}.0001$. The stud against which the screw abuts is removed from the yoke and a cylindrical piece corresponding to *C* of Fig. 61 is inserted. This is pushed against the end of the bar by the micrometer screw and returned by a spiral spring. To insure contact with uniform pressure the contact slide of the Coast Survey secondary bars (Fig. 57) is placed on the piece *C* in place of the contact level. This slide can be read to about $0^{in}.0002$. The two ends are mounted on a timber base and used for comparing the 4^m iced bar standard with the secondary base bars under field conditions.

The Coast Survey base bars are 4, 5, or 6 meters long, all end measures except the iced bar, and their old standard unit (§ 69) a 1^m French end measure. The Lake Survey Bache-Würdemann base bars are 15-foot end measures and their standard unit an English yard end measure. The method of finding the length of the base bar in each case has been to construct as many one-unit bars as there were units in the base bar; compare each with the standard; then place them end to end (holding the surfaces in contact by light springs) and compare the compound bar with a new bar of the length of the base bar. This new bar then becomes the standard for determining the length of the base bar, both in the office and in the field.

For *line measures*, micrometer microscopes are mounted on piers, or on a rigid frame if changes in distance are frequently required, and the difference in length obtained in terms of the micrometer screws.

In comparing commensurate units (e.g., finding the length of a 4-meter bar from a standard meter), the shorter is laid off on the longer, dividing it into equal parts, and microscopic lines are drawn; each length is then compared with the shorter unit and the results added as with end measures.

Fig. 62 shows a 1^m comparator used by the International Bureau of Standards of Paris. The two tanks are for determining coefficients of expansion, one bar being heated by circulating

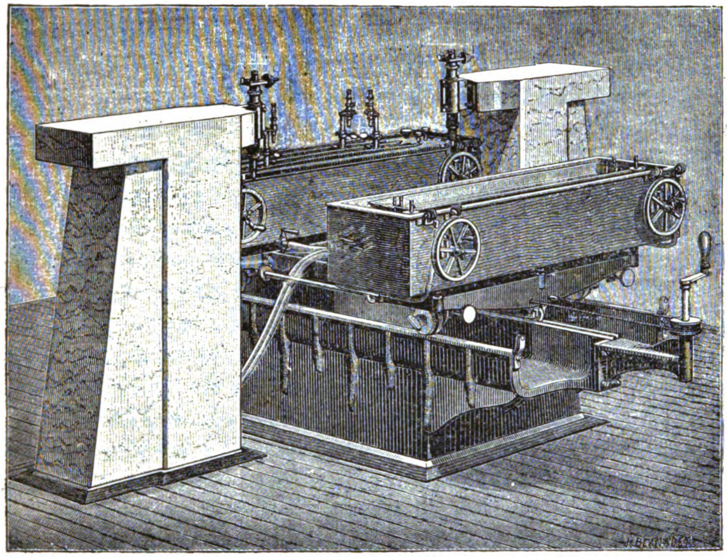


FIG. 62.

warm water through the pipes, while the other remains at a constant temperature. The microscopes shown on the tank are for reading the thermometers near or in contact with the bars. The transverse tracks allow of bringing first one bar, then the other, under the microscopes and nearly in focus by turning the crank shown at the right. The exact focusing, after comparisons have been begun, must, of course, be done by raising or lowering the ends of the bars by tangent screws so as not to disturb the positions of the microscopes.

71. Mercurial Thermometers.—Thermometers are divided into standard and auxiliary; the scales of the former include both the boiling and the freezing point of water, which allows of their being studied and standardized, each one independently; the scales of the latter do not contain both of these fixed points, and they can only be standardized by comparison with some other thermometer.

With glass, as with tempered steel, zinc and its alloys, and some other substances, the volume change lags behind the temperature change, giving rise to *residual expansion.* This is especially apparent with thermometers in the variations of the zero point, the volume of the bulb at the temperature of melting ice depending for some time upon the previous temperature of the thermometer. When the bulb remains expanded by exposure to a temperature above 0°, it is said that the zero is *depressed;* when exposed to 0° the zero of the thermometer rises, and when it has reached a stable position it is said to have *recovered.* The depression of the zero for a temperature t is the difference between its position after the thermometer has been exposed to $t°$ and the position which it will reach when the thermometer remains a long time at 0°. These variations of the zero were for a long time an obstacle to the accurate determination of temperature, but systematic study has shown how to use thermometers so as to become almost wholly free from these anomalies. However, the errors which remain are reduced in proportion as the residual expansion of the glass is reduced. The *verre dur* glass of the French and some numbers of the Jena glass of the Germans have very little residual expansion, and it disappears more rapidly than with the easily fusible crystal glass so commonly used.

The most important property of the movement of the zero is that at a given temperature the depressions are produced more rapidly than the recoveries; and further, that the rapidity of the two movements increases with the temperature. Thus when a thermometer of verre dur glass is heated from ordinary temperature to 100°, the stable condition is reached in a few minutes; when cooled, more than one-half of the residual expansion remains after twenty-four hours, and the stable condition is reached only after several weeks. With crystal glass, the stable condition at 100° is reached in about an hour, while months are required after cooling.

To utilize this property in the determination of temperature: the thermometer having been exposed for some time to the temperature to be measured, a reading is taken; it is then plunged into melting ice and read; the difference will give the temperature above 0° referred to the fundamental interval, 0° to 100°, the 100° point having been found by referring to 0° in the same way.

Small bulbs are often blown in the tubes of standard thermometers to allow of the 0° and 100° points on the scale without too long a tube.

The scales of the best thermometers are scales of equal parts etched on the stem. This requires tubes of very uniform bore to prevent large scale corrections.

The tube is *calibrated* by breaking off columns of mercury of different lengths and noting the length in scale divisions as they are moved from end to end of the tube (a small bulb at the top is necessary for this work).

The 100° point is computed from the observed temperature in steam under a given barometric pressure, and the 0° point by melting ice immediately after. This gives the fundamental interval which is to be divided into 100 equal parts for the centigrade scale. The calibration corrections refer these equal volume parts to the scale divisions, so that the scale divisions can be expressed in degrees. A perfect tube and scale within the errors of observation are thus secured, and residual expansion can be eliminated in use as indicated above. These corrected temperatures (including a correction for pressure on the bulb) are called mercurial thermometer temperatures, and they are usually accepted as standard, assuming the expansion of mercury in glass to be proportional to the temperature.

The International Bureau has adopted the hydrogen scale as standard, and by comparing the mercurial thermometer readings with the corresponding pressures of a constant volume of hydrogen, by Mariotte's law, they have derived correction tables for different kinds of glass. The Coast Survey has also adopted the hydrogen scale.

See *Thermométrie de Précision*, by Guillaume, Paris, 1889, for corrections for various kinds of glass.

The corrections for verre dur glass are as follows: *

t	Cor.	t	Cor.	t	Cor.	t	Cor.	t	Cor.	t	Cor.
−25°	+0.233	0°	0.000	+25°	−0.095	+45°	−0.106	+65°	−0.082	+ 85°	−0.038
20	0.172	+ 5	−0.02	30	0.102	50	0.103	70	0.072	90	0.026
15	0.119	10	0.052	35	0.106	55	0.097	75	0.062	95	0.013
10	0.073	15	0.070	40	0.107	60	0.090	80	0.050	100	0.000
5	0.034	20	0.085								

72. Length of Apparatus.—From what has been given in §§ 69–71 the method of finding the length of a base bar is evident. All the comparisons except field comparisons are made in a room so protected that the daily range of temperature is small; thermometers are placed in contact with the bars and a few readings at a time are taken quickly before the heat of the body causes a local disturbance of the temperature of the bars, the latter being protected by a case or cover. With bars of the same material the actual temperature need not be known very closely, but the exact difference is essential.

Since the probable error in bisecting a line with a micrometer microscope of a power of 25 under favorable conditions is given in § 32 as 0.25^{μ} upon the retina, $\frac{0.25}{0.075\times25}=0.13^{\mu}$ upon the scale, and $0^{\circ}.01^{C}$ changes the length of a steel bar 0.12^{μ} per meter, care should be taken to secure good temperature conditions, and to avoid the accumulation of constant errors. This will require changing the order of the readings, the positions of the bars, etc., for the different sets.

The determination of the coefficient of expansion requires great care, on account of the difficulty of getting all parts of the warm bar at a constant temperature and keeping them so long enough to read the thermometers and micrometer microscopes or other contact apparatus. The bar is usually immersed in water or glycerine, while its companion is surrounded by melting ice. In Fig. 62 the water is heated by a gas jet at a distance and circulated through the pipes shown; circulation in the tank is secured by turning the wheels at the ends. Readings are taken through the water.

* Tonnelot and Boudin of Paris are among the best foreign makers of precise thermometers. H. J. Green of Brooklyn is a maker in this country.

For a comparison of the European standards of lengths as a basis of study of the different triangulations for degree measurements, see *Comparisons of Standards of Length,* by Capt. A. R. Clarke, London, 1866. These standards were incommensurate, and some were line measures and some end measures. The purchase of the Repsold 4^m apparatus of § 63 by the Lake Survey involved a comparison of their end measure standard yard with the new Repsold standard line measure meter. (*Pri. Tri. U. S. L. Survey,* beginning p. 142.)

An end measure is converted into a line measure by bisecting the angle between the direct and reflected images of a spider thread stretched across the plane end surface; or by adding end pieces with micrometer lines close to the contact surfaces for comparison with the other bar and then determining the distance between the micrometer lines in terms of the micrometer microscope screw when the end measure is removed and the end pieces brought in contact. A comparison of incommensurate units involves a comparison of the subdivisions of the longer standard by the method of aliquot parts down to where either the length of the shorter or the difference in length can be directly compared within the range of the micrometer screw.

Ex. To find the length of secondary bar No. 1, the following comparisons with standard No. 2 and data are given (*C. S. Rept.*, 1868):

Length of standard bar No. 2 at 32° F. . . . . 5^m.999 982 33
One division of the scale of pyrometer. . . . . .0 .000 001 74
Coef. of expansion for F. scale. . . . . .0 .000 006 41
Thermometer attached to standard, too high, correction = −0°.7
" " " rod, correction = 0 .0

Standard Thermo.	No. 2 Div.	Rod Thermo.	No. 1 Div.
77°.3	21	76°.0	−10
78 .0	15	76 .4	+41
78 .5	18	77 .0	+55
77 .93	18	76 .47	28.67
−0 .70		77 .23	18.00
77 .23		+ .76	10.67

Computation.

0.76×0.000 006 41×6. . . . . = $+0^m$.000 029 23
10.67×0.000 001 74. . . . . = +0 .000 018 57

At 77°.23 No. 1 longer than standard. . . . 0 .000 047 80
At 77 .23 standard No. 2. . . . . 6 .001 721 88
At 77 .23 rod No. 1 . . . . 6 .001 769 68
At 75 " " 1 . . . . 6 .001 683 91

The pyrometer is a form of end measure comparator in which the measurement is indicated by the reflection of a graduated scale from a mirror in the same way as is done with some galvanometers.

73. Defects and Difficulties.—It is very difficult to find the temperature of a bar and its consequent length under field conditions. The Colby apparatus (§ 60), after being used in England, was taken to India and a large number of bases measured, but the compensation could not be relied upon and mercurial thermometers were substituted. The Bache-Würdemann 15-foot bar No. 1 made for the U. S. L. Survey gave a length at 10 P.M. Aug. 5, 1873, $0^{in}.00470$ shorter than at 9 A.M., as stated on p. 86 of the *Report,** it having been exposed to direct sunlight during the day. This would correspond to a difference of 1°.3 F. between the two components. In 1875, at the Buffalo base, its mean length for the eleven days of comparison was $0^{in}.00230$ greater at 1 P.M. than at 8 A.M., the comparisons being made in a tent.

From numerous comparisons made with the bars under field conditions and the brass standard in melting ice, it was found that the mean increase in length of the bars, between 8 A.M. and 5 P.M., over that when the components were at the same temperature, was proportional to the increase in temperature between 8 A.M. and 12 M. A correction was thus found and applied for the bars placed each day, thus eliminating a source of constant error which would escape detection by the usual method of remeasurement. The Repsold steel-zinc apparatus was studied in the same manner, and corrections found to apply for the different times of day. The bar was protected by an awning as used on the base. The maximum correction was for the interval from 2 to 5 P.M., reaching $+12^{\mu}$ per bar length.

The iced bar of § 68 was designed for experimental work with different forms of apparatus on the Holton base of the 39th parallel triangulation. Two methods of finding its length in terms of the standard meter, with the bars in melting ice, were tried with the office comparator: one by placing six micrometer microscopes 1^m apart and measuring the distance between the extreme ones with both the 1^m and the 5^m bars; the other by comparing the 1^m spaces of a 5^m bar with the 1^m standard and then comparing

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* *Pri. Tri. U. S. Lake Survey.*

the two 5^m bars. The results were not satisfactory, owing to instability of microscopes. These were carried by brackets from a $6^m.5$ I-beam supported at its ends, all wrapped with cotton batting, rather than by independent piers, and they appeared to have been disturbed by vibrations due to street traffic. A new comparator was then built in a vacant lot adjacent to the office and covered by a shed. Six brick piers were built for the microscopes on a heavy continuous concrete bed. Free air circulation was allowed, and the inside of the shed was whitewashed, giving light without artificial illumination. The distance between extreme microscopes was measured with the standard meter bar and with the two 5^m bars, all in melting ice.

For the 100^m comparator of the Holton base, § 68, the ends were marked by brass hemispherical-headed bolts set in stone monuments, the heads fitting the cut-off tube. This was used as a standard for the 5^m secondary base bars and for the 100^m tapes. On a test in August the bars showed (by their length) a lag of the thermometers of 0°.3 C. for both rising and falling temperatures. The thermometers, one at 1^m from each end, were in contact with the steel measuring rod and the wooden case was wrapped with cotton and canvas, while the comparator was covered with a shed. A later test in bright sunshine on a portion of the standard kilometer showed the lag to be insensible.

The final conclusions reached were that the absolute lag of the thermometers can be safely assumed between 0° and 0°.5 C. on the average in a day's measurement; that a correction for lag of 0°.25, plus for rising, minus for falling, applied to the mean temperature for the day will keep the error in length within 1/340 000; and that, if the day's measures are equally divided between rising and falling temperatures, or if they are made during hot weather in bright sunshine and with rising temperatures, no correction will be necessary. It is pointed out that where day measurement only is intended the bars should be protected as much as possible from change of temperature at night.

Ex. 1. With the Bessel apparatus, § 60, find the difference in temperature between the two components required to introduce an error of one millionth of the length of the iron bar in its computed length.

Ex. 2. Find the error in the observed temperature of the iron bar for the same error in length.

Ex. 3. With the Colby apparatus, § 60, find the effect upon the distance between the end dots due to a difference of 1° between the two components.

Ex. 4. Compare zinc-steel and brass-steel Borda thermometers with mercurial in the effect upon the length of the measuring bar of a difference of 1° between the two components or the component and the mercurial thermometer.

74. **Field Work.**—The following is condensed from the instructions issued to the chief of party for the measurement of nine bases along the 98th meridian in 1900 (App. 3, *Report*, for 1901): Two 100^m steel tapes, two 50^m tapes, and the duplex base bars are to be used in the measurement of the bases. A portion of each base, about 1^{km} in length, is to be measured with all five sets of apparatus, and of the remainder, a portion is to be measured with the tapes and a portion with the bars.

Very little increase in the average accuracy of the triangle sides will result from increasing the accuracy of the base measurement beyond that represented by a probable error of 1/500 000. The following limits of accuracy on each operation are selected with this in view. You will strive to keep as far within these limits as possible by the use of good judgment and skill, but you will restrict the time and money expended upon each operation substantially to that required to keep barely within these limits.

You will standardize each tape and the duplex bars at the beginning of the field season upon a 100^m comparator, which you will establish near Shelton base, by measuring the length of the comparator at least four times with each apparatus. The measurements by each apparatus must not all be made on a single day or night. If four measures do not suffice to reduce the p. e. of the mean for any apparatus below 1/300 000, additional measures will be made until the p. e. is reduced below that limit. The length of the comparator is to be found by measuring it with the iced bar at such times that the longest interval between any such measurement and any measure with a base apparatus shall not exceed thirty-six hours. In computing the length of the bars and tapes, the length of the comparator shall not be assumed constant, but will be assumed to be that given by the nearest measurement with the iced bar, or by the mean of the last preceding and first following measurements. The bars and tapes are to be restandardized in the same manner at or near the end of the field season. The conditions under which each form of apparatus is standardized should approximate as nearly as is feasible those under which it is used in the actual measurements of the base. One kilometer of each base, called the test kilometer, is to be measured with the duplex bars and with each of the tapes. If any one of these five measurements shows a residual from the mean of more than 17^{mm}, an additional measurement shall be made with that apparatus. Residuals of more than 17^{mm} on the test kilometer, which persist even after a remeasurement, shall be considered to indicate the desirability of restandardization, but in no case shall the apparatus be standardized without special authorization, except at the beginning and ending of the season as indicated above.

Of the remainder of each base not less than one-tenth nor more than one-third is to be measured twice with the duplex bars; not less than one-fifth nor more than two-thirds is to be measured once with each 50^{m} tape; and not less than one-fifth nor more than two-thirds is to be measured once with each 100^{m} tape. Additional measurements shall be made if the discrepancy between the two measurements on any section exceeds $20^{mm}K$ (K=length in kilometers) until two measures are obtained which agree within that limit. About one-fifth of the total length of all the bases after the test kilometers are deducted should be measured with the duplex bars, and about two-fifths each with the 50^{m} tapes and the 100^{m} tapes.

Such precaution should be taken to secure accurate horizontal and vertical alignment as is necessary to ensure that errors from this source on any section of the base shall be less than 1/1 000 000. It is not desirable, however, to use any more time than that necessary to keep well within this limit. This principle should also be applied to the determination of the tension on the tape while in use.

The party is made up of a chief, an assistant, two aids, a recorder, and five laborers.

In measuring the 100^{m} comparator with the *iced bar*, the first microscope is set over the starting point by means of the cut-off tube, reading its scale and level when both are parallel with the bar, both direct and reversed. The rear end of the bar is brought under the rear microscope, and the front end under the front one, moving either bar or microscope; the rear observer then moves the bar longitudinally by means of a lever until the graduation is accurately bisected by the micrometer wire without turning the micrometer screw. He then gives the signal "Read" to the front observer, who brings the micrometer wire to bisect the front graduation. The recorder then records the readings given, the rear observer turns the micrometer screw slightly and the observers change places. The rear observer brings the front graduation under the wire without disturbing the screw, and the front observer bisects the rear graduation with the screw. The recorder then records the readings. This eliminates the personal equation of the observers and checks mistakes, each observer making two bisections. The p. e. of a bisection is less than 1^{μ}.

While the bar is in position the third observer measures the distance of the front end from the reference line * and adjusts and reads the level sector. The rear microscopes and track sections are carried forward and reset as rapidly as relieved. The observers stand on platforms which rest on the ground about 3 feet from the microscope post. The ice is stirred up and fresh ice supplied at intervals of twenty to forty minutes, the amount required being from 7 to 11 pounds.

CUT-OFF MEASURES, Aug. 7, 1891.

AT WEST (LEFT-HAND) END OF COMPARATOR.

(Repsold cut-off cylinder.)

Time (p. m.).	End of Cut-off Scale, Right.	Scale Division Observed.	Microscope Reading.	Height of Scale.	Level Readings.	
					l	*r*
			Rev.	mm.		
3.46	*A*	29*I* [a]	18.55	840	5.5	9.0
	B	29*I*	20.83		4.5	10.0

AT EAST (RIGHT-HAND) END OF COMPARATOR.

(Coast and Geodetic Survey cut-off cylinder.)

Time (p. m.).	End of Cut-off Scale, Right.	Scale Division Observed.	Microscope Reading.	Height of Scale.	*l*	*r*
4.46	*A*	10*E* [a]	25.70	838	3.0	3.0
	B	11*E*	18.46		5.0	1.0

[a] *I* indicates that the scale is observed with the image inverted, *E* with the image erect.

1^d of level on Repsold cylinder = 1″.6. 1^d of level on C. & G. S. cylinder = 6″.0.

* A reference line is ranged out in advance with a transit and marked.

SPECIMEN OF RECORD.

Measure of 100^m Comparator Interval, Aug. 7, 1891.

No Bar Lengths.	Time, p. m.	Left-hand End of Bar.		Right-hand End of Bar.		$L-R.$	Height of Bar.
		No. Microscope.	Microscope Reading.	No. Microscope.	Microscope Reading.		
1	3.54	1	W 20.83	2	S 21.33		491
			S 20.83		W 21.33	− 3μ	504
2	.56	2	21.33	3	20.73		
			21.26		20.73	− 3	500
3	4.01	3	20.74	4	21.26		
			20.66		21.26	− 2	509
4	.03	4	20.26	1	21.22		
			20.21		21.22	− 4	502
5	.05	1	21.22	2	22.03		
			21.13		22.03	0	506
6	.08	2	22.03	3	21.02		
			22.03		21.02	− 4	505
7	.11	3	21.02	4	21.57		
			20.95		21.57	− 5	507
8	.12	4	21.57	1	20.60		
			21.47		20.60	− 4	505
9	.15	1	20.60	2	23.10		
			20.52		23.10	− 5	509
10	.17	2	23.10	3	21.30		
			23.00		21.30	− 2	514
11	.20	3	21.30	4	21.66		
			21.26		21.66	− 8	520
12	.25	4	21.66	1	20.81		
			21.50		20.81	− 5	516
13	.28	1	20.81	2	21.91		
			20.72		21.91	+ 1	522
14	.31	2	21.91	3	20.97		
			21.94		20.97	− 7	524
15	.34	3	20.97	4	21.74		
			20.84		21.74	+ 2	523
16	.36	4	21.74	1	21.49		
			21.77		21.50	+ 1	515
17	.40	1	21.50	2	22.01		
			21.52		22.01	− 5	523
18	.44	2	22.01	3	20.70		
			21.91		20.70	− 2	520
19	.48	3	20.70	4	21.21		
			20.66		21.21	+ 3	530
20	4.49	4	21.21	1	21.80		
			21.28		21.80	−97	524

The measurement was from left to right. The letters S and W in the fourth column refer to the front and rear observers, respectively. They changed positions for the second readings as already stated. The seventh column gives the differences in microns (1^R of the micrometer screw $=0.1^{mm}$, so that $0.01^R=1^{\mu}$) between the mean readings of any microscope at the two ends of the bar in the order left minus right or $(l-r)$. Thus the first value comes from the readings of microscope No. 2, namely $\frac{1}{2}(21.33^R+21.26^R)-\frac{1}{2}(21.33^R+21.33^R)=-3_{\mu}$. The last value in this column is derived from the readings of microscope No. 1 at the beginning and end of the measurement. The last column gives the height of the front end of the bar (and of the rear in starting) in place of the usual grade sector readings.

In using the *duplex bars* the front end of the zero bar, No. 15, was placed on the trestles, aligned and centered over the starting point in position "face up," using a transit direct and reverse at a distance of 30 feet at right angles for plumbing up. The temperature was noted and compared with 26.45° C., that at which the two components have the same length, and the brass component set by the scale normal, or backward or forward a sufficient amount to permit the measuring of 250^m if possible without another set-back or set-up. The first bar of the measure was then set, aligned, contact made, mercurial thermometers and grade sector read and recorded. This completed the placing of the first bar. Unless a set-up or set-back of the brass rod became necessary no more scale readings were taken until the end of the half section, usually after the laying of the 100th bar. The thermometer readings were taken for the first and second bars, the ninth and tenth, etc. The apparatus was used without shade or shelter.

75. Tape Measurements.—Some years ago M. Jäderin introduced a method of measurement with tapes for which he claimed an accuracy of 1/1 000 000, even when the work was done in sun and wind. He used two tapes, one steel, the other brass, each 25^m long, the ends resting upon portable tripods serving as pins to mark the tape lengths, while under a fixed tension applied through a spring balance. The differential expansion of the steel and brass is relied upon for the temperature correction for the steel.

The tape method was taken up by the Missouri River Commission for base lines, using a single steel tape 300 or 500 feet long, with firmly driven posts for markers and stakes for intermediate supports, the measures being made at night.

Later the tape method was taken up by the Coast Survey in connection with the Holton base. From the results obtained it was concluded that a p. e. of less than 1/1 000 000 as derived from the range of different measures could be secured, or including errors from all sources a result true to 1/500 000 could be reached. For an accuracy of 1/200 000 to 1/500 000 the cost was estimated to be about the same as with bars.

The mathematical theory of metallic tapes is developed in Supplement B, App. 8, *Report* 1892.

The tapes of § 74 have a cross-section of about 0.25 by 0.022 of an inch; the 50^m tapes are without intermediate graduations. They were supported at points 25^m apart under a tension of 15^{kg} applied by a spring balance. The markers were 4 by 4 inch posts with copper strips nailed on top parallel with the line and thick enough so that the top of the tape when held alongside would be flush with the surface, allowing the tape graduation to be marked off onto the slip with a sharp awl. The intermediate supports were nails driven into the sides of 2 by 4 inch stakes. A thermometer was tied to the tape about 1^m from each end with its metal back in contact with the tape.

The measurements were made at night, the supports having been prepared in advance. Eight men for the 50^m tapes and twelve for the 100^m were the regular number when available, but much of the work was done with six and eight men, respectively. There were two observers, one front, the other rear; one recorder, who also held the light for reading the tension; two men to apply the tension and to hold in the rear; and three men to handle lamps and carry the tape ahead. The thermometers were read by the observers after making contacts. The rear end of the tape was held by a staff. At the front end a tape stretcher was used consisting of a base of plank to which was attached by a universal joint a piece of gas pipe threaded at the proper height for the contact post. A gimbal was attached to the pipe, which could be raised or lowered by a nut. The spring balance was attached to the gimbal on one side and a counterweight, to hold it horizontal, on the other.

When the front end of the tape came too near one end of the copper strip, a set-up or set-back was made and noted, and its length measured by daylight with scale and dividers.

The length of the 100^m tape was determined by measuring the length of the 100^m comparator under field conditions, by bisecting the end graduations of the tape with the micrometer wires after making the readings on the cut-off. For the 50^m tapes a marker was used in the center for rear contact and the end graduation bisected as above. After reading, the observers changed places and read again, so that two sets of values for contact and temperature were secured.

Recently the Coast Survey has purchased eight nickel-steel 50^m tapes having a certified coefficient of expansion of 0.000 000 3 per degree Centigrade, under a pull of 10 kilograms between the temperatures of 0° and 30° Centigrade. The correctness of this coefficient was borne out by subsequent tests at the National Bureau of Standards. This coefficient is but $\frac{1}{35}$ that of the steel tapes above described. Two of these tapes are in use in comparison with the old steel tapes and it is found that better results are obtained working by daylight than with the old tapes at night. A tension of 15^{kg} is applied.

76. Correction Formulas.—For the *iced bar*, if a micrometer wire is moved to the left (measurement left to right) in making a rear end bisection the correction is subtractive, it having been added to the distance measured when it should not have been. Hence for readings increasing from right to left as with the micrometers used,

$$\text{Correction, } c' = \Sigma(L-R). \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Similarly, if L_c and R_c are the mean micrometer readings on the cut-off scales,

$$\text{Correction, } c'' = -(L_c - R_c). \quad . \quad . \quad . \quad . \quad (29).$$

If the cut-off scale readings are S_l and S_r,

$$\text{Correction, } c''' = S_l - S_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where S_l and S_r are plus or minus according as the image of the graduation mark appears inverted or erect.

If I_r and I_l denote the inclinations of the cut-off cylinders at the ends, both inclined towards the right, and h_r and h_l, the corresponding heights of the cut-off scales,

$$\text{Correction } c^{iv} = (h_l I_l - h_r I_r) \sin 1''. \quad . \quad . \quad . \quad (31)$$

Denoting the readings of the ends of the bubble by l', l'', r', r'', and the value of a level division by d,

$$I=\frac{1}{4}(l'+l''-r'-r'')d. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

For the correction c^{v}, for difference in elevation, Δh,

$$(\Delta h)^2+(5-c^{v})^2=5^2=25,$$

or

$$c^{v}=\frac{(\Delta h)^2}{10-c^{v}}=0.1(\Delta h)^2, \text{ nearly.}$$

With Δh in millimeters and c^{v} in microns,

$$c^{v}=0.1(\Delta h)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The correction for alignment would be of the same form.

Ex. Referring to the comparator measurement of § 74, the $L-R$ column foots up $-149^{\mu}=c'$.

The mean micrometer reading on the left-hand cut-off scale is 19.69; that for the right-hand is 22.08. From (29),

$$c''=-19.69+22.08=2^{r}.39.$$

The left-hand cut-off scale reading is 29I; the right-hand 10.5E. From (30)

$$c'''=29+10.5=39.5^{mm}.$$

For the cut-off levels, $I_l=-3.6''$; $I_r=6''$. From (31),

$$c^{iv}=(-3.6\times 840-6\times 838)\sin 1''=0^{mm}.039.$$

From the last column (32) gives $c^{v}=-0^{mm}.075$.
Adding results, and calling the iced bar B_{17},
Length of comparator from this measurement $=20B_{17}+39^{mm}.476$.

For the *duplex bars* a correction for expansion is required. The coefficient of expansion is 11.54^{μ} per meter for the steel and 18.45 for the brass, giving a difference of 6.91. Hence for every unit of differential expansion accumulated on the scale the steel measuring component has expanded $\frac{11.54}{6.91}=1.668$. This factor is called the *duplex coefficient*. The components have the same length at 26°.45 C., which is $5^{m}+0^{mm}.0715$.

Hence to find the expansion for the steel bar multiply the differential expansion accumulated on the scale by the duplex coefficient. Thus on the first measure of the west half of Sec. 8 of the El Reno base (*Coast Survey Rep.*, 1901, p. 250) the brass component lost $30^{mm}.84$ as compared with the steel, giving

$$\text{Length of half section} = 100(5^{m} + 0^{mm}.0715) - 30^{mm}.84 \times 1.668$$
$$- \text{grade correction}$$
$$= 499^{m}.9382,$$

the grade correction being $= 17^{mm}.45$.

For the *steel tape* a correction is necessary for expansion, for the set-ups or set-backs and for the differences in elevation of the ends or of the marker-posts, the intermediate supports being usually on the grade line between markers.

For the grade correction, calling h the difference in height, c the correction, b the length, and i the angle of inclination,

$$b - c = \sqrt{b^2 - h^2},$$

or

$$c = b - b\sqrt{1 - \frac{h^2}{b^2}}. \quad \dots \dots \quad (34)$$

Also,

$$\sin i = \frac{h}{b},$$

whence, form. 12], $c = b(1 - \cos i) = 2b \sin^2 \tfrac{1}{2}i$. . . . . . (35)

One of the above formulas would be used for steep grades.

For flat grades expand the radical by the binomial theorem,

$$c = b - b\left(1 - \frac{h^2}{2b^2} - \frac{h^4}{8b^4} - \right),$$

or

$$c = \frac{h^2}{2b} + \frac{h^4}{8b^3}. \quad \dots \dots \dots \quad (36)$$

For a $\left.\begin{matrix}100^{m}\\50^{m}\end{matrix}\right\}$ tape, $\frac{h^4}{8b^3} < 0.01^{mm}$ for $h < \left.\begin{matrix}2.9^{m}\\1.7^{m}\end{matrix}\right\}$.

Hence within these limits, for the correction in millimeters, h in meters, and the tape length,

$$\left.\begin{matrix}50^{m}\\100^{m}\end{matrix}\right\}, \quad c = \left.\begin{matrix}10h^2\\5h^2\end{matrix}\right\}. \quad \dots \dots \quad (37)$$

Correction for index error of spring balance.—The index error with the balance in a horizontal position will be slightly different from that in a vertical, while the index error in either position is liable to change with use. The tape length is quite sensitive to tension, so that it is necessary to be able to apply a correction to length due to error in tension.

To determine the actual tension T with balance in horizontal position, let r = index correction for vertical position, = reading with hook end down and unloaded, minus when the index reads greater than zero, R = reading when balance is suspended by hook, hook end up, W = total weight found from another balance, and T_1 = observed or face reading when horizontal.

$$T = T_1 + r + \tfrac{1}{2}(W - R - r),$$

or

$$T = T_1 + \tfrac{1}{2}(W - R + r). \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The relation between change in length and change in tension is derived in the theory of tapes referred to in § 75. It can be more satisfactorily determined experimentally by varying the tension when under standard conditions and measuring the corresponding movement of the end graduation of the tape. For the 100^{m} tapes used on the Holton base this change was 0.05^{mm} per ounce of change in tension.

77. Reduction to Sea-level.—Base lines are usually reduced to sea-level so that all the computed triangle sides will be arcs of the spheroid whose surface is that of the sea produced under the land.

Let B' = the reduced horizontal length of the base at an average height h; B = the sea-level length; y = the correction; R_α = radius of curvature of the plane section through the base (see Table VI).

Then since arcs are to each other as their radii,

$$\frac{B}{B'} = \frac{R_\alpha}{R_\alpha + h}; \quad B = \frac{B'R_\alpha}{R_\alpha + h}; \quad y = B' - B = B' - \frac{B'R_\alpha}{R_\alpha + h};$$

or

$$y = \frac{B'h}{R_\alpha + h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Unless h is large, or extreme accuracy is desired, the h of the denominator may be omitted.

CHAPTER V.

TRIGONOMETRIC AND PRECISE LEVELING.

TRIGONOMETRIC LEVELING.

78. Observations.—In triangulation work the zenith distances of other stations, or their vertical angles, are usually measured when the station is occupied for horizontal angles. If the theodolite has no vertical circle and the micrometric method is not available an altazimuth or other vertical circle instrument is set up near the station for the vertical measurements. If the circle is small, or is read by verniers, increased accuracy can be secured by using the different horizontal wires (e.g., the stadia wires) for the different measurements of an angle, thus securing the advantages of the repeating circle in increasing the fineness of the reading.

The height of the telescope above the station mark should be measured. If the eccentricity of the instrument will have any appreciable effect upon the vertical angles it should be measured in distance and direction so that the angles can be reduced to center. The part of the signal sighted upon should be noted each time so that its height can be considered in finding the difference of elevation. A line of spirit levels is usually run from tide water, or from a station of known height above tide, to one of the stations.

Refraction is least and most nearly constant during the middle of the day, and greatest and most variable at night and morning. The best time for observing is thus usually from 9 A.M. to 3 P.M., and the worst at sunrise and sunset. Simultaneous observations at the two stations will give the best results. If not simultaneous they should be distributed over several days to get an average value for refraction. Instrumental errors are small in comparison with those from refraction, so that with good instruments it is not necessary to eliminate errors of graduation by shifting the position of the circle.

The form of record shown on the opposite page is taken from the Coast Survey.

Station , Date , Observer , Recorder , Instrument.

Object Observed.	Time.	Tel.	Level.		Circle.				Level Cor.	Corrected Reading.	Zenith Distance.	Remarks.
			E	O	° ′	d	d	Mean.				
								″	″			
Spear	$6^h\ 30^m$	D	17	9	163 20	34.6	34.9					
	p.m.					48.4	48.8					
						41.5	41.8	83.3	+5.4	163° 21′ 28″.7		
		R	17	8.5	342 40	33.8	34.0					
						61.0	59.9				90° 20′ 0″.0	
						47.4	47.0	94.4	−5.7	342 41 28 .7		
		D	14	11.5	163 20	39.2	38.7				3.3	
						55.0	54.3					
						47.1	46.5	93.6	+1.7	163 21 35 .3		

Telescope 7.10 ft. above bolt. 1^d of level $= 1''.35$. 1^d of microm. $= 2''$. Heliotrope 6.18 ft. above bolt. Buffalo tripod head 26.1 ft. above station.

79. Reduction for Difference in Height of Telescope and Object Above Station Mark.—If the observations for the zenith distance of a station taken on different days are to different parts of the signal, as tripod head, top of pole, etc., on account of better visibility, it is desirable to reduce them to the same point in order to take the mean. If reciprocal observations are worked up as such, they must be reduced to the line joining the station points.

To find the correction, x, let z_1, z_2 be the measured zenith distances, corrected for difference of height above station mark of telescope and object; h_1, h_2 the heights of the stations above mean tide; s the horizontal distance in meters at sea-level; R_α the radius of curvature; C the central angle; m_1, m_2 the coefficients of refraction; mC the angle of refraction. Assuming the angle between the tangent FA to the line of sight and the chord AD, Fig. 63, proportional to the distance is equivalent to assuming the line of sight an arc of a circle, although the actual curvature is usually irregular.

In Fig. 63, B is as high above the station point as the telescope at A, E is at the same height above sea-level as A, while D, the point sighted at, is at the height q above B and requires the correction x to the observed zenith distance. Then

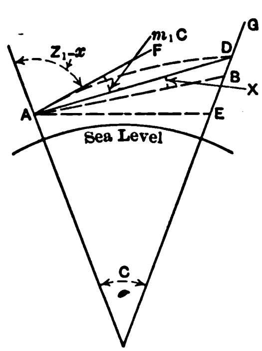


FIG. 63.

$$\sin x = \frac{q \sin ABD}{AD} = \frac{q \sin ADE \sin ABD}{AE \sin AED}.$$

But $\sin ADE = \sin (z_1 - x + m_1C - C)$,

$\sin ABD = \sin (z_1 + m_1C - C)$

(since $ABE = ADE + x$).

Also $\sin AED = \sin (90° + \frac{1}{2}C)$.

Substituting,

$$\sin x = \frac{q}{AE} \cdot \frac{\sin (z_1 - x + m_1C - C) \sin (z_1 + m_1C - C)}{\sin (90° + \frac{1}{2}C)}.$$

Now, $\sin (90° + \frac{1}{2}C)$ is nearly unity, x and m_1C are small, C is small for short distances and $(z_1 - C) = 90°$ nearly for long distances, and s, the distance between stations, $= AE$ nearly.

$$\left.\begin{array}{ll}\text{Hence, for short distances,} & x''=\dfrac{q\sin^2 z_1}{s\sin 1''}\\ \text{and for long distances,} & x''=\dfrac{q}{s\sin 1''}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

80. Difference of Height From Observed Zenith Distances.—(*a*) *Non-simultaneous Observations.* In the triangle *ABC*, by formula 20], we have

$$h_2+R_\alpha-(h_1+R_\alpha)=(h_2+h_1+2R_\alpha)\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}. \quad . \quad (A)$$

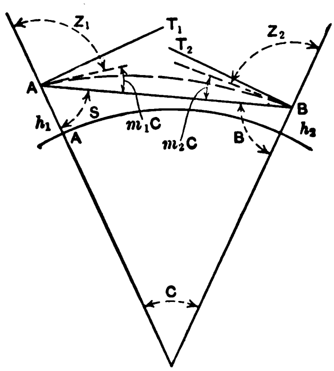


FIG. 64.

But $\quad A=180°-z_1-m_1C;\quad B=180°-z_2-m_2C,$

giving $\quad \frac{1}{2}(A-B)=\frac{1}{2}(z_2-z_1)+\frac{1}{2}C(m_2-m_1).$

Furthermore, $\quad \frac{1}{2}(A+B)=90°-\frac{1}{2}C.$

Substituting,

$$h_2-h_1=\tan\left[\tfrac{1}{2}(z_2-z_1)+\tfrac{1}{2}C(m_2-m_1)\right](h_2+h_1+2R_\alpha)\tan\tfrac{1}{2}C;$$

$\frac{1}{2}C=\frac{s}{2R_\alpha}$, being small, $\tan\frac{1}{2}C=\frac{s}{2R_\alpha}+\frac{s^3}{24R_\alpha^3}$, by formula 15];

$\frac{1}{2}C(m_2-m_1)$ being very small, formula 5] gives

$$\tan\left[\tfrac{1}{2}(z_2-z_1)+\tfrac{1}{2}C(m_2-m_1)\right]=\tan\tfrac{1}{2}(z_2-z_1)+\frac{s}{2R_\alpha}(m_2-m_1).$$

Substituting,

$$h_2-h_1=[s \tan \tfrac{1}{2}(z_2-z_1)+\frac{s^2}{2R_\alpha}(m_2-m_1)]\left(1+\frac{h_2+h_1}{2R_\alpha}+\frac{s^2}{12R_\alpha^2}\right). \quad (41)$$

In equations (41) and (42) below, the terms $\frac{h_2+h_1}{2R_\alpha}$ and $\frac{s^2}{12R_\alpha^2}$ are small fractions; and the equations may be most conveniently solved by neglecting these terms and obtaining an approximate value of the unknown h_2 or h_1. This value may then be substituted in the second factor of the equations and an accurate value of h obtained.

(*b*) *Simultaneous Observations.*—The refraction is assumed the same at both stations. Placing $m_1=m_2$ in (41), we have

$$h_2-h_1=s \tan \tfrac{1}{2}(z_2-z_1)\left(1+\frac{h_1+h_2}{2R_\alpha}+\frac{s^2}{12R_\alpha^2}\right). \quad . \quad . \quad (42)$$

(This is the formula employed by the U. S. Coast and Geodetic Survey.)

(*c*) *Zenith distance at one station only.*

$$A=180°-z_1-m_1C, \text{ as before.} \quad B=z_1+m_1C-C,$$

giving $$\tfrac{1}{2}(A-B)=90°-[z_1+(m_1-\tfrac{1}{2})C].$$

Substituting in formula (A) and following the same line of transformation as in sec. (*a*),

$$h_2-h_1=s \cot\left(z_1+(m_1-\tfrac{1}{2})\frac{s}{R_\alpha \sin 1''}\right)\left(1+\frac{h_2+h_1}{2R_\alpha}+\frac{s^2}{12R_\alpha^2}\right). \quad (43)$$

By calling the second factor unity and expanding the cotangent by formula 5], (43) can be reduced to another form which is sometimes given. $(m_1-\frac{1}{2})C$ is so small that for $\tan(m_1-\frac{1}{2})C$, $(m_1-\frac{1}{2})C$ can be used.

$$\frac{1}{\tan\,[z_1+(m_1-\frac{1}{2})C]}=[1-(m_1-\tfrac{1}{2})C \tan z_1][\tan z_1+(m_1-\tfrac{1}{2})C]^{-1}$$
$$=\cot z_1+(\tfrac{1}{2}-m_1)C+(\tfrac{1}{2}-m_1)C \cot^2 z_1, \text{ by form. 32].}$$

Substituting in (43),

$$h_2-h_1=s \cot z_1+(\tfrac{1}{2}-m_1)\frac{s^2}{R_\alpha}+(\tfrac{1}{2}-m_1)\frac{s^2}{R_\alpha}\cot^2 z_1. \qquad (44)$$

If the line is sighted from the other end, a second value will be obtained, and the weighted mean will give the required result.

81. Coefficient of Refraction.—For reciprocal observations from Fig. 64,

$$z_1+m_1C+z_2+m_2C=180°+C,$$

or

$$m_1+m_2=[180°-(z_1+z_2)]\frac{R_\alpha}{s}\sin 1''+1. \quad . \quad . \quad . \quad (45)$$

The refraction coefficients are thus indeterminate from any number of reciprocal observations, since two unknowns are introduced for each equation. If the observations are simultaneous, m_1 is usually assumed equal to m_2; each line will then give a value for m, and the average for the whole area can thus be found by taking the weighted mean. Thus

$$m=[180°-(z_1+z_2)]\frac{R_\alpha}{2s}\sin 1''+\tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad (46)$$

If not simultaneous, the coefficients for the lines radiating from each station may be taken the same, so that in a system of l lines joining p points, there would be p unknown coefficients with l observation equations of the form (45). If $l>p$, the coefficients would be found by a least square adjustment. If the weight of each z be taken proportional to the number of observations, n, then by Part II, (45) would have a weight w given by

$$\frac{1}{w}=\frac{R_\alpha^2\sin^2 1''}{s^2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right), \quad . \quad . \quad . \quad . \quad . \quad (47)$$

that is the weight would be proportional to $\frac{n_1n_2s^2}{n_1+n_2}$.

Bessel assigns weights by the arbitrary formula,

$$w=\frac{n_1n_2\sqrt{s}}{n_1+n_2},$$

on the ground that errors arising from variations in m are of more importance than those from errors in z.

The average value of m as found by the Coast Survey is:

Across parts of the sea, near the coast	0.078
Between primary stations	0.071
In the interior of the country, about	0.065

Clarke, *Geodesy*, p. 281, gives the range in India from -0.09 to $+1.21$.

82. Observed Angle of Elevation in Seconds.—If $\alpha =$ the elevation angle (supposed small) $=90°-z$, (44) becomes

$$h_2-h_1=s\tan\alpha+(\tfrac{1}{2}-m)\frac{s^2}{R_\alpha}=s\,\alpha''\tan 1''+(\tfrac{1}{2}-m)\frac{s^2}{R_\alpha},$$

where α'' is in seconds.

Substituting for R_α and m average values,

$$(\tfrac{1}{2}-m)\frac{s^2}{R_\alpha}=0.000\,000\,0667\ s^2,\ \text{log. const.}=2.82413$$
$$\tan 1''=0.000\,004\,85 \qquad \text{"} \qquad \text{"} \quad =4.68574$$

giving in metric units, α in seconds,

$$h_2-h_1=0.000\,00485\ s\,\alpha''+0.000\,000\,0667\ s^2. \quad . \quad . \quad (48)$$

For s and h_2-h_1 in feet, the last term becomes $0.000\,000\,0202\ s^2$.

It is claimed on the Coast Survey that for $\alpha<5°$ and $s<15$ miles, (48) will give results within the uncertainty of the refraction.

83. Zenith Distance of Sea Horizon.—The line AB, Fig. 64, will be drawn tangent to the sea-level surface at B, giving in the right-angled triangle ABC,

$$R_\alpha+h_1=\frac{R_\alpha}{\cos C},$$

or

$$h_1=R_\alpha\frac{1-\cos C}{\cos C}.$$

By form. 12],

$$h_1=\frac{2R_\alpha\sin^2\tfrac{1}{2}C}{\cos C}=\frac{2R_\alpha\sin^2\tfrac{1}{2}C}{\sin C}\,\frac{\sin C}{\cos C}.$$

By form. 10],

$$h_1=R_\alpha\frac{\sin\tfrac{1}{2}C}{\cos\tfrac{1}{2}C}\,\frac{\sin C}{\cos C}=R_\alpha\tan\tfrac{1}{2}C\tan C,$$
$$=\tfrac{1}{2}R_\alpha\tan^2 C,\ \text{nearly}.$$

But
$$z+mC=90°+C,$$

or
$$C=\frac{z-90°}{1-m}.$$

Substituting, $$h_1=\frac{R_\alpha}{2(1-m)^2}\tan^2(z-90),\ \text{nearly}. \quad . \quad . \quad . \quad (49)$$

Ex. 1. At △ Lewis, a zenith distance of 88° 54′ 16″.7 is read upon △ Morehead, distant 6845m.32. The height of instrument at Lewis is 4′.9 and the point observed at Morehead viz. the bottom of the signal pole, is 24′.7 above the station. Find the correction necessary to reduce the observed zenith distance to the station point. If the azimuth of the line of sight is 222° 15′ and the latitude of Lewis is 42° 41′ 10″.0, find the difference in elevation of the stations, assuming $m = 0.07$.

Solution. Let z_1' = the observed zenith distance uncorrected = 88° 54′ 16″.7.

Then, by (40), the correction, $x'' = \frac{q \sin^2 z_1'}{s \sin 1''}$.

Now $q = 24.7 - 4.9 = 19'.8 = 6^m.035$.

$s = 6845^m.32$	$\sin^2 z_1' = 9.99984$
$\log s = 3.83539$	$\log q = 0.78068$
$\sin 1'' = 4.68558$	
	0.78052
8.52097	8.52097
	$\log x'' = 2.25955$
	$x'' = 181.78$
	$x = 03' \, 01''.8$
	88° 54′ 16″.7
	$z_1 = 88° \, 57' \, 18''.5$

In the direction of the line of sight, R_α, the mean radius of the earth's curvature is taken from Table VI as its logarithm, 6.80450. We can now by substitution in formula (44) obtain the desired difference of elevation.

$$\text{Formula: } h_2 - h_1 = s \cot z_1 + (\tfrac{1}{2} - m_1)\frac{s^2}{R_\alpha} + (\tfrac{1}{2} - m_1)\frac{s^2}{R_\alpha}\cot^2 z_1$$

$$\tfrac{1}{2} - m_1 = 0.5 - 0.07 = 0.43.$$

$\log s = 3.83539$	$\log s^2 = 7.67078$	
$\cot z_1 = 8.26098$	$\text{colog } R_\alpha = 3.19550$	$\cot^2 z_1 = 6.5219$
$\log (\text{I}) = 2.09637$	$\log 0.43 = 9.6334$	
$(\text{I}) = 124.84$	$\log \text{II} = 0.4997$	0.4997
$\text{II} = 3.160$	$\text{II} = 3.160$	
$\text{III} = 0.001$		$\log (\text{III}) = 7.0216$
$h_2 - h_1 = 128^m.00$		$\text{III} = 0.0011$

The required difference of elevation is 128m.00. The problem could also have been solved by formula (43).

Ex. 2. At △ Warren, in latitude 39° 54′ 17″.0, the zenith distance of △ Bald Top is 89° 12′ 47″.6; at Bald Top, in latitude 40° 06′ 32″.9, the zenith distance of △ Warren is 91° 04′ 22″.4, the observations being simultaneous and being reduced to station points. The distance between stations is 35076m.2 and the elevation of △ Warren is 2104m.6 above sea-level. What is the elevation of Bald Top? The azimuth for determining R_α can be determined closely enough by considering the surface a plane and converting difference of latitude into distance by Table V.

PRECISE LEVELING.

84. Instruments.—Precise spirit, or geodetic, leveling is distinguished from ordinary spirit leveling by the use of better instruments and methods and more care in observing.

Some of the more common instruments in use are shown.

With the instruments shown in Figs. 65, 66, and 67 the level is used as a striding level giving greater facility of adjustment for both level tube and collimation, and opportunity to eliminate both errors by reversals in observing. The rear wye can be raised or lowered by a micrometer screw, giving a delicate means of releveling when pointing at the rod. With the Mendenhall level of Fig. 67 this slight releveling cannot affect the height of the telescope as with the others, the pivot being over the vertical axis.

With the new Coast Survey level shown in Fig. 68, the level tube is dropped into the telescope tube down to the cone of sight rays, in order to diminish the lack of parallelism of the two tubes due to locally heating either end of the instrument, thus sacrificing the striding level. The two tubes are cast from an iron-nickel alloy having a coefficient of expansion = 0.000 004 per 1° C., about one-fifth that of brass. The motion with micrometer screw is retained.

In Figs. 66 and 68 the mirror for reflecting the bubble to the observer at the eye end is replaced by a system of prisms which eliminates parallax by giving vertical sight rays upon both ends of the bubble.

The French level, Fig. 66, has a quick leveling ball and socket tripod head which is very stable. The Kern or Swiss level is shown in Fig. 65.

The focusing slide of the telescope should be long and well fitted to preserve parallelism with the line of collimation when sighting at different distances.

The principal instrument constants are:

Fig.	Focal Length.	Diameter of Objective.	Power.	Stadia Ratio.	2 mm. Div. of Level.
65	14″.5	1.5 in.	50	1:231	1″.7 to 3″.4
66	14	1.4	25	1:100–1:200	8.3
67	16	1.5	50	1:200	2.2
68	16	1.7	43 and 32	1:333	1.9

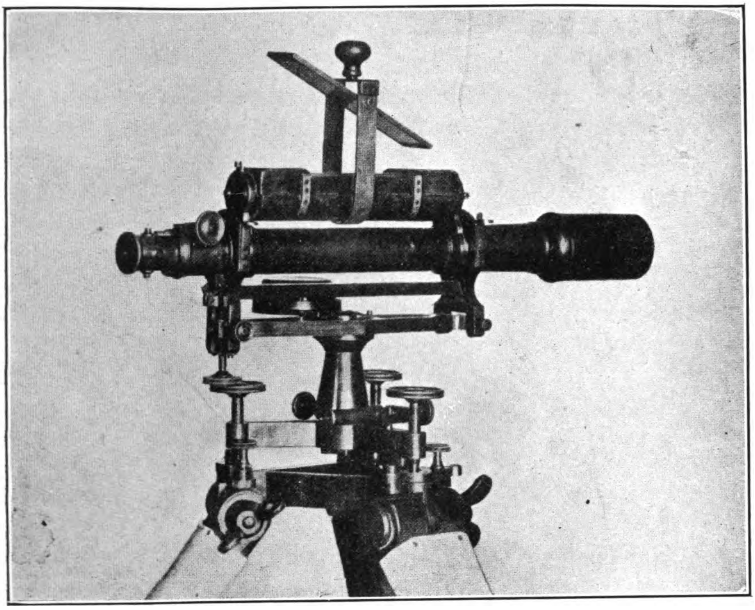


Fig. 65.

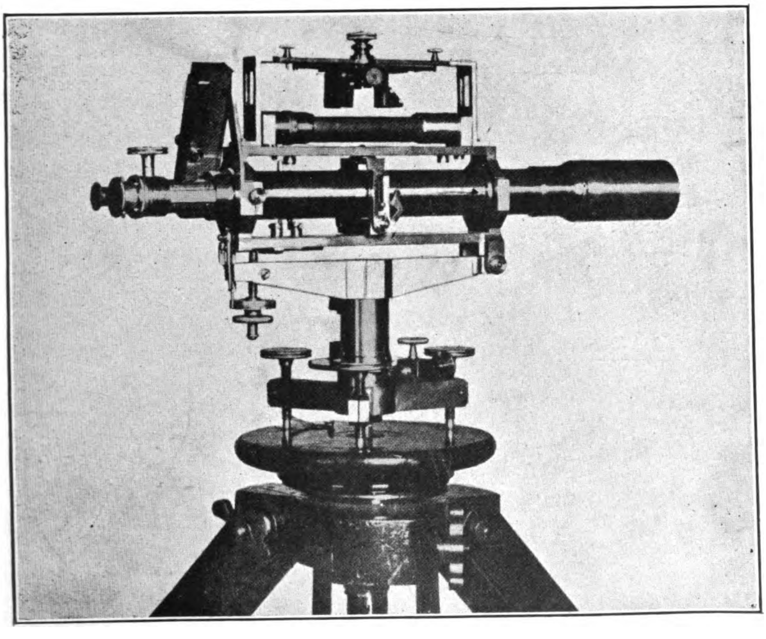


Fig. 66.

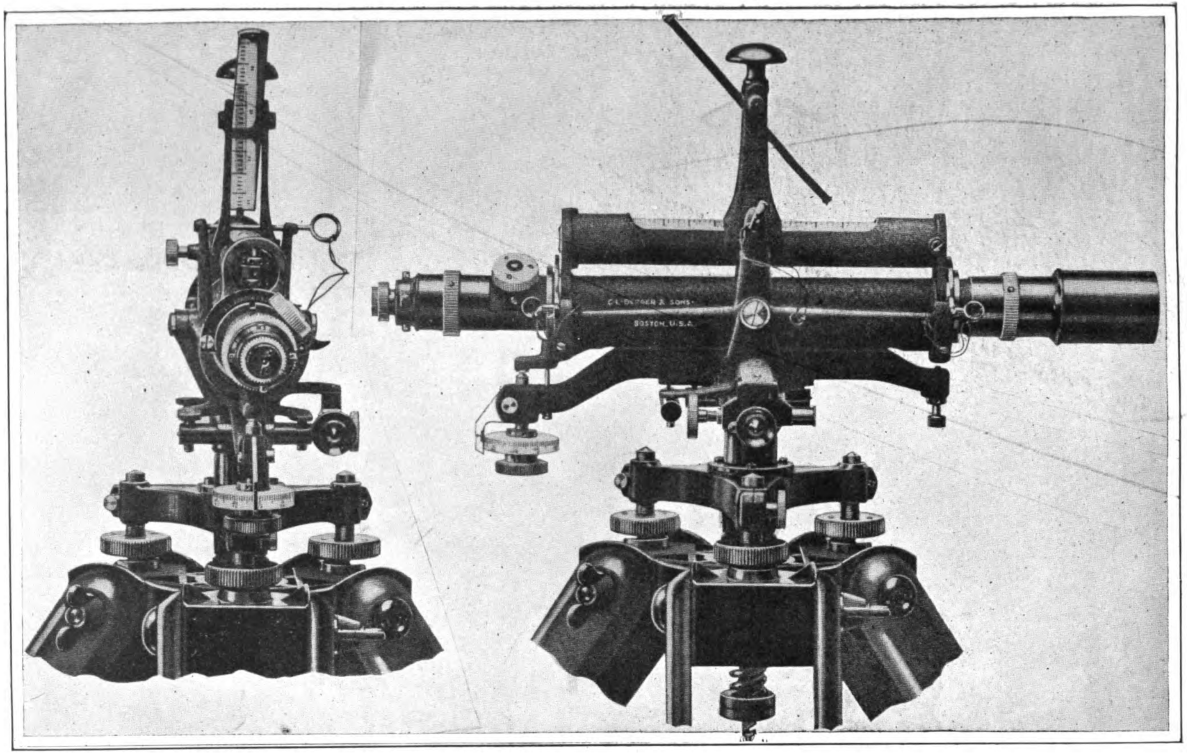


FIG. 67.

It will be noted that these values do not differ materially from those for ordinary levels, except in the sensitiveness of the level tube and in the magnifying power.

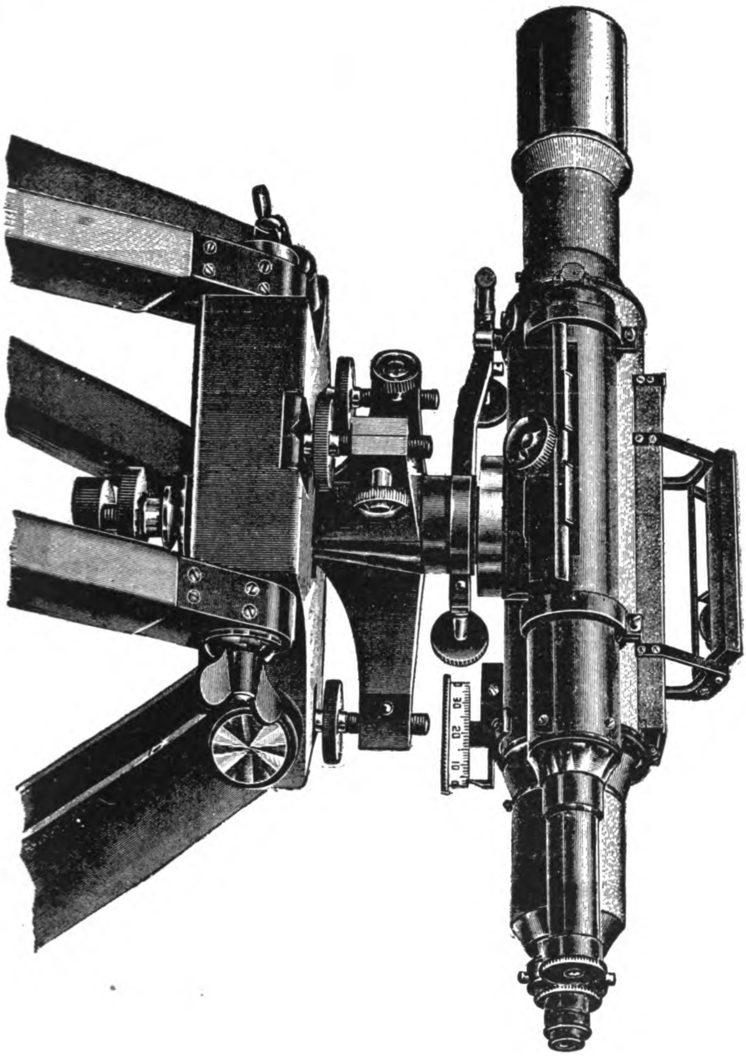


FIG. 68.

85. Rods.—Both target and speaking non-extensible rods are used.

The Kern or Swiss rod is shown in Fig. 69. This is used by the U. S. Corps of Engineers with the Kern level (Fig. 65). The smallest graduations are centimeters, while readings are estimated to millimeters.

The French rod is shown in Fig. 70. It has a line graduation to 2^{mm} printed upon paper and pasted to the rod. The rod is rather flexible. To determine changes in length due to changes

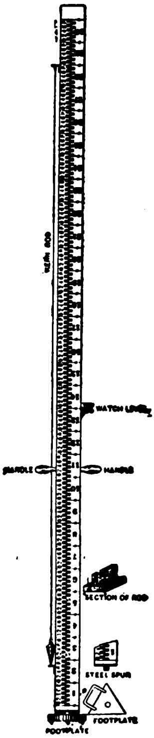


FIG. 69.

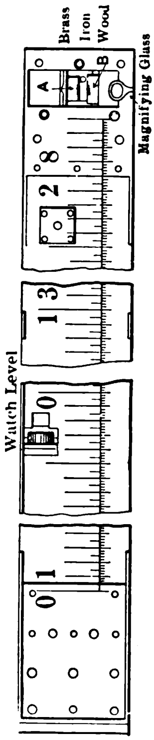
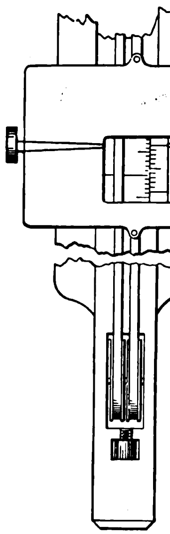
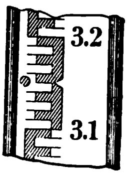
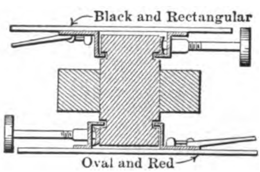


FIG. 70.





3-10 Size.



1-5 Size.

FIG. 71. FIG. 71a.

in temperature and moisture an iron and a brass bar are inserted side by side near the center line and fastened to the base plate, while at the top a scale is attached to the brass and one to the wood, each being read by an index on the iron.

The brass scale is so graduated that each division represents an expansion of 1^{cmm} * per meter of the iron bar; each division of the scale on the wood gives an expansion of 1^{cmm} per meter of

* 1cmm = one centimillimeter = .01 millimeter.

the wood. The sum of the two readings (A and B) will thus give the total change in length of the wooden rod.

The U. S. Geolcgical Survey rod shown in Fig. 71 is a double target rod made by W. and L. E. Gurley, of white pine impregnated with boiling paraffine to a depth of $\frac{1}{8}''$. It is graduated on both sides and each side has a target, one oval and red, the other rectangular and black. The targets are handled by endless tapes as shown, the length of the rod being a little over 10 feet. The steel base shoe has an area of half a square inch. The two targets are for use on "double-rodded lines," where two sets of turning points and two sets of notes are carried through with one instrument, the instrument man setting the rear- and front-rod targets as usual for the first set of turning points, then the front- and rear-rod targets of the other faces for the second set; afterwards checking both sets of target readings as the instrument man and rear rodman move forward past each other.

The Geological Survey speaking rod is shown in Fig. 71*a*. The unit, which is 0.2 foot, is divided to tenths and these divisions are read to fifths, or to 0.004 foot. The notes are kept on the 2-foot basis to correspond, requiring all derived elevations to be doubled. The shaded portion is red, the other portions black, on a white ground.

The Coast Survey rod is shown in Fig. 72. The centimeter graduations are on the edge of the +, which is $2^{cm}.2$ wide. Fig. 72*a* shows the foot plate and foot pin which are used to support the rod at turning points. The latter is the more frequently used.

FIG. 72.

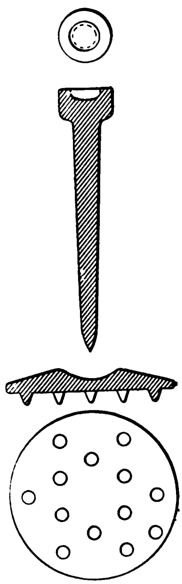


FIG. 72*a*.

The center of the bell-metal foot is in the plane of the graduation. Silver-faced plugs are placed 1^m apart and the distances between them checked by steel tape for field comparisons. A thermometer is attached for temperature, and a disk level for plumbing as with the others. The pine is soaked in boiling paraffine for its entire thickness, then all but about 19% is driven out, which increases the weight and does away with moisture changes. The coefficient of expansion is 0.000 0042 per 1° C., a value nearly the same as for untreated wood.

86. U. S. Corps of Engineers' Method.—In observing by this method the instrument is leveled and pointed at the rear rod; both ends of the bubble are read, the three wires and the level again, for the back sight; similarly for the front sight. The length of sight is limited to 100^{m}, and the difference between front and back sight to 10^{m}. A heavy canvas umbrella is used to protect the instrument from the sun, or sometimes a tent if the weather is windy.

Each rod reading is corrected for observed inclination by the formula

$$\text{Correction} = 4id\,\frac{lm \tan 1''}{4} = 4idA, \quad . \quad . \quad . \quad . \quad (50)$$

where $4i = E + E' - (O + O')$, the sum of the two eye-end readings minus the sum of the two object-end readings of the level; $d=$ stadia interval; $l=$value of 1^{d} of level; $m=$stadia constant; $A = \frac{1}{4}lm \tan 1''$, a constant.

The difference in elevation between two bench marks is corrected by the formula

$$\text{B.M. cor.} = (d_1 - d_2)cm \tan 1'', \quad . \quad . \quad . \quad . \quad (51)$$

where $d_1=$sum of stadia intervals for back sights; $d_2=$sum of stadia intervals for front sights; $c=$inclination of line of collimation in seconds when the bubble is in the center (+ if object end low); c must include inequality of pivots, level error and collimation error.

The level tube is adjusted until within two divisions and the collimation until the mean of the three wires for direct and for reverse position upon a rod at a distance of 50^{m} do not differ more than $2^{mm}.5$. Readings are then taken every morning, and at other times when there is reason to suspect disturbances, for the level tube and collimation errors to use in (51). By keeping the sums of the stadia intervals, as in the record shown, these can be made equal in closing on a B.M., so that the correction (51) will disappear.

Steel pins are frequently used for turning points instead of the foot plate shown in Fig. 69.

FORM OF RECORD.

Locality,

Date,

Direction,

BACK SIGHT.

Wire Readings.	Means.	Wire Intervals.	Bubble.		Rod.	Remarks.
			E.	O.		
1009 1184 1362	1185.0	175 178 353	11.5	11.5	10	T.B.M. 30

Observer, Level No.

Recorder, Tube No.

FRONT SIGHT. *Upon opposite page.*

Wire Readings.	Means.	Wire Intervals.	Bubble.		Rod.	Remarks.
			E.	O.		
1686 1873 2063.5	1874.2	187 190.5 377.5	11.5	11.5	13	P.B.M 20

87. French Government Method.—In this method the bubble is kept in the center when sighting; the three wires are read on the back sight and also on the front sight; the level is reversed, the telescope rotated 180° about the line of collimation, and the recorder reads the middle hair on the front rod and then upon the back rod. These reversals tend to eliminate the error of level and of collimation and those portions of the errors of refraction and instability which are proportional to time. If the discrepancy between the first and last readings exceeds a certain amount, both sets are repeated.

The longest sight is limited to 100^m and the greatest difference to 10^m. Wooden hubs are used for turning points, and the same ones are intended to be used on the return line between each two bench marks.

The only corrections required are for rod errors, but since these include scale errors (paper scales), they have to be made separately for each set up, taking into account the change in length of the rods as shown by comparisons with the inclosed steel bar.

FORM OF RECORD.

Back Sight.				Front Sight.				Rod Corrections.		Name of Rod.	B.M. or Turning Point.	Remarks.	Atmospheric Conditions.
Stadia Interval.	Stadia Threads.	Center Thread.	Difference First and Second.		Center Thread.	Stadia Threads.	Stadia Interval.	+	−				
Br't forward		109 580			427 555			5	340	2	B.M. 8	Began $7^h\ 15^m$ a.m.	Rain
367	20 36					08 25	366						
		+24 03			−11 91			40				Rod 1	
366	27 69		−2	+2		15 56	365					$A=66.5$; $B=36$	
		−24 05			+11 93					1	T.P. 9	$A+B=102.5$	
371	03 15					15 04	357					Rod 2	
		+06 86			−18 61				5			$A=87$, $B=35$	
372	10 58		+3	−4		22 18	357					$A+B=122$	
		−06 83			+18 57								Sun
										2			
Correc.		+171 350	Totals		−488 575	Correc.		45	345				
$D+e>0$			$2D=$		−317 225	$D+e<0$							
			$D=$		−158 613	$-15^m.8913$		$e=-300$					

Rod readings to millimeters; rod corrections to decimillimeters.

The change of observers adds to cost in requiring a good observer for recorder, and it adds to delay and instability, especially if the cross-hairs have to be re-focused on account of the change.

88. U. S. Geological Survey Method. — For double-rodded lines and the double target rods of Fig. 71, the rear rodman holds on the turning point of line *A* and clamps his red target when covered by the cross-hair; the front rodman then holds on the next turning point of line *A* and clamps his red target at the proper height; he then holds on the turning point of line *B* and clamps his black target; the rear rodman then holds on the rear turning point of line *B* and clamps his black target.

Separate notes are kept for the two lines (which are claimed to be equivalent to lines run in opposite directions), while the instrument man checks all four rod readings as he and the rear rodman move forward.

The bubble is kept in the center when sighting. Steel pegs are preferred for turning points. The level is adjusted daily, or oftener when necessary.

Attention is called to the fact that the length of sight should be kept so nearly constant that the focus of the telescope will not require changing for front or back sight during the day, and that if it should require changing on account of grades or atmospheric disturbance requiring shorter sights, then the level should be readjusted for the new position of the slide. It does not appear, however, that this restriction is enforced, nor does it appear necessary with a well-made precise-level telescope.

With target rods the rodman is usually required to keep a separate set of notes.

89. Coast Survey Methods.—In the old method (*Report*, 1879, App. 15) the Vienna or Stampfer level, slightly modified, was used. Its general construction is like the Kern. The rod is a non-extensible pine rod graduated to centimeters on the front edge of the + as a speaking rod, and on a brass scale on the side of the front portion for the target. The target is moved by a chain similar to the tape of Fig. 71.

To take a reading: *a*. The bubble is brought near the center and the target clamped to correspond; the bubble is then accurately centered and the micrometer screw of the rear wye read;

the target is bisected by turning the micrometer screw and the screw again read. *b*. The level is reversed, the bubble brought to the center, and the target bisected, and both screw readings taken. *c*. The telescope is rotated 180° about the optical axis, the bubble brought to the center, and the target bisected, and both screw readings recorded. *d*. The level is made direct, the bubble brought to the center, and the target bisected, and both screw readings recorded.

FORM OF RECORD.

BACK SIGHT.

No. of Station.	Telescope.	Level.	Micrometer. Horizon.	Micrometer. Target.	Dist. Wire.	Edges of Target.	Rod Reading and Temp.
	Runn	ing on ro	ad. Mud	dy. Rod	*F* on B.M.	No. 61.	
9	*I*	*R*	17.102	17.113			
		D	.126	.117			
	E	*D*	.127	.107		0.860	
		R	.117	.111	2.240	0.810	0.833
			17.118	17.112	140.5	0.835	
			−0.6				

FRONT SIGHT.

No. of Station.	Telescope.	Level.	Micrometer. Horizon.	Micrometer. Target.	Dist. Wire.	Edges of Target.	Rod Reading and Temp.
		Weather	cloudy.	Brisk wind	. Rod *E*.		
9	*I*	*D*	17.107	17.102			39°
		R	.097	.102			
	E	*R*	.095	.098		1.332	
		D	.107	.092	2.710	1.282	1.3072
			17.101	17.098	140.3	1.307	
			−0.3				

The stadia hairs and the edges of the target are then read by the observer, while the target and the rod thermometer are read by the rodman. Having the value of 1^d of the micrometer screw, and the distance to the rod, the rod correction for each of the four readings can be computed by a formula similar to (50); the average of the four added to the target reading will give the corrected rod reading.

The method of double rodding is in use, as also that of running a single line through and checking back.

For the new method, introduced in 1899 and slightly modified in 1900, the following abstract has been taken from the general instructions given in the *Report* for 1903, App. 3.

Except when specific instructions are given to proceed otherwise, all lines are to be leveled independently in both directions. The distance between successive bench marks shall be less than 15^{km}. The line of levels is to be broken up by temporary bench marks from 1^{km} to 2^{km} apart, unless special conditions make shorter distances advisable. The observer is to secure as much difference of conditions for the measurements in the two directions as is possible without materially delaying the work, it being especially desirable to make the one in the afternoon if the other was made in the forenoon, and *vice versa*. If the measurements in opposite directions between two bench marks differ by more than $4^{mm}\sqrt{\text{distance in kilometers}}$, both are to be repeated until two such measurements fall within the limit. If any measurement differs more than 6^{mm} from the mean of all the measurements over the section, it shall be rejected. If a measurement differing by a less amount is rejected the reason for the rejection must be fully stated in the record. Whenever a blunder is discovered and corrected, the measurement may be retained if there are at least two others over the same section not subject to any such uncertainty. In reading the rod, all three hairs are to be used and the bubble kept in the center. At odd-numbered instrument stations the rear rod is read first, at even-numbered the front rod, the time interval between readings to be as small as possible. The rod thermometer is to be read at each station. The maximum length of sight is placed at 150^{m}, and this only under the most favorable circumstances. The maximum difference between the front sight and the corresponding back sight is limited to 10^{m}. The actual difference is to be made as small as is feasible by the use of good judgment, without taking extra time for the purpose. The stadia intervals are recorded and their continuous sums computed for both the front sights and back sights. These sums are kept as nearly equal as is feasible without spending extra time for the purpose. The greatest difference is limited to a quantity corresponding to 20^{m}.

The error of level is determined each day by using a regular front sight reading, with a special back sight reading with the rod at a distance of about 10^m, then setting up about 10^m behind the front rod and reading both rods again. The correction constant, which is the ratio of the required rod correction to the corresponding stadia interval, is *

$$C=\frac{(\text{sum of near rod readings})-(\text{sum of distant rod readings})}{(\text{sum of dist. stadia interv.})-(\text{sum of near stadia interv.})}. \quad (52)$$

The level should not be adjusted for $C<0.005$. The observer is advised not to adjust for $C<0.010$ and required to adjust for $C>0.010$. The instrument is shaded from the direct rays of the sun during the observations and the movement from station to station.

Notes for future use in studying leveling errors should be inserted in the record, indicating the time of beginning and ending for each section, the weather conditions especially as to sun and wind, whether the line is run towards or away from the sun, etc.

The 3^m interval between metallic plugs is measured for each rod with a steel tape twice each month and at the beginning and end of the field season with a steel tape kept by the party for the purpose. The object is to detect any changes in length and not to determine the actual length.

The accidental errors appear to increase rapidly when the length of sight is pushed beyond the limit at which it is difficult to estimate to millimeters accurately, and considerable re-running will be necessary. The observers are urged to lengthen sights for speed up to the limit fixed by too much waste of time in having to re-run lines. The maximum speed apparently is reached when from 5 to 15 per cent of the lines have to be re-run. The

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* Let n_1, n_2 be the near rod readings; d_1, d_2 the distant rod readings; s_1, s_2 the near stadia intervals; s_1', s_2' the distant stadia intervals. The true difference in elevation,

From the first, $=n_1+Cs_1-(d_1+Cs_1')$.

From the second, $=d_2+Cs_2'-(n_2+Cs_2)$.

Placing these values equal and solving for C,

$$C=\frac{(n_1+n_2)-(d_1+d_2)}{(s_1'+s_2')-(s_1+s_2)},$$

as given above.

observers have learned to shorten sights whenever the upper and lower stadia intervals differ frequently by more than a selected limit, this limit being fixed by each observer from his own experience. The observers are urged to watch the rate of divergence between forward and backward lines and to make an effort to keep it small. It has been the opinion of those in charge and of many of the observers, that a large steady rate of divergence is in general due to a systematic rising or settling of rod supports, either foot plates or pins. The usual practice has been for the observer, when he noticed a steady tendency of his lines to diverge rapidly, to study the manner in which the foot plates or pins were handled and to make tentative changes until the tendency disappeared.

Correction is made for curvature and refraction, and for level when the stadia intervals differ for front and back sights; also for length of rod.

FORM OF RECORD.

Number of Station.	Thread Reading, Back Sight.	Mean.	Thread Interval.	Sum of Intervals.	Rod and Temp.	Thread Reading, Fore Sight.	Mean.	Thread Interval.	Sum of Intervals.
Time 2 P.M.									
43	0674		99		V	2683		99	
	0773	0773	99		38	2782	2782.3	100	
	0872					2882			
			198					199	
44	0925		106		W	2415		103	
	1031	1030.3	104		35	2518	2518.0	103	
	1135					2621			
			210	408				206	405

Name and temp. of rear rod given.

The corrections between bench marks are summed from tables or slide rule and entered on the computation sheet separately.

90. Inequality of Pivots.—The level is set up on a pier or other firm support where it is protected from air currents and from sudden changes of temperature and the bubble brought to the center. The telescope is changed end for end in the wyes and the bubble read without reversal. The out of level, if any, must be twice (within the errors of observation) the inequality of pivots referred to the supporting wyes, or four times the error referred to the telescope axis on the basis of circular collars. The observations should be repeated until the desired accuracy is secured.

Ex. 1

FAUTH PRECISE LEVEL.

May 15, 1890. Landers } obs.
Wadsworth

Eye End of Telescope.	Level.			East End High.		Eye End Large.
	D or R.	East.	West.			
W	D	41.4	6.5	$-1^d.05$	-1.27	
	R	9.	44.	$-1\ .50$		$0^d.91$
E	D	7.2	42.	$0\ .4$	0.55	
	R	43.2	8.2	$0\ .7$		1 .18
W	D	40.8	5.8	$-1\ .7$	-1.82	
	R	9.5	44.4	$-1\ .95$		0 .75
E	D	7.8	42.8	$-0\ .3$	-0.32	
	R	42.8	7.9	$-0\ .35$		0 .80
W	D	40.5	5.5	$-2\ .$	-1.92	
	R	9.3	44.4	$-1\ .85$		1 .06
E	D	7.6	42.6	$-0\ .1$	0.2	
	R	43.	8.	$0\ .5$		1 .21
W	D	40.4	5.4	$-2\ .1$	-2.22	
	R	9.8	44.9	$-2\ .35$		1 .18
E	D	7.7	42.7	$-0\ .2$	0.15	
	R	43.	8.	$0\ .5$		1 .00
W	D	40.8	5.8	$-1\ .7$	-1.85	
	R	9.5	44.5	$-2\ .$		1 .03
E	D	7.5	42.5	$0\ .$	0.2	
	R	42.9	7.9	$0\ .4$		Mean = 1 .01

Level graduated from eyepiece toward objective with 25 for center. Value 1^d of level $= 3''.8$.

Referred to telescope axis. Eye end large $= \frac{1}{2} \times 1.01 \times 3.8 = 1''.92$. Correction to rod reading negative.

Ex. 2. If the collars are 10 inches apart and the wye angle for telescope and level is 90°, find the inequality of the collars in inches for the value $1''.92$ given above.

91. Rod Correction.—For the paraffined rods and those where a brass scale is used, the temperature at which the rod is standard can be found by comparison with a standard. A table of double entry can then be made out, or a slide rule used, for the correction for any observed temperature and rod reading, it being the product of the temperature increment, the rod reading, and the coefficient of expansion, and positive when the rod is too long or the reading too small.

For the Kern rod which changes length with moisture as well as with temperature the actual error per unit length can be determined from day to day by comparison with a standard tape and the corresponding correction applied if appreciable.

For the French rod the paper scales require correction for graduation errors, and the wooden rods corrections for changes in length as denoted by the A and B readings. This is accomplished by comparing the rods with a standard and at the same time reading the scales A and B. The corrections are platted on cross-section paper as ordinates with rod readings for abscissas and the correction curve drawn. An equalizing line is also drawn through the origin, which separates the correction into two parts, one proportional to the rod reading and the other a local graduation correction. It is assumed that only the first is affected by a change in the length of the rod.

To obtain the correction graphically, the straight line correction, say 150^{cmm} per 1^{m} for $A+B\doteq 135$, is laid off from D on the vertical DE to E, Fig. 73, to a scale of 15 to 1 for the rod length $2^{m}.8$. This would be $150 \times 2.8 \times 15 = 6300^{cmm} = 63^{mm}$. An arm DF is pivoted at D and rotated until DF makes an angle with DE large enough to allow for the greatest anticipated value of $A+B$, as explained later.

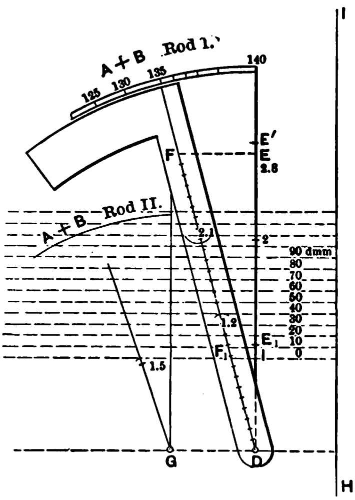

Fig. 73.

Project E horizontally to F and divide DF into twenty-eight equal parts. These correspond to the decimeter divisions of the rod. With these points as centers, and the local graduation corrections as radii (to scale) describe arcs, each from its proper center, above if positive and below if negative, and mark each with the decimeter division to which it belongs. The vertical distances from D to horizontal tangents to these arcs will give the total corrections for the corresponding divisions.

If the rod expands so that $A+B=136$, the correction for each division will be increased at the rate of 1^{cmm} per 1^{m}. Hence lay off 2.8^{cmm} (to scale$=0.42^{mm}$) above E to E' and rotate the arm until F is on a horizontal through E'. The straight-line corrections will thus be increased in the proper ratio as projected by horizontal tangents on DE, while the local or graduation errors proper will not be affected, all tangents to a circle being at the same distance from the center. By repeating this process the $A+B$ arc can be graduated as shown. A better method is by finding the different values of $\alpha=FDE$ from its cosines and laying them off with a protractor. For the assumed correction of 150^{cmm}, α was taken $=24°$. Hence the radius $DF_1=\dfrac{150}{\cos 24°}=164.2$; $\dfrac{1}{164.2}=0.00609$, the increment to the cosine for each unit added to $A+B$. Cos $24°=0.91355$, so that the different values of $A+B$ are readily found.

The correction diagram for the other rod is placed on the same sheet with the center at G on the horizontal through D. A celluloid sheet is ruled with horizontal lines 5^{dmm} apart (shown dotted in diagram) to the scale 15/1 and kept in position by the guide strip HI.

Both rods are too short, giving plus corrections. Hence the correction for a set-up is plus when the rear rod correction is the greater, and vice versa. To take out a correction for a set-up: set each arm to the correct $A+B$ reading; slide the celluloid sheet until the zero coincides with rod II reading and read the scale for rod I reading. Thus if $(A+B)_{\text{I}}=135$, $(A+B)_{\text{II}}=118$; the correction for a back-sight reading of 2.1^m on I and a front-sight reading of 1.5 on rod II would$=+102^{dmm}$.

92. Accuracy and Cost of Results.—The authors of *Lever des Plans et Nivellement* (Paris, 1889) estimate the probable error for a set-up with the French level for sights 75^m long as follows:

1. *Error of level.* The eye can detect a difference of $\frac{1}{2}^{mm}$ in the readings of the ends of the bubble with the 3^{mm} divisions on the tube. This gives a probable inequality of about 2^{dmm}, or a probable out of level of 1^{dmm}. This would give the same uncertainty for a rod reading at a distance equal the radius of curvature, or 50^m, or 1.5^{dmm} at 75^m.

2. *Error of estimation.* With a power of 25, the centimeters of the rod at 75^m appear of the same size as millimeters at 0.3^m.

Under these conditions tenths can be easily estimated with a probable error of about 0.33^{dmm}, giving 3.3^{dmm} when referred back to the rod.

3. *Errors due to temperature changes.* Experience has shown these to be as great as those of (2).

Combining, the total for a reading (see Part II),

$$r=\sqrt{(1.5)^2+(3.3)^2+(3.3)^2}=5^{dmm}.$$

For a set-up, T.P. to T.P.,

$$r'=\sqrt{r^2+r^2}=r\sqrt{2}.$$

With 75^m sights there are $6\frac{2}{3}$ set-ups per 1^{km}, which with the four observed differences between each pair of T.P.'s would give the resulting probable error per 1^{km},

$$r_k=r\sqrt{\frac{2\times 6.66}{4}}=9^{dmm},$$

which agrees with the results found for the fundamental French lines.

The above supposes all constant or systematic errors eliminated by the methods of observation or by applying computed corrections.

The principal constant errors recognized are:

1. *The variation of gravity with latitude.* This results in making the distance between two level surfaces vary inversely with g, the work required to raise a unit mass from one to the other, or hg, being constant. The observed difference in height of two points would thus depend on the height of the line of levels run between them. Heights above sea level obtained by direct measurement are called orthometric; obtained on the basis of work done in raising a unit mass, dynamic; the differences are usually within the errors of observation, but in rugged country they may be greater. For a full discussion, see Helmert, *Hoheren Geodäsie*, or *Lever des Plans* . . . .

2. *Variations of refraction* with height of line of sight, with character of ground surface over which the line passes, and with

the time of day. In ascending or descending long grades this becomes cumulative and may easily exceed the accidental errors unless short sights are taken.

3. *Change in height of instrument or turning point* due to settlement or springing up of ground. This has long been one of the reasons assigned for greater discrepancies between lines run in opposite directions as compared with those run in the same direction.

4. *Change in collimation and level error* due to heating the end of the telescope nearest the sun. This is the principal reason assigned by the Coast Survey for the change in method introduced in 1899.

In *Proc. Am. Soc. Civ. Engrs.*, Vol. 26, p. 888, the probable error per kilometer is given for some 1200 miles of Coast Survey levels averaging 1.07^{mm}, and for some 1500 miles of U. S. Engr. Corps levels averaging 0.69^{mm}. These apparently are from circuit closures.

In checking forward and backward between benches the limit is placed $=4^{mm}\sqrt{\text{kilometers}}$, as already stated.

The cost is estimated by Mr. Molitor (*Proc. Am. Soc. Civ. Engrs.*, Vol. 26, p. 897) at $24 per mile for a double line with permanent bench marks about 0.6 mile apart. On p. 1160 it is stated by Mr. Hayford that the total cost of the 1899 work of the Coast Survey was $13.55 per mile.

Seven minutes per station is given as about the average time for the 1899 work, with a record of 111 stations in $9^h\ 20^m$ on June 20, with 40^m to 80^m sights, and of 10.3 miles July 14 in 7.4 hours, with 80^m to 110^m sights. Later he gives the cost during the period 1900–1904 (a total of 3900 miles) at $10 per completed mile, including salaries, transportation, and bench marks. The rate of progress was 64 miles of completed line per month for each observer. The error of closure from the Atlantic and Gulf to the Pacific, a length of 4800 miles of levels from the first and 3500 from the second, was 0.615 foot. It is assumed that this discrepancy is probable due to inaccuracy in leveling rather than to a difference in sea-level.

93. **Datum.**—Mean sea-level is the ultimate datum to which all land levels should be referred. It can be obtained approximately from the mean of two consecutive high tides and the

intermediate low tide. For more accurate results a permanent bench mark and a tide gage should be established and readings taken for a semi-lunation, or longer,—a year or more being necessary for a datum for precise levels.

The zero of the tide gage should be occasionally referred to the bench mark to guard against disturbance.

The total range in the six annual means taken by the Coast Survey at Sandy Hook, N. J., with a self-recording gage was 0.322 foot, and in the annual means at Biloxi, Miss., was 0.100 foot.

CHAPTER VI.

FIGURE OF THE EARTH.

94. Meridian Section, Coordinates of Point.—In reducing geodetic data the earth is usually assumed to be an ellipsoid of revolution. The dimensions given in Table I for the Clarke ellipsoid best satisfy the degree measurements which had been made at the time (1866) when they were derived.

In the meridian section, through M, Fig. 74, let MH, the normal extended to the axis of rotation, $=N$; MG, the normal extended to the major axis, $=n$; MGD, the geographic latitude, $=\phi$; MCD,

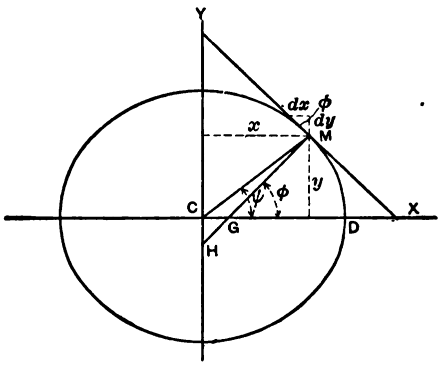

FIG. 74.

the geocentric latitude, $=\psi$; $R=$ radius of curvature of the meridian; x and $y=$ coordinates; a and $b=$ semi-axes.

The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \ldots \ldots \ldots \quad (a)$$

Differentiating, $$\frac{2xdx}{a^2}+\frac{2ydy}{b^2}=0,$$

or $$\frac{dy}{dx}=-\frac{b^2x}{a^2y}. \quad \quad (b)$$

From the differential triangle, Fig. 74,

$$\frac{dy}{dx}=-\cot\phi=-\frac{\cos\phi}{\sin\phi}. \quad \quad (c)$$

Equating (*b*) and (*c*),

$$\frac{b^2x}{a^2y}=\frac{\cos\phi}{\sin\phi},$$

or $$\frac{x}{y}=\frac{a^2\cos\phi}{b^2\sin\phi}.$$

Multiplying, $$\frac{b^2x^2}{a^2y^2}=\frac{a^2\cos^2\phi}{b^2\sin^2\phi}. \quad \quad (d)$$

If the eccentricity $=e$, form. 22],

$$e^2=\frac{a^2-b^2}{a^2},$$

or $$b^2=a^2(1-e^2).$$

Substituting in (*a*),

$$x^2(1-e^2)+y^2=a^2(1-e^2). \quad \quad (e)$$

From (*d*), $$x^2(1-e^2)^2\sin^2\phi-y^2\cos^2\phi=0. \quad \quad (f)$$

Multiplying (*e*) by $\cos^2\phi$ and adding to (*f*),

$$x^2(1-e^2)\ (\cos^2\phi+\sin^2\phi-e^2\sin^2\phi)=a^2(1-e^2)\cos^2\phi,$$

or $$x^2=\frac{a^2\cos^2\phi}{1-e^2\sin^2\phi}. \quad \quad (53)$$

Multiplying (*e*) by $(1-e^2)\sin^2\phi$ and subtracting (*f*),

$$y^2=\frac{a^2(1-e^2)^2\sin^2\phi}{1-e^2\sin^2\phi}. \quad \quad (54)$$

Putting $$1-e^2 \sin^2 \phi = r^2,$$

$$x = a\frac{\cos \phi}{r}, \quad y = \frac{a(1-e^2)\sin \phi}{r}. \quad . \quad . \quad . \quad . \quad (55)$$

95. Principal Radii of Curvature.—Since arcs subtending the same angle are to each other as their radii, the radius of curvature of the meridian

$$R = \frac{ds}{d\phi} = -\frac{dx}{\sin \phi d\phi}. \quad \text{(See Fig. 74.)}$$

dx is negative because x decreases as ϕ increases.

Differentiating (55),

$$\frac{dx}{d\phi} = \frac{-ar \sin \phi + ar^{-1}e^2 \sin \phi \cos^2 \phi}{r^2}.$$

Multiplying the second member by $\frac{r}{r}$, and substituting the value of r^2,

$$\frac{dx}{d\phi} = -\frac{a(1-e^2)\sin \phi}{r^3}.$$

Substituting, $$R = \frac{a(1-e^2)}{r^3} = \frac{a(1-e^2)}{(1-e^2 \sin^2 \phi)^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad (56)$$

The section by a plane through the normal MH and perpendicular to the meridian is called the prime vertical. It is tangent to the parallel of latitude at M, and its center of motion, or of curvature, is on the axis at H as the point M moves past the meridian plane. Hence, if ρ = radius of curvature of prime vertical and N = the normal ending at minor axis, we have, from Fig. 74 and (55),

$$\rho = N = \frac{x}{\cos \phi} = \frac{a}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Dividing (57) by (56),

$$\frac{N}{R} = \frac{r^2}{1-e^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

This ratio is often of value as indicating the deviation of the surface at any point from that of a sphere. Employing the Clarke ellipsoid of 1866, we have:

For $\phi=$ 0°	$\frac{N}{R}=1.0067$	$\phi=45°$	$\frac{N}{R}=1.0034$
15°	1.0058	60°	1.0017
30°	1.0050	90°	1.0000

The geometrical mean of N and R is taken for the mean radius of curvature at the point, i.e.,

Mean radius of curvature,

$$R'=\sqrt{NR}. \quad \ldots \ldots \ldots \quad (59)$$

We also have the following:

Radius of parallel,

$$R_p=x=\frac{a \cos \phi}{r}=N \cos \phi. \quad \ldots \ldots \quad (60)$$

Normal ending at major axis,

$$n=\frac{y}{\sin \phi}=\frac{a(1-e^2)}{r}=(1-e^2)N. \quad \ldots \quad (61)$$

Geocentric latitude (Fig. 74),

$$\tan \psi=\frac{y}{x}=(1-e^2) \tan \phi. \quad \ldots \ldots \ldots \quad (62)$$

$\phi-\psi$ varies from 0° at the equator to 11′ 40″ in latitude 45° and 0° again at the pole.

96. Radius of Curvature for a Given Azimuth.—A plane through the normal at any point of the ellipsoid will cut the surface in an ellipse. Its equation is found by taking a point on the curve and expressing its coordinates in the equation of the surface in terms of the coordinates of the curve. In Fig. 75 the plane is passed through the normal MG in latitude ϕ, and it cuts the surface in the curve PM. It makes an angle α with the meridian plane $Z'X'$. The coordinates for the surface are x', y', z', with origin at O'. Those for the plane curve are x and y, with origin at G and axes GX and GY.

The equation of the surface is

$$a^2z'^2+b^2(x'^2+y'^2)=a^2b^2. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Noting that

$$O'G=(N-n)\cos\phi$$
$$=\frac{a}{r}\Big[1-(1-e^2)\Big]\cos\phi$$
$$=\frac{a}{r}e^2\cos\phi$$
$$=Ne^2\cos\phi,$$

we have for the point P,

$$x'=O'G+GS-NA=Ne^2\cos\phi+y\cos\phi-x\cos\alpha\sin\phi;$$
$$y'=PN \qquad =x\sin\alpha;$$
$$z'=SQ+QA=y\sin\phi+x\cos\alpha\cos\phi.$$

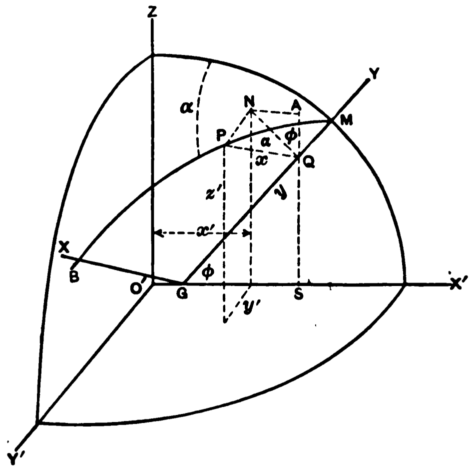

FIG. 75.

Substituting in (a) and using for b^2,

$$a^2(1-e^2),$$

$$x^2[1-e^2(1-\cos^2\alpha\cos^2\phi)^*]+y^2(1-e^2\cos^2\phi)+xy(2e^2\sin\phi\cos\phi\cos\alpha)$$
$$-2x(1-e^2)Ne^2\cos\phi\sin\phi\cos\alpha+y\,2e^2(1-e^2)N\cos^2\phi$$
$$=(1-e^2)(a^2-N^2e^4\cos^2\phi),$$

* Form. 1] is employed in transforming the coefficient of x^2.

which is the equation of the oblique section through the normal with origin at G. For this we may write

$$Ax^2+By^2+Cxy+Dx+Ey=F. \quad . \quad . \quad . \quad . \quad (b)$$

By formula 35], $$R_\alpha=-\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.^{*}$$

Differentiating (b),

$$\frac{dy}{dx}=-\frac{2Ax+Cy+D}{2By+Cx+E};$$

$$\frac{d^2y}{dx^2}=-\frac{2A+2B\left(\frac{dy^2}{dx^2}\right)+2C\frac{dy}{dx}}{2By+Cx+E}.$$

For point M, $\quad x=0, \quad y=n=(1-e^2)N.$

Substituting, and restoring the values of A, B, C, D, and E,

$$\frac{dy}{dx}=0,$$

$$\frac{d^2y}{dx^2}=-\frac{1-e^2(1-\cos^2\alpha\cos^2\phi)}{(1-e^2\cos^2\phi)(1-e^2)N+e^2\cos^2\phi(1-e^2)N}$$

$$=-\frac{(1-e^2)(\sin^2\alpha+\cos^2\alpha)+e^2\cos^2\alpha(1-\sin^2\phi)}{N(1-e^2)}\cdot\frac{R}{R}$$

$$=-\frac{R\sin^2\alpha+N\cos^2\alpha}{NR},$$

or from 35], since $\frac{dy}{dx}=0$,

$$R_\alpha=\frac{NR}{N\cos^2\alpha+R\sin^2\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

If $\alpha=0$,

$R_\alpha=\frac{NR}{N}=R$, the radius of curvature of the meridian.

If $\alpha=90°$,

$R_\alpha=\frac{NR}{R}=N$, the radius of curvature of the prime vertical.

---

* R_α is taken negative, because the direction of bending as represented by d^2y/dx^2 is negative.

The geometrical derivation of R_α is simpler than the above. Thus in Fig. 76 draw a tangent plane at M and a parallel plane at the infinitesimal distance c from it. The latter will cut an ellipse as shown in plan. The three points $B\ M\ B'$ are consecutive points in the prime vertical or three points in the osculating circle with radius N. Similarly for the meridian with radius of curvature R.

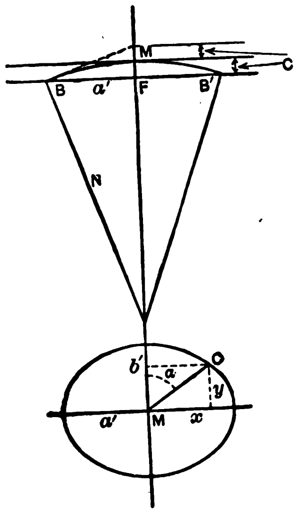

FIG. 76.

Let a' and b' denote the semi-axes of the ellipse through BB' and s the semi-diameter making the angle α with the meridian. From similar triangles

$$\frac{2c}{a'}=\frac{a'}{N}, \quad \text{or} \quad c=\frac{a'^2}{2N}.$$

Similarly, $$c=\frac{b'^2}{2R}=\frac{s^2}{2R_\alpha}.$$

Equating the values of c,

$$\frac{s^2}{a'^2}=\frac{R_\alpha}{N}, \quad \frac{s^2}{b'^2}=\frac{R_\alpha}{R}. \quad \ldots\ldots \quad (a)$$

The coordinates of C are $x=s \sin \alpha$ and $y=s \cos \alpha$. Substituting in the equation of the ellipse,

$$\frac{s^2 \sin^2 \alpha}{a'^2}+\frac{s^2 \cos^2 \alpha}{b'^2}=1.$$

From (a), $$\frac{R_\alpha}{N} \sin^2 \alpha+\frac{R_\alpha}{R} \cos^2 \alpha=1,$$

or $$R_\alpha=\frac{NR}{R \sin^2 \alpha+N \cos^2 \alpha}. \quad \ldots\ldots \quad (64)$$

Table VI is computed from (64).

97. Length of Meridional Arc.—The radius of curvature, R, changes slowly with ϕ, so that for arcs of less than one degree

$$ds=R\Delta\phi'' \sin 1'', \quad \ldots\ldots \quad (65)$$

where $\Delta\phi$ is in seconds and R is for the middle latitude.

For long arcs (65) must be integrated. Substituting the value of R from (56),

$$ds = a(1-e^2)d\phi(1-e^2\sin^2\phi)^{-\frac{3}{2}}.$$

Expanding by the binomial theorem,

$$ds = a(1-e^2)(1+\tfrac{3}{2}e^2\sin^2\phi+\tfrac{15}{8}e^4\sin^4\phi+\tfrac{35}{16}e^6\sin^6\phi\ldots)d\phi;$$

$$s = a(1-e^2)\int_{\phi'}^{\phi''}(1+\tfrac{3}{2}e^2\sin^2\phi+\tfrac{15}{8}e^4\sin^4\phi+\tfrac{35}{16}e^6\sin^6\phi\ldots)d\phi.$$

By formulas 11] and 12]

$$\sin^2\phi = \tfrac{1}{2}(1-\cos 2\phi).$$
$$\sin^4\phi = \tfrac{1}{8}(3-4\cos 2\phi+\cos 4\phi).$$

By formula 9],

$$\sin^6\phi = \sin^2\phi\sin^4\phi = \tfrac{1}{32}(10-15\cos 2\phi+6\cos 4\phi-\cos 6\phi).$$

Substituting and putting,

$$\begin{array}{llll}
A = & 1+\tfrac{3}{4}e^2+\tfrac{45}{64}e^4+\tfrac{175}{256}e^6\ldots & = 1.005\,109\,3 & \log = 0.002\,2133 \\
B = & \tfrac{3}{4}e^2+\tfrac{15}{16}e^4+\tfrac{525}{512}e^6\ldots & = 0.005\,120\,2 & \text{''} \;\; = 7.709\,287 \\
C = & \tfrac{15}{64}e^4+\tfrac{105}{256}e^6\ldots & = 0.000\,010\,8 & \text{''} \;\; = 5.033\,42 \\
D = & \tfrac{35}{512}e^6\ldots & & \quad\;\; = 2.326
\end{array}$$

$$s = a(1-e^2)\int_{\phi'}^{\phi''}(A-B\cos 2\phi+C\cos 4\phi-D\cos 6\phi\ldots)d\phi$$

$$= a(1-e^2)\Big(A\phi-\tfrac{1}{2}B\sin 2\phi+\tfrac{1}{4}C\sin 4\phi-\tfrac{1}{6}D\sin 6\phi\ldots\Big)_{\phi'}^{\phi''}.$$

Substituting the limits, and putting $\phi''-\phi'=\gamma$, $\phi''+\phi'=\delta$, we have, by formula 8],

$$s = (1.489\,5369)\gamma''-(4.511\,036)\sin\gamma\cos\delta+(1.534\,14)\sin 2\gamma\cos 2\delta - (8.651)\sin 3\gamma\cos 3\delta, \quad \ldots \quad (66)$$

where s is in meters and the numbers in parentheses are the *logarithms* of the constant factors.

Equation (66) is correct to seven decimal places. If more are desired, the next term for A is $(11025/16384)e^8$; for B, (2205/2048)e^8; for C, (2205/4096)e^8; for D, (315/2048)e^8, while an E term is added = (315/16384)e^8.

Ex. 1. Find the meridional distance between the parallels of latitude 32° 15′ 40″.21 and 36° 44′ 12″.62.

Solution.

By equation (66):

$$s = (1.4895369)\ \gamma'' - (4.511036) \sin\gamma \cos\delta + (1.53414) \sin 2\gamma \cos 2\delta - (8.651) \sin 3\gamma \cos 3\delta.$$

$$\gamma = \phi'' - \phi' = 4^\circ\ 28'\ 32''.41 = 16112''.41.$$

$$\delta = \phi'' + \phi' = 68^\circ\ 59'\ 52''.83.$$

$2\gamma = 8^\circ\ 57'\ 04''$. $3\gamma = 13^\circ\ 25'$.

$2\delta = 137^\circ\ 59'\ 45''$. $3\delta = 206^\circ\ 59'$.

(1.4895369)	−(4.511036)
log γ'' = 4.2071606	sin γ = 8.892293
log (I) = 5.6966975	cos δ = 9.554369
(I) = 497390.57	log(II) = 2.957698$_n$
(IV) = 0.01	(II) = −907.190
497390.58	III = − 3.96
911.15	−911.15
s = 496479.43 meters	
= 496^{km}.4795	
+(1.53414)	−(8.651)
sin 2γ = 9.19200	sin 3γ = 9.366
cos 2δ = 9.87104$_n$	cos 3δ = 9.949$_n$
log (III) = 0.59718$_n$	log (IV) = 7.966
(III) = −3.955	(IV) = 0.0093

Ex. 2. At Fuertes Observatory, Ithaca, N. Y., $\phi = 42^\circ\ 26'\ 47''.28$. Find the meridional distance in miles from it to the parallel of the Naval Observatory at Washington, $\phi = 38^\circ\ 55'\ 17''.03$.

98. Areas on the Ellipsoid.—The surface is divided into elementary frustums of cones by parallels of latitude, of width $Rd\phi$, and circumference $2\pi N \cos\phi$.

Hence the differential area,

$$dA = 2\pi NR \cos\phi d\phi.$$

Substituting for N and R their values from (57) and (56) with b^2 for $a^2(1-e^2)$,

$$dA = 2\pi b^2 \cos\phi d\phi (1 - e^2 \sin^2\phi)^{-2}.$$

By form. 32],

$$(1-e^2\sin^2\phi)^{-2}=1+2e^2\sin^2\phi+3e^4\sin^4\phi+4e^6\sin^6\phi+5e^8\sin^8\phi\ldots.$$

Substituting, the expression to be integrated becomes

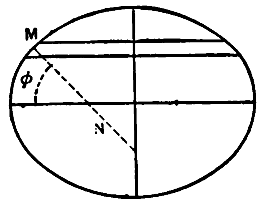

Fig. 77.

$$\int\cos\phi\,\sin^n\phi\,d\phi=\frac{1}{n+1}\sin^{n+1}\phi,$$

which gives

$$A_{\phi'}^{\phi''}=2b^2\pi\,(\sin\phi+\tfrac{2}{3}e^2\sin^3\phi+\tfrac{3}{5}e^4\sin^5\phi+\tfrac{4}{7}e^6\sin^7\phi+\ldots)_{\phi'}^{\phi''}.$$

Substituting the limits,

$$A_{\phi'}^{\phi''}=2b^2\pi\,[(\sin\phi''-\sin\phi')+\tfrac{2}{3}e^2(\sin^3\phi''-\sin^3\phi')+\tfrac{3}{5}e^4(\sin^5\phi''-\sin^5\phi')+\tfrac{4}{7}e^6(\sin^7\phi''-\sin^7\phi')+\ldots]. \quad (67)$$

To put in convenient form for computation, by forms. 12] and 8],

$$\sin^3\phi=\tfrac{3}{4}\sin\phi-\tfrac{1}{4}\sin 3\phi,$$
$$\sin^5\phi=\tfrac{5}{8}\sin\phi-\tfrac{5}{16}\sin 3\phi+\tfrac{1}{16}\sin 5\phi,$$
$$\sin^7\phi=\tfrac{35}{64}\sin\phi-\tfrac{21}{64}\sin 3\phi+\tfrac{7}{64}\sin 5\phi-\tfrac{1}{64}\sin 7\phi.$$

Substituting in (67), we have by form. 8], with $\phi''-\phi'=2\gamma'$ and $\phi''+\phi'=2\delta'$,

$$A_{\phi'}^{\phi''}=4b^2\pi(B\sin\gamma'\cos\delta'-C\sin 3\gamma'\cos 3\delta'+D\sin 5\gamma'\cos 5\delta'-E\sin 7\gamma'\cos 7\delta'\ldots), \quad (68)$$

where
$B=1+\tfrac{1}{2}e^2+\tfrac{3}{8}e^4+\tfrac{5}{16}e^6+\tfrac{35}{128}e^8=1.003\,4016$ log $=0.001\,4748$
$C=\tfrac{1}{6}e^2+\tfrac{3}{16}e^4+\tfrac{3}{16}e^6+\tfrac{35}{192}e^8=0.001\,1368$ 7.055 68
$D=\tfrac{3}{80}e^4+\tfrac{1}{16}e^6+\tfrac{5}{64}e^8=0.000\,0017$ 4.230 4
$E=\tfrac{1}{112}e^6+\tfrac{5}{256}e^8=0.000\,0000$
$F=\tfrac{5}{2304}e^8=0.000\,0000$

If (68) be divided by 360 and $\phi''-\phi'=2\gamma'=1°$, the area for 1° square,

$$G=\frac{b^2\pi}{90}(B\sin 30'\cos\delta'-C\sin 1°\,30'\cos 3\delta'+D\sin 2°\,30'\cos 5\delta'-E\sin 3°\,30'\cos 7\delta'\ldots). \quad (69)$$

The values of B, C, etc., should be carried to more than seven places for accurate results with the Clarke ellipsoid, although the above values are carried as far as the data will warrant when applied to the earth.

99. Spherical Excess.—By Legendre's theorem in Spherical Geometry it is proved that in a spherical triangle whose sides are short compared with the radius R of the sphere and in a plane triangle with sides of equal length the corresponding angles differ by the same quantity which is one-third the spherical excess.

Thus, let A, B, C = the angles of the spherical and A', B', C' those of the plane triangle; a, b, c (π-measure) and a', b', c' the corresponding sides; $aR=a'$; $bR=b'$; $cR=c'$.

Plane triangle. By form. 19],

$$\cos A' = \frac{b^2+c^2-a^2}{2bc}\cdot\frac{R^2}{R^2}. \quad \ldots \ldots \ldots \ldots \quad (a)$$

By form. 1],

$$\sin^2 A' = 1-\cos^2 A' = \frac{4b^2c^2-(b^2+c^2-a^2)^2}{4b^2c^2}\cdot\frac{R^4}{R^4},$$

or

$$\sin^2 A' = \frac{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}{4b^2c^2}. \quad \ldots \ldots \quad (b)$$

Spherical triangle. By form. 27],

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

By forms. 13] and 14], omitting terms of higher power than the fourth,

$$\cos A = \frac{1-\frac{a^2}{2}+\frac{a^4}{24}-\left(1-\frac{b^2}{2}+\frac{b^4}{24}\right)\left(1-\frac{c^2}{2}+\frac{c^4}{24}\right)}{\left(b-\frac{b^3}{6}\right)\left(c-\frac{c^3}{6}\right)}$$

$$= \frac{\frac{1}{2}(-a^2+b^2+c^2)-\frac{1}{24}(b^4+c^4-a^4)-\frac{1}{4}b^2c^2}{bc[1-\frac{1}{6}(b^2+c^2)]}.$$

By form. 32],

$$=[\tfrac{1}{2}(-a^2+b^2+c^2)-\tfrac{1}{24}(b^4+c^4-a^4)-\tfrac{1}{4}b^2c^2]\frac{1+\frac{1}{6}(b^2+c^2)}{bc}$$

$$=\frac{b^2+c^2-a^2}{2bc}-\frac{b^4+c^4-a^4+6b^2c^2}{24bc}+\frac{-a^2b^2+b^4+2b^2c^2-a^2c^2+c^4}{12bc};$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}-\frac{1}{6}\frac{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}{4bc}. \quad . \quad (c)$$

From (a), (b), and (c),

$$\cos A=\cos A'-\tfrac{1}{6}bc\sin^2 A'. \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Since b and c are very small, the difference between A and A' must be small. Putting this difference $=x$,

$$\cos x=1, \quad \sin x=x''\sin 1''.$$

By form. 4],

$$\cos A=\cos(A'+x)$$

$$=\cos A'-x''\sin 1''\sin A';$$

by (d),

$$=\cos A'-\tfrac{1}{6}bc\sin^2 A',$$

or

$$x''=\frac{bc\sin A'}{6\sin 1''}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

where b and c are in π-measure. If b and c are in units of length on the sphere of radius R,

$$x''=\frac{bc\sin A'}{6R^2\sin 1''}$$

$$=\frac{\text{area of triangle}}{3R^2\sin 1''}$$

The same result can be found for $B-B'$ and $C-C'$. Consequently the spherical excess of the triangle must be the sum of the three equal excesses of its three angles, or

$$s = 3x = \frac{\text{area of triangle}}{R^2 \sin 1''} \quad . \quad . \quad . \quad . \quad . \quad (70)$$

$$= \tfrac{1}{2} bc \sin A' \frac{1}{R^2 \sin 1''},$$

or

$$s = mbc \sin A', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

where $m = \frac{1}{2R^2 \sin 1''} = \frac{1}{2NR \sin 1''}$, by (59). It is given in Table VII in metric units.

Ex. 1. In the triangle Vinton-Woodstock-Norwalk the following angles are observed and latitudes determined:

Vinton,	71° 43′ 49″.66, . . .	$\phi = 42° 30'$
Norwalk,	62 33 55 .52	$\phi = 42 \ 23$
Woodstock,	45 42 23 .00	$\phi = 41 \ 59$

The length of the side, Norwalk to Vinton, is 45128m.45; its log = 4.6544504. What is the spherical excess of the triangle?

Solution. By formula (71) the excess $= s = mbc \sin A'$, where m is a factor depending upon the mean latitude of the triangle, and $bc \sin A' =$ twice the area of the plane triangle, approximately the area of the spherical triangle.

Now $\phi_m =$ mean latitude $= \frac{1}{3}[(42° 30') + (42° 23') + (41° 59')] = 42° 17'$.

For which, by Table VII, $\log m = 1.40428$.

By considering the triangle plane with the side NV known as 45128m, and the angles given to seconds, we compute by the law of sines, $\log WV$ and $\log WN = 4.74786$ and 4.77722, respectively.

Using two solutions to check the result we have

$\log m = 1.40428$	$\log m = 1.40428$
$\log NV = 4.65445$	$\log VN = 4.65445$
$\log NW = 4.77722$	$\log VW = 4.74786$
$\sin N = 9.94818$	$\sin V = 9.97754$
$\log s = 0.78413$	$\log s = 0.78413$
$s = 6''.08$	$s = 6''.08$

Ex. 2. In the triangle Milo-Clyde-Niles the length of side Clyde-Niles is 44884m.27 (log = 4.6520942) and the following angles were observed and latitudes determined:

Milo,	49° 24′ 22″.71	$\phi = 42° 35'$
Clyde,	66 19 09 .25	$\phi = 43 \ 03$
Niles,	64 16 31 .64	$\phi = 42 \ 48$

Find the spherical excess.

100. Effect of Height upon Horizontal Angles.—The observer at A, Fig. 78, at sea level sights upon M at the height h above sea level at B. The vertical plane of collimation at A projects M to B on the line drawn to H, where the normal at A meets the earth's axis, while the true projection is at B' on the normal MH' to the surface at M. The angle between the normals is δ. This makes an error x in the horizontal angle at A due to the height h.

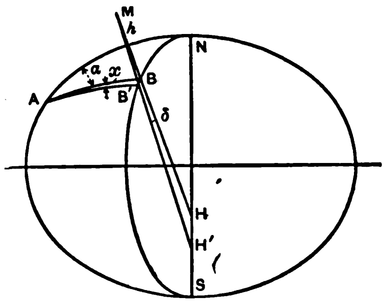

Fig. 78.

First we find the angle between the two projecting lines at M. In Fig. 79 let C be the intersection of the normals at M_1 and M_2 (M_1 is at the latitude of A and M_2 at that of B), both in the same meridian. If $\varDelta s$ is small C will also be the center of curvature for the arc M_1M_2, and

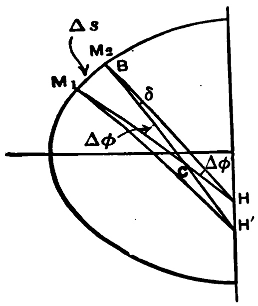

Fig. 79.

$$\varDelta s = R\varDelta\phi. \quad . \quad . \quad . \quad . \quad (a)$$

If M_1C be produced to meet the axis at H, and the reduced difference in latitude M_1HM_2 be called $\varDelta\phi'$, $(M_1H=N)$,

$$\varDelta s = N\varDelta\phi'. \quad . \quad . \quad . \quad . \quad (b)$$

From (a) and (b),

$$\frac{\varDelta\phi'}{\varDelta\phi} = \frac{R}{N}.$$

But

$$\delta = \varDelta\phi - \varDelta\phi' = \varDelta\phi\left(1 - \frac{\varDelta\phi'}{\varDelta\phi}\right) = \varDelta\phi\left(1 - \frac{R}{N}\right).$$

From the values of R and N, (56) and (57),

$$\frac{R}{N} = \frac{1-e^2}{1-e^2\sin^2\phi}.$$

Substituting,

$$\delta=\Delta\phi\left(1-\frac{1-e^2}{1-e^2\sin^2\phi}\right)=\frac{\Delta\phi e^2\cos^2\phi}{1-e^2\sin^2\phi}.$$

From Fig. 78,

$$R\Delta\phi=-s\cos\alpha, \text{ nearly,}$$

where α is counted clockwise from the south and s is the distance between stations (latitude difference = distance into cosine of bearing).

Hence,

$$\delta=-\frac{se^2\cos^2\phi\cos\alpha}{(1-e^2\sin^2\phi)R}=-\frac{se^2\cos^2\phi\cos\alpha}{(1-e^2)N}.$$

$$BB'=h\delta=-\frac{hse^2\cos^2\phi\cos\alpha}{(1-e^2)N}.$$

The corresponding horizontal angle error at A in π-measure,

$$x=-\frac{BB'\sin\alpha}{s}=\frac{he^2\cos^2\phi\sin\alpha\cos\alpha}{(1-e^2)N},$$

$$x''=\frac{he^2\cos^2\phi\sin\alpha\cos\alpha}{(1-e^2)N\sin 1''}. \quad \ldots \ldots \ldots \quad (72)$$

This will be a maximum for $\alpha=45°$. If ϕ also $=45°$,

$$x''=0.000\,055h, \quad \ldots \ldots \quad (73)$$

where h is in meters.

For a height of 1000^m this would give $0''.05$. The probable error in the value of a primary angle is seldom less than $0''.25$, so that the above correction would be negligible except for very high altitudes.

101. Triangle Side Computations.—The triangles of a triangulation net are strictly *spheroidal*, but by § 100 the three vertices of a triangle can be projected down to sea level by lines drawn to the center of a sphere tangent to the ellipsoid at the center of gravity of the triangle and having $\sqrt{NR}$ for radius, without introducing horizontal angle errors as great as the errors of observation.

The sides of these projected triangles have the same lengths, *within the errors of measurement*, upon the tangent spheres as upon the ellipsoid.

The triangles can thus be considered spherical, and, by Legendre's theorem, computed as plane by subtracting one-third the spherical excess from each spherical angle.

In simple systems, and where the greatest accuracy is not desired, if the sum of the observed angles in any triangle does not equal 180° + the spherical excess, or the sum of those about a point 360°, the error is distributed equally among the angles, or sometimes inversely as the number of repetitions.

But in complicated systems, or where extreme accuracy is desired, the errors are distributed by the method of least squares. The following is a convenient form for computation.

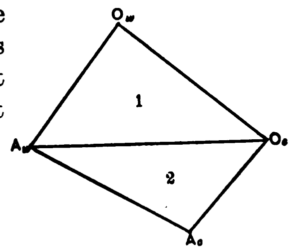

FIG. 80.

Base $O_wO_c = 6410.66$ ft.

O_wO_c	3.8069028		
sin A_w^1	9.9907935		
	3.8161093		3.8161093
sin O_w^1	9.8593280	sin O_c^1	9.9156182
	3.6754373		3.7317275
$A_wO_c = 4736.28$		$O_wA_w = 5391.72$	
	3.6754373		
sin A_c^2	9.8135605		
	3.8618768		3.8618768
sin O_c^2	9.9999957	sin A_w^2	9.8819486
	3.8618725		3.7438254
$A_wA_c = 7275.66$		$O_cA_c = 5544.03$	

The form of triangle computation employed by the Coast Survey is as follows, using for illustration the triangle of § 99, Ex. 1:

Stations.	Observed Angles.	Cor-rec-tions.	Spher-ical Angles.	Spher-ical Excess	Plane Angles and Distances.	Loga-rithms.
(1) Norwalk to Vinton. . .		"	"	"	45128.45	4.6544504
(2) Woodstock.	45° 42′ 23″.00	−0.70	22.30	2.02	20″.28	0.1452317
(3) Norwalk.	62 33 55 .52	−0.70	54.82	2.03	52 .79	9.9481837
(4) Vinton	71 43 49 .66	−0.70	48.96	2.03	46 93	9.9775353
(5) Woodstock to Vinton.					55958.47	4.7478658
(6) Woodstock to Norwalk					59871.12	4.7772174
	180° 00′ 08″.18			6.08		

In the station column, after stating in line (1) the known side of the triangle, the station opposite this side is set down, and then the two other stations, proceeding around the triangle in the clockwise direction. In line (5) is placed the side from the station opposite the known side to the last-named station, viz., that in line (4). Finally, in line (6) is set down the remaining unknown side.

The observed angles, summing up 180° 00′ 08″.18, are set in the second column. Since the spherical excess by § 99 is 6″.08, the sum of the observed angles is too great by 180° 00′ 08″.18 − 180° 00′ 06″.08 = 2″.10. This discrepancy is commonly distributed among the angles by the method of least squares, but we here distribute the discrepancy evenly between the angles and obtain the spherical angles. By subtracting one-third the spherical excess from each triangle, the plane angles are obtained from which the triangle is solved. In the logarithm column, term (2) is the colog sin of the angle, while (3) and (4) are the log sines; (5) is the sum of (1), (2), and (3); and (6) is the sum of (1), (2), and (4).

Ex. Compute according to the above form the triangle of Ex. 2, § 99, employing seven-place logarithms.

CHAPTER VII.

GEODETIC POSITIONS.

102. Difference of Latitude.—It is usual to find the latitude and the longitude of one or more of the triangulation vertices by astronomical observation, as also the azimuth of one or more of the sides, and from these data to compute the positions of the other vertices and the azimuths of the other sides. The formulas for computation are obtained below.

In Fig. 81, P' is the pole of the ellipsoid and P that of a tangent sphere. The latitude ϕ of A and the azimuth α (measured clockwise from the south), and distance s to B, are given.

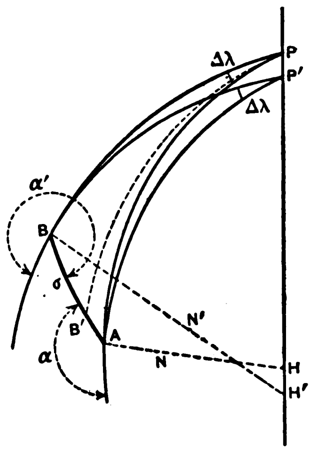

FIG 81.

Since σ, the subtending angle of s, is always small, being usually less than 1°, we have by Maclaurin's theorem, form. 33],

$$\phi' = f(\sigma) = (\phi)_{\sigma=0} + \left(\frac{d\phi}{d\sigma}\right)_{\sigma=0}\sigma + \left(\frac{d^2\phi}{d\sigma^2}\right)_{\sigma=0}\frac{\sigma^2}{2} + \left(\frac{d^3\phi}{d\sigma^3}\right)_{\sigma=0}\frac{\sigma^3}{6} + \dots$$

In the differential triangle PAB', form. 27],

$$\cos PB' = \cos PA \cos AB' + \sin AP \sin AB' \cos PAB'.$$

Noting that $PA = 90° - \phi$ and $PB' = 90° - (\phi + d\phi)$, this becomes

$$\sin(\phi + d\phi) = \sin\phi \cos d\sigma - \cos\phi \sin d\sigma \cos\alpha.$$

Expanding the first member,

$$\sin \phi + d\phi \cos \phi = \sin \phi - d\sigma \cos \phi \cos \alpha,$$

since $\sin d\phi = d\phi$ and $\cos d\phi = 1$, etc., or

$$\frac{d\phi}{d\sigma} = -\cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

Differentiating,

$$\frac{d^2\phi}{d\sigma^2} = -\frac{d}{d\alpha} \cos \alpha \frac{d\alpha}{d\sigma} = \sin \alpha \frac{d\alpha}{d\sigma}. \quad . \quad . \quad . \quad . \qquad (b)$$

To find $\frac{d\alpha}{d\sigma}$: from the differential triangle PAB', form. 29],

$$\cot PB'A = \frac{\sin AB' \cot PA - \cos AB' \cos PAB'}{\sin PAB'},$$

or

$$\cot (\alpha + d\alpha) = \frac{d\sigma \tan \phi + \cos \alpha}{\sin \alpha}.$$

By forms. 2], 3], and 4],

$$\sin \alpha \ (\cos \alpha - d\alpha \sin \alpha) = (\sin \alpha + d\alpha \cos \alpha)(d\sigma \tan \phi + \cos \alpha),$$

$$\sin \alpha \cos \alpha - d\alpha \sin^2 \alpha = d\sigma \sin \alpha \tan \phi + \sin \alpha \cos \alpha + d\alpha \cos^2 \alpha$$

(neglecting the term of higher order).

$$\frac{d\alpha}{d\sigma} = -\sin \alpha \tan \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (c)$$

From (b),

$$\frac{d^2\phi}{d\sigma^2} = -\sin^2\alpha \tan \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (d)$$

Differentiating,

$$\frac{d^3\phi}{d\sigma^3} = \frac{d(-\sin^2\alpha \tan \phi)}{d\sigma}$$

$$= -2 \sin \alpha \cos \alpha \tan \phi \frac{d\alpha}{d\sigma} - \sin^2 \alpha \sec^2 \phi \frac{d\phi}{d\sigma}.$$

From (a) and (c),

$$\frac{d^3\phi}{d\sigma^3}=2\sin^2\alpha\cos\alpha\tan^2\phi+\sin^2\alpha\cos\alpha\,(1+\tan^2\phi)$$
$$=\sin^2\alpha\cos\alpha(1+3\tan^2\phi).$$

Substituting now in Maclaurin's theorem above, we have

$$\phi'-\phi=-\sigma\cos\alpha-\tfrac{1}{2}\sigma^2\sin^2\alpha\tan\phi$$
$$+\tfrac{1}{6}\sigma^3\sin^2\alpha\cos\alpha\,(1+3\tan^2\phi),\qquad(74)$$

where $\phi'-\phi$ and σ are in π-measure.

For sphere of radius N, $\sigma=\frac{s}{N}$, where s is the distance in linear units.

Substituting,

$$\phi'-\phi=-\frac{s\cos\alpha}{N}-\frac{s^2}{2N^2}\sin^2\alpha\tan\phi+\frac{s^3}{6N^3}\sin^2\alpha\cos\alpha(1+3\tan^2\phi).$$

If the center of the sphere is taken at H the sphere will be tangent to the ellipsoid along the parallel through A so that ϕ will be the same for both, as also s and α. The linear difference in latitude will be the same for each surface, i.e.,

$$(\phi'-\phi)N=\Delta\phi R_m\sin 1'',$$

or

$$\Delta\phi=(\phi'-\phi)\frac{N}{R_m\sin 1''},\qquad(e)$$

where $\Delta\phi$ = difference in latitude in seconds for the ellipsoid, and R_m is the radius of curvature of the meridian for the middle latitude.

Substituting,

$$-\Delta\phi=\frac{s\cos\alpha}{R_m\sin 1''}+\frac{s^2\sin^2\alpha\tan\phi}{2NR_m\sin 1''}-\frac{s^3\sin^2\alpha\cos\alpha(1+3\tan^2\phi)}{6N^2R_m\sin 1''}.\qquad(75)$$

It is inconvenient to look out R_m for the middle latitude, which is at first unknown. If R for vertex A is used the resulting difference in latitude $\delta\phi$ will have to be changed in inverse ratio to the radius by (e); i.e.,

$$\Delta\phi=\delta\phi\frac{R}{R_m}=\delta\phi\left(1-\frac{R_m-R}{R_m}\right)=\delta\phi\left(1-\frac{dR_m}{R_m}\right);$$

i.e., the true value can be found by subtracting $\delta\phi\,\frac{dR_m}{R_m}$ from the approximate value.

From (56), $$R=\frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}}.$$

Hence, $$dR=\frac{a(1-e^2)3e^2\sin\phi\cos\phi\,d\phi}{(1-e^2\sin^2\phi)^{\frac{5}{2}}}.$$

Since dR is the change from the starting point to the middle latitude,

$$\frac{d\phi}{\sin 1''}=\frac{\delta\phi}{2}.$$

Hence,

$$\delta\phi\frac{dR}{R}=\frac{3e^2\sin\phi\cos\phi\sin 1''}{2(1-e^2\sin^2\phi)}(\delta\phi)^2.$$

Placing $$D=\frac{3e^2\sin\phi\cos\phi\sin 1''}{2(1-e^2\sin^2\phi)}.$$

$$\text{The corrective term}=(\delta\phi)^2D. \quad . \quad . \quad . \quad . \quad (76)$$

For convenience in computation place

$\frac{1}{R\sin 1''}=B$; $\frac{\tan\phi}{2NR\sin 1''}=C$; $\frac{s\cos\alpha}{R\sin 1''}=h$, the first term of (75); $\frac{1+3\tan^2\phi}{6N^2}=E$. Substituting h and E in the third term, it becomes $hs^2\sin^2\alpha E$.

Substituting in (75),

$$-\Delta\phi=sB\cos\alpha+s^2C\sin^2\alpha+(\delta\phi)^2D-hs^2E\sin^2\alpha. \quad . \quad (77)$$

B, C, D, and E are given in Table II, the unit being the meter.

The fourth term is insensible for $\log s<4.23$. The third term is insensible for $\log h<2.31$, while h^2 may be used for $(\delta\phi)^2$ for $\log s<4.93$.

The fourth differential coefficient, $\frac{d^4\phi}{d\sigma^4}\frac{\sigma^4}{24}$, $<0''.002$, for $\sigma=1°$ or $s=100^{\text{km}}$ and may be neglected. Its value may be found in App. 9, Coast Survey *Report*, 1894.

103. Difference of Longitude.—In the triangle PBA of Fig. 81, by form. 26],

$$\sin \Delta\lambda = \frac{\sin \sigma \sin \alpha}{\cos \phi'}.$$

Referring to a sphere tangent at B since we are now dealing with ϕ', its center at H', α, ϕ', and $\Delta\lambda$ are the same as for the ellipsoid, while $\sigma = \frac{s}{N'}$.

Substituting,

$$\sin \Delta\lambda = \sin \frac{s}{N'} \cdot \frac{\sin \alpha}{\cos \phi'}. \quad . \; . \; . \; . \; . \; (78)$$

It is more convenient to assume*

$$\Delta\lambda = As \sin \alpha \sec \phi', \quad . \; . \; . \; . \; . \; (79)$$

where $A = \frac{1}{N' \sin 1''}$, and correct for the difference between arc and sine for both $\Delta\lambda$ and σ.

Form. 13],

$$\sin x = x - \frac{x^3}{6} \; . \; . \; . = x\left(1 - \frac{x^2}{6} \; . \; . \; .\right).$$

$$\log \sin x = \log x + \log\left(1 - \frac{x^2}{6}\right).$$

Form. 37],

$$\log x - \log \sin x = -\log\left(1 - \frac{x^2}{6}\right) = M\frac{x^2}{6},$$

where M is the modulus of the common system of logs.

$$\log\,(\log x - \log \sin x) = \log\left[M \frac{x''^2}{6 \sin^2 1''}\right]$$
$$= 8.2308 + 2 \log x''.$$

Hence for $\Delta\lambda$,

$$\log\,(\log \text{ difference}) = 8.2308 + 2 \log \Delta\lambda''. \quad . \; . \; (80)$$

---

* We wish a value of $\Delta\lambda$ in seconds of arc directly. The arc is greater than the sine, so we must add a correction for $\Delta\lambda$ to the expression for the sine of $\Delta\lambda$ to give us the arc. Then if we use in this expression $\frac{s}{N'}\left(\frac{\text{arc}}{\text{rad}}\right)$, instead of $\sin \sigma$, we must subtract a correction for s. These corrections are in the form of factors given by logs.

For σ,

$$\log (\log \text{difference}) = 8.2308 + 2 \log s + 2 \log A,$$

Using an average value (8.5090) for $\log A$,

$$\log (\log \text{difference}) = 5.2488 + 2 \log s. \quad . \quad . \quad . \quad (81)$$

For convenience in tabulating place

$$8.2308 + 2 \log \Delta\lambda = 5.2488 + 2 \log s,$$

which gives

$$\log s - \log \Delta\lambda = 1.4910 \text{ for the same log difference.} \quad . \quad (82)$$

The correction for $\log s$ is negative and for $\log \Delta\lambda$ positive. The values are given in Table III.

In using the table, take out the corrections with $\log s$ and $\log \Delta\lambda$ as arguments and add their algebraic sum to $\log \Delta\lambda$.

104. Convergence of Meridians.—In the triangle *PBA* of Fig. 81, by form. 28],

$$\tan \tfrac{1}{2} (A+B) = \cot \tfrac{1}{2}\Delta\lambda \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)},$$

or

$$-\cot \tfrac{1}{2}\Delta\alpha = \cot \tfrac{1}{2}\Delta\lambda \frac{\cos \frac{1}{2}(\phi-\phi')}{\sin \frac{1}{2}(\phi+\phi')},$$

where $-\Delta\alpha$ is the change in azimuth from A to B due to the convergence of the meridians.

Inverting,
$$-\tan \tfrac{1}{2}\Delta\alpha = \tan\tfrac{1}{2}\Delta\lambda \frac{\sin \frac{1}{2}(\phi+\phi')}{\cos \frac{1}{2}(\phi-\phi')}. \quad . \quad . \quad . \quad . \quad (83)$$

Expressing $\frac{1}{2}\Delta\alpha$ in terms of the tangent and $\tan \frac{1}{2}\Delta\lambda$ in terms of the arc, by forms. 17] and 15], and denoting the middle latitude by ϕ_m,

$$-\Delta\alpha = \Delta\lambda \frac{\sin \phi_m}{\cos \frac{1}{2}\Delta\phi} + \tfrac{1}{12}(\Delta\lambda)^3 \left(\frac{\sin \phi_m}{\cos \frac{1}{2}\Delta\phi} - \frac{\sin^3 \phi_m}{\cos^3 \frac{1}{2}\Delta\phi}\right).$$

Placing $\cos \frac{1}{2}\Delta\phi = 1$ in the corrective terms,

$$\left.\begin{aligned} -\Delta\alpha'' &= \Delta\lambda'' \frac{\sin\phi_m}{\cos\frac{1}{2}\Delta\phi} + \tfrac{1}{12}(\Delta\lambda'')^3 \sin\phi_m \cos^2\phi_m \sin^2 1'' \\ &= \Delta\lambda'' \sin\phi_m \sec\tfrac{1}{2}\Delta\phi + (\Delta\lambda'')^3 F \end{aligned}\right\}, \quad (84)$$

where $F = \frac{1}{12} \sin\phi_m \cos^2\phi_m \sin^2 1''$, tabulated in Table II.

The last term is only $0''.01$ for $\log \Delta\lambda'' = 3.36$ and need never be used for secondary triangulation.

The inverse azimuth,

$$\alpha' = \alpha + \Delta\alpha + 180°. \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

For computing $\Delta\phi$, $\Delta\lambda$, and $\Delta\alpha$ the adjusted spherical angles must be taken and not the plane ones used in computing the triangle sides. For each triangle, starting from the known side, the latitude and longitude of the required point must be the same computed from each of the two sides, while the inverse azimuths of these two sides must differ by the third angle, thus checking the work.

105. Form of Computation.—As shown in the three preceding sections the equations for computing geodetic positions in primary triangulation are:

$$-\Delta\phi = sB\cos\alpha + s^2C\sin^2\alpha + (\delta\phi)^2D - hs^2E\sin^2\alpha,$$
$$\Delta\lambda = As\sin\alpha\sec\phi',$$
$$-\Delta\alpha = \Delta\lambda\sin\tfrac{1}{2}(\phi+\phi')\sec\tfrac{1}{2}\Delta\phi + (\Delta\lambda)^3F.$$

A convenient method of performing this computation is that adopted by the Coast Survey, an example of which is shown on page 180. The geographic position of Station Mount Blue is known and the distance and azimuth at Mount Blue of the line joining it to Ragged Mountain are given by the triangulation. It is desired to find ϕ' and λ', the geographic coordinates of Ragged Mountain and α', the back azimuth of the connecting line.

$\Delta\phi$ must be computed first. B, C, D, and E are taken from Table II with ϕ as argument, and the first and second terms of the formula are computed. The sum of these two is the $\delta\phi$ involved in the third term, while h in the last term is the value of the first term alone. These terms are combined in a manner readily followed in the form.

α	Mount Blue to Mount Pleasant.	26° 19′ 28″.69
$\angle$	Ragged Mountain and Mount Pleasant.	−85 35 25 .78
α	Mount Blue to Ragged Mountain.	300 44 02 .91
$\Delta\alpha$		+50 03 .88
		180
α'	Ragged Mountain to Mount Blue.	121 34 06 .79

ϕ	44° 43 41″.437	Mount Blue.	λ	70° 20′ 33″.157
$\Delta\phi$	− 30 56 .052	s = 110,743.7 metres.	$\Delta\lambda$	−1 11′ 27 .830
ϕ'	44 12 45 .385	Ragged Mountain.	λ'	69 09 05 .327

	FIRST.		SECOND.		THIRD.		FOURTH.
s	5.0443191	s^2	10.08864	$(\partial\phi)^2$	6.5372	h	3.2633
$\cos\alpha$	9.7084678	$\sin^2\alpha$	9.86854	D	2.3926	$s^2\sin^2\alpha$	9.9572
B	8.5104887	C	1.39991			E	6.2069
					8.9298		
h	3.2632756		1.35709				9.4274
1st term	+1833″.478	3d term	+0″.0851			$(\Delta\lambda)^3$	10.897_n
2d term	+ 22 .756	4th term	−0 .2675			F	7.844
3d (+4th)	+1856 .234		−0 .1824				8.741_n
terms	− 0 .182						
$-\Delta\phi$	+1856 .052						
		s	5.0443191				
		$\sin\alpha$	9.9342701_n	Arg.		$\Delta\lambda$	3.632237_n
$\frac{1}{2}(\phi+\phi')$	44° 28′ 13″.4	A	8.5090107	s	−218	$\sin\frac{1}{2}(\phi+\phi')$	9.845433
$\frac{1}{2}\Delta\phi$	0 15 28 .0	sec. ϕ'	0.1446280	$\Delta\lambda_u$	+314	$\sec\frac{1}{2}(\Delta\phi)$	0.000004
			3.6322375_n		+ 96		3.477674_n
		$\Delta\lambda$	−4287″.830				−3003″.82
							− 0 .06
						$-\Delta\alpha$	−3003 .88

With $\Delta\phi$ determined, ϕ' is known and $\Delta\lambda$ may be computed, A being taken from Table II with ϕ' as argument. The result obtained by the formula must be corrected for the difference between arc and sine as shown in § 103. By referring to Table III the corrections for log s and log $\Delta\lambda$ may be found by using these numbers as arguments. The algebraic sum of the two corrections is in this case 0.000 0096 and is included in the final value of $\Delta\lambda$.

Having determined $\Delta\lambda$, $\Delta\alpha$ is found last, F being taken from Table II with $\phi_m = \frac{1}{2}(\phi+\phi')$ as argument.

In secondary triangulation the fourth term of the $\Delta\phi$ formula may generally be omitted, as may also the two corrections to the longitude and azimuth differences, while $\sec\frac{1}{2}\Delta\phi = 1$.

106. Polyconic Map Projection.—This projection is the one most generally used in platting geodetic and topographic surveys. It supposes each parallel of latitude to be developed upon its own cone, the vertex of which is on the axis at its intersection with the tangent to the meridian at the parallel.

In Fig. 82 the side of the tangent cone, or radius of the developed parallel,

$$r = N \cot \phi. \qquad (86)$$

If an arc of the parallel subtend the angle $\Delta\lambda$ before development, and θ after development,

$$\theta = \Delta\lambda \frac{R_p}{r} = \Delta\lambda \frac{N \cos \phi}{N \cot \phi},$$

or

$$\theta = \Delta\lambda \sin \phi. \qquad (87)$$

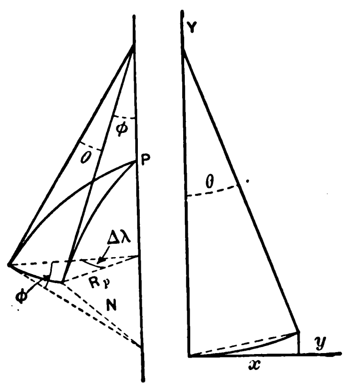

FIG. 82.

The radii of the developed parallels are so great that the parallels are plotted by coordinates.

$$\left. \begin{aligned} x &= r \sin \theta = N \cot \phi \sin (\Delta\lambda \sin \phi), \\ y &= r \operatorname{vers} \theta = x \tan \tfrac{1}{2}\theta = x \tan \tfrac{1}{2}(\Delta\lambda \sin \phi). \end{aligned} \right\} \qquad (88)$$

In platting, a central meridian is drawn as a straight line upon the map, and the true distances between parallels are laid off from Table IX. Perpendiculars, by describing arcs with a beam compass, are carefully drawn through these points for the x coordinate axes of the parallels. The x coordinates are then laid off on each for the different longitudes from the table. Perpendiculars are drawn through these points and the y coordinates laid off from the table. The meridians join the points of the same longitude and the parallels those of the same latitude.

A glance at Fig. 82 will show that, starting from the pole where the radius of the developed parallel is zero, the radius increases more rapidly than the distance from the pole, becoming infinity at the equator; the developed parallels will then not be concentric circles but the distances between them will increase with the longitude from the central meridian; distances in latitude will then be stretched out as we leave the central meridian, distorting the map since the longitude scale is constant.

The triangulation stations must then be plotted by latitude and longitude, interpolating between the nearest meridians and parallels and using the triangle sides for checks only.

107. Mercator's Map Projection.—This projection is used by navigators on account of the facility in obtaining directions for constant bearing or rhumb-line sailing. A tangent cylinder is drawn at the equator; the meridional planes are produced to meet the cylinder in elements, and the cylinder is then developed. The meridians thus become parallel straight lines at distances apart equal to the true distances at the equator. This enlarges the scale in longitude in the ratio a/R_p.

To preserve local bearings the latitude scale is increased in the same ratio; the *loxodrome* or curve of constant bearing at sea thus becomes a straight line with the same bearing on the map.

To find a sailing course between any two points, the navigator joins them on the map with a straight line, measures the angle made by this line with a meridian and allows for the magnetic variation.

In the differential triangles LCP, lcp, Fig. 83,

$$\frac{dm}{ds}=\frac{LP}{lp}=\frac{ee'}{lp}=\frac{a}{R_p}.$$

Substituting for $ds=Rd\phi$ and for $R_p=N\cos\phi$,

$$dm=\frac{aRd\phi}{N\cos\phi}. \qquad (89)$$

Substituting the values of R and N and integrating between the limits ϕ_1 and ϕ_2 will give the distance on the map between the corresponding parallels.

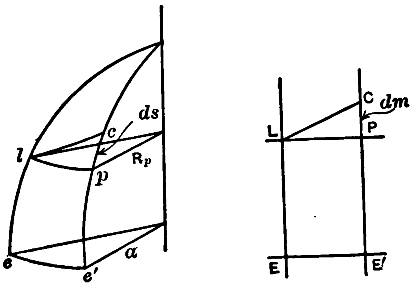

FIG. 83.

108. Location of Great Arcs.—If the two extremities of the line are given, the latitude and longitude of each is accurately determined by astronomical observation. The azimuth and length of the line can then be found by (77) and (79).

Thus,

$$s \sin \alpha = y = \frac{\Delta\lambda \cos \phi'}{A},$$

$$s \cos \alpha = x = -\frac{1}{B}[\Delta\phi + Cy^2 + D(\Delta\phi)^2 - E(\Delta\phi)y^2],$$

whence

$$\tan \alpha = \frac{y}{x} \quad \text{and} \quad s = x \sec \alpha = y \operatorname{cosec} \alpha. \quad . \quad . \quad . \quad (90)$$

If the distance is great it may be necessary to employ several triangles in locating the arc, or to test the direction by an observed azimuth at an intermediate point.

A lack of agreement of a few seconds in the astronomical determination of the difference of latitude or of longitude, due to "station error," which is not uncommon in rough country for fairly moderate distances, would affect the relative positions by at least that number of hundreds of feet. The line run would thus correspond to a random line which would probably require a small correction in azimuth and distance to reach the required point.

109. Location of Parallels.—Great arcs and parallels of latitude are both used as boundary lines between countries and states. First to find any point A of the parallel, a station A' as near as may be is occupied and its latitude determined by observation; the difference between it and that of the required parallel will give the arc $\Delta\phi$ to move either north or south on the meridian to reach the parallel, or, in distance at sea level,

$$s' = R\Delta\phi'' \sin 1''. \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

If $\Delta\phi$ is large the latitude of A should be determined by a new set of observations on account of the danger of station error.

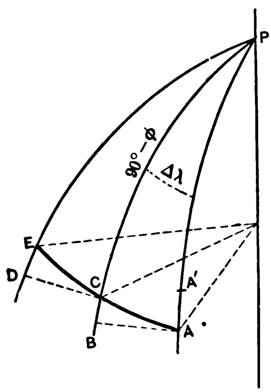

Fig. 84.

Having one point A, the parallel can be determined by offsets from the prime vertical AB.

In the right triangle PAB, form. 31],

$$\tan \sigma = \tan \Delta\lambda \cos \phi.$$

Expressing σ in terms of the value of its tangent and $\tan \Delta\lambda$ in terms of the angle, forms. 17] and 15], as in § 104,

$$\sigma = \Delta\lambda \cos \phi + \tfrac{1}{3}(\Delta\lambda \cos \phi)^3 \tan^2 \phi,$$

or

$$s = \sigma N = N \sin 1''\Delta\lambda'' \cos \phi + \tfrac{1}{3}N(\Delta\lambda'' \cos \phi \sin 1'')^3 \tan^2 \phi. \quad (92)$$

Placing $\alpha = 90°$ in (75) for the prime vertical,

$$-\Delta\phi'' = \frac{s^2 \tan \phi}{2NR \sin 1''}.$$

Hence the offset from the prime vertical to the parallel, for any distance s from the tangent point,

$$P = -\Delta\phi''R \sin 1'' = \frac{s^2 \tan \phi}{2N}. \quad . \quad . \quad . \quad . \quad (93)$$

Since P varies as s^2, if s be divided into n equal parts, the ordinates to the parallel will be

$$\left(\frac{1}{n}\right) P, \quad \left(\frac{2}{n}\right)^2 P, \quad \left(\frac{3}{n}\right)^2 P. \quad . \quad . \quad . \quad . \quad . \quad (94)$$

The direction angle

$$PBA = 90° + \Delta\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

while those of the $n-1$ ordinates, assuming s to increase proportionately to $\Delta\lambda$, will be

$$90° + \frac{\Delta\alpha}{n}, \quad 90° + \frac{2\Delta\alpha}{n}, \quad 90° + \frac{3\Delta\alpha}{n}, \quad . \quad . \quad . \quad (96)$$

If the parallel to be located is long $\Delta\lambda$ should be divided into sections and each one located from a new prime vertical to avoid long offsets.

Errors of direction may be prevented from accumulating, and station errors may be detected, by observations for azimuth and latitude at the beginning of each new prime veıtical.

In locating the 49th parallel west of the Lake of the Woods (*U. S. Northern Boundary Survey,* Washington, 1878) astronomical observations for latitude and azimuth were taken at points about twenty miles apart. At first four of the stations were observed jointly by the United States and British commissions, afterward each commission occupied alternate stations. The uncertainty in latitude is estimated at about 7 feet.

For azimuth a bull's-eye lantern was set up near the prime vertical at a distance of from 1 to 3 miles and its azimuth determined by observations on pole stars near elongation; the difference between the observed azimuth and 90° was converted into distance for the offset from the lantern to the prime vertical. In ranging out the prime vertical the hubs were double pointed to eliminate error of collimation. At the end of the line the inverse azimuth was observed. The average azimuth error was 20″ in 18 miles; when less than this the line was considered correct, if greater it was adjusted in computing the offsets to the parallel.

The difference between the computed offsets irom the tangent to the parallel and the measured distance to the astronomical mound or monument was taken to be the station error. The station error between two adjacent stations was over 7″ in one case and about 4″ in several others, corresponding to about 700 feet and 400 feet respectively. The astronomical parallel was the one located on the ground and monumented. The total offset from the tangent to the parallel was made up of the computed offset from the prime vertical to the parallel, increasing as the square of the distance, ± the proportional part of the station error, ± the error of the initial point north or south of 49°, ± the correction due to azimuth error.

Ex. Required the data for locating the 42d parallel between New York and Pennsylvania from the Delaware River (approx. longitude 1° 30′ E.) to the west end of the state (approx. longitude 2° 54′ W.), total distance 4° 24′ in longitude.

Dividing into three equal parts, we have $\Delta\lambda = 1° 28'$. Apply form. (92),

$5280'' = \Delta\lambda$		$\phi = 42°$.	
log sin 1″	4.685 5749	log $(\sin 1'' \, \Delta\lambda \cos\phi)^3$	4.83785
$\Delta\lambda$	3.722 6339	N	6.80536
$\cos\phi$	9.871 0735	1/3	9.52288
	8.279 2823	$\tan^2\phi$	9.90887
N	6.805 3577	2d term = 11.9	1.07496
1st term = 121517.8	5.084 6400	121517.8	
		$s =$ 121529.7 meters	
		From (84), since $\cos\frac{1}{2}(\Delta\phi) = 1$, we have	
s^2	10.169 3648	$\Delta\lambda$	3.72263
$\tan\phi$	9.954 4374	$\sin\phi$	9.82551
	10.123 8022	$-\Delta\alpha = 3533''.0$	3.54814
$2N$	7.106 3877	$-58' \, 53''$	
$CB = 1040.9^m$	3.017 4145	90°	
		$v =$ 89° 01′ 07″ = ∠CBA	

The ordinates and direction angles for intermediate points can be found by (94) and (96).

110. Parallels by Solar Compass.—From (84), the convergence of the meridians for short lengths of the parallel,

$$-\Delta\alpha'' = \Delta\lambda'' \sin\phi.$$

From (79), since $\sin\alpha = 1$, nearly,

$$\Delta\lambda'' = \frac{As}{\cos\phi}. \quad \ldots \ldots \ldots \quad (97)$$

Substituting,

$$-\Delta\alpha'' = As \tan\phi. \quad \ldots \ldots \ldots \quad (98)$$

The first instrument point being upon the parallel, the solar will give the meridian, from which to turn $90 - \frac{1}{2}\Delta\alpha$ from the south to reach the parallel. In our latitude, 42° 30′, the convergence for the width of a township (6 miles) is $4\frac{3}{4}$ minutes. This would give 90° $2\frac{1}{2}'$ for $90 - \frac{1}{2}\Delta\alpha$.

The difference in length, d, between the north and south lines of a township will be the distance s' between them into the convergence in seconds times tan 1″.

$$d = -s'\Delta\alpha \tan 1''. \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

For long distances the difference should be found by computing the arc of the parallel for each latitude and subtracting.

111. **Rectangular Spherical Coordinates.***—In Europe the positions of triangulation points have been found more convenient for use by local surveyors when expressed as coordinates than as latitudes and longitudes. In the rectangular system the x coordinates are measured on the meridian through the origin O, Fig. 85,

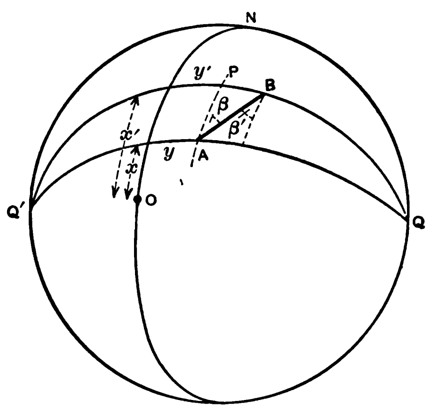

FIG. 85.

and the y coordinates on great circles through the poles QQ' of the meridian. Thus the coordinates of A are x and y, and of B, x' and y', positive to the north and east.

The bearing or direction angle β is the angle made, not with the meridian through A, but with the parallel AP through A having the same poles QQ' as the meridian.

To find *the coordinates and direction angle at B from those at A.* In the triangle ABQ, the three sides are known, as also the angles at $Q\left[=(x'-x)\frac{1}{R}\right]$ and at A $[=90°-\beta]$.

* See Jordan, *Handbuch der Vermessungskunde*, Vol. III.

Hence for y', form. 27],

$$\cos BQ=\cos AB \cos AQ+\sin AB \sin AQ \cos A.$$

$$\sin\frac{y'}{R}=\cos\frac{s}{R}\sin\frac{y}{R}+\sin\frac{s}{R}\cos\frac{y}{R}\sin\beta. \quad . \quad . \quad (100)$$

For x', form. 26],

$$\sin Q=\frac{\sin AB\sin A}{\sin BQ},$$

$$\sin\frac{x'-x}{R}=\sin\frac{s}{R}\cos\beta\sec\frac{y'}{R}. \quad . \quad . \quad . \quad . \quad (101)$$

For β', form. 28],

$$\tan\tfrac{1}{2}(A+B)=\cot\tfrac{1}{2}Q\cos\tfrac{1}{2}(AQ-BQ)\sec\tfrac{1}{2}(AQ+BQ),$$

$$\cot\tfrac{1}{2}(\beta-\beta')=\cot\frac{1}{2R}(x'-x)\cos\frac{1}{2R}(y'-y)\operatorname{cosec}\frac{1}{2R}(y'+y),$$

$$\tan\tfrac{1}{2}(\beta-\beta')=\tan\frac{1}{2R}(x'-x)\sin\frac{1}{2R}(y'+y)\sec\frac{1}{2R}(y'-y). \quad (102)$$

Replacing the functions of the small angles by the developments in series (100) becomes (see Table I)

$$y'-\frac{y'^3}{6R^2}=\left(1-\frac{s^2}{2R^2}\right)\left(y-\frac{y^3}{6R^2}\right)+\left(s-\frac{s^3}{6R^2}\right)\left(1-\frac{y^2}{2R^2}\right)\sin\beta$$
$$=y\left(1-\frac{s^2}{2R^2}-\frac{y^2}{6R^2}\right)+s\sin\beta\left(1-\frac{s^2}{6R^2}-\frac{y^2}{2R^2}\right).$$

Since y'^3 has a large divisor, the approximate value, $y+s\sin\beta$, found by neglecting all terms containing $\frac{1}{R^2}$ can be used for y', giving

$$y'-\frac{1}{6R^2}(y+s\sin\beta)^3=y+s\sin\beta+y\left(-\frac{s^2}{2R^2}-\frac{y^2}{6R^2}\right)+s\sin\beta\left(-\frac{s^2}{6R^2}-\frac{y^2}{2R^2}\right),$$

$$y'=y+s\sin\beta-\frac{1}{6R^2}(3s^2y-3s^2y\sin^2\beta+s^3\sin\beta-s^3\sin^3\beta),$$

$$y'=y+s\sin\beta-\frac{1}{2R^2}s^2y\cos^2\beta-\frac{1}{6R^2}s^3\sin\beta\cos^2\beta. \quad . \quad . \quad (103)$$

From (101),

$$(x'-x)-\frac{(x'-x)^3}{6R^2}=\left(s-\frac{s^3}{6R^2}\right)\cos\beta\left(1-\frac{y'^2}{2R^2}\right)^{-1}$$

$$=s\cos\beta\left(1-\frac{s^2}{6R^2}+\frac{y'^2}{2R^2}\right).$$

For a first approximation,

$$x'-x=s\cos\beta.$$

Substituting,

$$x'-x=\frac{1}{6R^2}(s\cos\beta)^3+s\cos\beta-\frac{1}{6R^2}s^3\cos\beta+\frac{1}{2R^2}sy'^2\cos\beta,$$

or

$$x'=x+s\cos\beta+\frac{1}{2R^2}sy'^2\cos\beta-\frac{1}{6R^2}s^3\cos\beta\sin^2\beta. \quad . \quad . \quad (104)$$

From (102),

$$\beta-\beta'=\frac{1}{2R^2}(x'-x)(y'+y). \quad . \quad . \quad . \quad . \quad . \quad (105)$$

Substituting for y', $y+s\sin\beta$,

$$\beta-\beta'=\frac{1}{R^2}y(x'-x)+\frac{1}{2R^2}(x'-x)s\sin\beta. \quad . \quad . \quad (106)$$

If $s\sin\beta=n$, and $s\cos\beta=m$, (103), (104), and (106) become

$$\left.\begin{aligned} y'&=y+n-m^2y\frac{1}{2R^2}-m^2n\frac{1}{6R^2},\\ x'&=x+m+my'^2\frac{1}{2R^2}-mn^2\frac{1}{6R^2}.\\ \beta-\beta'&=\frac{my}{R^2\sin 1''}+\frac{mn}{2R^2\sin 1''};\\ \text{also,}\quad \beta-\beta'&=\frac{m(y+y')}{2R^2\sin 1''}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (107)$$

The terms containing $1/R^2$ in the values of y' and x' are the small corrections to the values which would be found for plane coordinates. For the ellipsoid $1/R^2$ is replaced by $1/RN$ as in § 95.

Table XI gives the values of $\frac{m^2n}{2RN}$ (the form of each corrective term for the coordinates) and $\frac{mn}{RN \sin 1''}$ (the form of each corrective term for the convergence) for values of m and n from 10^{km} to 100^{km} for latitude 40°. The variation with latitude is small. Thus the values for $m=100^{km}$ and $n=100^{km}$ change from $12^m.31$ and $50''.8$ at 40° to $12^m.28$ and $50''.6$ for latitude 50° and to $12^m.30$ and $50''.9$ for latitude 30°.

Mental interpolation can be used for small distances, but slide rule interpolation would be more convenient for large ones. The values in the columns of the first part of the table increase as m^2, and in the second part as m, while the values for the horizontal lines for both parts increase with n. For the terms $\frac{m^2n}{6RN}$ and $\frac{mn}{2RN \sin 1''}$, the tabular quantities can be divided by 3 and 2 respectively in recording.

112. Mapping Spherical Coordinates.—In mapping, the y coordinates are laid off perpendicular to the central meridian. This enlarges the latitude scale as the distance from the central meridian increases.

From (103), replacing $s \cos \beta$ and $s \sin \beta$ by their approximate values, $x'-x$, and $y'-y$,

$$s \sin \beta = (y'-y) + \frac{1}{2R^2}(x'-x)^2 y + \frac{1}{6R^2}(x'-x)^2(y'-y).$$

Similarly from (104),

$$s \cos \beta = (x'-x) - \frac{1}{2R^2}(x'-x)y'^2 + \frac{1}{6R^2}(x'-x)(y'-y)^2.$$

Squaring and adding,

$$s^2 = \left[(y'-y) + \frac{1}{2R^2}(x'-x)^2 y + \frac{1}{6R^2}(x'-x)^2(y'-y)\right]^2$$
$$+\left[(x'-x) - \frac{1}{2R^2}(x'-x)y'^2 + \frac{1}{6R^2}(x'-x)(y'-y)^2\right]^2.$$

Placing the distance for plane coordinates, or the map distance $= s_0$,

$$s^2 = s_0{}^2 + \frac{1}{R^2}(x'-x)^2(y'-y)y + \frac{1}{3R^2}(x'-x)^2(y'-y)^2$$

$$-\frac{1}{R^2}(x'-x)^2y'^2 + \frac{1}{3R^2}(x'-x)^2(y'-y)^2$$

$$= s_0{}^2 + \frac{1}{3R^2}(x'-x)^2[3y(y'-y) + 2(y'-y)^2 - 3y'^2]$$

$$= s_0{}^2 - \frac{1}{3R^2}(x'-x)^2(y^2 + yy' + y'^2)$$

$$= s_0{}^2\left[1 - \frac{1}{3R^2}\cos^2\beta(y^2 + yy' + y'^2\right],$$

or

$$s = s_0\left[1 - \frac{1}{3R^2}\cos^2\beta(y^2 + yy' + y'^2)\right]^{\frac{1}{2}}.$$

Expanding by the binomial theorem,

$$s = s_0\left[1 - \frac{1}{6R^2}\cos^2\beta(y^2 + yy' + y'^2)\right]. \quad . \quad . \quad . \quad (108)$$

Putting the map magnification $= G$, and expanding by the binomial theorem,

$$G = \frac{s_0}{s} = 1 + \frac{\cos^2\beta}{6R^2}(y^2 + yy' + y'^2). \quad . \quad . \quad . \quad (109)$$

For short lines $y = y'$, nearly, giving

$$G = 1 + \frac{1}{2R^2}y^2\cos^2\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

This becomes unity for $\beta = 90°$, the map giving true differences of longitude, and a maximum of

$$G = 1 + \frac{y^2}{2R^2} \text{ for } \beta = 0.$$

CHAPTER VIII.

DETERMINATION OF THE DIMENSIONS OF THE ELLIPSOID.

113. The Meridian from Two Latitude-degree Measurements. These arcs may be on the same meridian, or on different ones if the earth is assumed to be an ellipsoid of rotation. The arc s

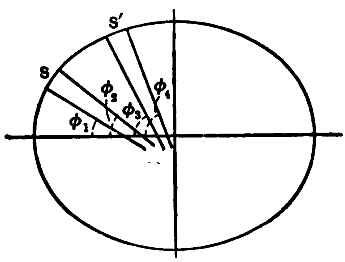

FIG. 86.

is measured, as also the latitudes of its extremities for each case.

Placing

$$\phi_2-\phi_1=\Delta\phi,\ \tfrac{1}{2}(\phi_2+\phi_1)=\phi,\quad \phi_4-\phi_3=\Delta\phi',\quad \tfrac{1}{2}(\phi_4+\phi_3)=\phi',$$
$$s=\Delta\phi\, R \sin 1'',\qquad s'=\Delta\phi' R' \sin 1''. \quad . \quad . \quad . \quad (111)$$

Dividing and substituting the value of R from (56),

$$\left(\frac{s\Delta\phi'}{s'\Delta\phi}\right)^{\frac{2}{3}}=\frac{1-e^2\sin^2\phi'}{1-e^2\sin^2\phi}=q^2.$$

Solving for e^2,

$$e^2=\frac{1-q^2}{\sin^2\phi'-q^2\sin^2\phi}. \quad . \quad . \quad . \quad . \quad . \quad (112)$$

Substituting the value of R from (56) in (111),

$$a=\frac{s(1-e^2\sin^2\phi)^{\frac{3}{2}}}{\Delta\phi(1-e^2)\sin 1''}=\frac{s'(1-e^2\sin^2\phi')^{\frac{3}{2}}}{\Delta\phi'(1-e^2)\sin 1''}. \quad . \quad . \quad (113)$$

Having the semi-major axis,

$$\text{semi-minor axis, } b=a\sqrt{1-e^2}. \quad . \quad . \quad . \quad . \quad (114)$$

The entire quadrant can be found from (66) if desired.

114. Reduction of a Measured Arc to the Meridian.—The arc is supposed to make only a small angle with the meridian.

From (75), MN (Fig. 87) or

$$\begin{aligned} K &= \Delta\phi R \sin 1'' \\ &= -s\cos\alpha - \frac{1}{2N}s^2\sin^2\alpha\tan\phi \\ &\quad + \frac{1}{6N^2}s^3\sin^2\alpha\cos\alpha\,(1+3\tan^2\phi). \quad (115) \end{aligned}$$

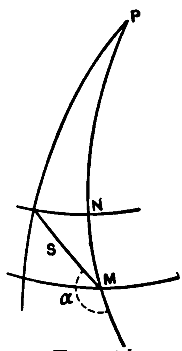

FIG. 87.

The second and third terms of the last member are so small that an approximate value can be used for N.

With two oblique arcs this case can thus be reduced to the preceding, or with a chain of triangles side after side can be projected for the whole length of the chain.

115. The Meridian from Several Latitude-degree Measurements.*—This involves the formation of observation equations between the observed latitudes and the projected, or directly measured, meridional arcs. The simplest relation is (111), which can be used for a $\Delta\phi$ of several degrees on account of the large probable error of a latitude determination, some 0.04″ or 4 feet, aside from the station error.

For longer arcs a correction for (111) will be required.

From the formula just above (66), Chap. VI, with the values of A and B restored,

$$s=a(1-e^2)[(1+\tfrac{3}{4}e^2)(\phi''-\phi')-\tfrac{3}{8}e^2(\sin 2\phi''-\sin 2\phi')\ldots]$$

---

* This article and the next should be omitted until Part II has been studied.

Eliminating $\sin 2\phi'' - \sin 2\phi'$ by form. 8],

$$s = a(1-e^2)[(1+\tfrac{3}{4}e^2)\Delta\phi - \tfrac{3}{4}e^2 \sin \Delta\phi \cos 2\phi \ldots]$$

Replacing $\sin \Delta\phi$ by $\Delta\phi - \tfrac{1}{6}(\Delta\phi)^3$,

$$s = a\Delta\phi(1-e^2)(1+\tfrac{3}{4}e^2 - \tfrac{3}{4}e^2 \cos 2\phi + \tfrac{1}{8}e^2(\Delta\phi)^2 \cos 2\phi \ldots) \quad \textbf{(116)}$$

Expanding the value of R from (56), form 32],

$$R = a(1-e^2)(1+\tfrac{3}{2}e^2 \sin^2 \phi)$$
$$= a(1-e^2)(1+\tfrac{3}{4}e^2 - \tfrac{3}{4}e^2 \cos 2\phi).$$

Hence the approximate value (111) for s,

$$s_1 = \Delta\phi R = a\Delta\phi(1-e^2)(1+\tfrac{3}{4}e^2 - \tfrac{3}{4}e^2 \cos 2\phi).$$

Subtracting this value from the true value (116) will give the correction δs to apply to the approximate value, or

$$s - s_1 = \delta s = a\Delta\phi(1-e^2)\tfrac{1}{8}e^2(\Delta\phi)^2 \cos 2\phi,$$

or, with $\Delta\phi$ in seconds,

$$\delta s = \tfrac{1}{8}ae^2 (\Delta\phi'' \sin 1'')^3 \cos 2\phi. \quad . \quad . \quad . \quad . \quad . \quad (117)$$

The correction for $\Delta\phi = 1°$ reduces to $-0^m.028$ in latitude 0°; $+0^m.014$ for $\phi = 30°$; $0.^m000$ for $\phi = 45°$; $+0^m.014$ for $\phi = 60°$; $+0^m.028$ for $\phi = 90°$.

Jordan gives the following data: *

	Station.	Latitude Degree Measures in Europe. Latitude ϕ.	$\Delta\phi$.	Meridian Arc s.
French	Formentera	ϕ_1 = 38° 39′ 56.1″		
	Barcelona	ϕ_2 = 41 22 47.9	2° 42′ 51.8″	301 354
	Carcassonne	ϕ_3 = 43 12 54.3	4 32 58.2	505 137
	Pantheon	ϕ_4 = 48 50 49.4	10 10 53.3	1 131 050
	Dunkirk	ϕ_5 = 51 02 08.8	12 22 12.7	1 374 572
English	Dunnose	ϕ_6 = 50 37 07.6		
	Greenwich	ϕ_7 = 51 28 39.0	0 51 31.4	95 620
	Arburyhill	ϕ_8 = 52 13 28.0	1 36 20.4	178 720
	Clifton	ϕ_9 = 53 27 31.1	2 50 23.5	315 892
Hanover	Göttingen	ϕ_{10} = 51 13 47.8		
	Altona	ϕ_{11} = 53 32 45.3	2 00 57.5	224 458
Prussian	Trunz	ϕ_{12} = 54 13 11.5		
	Königsberg	ϕ_{13} = 54 42 50.5	0 29 39.0	54 985
	Memel	ϕ_{14} = 55 43 40.4	1 30 28.9	167 962
Russian	Belin	ϕ_{15} = 52 02 40.9		
	Jakobstadt	ϕ_{16} = 56 30 C4.6	4 27 23.7	496 114
	Dorpat	ϕ_{17} = 58 22 47.3	6 .20 06.4	705 209
	Hochland	ϕ_{18} = 60 05 09.8	8 02 28.9	895 315
Swedish	Malörn	ϕ_{19} = 65 31 30.3		
	Pahtawara	ϕ_{20} = 67 08 49.8	1 37 19.5	180 828

* *Vermessungskunde*, Vol. III, 1890, p. 507.

From the first two latitudes,

$$\phi_2-\phi_1=\frac{s}{R\sin 1''}-\frac{\delta s}{R\sin 1''}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

where

$$\frac{1}{R}=\frac{(1-e^2\sin^2\phi)^{\frac{3}{2}}}{a(1-e^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

Since a and e^2 are unknown or required quantities, we substitute for them approximate values with corrections.

$a=a_0+\delta a$, $e^2=e_0{}^2+\delta e^2$, and $R_0=$ the value of R resulting from the approximate a_0 and $e_0{}^2$.

Expanding by Taylor's theorem (see § 122 for extension to a function of two variables),

$$\frac{1}{R}=\frac{1}{R_0}+\frac{d\left(\frac{1}{R}\right)}{da}\delta a+\frac{d\left(\frac{1}{R}\right)}{de^2}\delta e^2. \quad . \quad . \quad . \quad . \qquad (c)$$

But

$$\frac{d\left(\frac{1}{R}\right)}{da}=-\frac{(1-e^2\sin^2\phi)^{\frac{3}{2}}}{(1-e^2)a^2},$$

$$=-\frac{1}{a^2} \text{ by neglecting all terms containing } e^2.$$

$$\frac{d\left(\frac{1}{R}\right)}{de^2}=\frac{-(1-e^2)\frac{3}{2}(1-e^2\sin^2\phi)^{\frac{1}{2}}\sin^2\phi+(1-e^2\sin^2\phi)^{\frac{3}{2}}}{a(1-e^2)^2}$$

$$=\frac{1}{a}(1-\tfrac{3}{2}\sin^2\phi), \text{ by neglecting } e^2 \text{ terms.}$$

Substituting in (c),

$$\frac{1}{R}=\frac{1}{R_0}-\frac{1}{a_0{}^2}\delta a+\frac{1}{a_0}(1-\tfrac{3}{2}\sin^2\phi)\delta e^2.$$

Hence (a) becomes

$$\phi_2=\phi_1+\frac{s}{\sin 1''}\left(\frac{1}{R_0}-\frac{\delta a}{a_0^2}+(1-\tfrac{3}{2}\sin^2\phi)\frac{\delta e^2}{a_0}\right)-\frac{\delta s}{R_0 \sin 1''}. \qquad (d)$$

The value of δs is given in (117).

The measured meridional arcs are considered accurate or perfect in comparison with the observed latitudes. The observed latitudes require corrections v_1, v_2, so that (d) becomes, with the last term omitted,

$$\phi_2+v_2=\phi_1+v_1+\frac{s}{\sin 1''}\left(\frac{1}{R_0}-\frac{\delta a}{a_0^2}+(1-\tfrac{3}{2}\sin^2\phi)\frac{\delta e^2}{a_0}\right). \qquad (e)$$

In the coefficients of δa and δe^2 all terms containing e^2 were omitted. This is equivalent to placing $R=a$, and $\frac{s}{a \sin 1''}=\phi_2-\phi_1$.

Substituting in (e),

$$v_2-v_1=-\frac{\phi_2-\phi_1}{a_0}\delta a$$
$$+(1-\tfrac{3}{2}\sin^2\phi)(\phi_2-\phi_1)\delta e^2+\frac{s}{R_0 \sin 1''}-(\phi_2-\phi_1). \qquad (f)$$

For convenience place

$$x=\frac{\delta a}{1000}, \quad y=1000\delta e^2. \quad . \quad . \quad . \quad . \quad . \qquad (g)$$

Substituting in (f),

$$a_1x+b_1y+l_1=v_2-v_1,$$

where

$$a_1=-1000\frac{\phi_2-\phi_1}{a_0}, \quad b_1=\frac{\phi_2-\phi_1}{1000}(1-\tfrac{3}{2}\sin^2\phi),$$

$$l_1=\frac{s_1}{R_0 \sin 1''}-(\phi_2-\phi_1).$$

For a_0 and e_0^2 the Bessel ellipsoid is taken, for which

$$a_0=6\,377\,397.155^{\mathrm{m}}, \quad e^2=0.006\,674\,372.$$

Then

$$R_0=\frac{a_0(1-e_0^2)}{(1-e_0^2\sin^2\phi)^{\frac{3}{2}}}, \quad \log R_0=6.803\,5358 \text{ for the first arc.}$$

These values give

$$a_1=-1.53, \quad b_1=+3.71, \quad l_1=-0''.2.$$

Similarly the coefficients and absolute terms for the other equations are found.

These do not correspond to observation equations, as there are two residuals or corrections in the second members. This is due to a correction for the initial latitude as well as for each of the others, and corresponds to the case of direction observations, where for each new position of the circle a correction is required for the initial reading as well as for each of the others.

Adding an observation equation for each initial latitude and placing the v for each in the first member as a correction will give twenty observation equations as follows:

OBSERVATION EQUATIONS.

$$\left.\begin{array}{lr}
v_1 & =v_1 \\
v_1 \quad -1.53x+\ 3.71y-0.2'' & =v_2 \\
v_1 \quad -2.57x+\ 5.83y-1.4 & =v_3 \\
v_1 \quad -5.75x+10.36y-2.1 & =v_4 \\
v_1 \quad -6.98x+11.31y+1.2 & =v_5 \\
v_6 & =v_6 \\
v_6 \quad -0.48x+\ 0.29y+3.2 & =v_7 \\
v_6 \quad -0.91x+\ 0.48y+3.2 & =v_8 \\
v_6 \quad -1.60x+\ 0.69y-1.9 & =v_9 \\
v_{10} & =v_{10} \\
v_{10} \quad -1.14x+\ 0.40y+5.0 & =v_{11} \\
v_{12} & =v_{12} \\
v_{12} \quad -0.28x+\ 0.01y-0.5 & =v_{13} \\
v_{12} \quad -0.85x-\ 0.03y+3.3 & =v_{14} \\
v_{15} & =v_{15} \\
v_{15} \quad -2.52x+\ 0.18y+3.6 & =v_{16} \\
v_{15} \quad -3.58x-\ 0.27y+0.7 & =v_{17} \\
v_{15} \quad -4.54x-\ 0.94y+2.3 & =v_{18} \\
v_{19} & =v_{19} \\
v_{19} -0.92x-\ 1.51y-1.1 & =v_{20}
\end{array}\right\} \quad \ldots \quad (h)$$

The normal equations are formed as usual.

NORMAL EQUATIONS.

$$\begin{array}{l}
+5v_1 \qquad -16.83x+\ 31.21y-\ 2.50=0 \\
+4v_6 \qquad -\ 2.99x+\ 1.46y+\ 4.50=0 \\
+2v_{10} \qquad -\ 1.14x+\ 0.40y+\ 5.00=0 \\
+3v_{12} \qquad -\ 1.13x-\ 0.02y+\ 2.80=0 \\
+4v_{15} \qquad -10.64x-\ 1.03y+\ 6.60=0 \\
+2v_{19}-\ 0.92x-\ 1.51y-\ 1.10=0 \\
-16.83v_1-2.99v_6-1.14v_{10}-1.13v_{12}-10.64v_{15}-0.92v_{19} \\
\qquad +137.07x-155.11y-23.18=0 \\
+31.21v_1+1.46v_5+0.40v_{10}-0.02v_{12}-\ 1.03v_{15}-1.51v_{19} \\
\qquad -155.11x+287.21y-14.08=0
\end{array}$$

Expressing the v in each of the first six equations in terms of x and y and substituting in the last two, we find

$$x= +0.4023, \quad y= +0.2347.$$

Substituting in (g),

$$\delta a = 1000x = +402^{m}.3, \qquad \delta e^2 = 0.001y = +0.000\,2347,$$
$$a = a_0 + \delta a = 6\,377\,397.2 + 402.3 = 6\,377\,800,$$
$$e^2 = e_0^2 + \delta e^2 = 0.006\,6744 + 0.000\,2347 = 0.006\,9091.$$

Substituting the values of x and y in the first six normal equations will give v_1, v_6, v_{10}, v_{12}, v_{15}, v_{19}. These values together with those for x and y substituted in the observation equations (h) will give the other v's.

Squaring each v and adding,

$$[v^2] = 52.$$

$$\therefore\ \epsilon = \sqrt{\frac{[v^2]}{m-n}} = \sqrt{\frac{52}{12}} = 2''.1$$

for the mean square error of a latitude determination referred to the ellipsoid. This is very much greater than the m.s.e. of a latitude determination, showing that an ellipsoid of revolution will not fit the data without large station errors or local deviations of the plumb line.

The v's for each group, i.e., French, English, etc., foot up zero within $0''.01$.

116. The Ellipsoid from a Degree Measurement Oblique to the Meridian.—The observed data are the latitude and azimuth at each end of the line, the difference in longitude, and the linear distance.

Each observation will give an equation of the form

$$f(X, Y, Z, \ldots) - M_1 = v_1, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where the required quantities X, Y, Z, . . . are the most probable values for the observed quantities, ϕ_1, ϕ_2, α_1, α_2, $\Delta\lambda$, s, and for the dimensions of the ellipsoid, a and e^2. The corrections for the observed or assumed values are denoted by δ.

For ϕ_1,

$$f_1(\phi_1 + \delta\phi_1) - \phi_1 = v_1, \quad \text{or} \quad \delta\phi_1 + 0 = v_1. \quad . \quad . \quad . \quad (b)$$

For ϕ_2,

$$f_2(\phi_1 + \delta\phi_1,\ a_0 + \delta a,\ e_0^2 + \delta e^2) - \phi_2 = v_2,$$

or, expanding by Taylor's theorem,

$$f_2(\phi_1, a_0, e_0^2) - \phi_2 + \frac{df_2}{d\phi_1}\delta\phi_1 + \frac{df_2}{da}\delta a + \frac{df_2}{de^2}\delta e^2 = v_2. \quad . \quad (c)$$

The quantity $f_2(\phi_1, a_0, e_0^2)$, the computed value of ϕ_2, can be found by (75). Place this computed value less $\phi_2 = l_2$. For the differential coefficients only the first term of the second member of (75) need be used, i.e.,

$$f_2(\phi_1 + \delta\phi_1, a_0 + \delta a, e_0^2 + \delta e^2) = \phi_1 - \frac{1}{R} s \cos \alpha$$

$$= \phi_1 - \frac{a(1-e^2)}{r^3} s \cos \alpha, \quad . \quad (d)$$

where $r = (1 - e^2 \sin^2\phi)^{\frac{1}{2}}$.

Differentiating,

$$\frac{df_2}{d\phi_1}\delta\phi_1 = \delta\phi_1, \quad \frac{df_2}{da}\delta a = -\frac{1-e_0^2}{r_0^3} s \cos \alpha \delta a,$$

$$\frac{df_2}{de^2}\delta e^2 = \frac{r_0^2 - \frac{3}{2}(1-e_0^2)\sin^2\phi}{r_0^5} a_0 s \cos \alpha \delta e^2.$$

Substituting in (*c*),

$$\delta\phi_1 - \frac{1-e_0^2}{r_0^3} s \cos \alpha \delta a + \frac{r_0^2 - \frac{3}{2}(1-e_0^2)\sin^2\phi}{r_0^5} a_0 s \cos \alpha \delta e^2 + l_2 = v_2. \quad (e)$$

For $\Delta\lambda$. By (79),

$$f_3(a_0 + \delta a, e_0^2 + \delta e^2) = \frac{s \sin \alpha_1}{N' \cos \phi_2} = \frac{r' s \sin \alpha_1}{a \cos \phi_2}, \quad . \quad . \quad . \quad (f)$$

since $\frac{1}{N'} = \frac{r'}{a}$.

Differentiating,

$$\frac{df_3}{da}\delta a = -\frac{r'_0 s \sin \alpha_1}{a_0^2 \cos \phi_2}\delta a, \quad \frac{df_3}{de^2}\delta e^2 = -\frac{\sin^2 \phi_2 s \sin \alpha_1}{2a_0 r'_0 \cos \phi_2}\delta e^2.$$

Substituting in (*f*),

$$-\frac{r'_0 s \sin \alpha_1}{a_0^2 \cos \phi_2}\delta a - \frac{\sin^2 \phi_2 s \sin \alpha_1}{2a_0 r'_0 \cos \phi_2}\delta e^2 + l_3 = v_3. \quad . \quad . \quad . \quad (g)$$

For α_1,

$$f_4(\alpha_1+\delta\alpha_1)-\alpha_1=v_4, \quad \text{or} \quad \delta\alpha_1+0=v_4. \quad . \quad . \quad . \quad (h)$$

For α_2,

$$f_5(\alpha_1+\delta\alpha_1,\ a_0+\delta a,\ e_0^2+\delta e^2)-\alpha_2=v_5. \quad . \quad . \quad . \quad . \quad (i)$$

By (84) and (85),

$$\alpha_2=f_5=180°+\alpha_1-\Delta\lambda \sin\phi_m$$

$$=180°+\alpha_1-\frac{s \sin\alpha_1 \sin\phi_m}{N' \cos\phi_2}$$

$$=180°+\alpha_1-\frac{r's \sin\alpha_1 \sin\phi_m}{a \cos\phi_2},$$

$$\frac{df_5}{d\alpha_1}\delta\alpha_1=\delta\alpha_1, \quad \frac{df_5}{da}\delta a=+\frac{r_0's \sin\alpha_1 \sin\phi_m}{a_0^2 \cos\phi_2}\delta a,$$

$$\frac{df_5}{de^2}\delta e^2=+\frac{\sin^2\phi_2 s \sin\alpha_1 \sin\phi_m}{2a_0r_0' \cos\phi_2}\delta e^2.$$

Substituting in (i),

$$\delta\alpha_1+\frac{r_0's}{a_0^2}\sin\alpha_1 \tan\phi \,.\, \delta a+\frac{s}{2a_0r_0'}\sin\alpha_1 \sin^2\phi \tan\phi \,.\, \delta e^2+l_5=v_5. \quad (j)$$

Collecting equations (b), (e), (g), (h), and (j), and denoting the coefficients of δa and e^2 by a and b,

$$\left.\begin{array}{r}
\delta\phi_1 \qquad\qquad\qquad\qquad +l_1=v_1, \\
\delta\phi_1 \qquad a_2\delta a+b_2\delta e^2+l_2=v_2, \\
-a_3\delta a-b_3\delta e^2+l_3=v_3, \\
\delta\alpha_1 \qquad\qquad\qquad +l_4=v_4, \\
\delta\alpha_1+a_5\delta a+b_5\delta e^2+l_5=v_5.
\end{array}\right\} \quad . \quad . \quad . \quad . \quad (118)$$

Weights can be introduced if desired.

If s is large or poorly measured so that its m.s.e. is appreciable in comparison with those for ϕ, $\Delta\lambda$, and α, another equation should be added, viz.,

$$f_6(a_0+\delta a,\ e_0^2+\delta e^2)-s=v_6.$$

From (90),

$$f_6=\frac{N'\Delta\lambda\cos\phi_1}{\sin\alpha_1}=\frac{a\Delta\lambda\cos\phi_1}{r'\sin\alpha_1},$$

$$\frac{df_6}{da}\delta a=\frac{\Delta\lambda\cos\phi_1}{r_0'\sin\alpha_1}\delta a,\quad \frac{df_6}{de^2}\delta e^2=\frac{a_0\Delta\lambda\cos\phi_1\sin^2\phi_1}{2r_0^3\sin\alpha_1}\delta e^2,$$

giving

$$\frac{\Delta\lambda\cos\phi_1}{r_0'\sin\alpha_1}\delta a+\frac{a_0\Delta\lambda\cos\phi_1\sin^2\phi_1}{2r_0^3\sin\alpha_1}\delta e^2+l_6=v_6.$$

Hence

$$a_6\delta a+b_6\delta e^2+l_6=v_6$$

is the equation to be added to (118).

If a second line starts from the initial station and its azimuth is computed from the observations which gave α_1, there would be added to (118)

$$\delta\phi_1-a_7\delta a+b_7\delta e^2+l_7=v_7 \text{ from } \phi_3,$$
$$-a_8\delta a-b_8\delta e^2+l_8=v_8 \quad \text{"} \quad \Delta\lambda_{1-3},$$
$$\delta\alpha_1+a_9\delta a+b_9\delta e^2+l_9=v_9 \quad \text{"} \quad \alpha_3,$$
$$a_{10}\delta a+b_{10}\delta e^2+l_{10}=v_{10} \text{ if } s_{1-3} \text{ is considered.}$$

The distance s can be greater than a triangle side by solving for an approximate α by (90); computing through the chain of triangles with two angles and the included side given each time

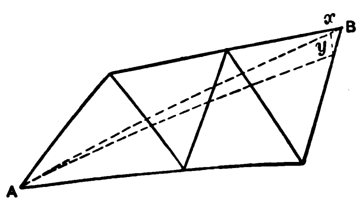

FIG. 88.

to find the third angle and the second side; calling the change in direction of s at each intersection 180°. Then α_1, α_2, and s as found for the total distance can be corrected for the error in closure at B by adding x to s and dividing y by $s \sin 1''$ for the correction to α.

For the more general treatment for an astronomical geodetic net, taking into account station error in its effect upon latitude, longitude, and azimuth, see Helmert's *Höheren Geodäsie.*

117. Results of Degree Measures.—The actual shape of the earth as compared with a spheroid of revolution can best be appreciated by comparing the dimensions obtained for the spheroid from the measurements of different arcs. The following are taken from the Coast Survey *Special Publications No. 4* and *No. 7*.

SPHEROIDS OF REVOLUTION FROM DIFFERENT DEGREE MEASURES.

Spheroid of—	Amplitude.	No. of astronomic stations.	a in meters.	b in meters.	$a-b$.
Bessel, 1841	$\Sigma=50° 35'.4$	38 ϕ's	6 377 397	6 356 079	21 318
Clarke,* 1858		75	6 378 494	6 355 746	22 748
Clarke, 1866	$\Sigma=76\ 35\ .0$	40 ϕ's	6 378 206	6 356 584	21 622
Clarke,* 1880	$\Sigma=88\ 59\ .8$		6 378 249	6 356 515	21 734
U. S. parallel 39° / L. Superior merid.	48° 16.8 cos ϕ / 9 23.8	28 λ's / 10 ϕ's	6 377 912	6 356 309	21 603
U. S. parallel 39° / Peruvian meridian	48 16.8 cos ϕ / 3 07.1	28 λ's / 2 ϕ's	6 378 027	6 356 819	21 208
U. S. oblique arc	$\Sigma=23° 31'.0$	84	6 378 157	6 357 210	20 947
Harkness,† 1891			6 377 972	6 356 727	21 245

* For converting English feet to meters, Clarke's value for 1866 was used, viz., 1 ft. = $0.304\,797\,27^{m}$.

† From *The Solar Parallax and Related Constants*, Washington, 1891, p. 138.

The Clarke 1858 spheroid is a special one for Great Britain and Ireland. It contains a range of 12° in latitude and the same in longitude. If this is omitted, the range in values for the semi-major axis is 852^{m} or about 1/7500, and for the semi-minor 1131^{m} or about 1/5600.

In the United States and connected by continuous triangulation there are 265 astronomical determinations of latitude, 79 of longitude, and 163 of azimuth. This gave an excellent opportunity to study the distribution and amount of the deflections of the plumb line and to eliminate or largely reduce their effect in computing the geodetic positions and azimuths for use in finding the figure of the earth.

The geodetic positions and azimuths are obtained by computing the positions and directions of the astronomical stations and observed azimuths from assumed but nearly correct data. The differences, astronomical minus geodetic $(A-G)$, are taken for latitude, for longitude, and for azimuth. A correction is then found for the initial data so that $\Sigma(A-G)=0$, or nearly so, for latitude, longitude, and azimuth. The correction due to placing $\Sigma(A-G)=0$ will depend upon the distribution of the astronomical stations over the area; groups of stations affected by the same local or general station error should be replaced by a single representative one. The extension over new territory or a change of the reference ellipsoid will also affect this correction. Hence geodetic positions and azimuths must always be regarded as more or less tentative.

In view of the statements made in § 102 about the equality of distances and azimuths on the spheroid and tangent sphere it may be interesting to note that the shortest line between Washington, D. C., and Eastport, Me., is about ·1/1000 inch shorter than a plane curve joining the points.*

The Coast Survey recently traced the boundary between California and Nevada running from a point in Lake Tahoe, about 45° east of south, to Fort Mohave, a distance of 404.5 miles. The difference between the shortest line and the plane curve is about 1/6000 inch. The angle at either end between the sight lines between the two stations, the one with the instrument at Lake Tahoe and the other with the instrument at Fort Mohave, due to the normals not intersecting the axis at the same point, is about 2″.2.

118. Local Deflections of the Vertical.—Local deflections of the plumb line are explained by the attraction of mountains or elevated plateaus and by variations in the density of the earth's crust. The attraction of two small masses m_1 and m_2 at the distance apart d is proportional to $\frac{m_1 m_2}{d^2}$.

For a point on the surface of the earth, if regarded as a sphere, the attraction is the same as if the mass were concentrated at the center. If a small mass is horizontally distant d from this point, the relative attractions of the earth and the small mass upon the point are M/R^2 and m/d^2. Hence if θ = the deflection of the plumb line due to m,

$$\tan\theta = \frac{mR^2}{Md^2}.$$

The volume of the earth $= \frac{4}{3}\pi R^3$, and its mean density is

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* *Recent Progress in Geodesy*, by F. D. Preston. Phil. Soc. of Wash. Bul. Vol. XIII.

about 5.63 (as compared with 2.3 for rock), while in round numbers $R = 6\,370\,000^{m}$.

Denoting the volume of the mass m by v and its density by δ, $m = v\delta$.

Substituting,

$$\theta'' = 0.001\,373 \frac{v\delta}{d^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

The effect of elevated masses can thus be computed if the topography is known. If the mass is large in proportion to distance, it should be divided up and $v\delta/d^2$ found for each part and added, or a more general expression found by calculus.*

After correcting for the topography, deflections still remain, and these are accounted for by variations in density. Investigations under way by the Coast Survey, taking into account the deflections of the vertical due to topography, and the contrary deflections due to defects and excesses of density below the surface, lead to the conclusion that the elevated masses are in the main supported by material of deficient density and that this deficiency compensates within a depth of about 70 miles.

Thus "The United States is not maintained in its position above sea level by the rigidity of the earth, but is, in the main, buoyed up, floated, upon underlying material of deficient density. . . . It is certain that for the United States and adjacent regions, including oceans, this compensation is more than two-thirds complete—perhaps much more." †

* See Coast Survey *Special Publication No. 4,* p. 848.

† J. F. Hayford, *Proc. Wash. Acad. Sciences,* Vol. VIII, p. 32.

PART II.

LEAST SQUARES.

CHAPTER IX.

THE METHOD OF LEAST SQUARES.

119. Introduction.—The methods of observing for geodetic work have been quite fully described in Part I. Attention is called to the care and skill required to secure good results and the tendency to check results by additional observations. In many cases these check observations are additional measurements of the same quantity. In this case the arithmetic mean is taken unless the observations are not considered equally reliable.

In other cases the check observations are indirect, that is, taken on some function of the first observed quantity or on some other function of the required quantity or quantities. In this case the propagation of error through the function from the measured to the required quantity or quantities will vary and the method of least squares is necessary to obtain the best values for the unknowns. Thus the distance across a stream may be measured directly and the measurements repeated as a check and to increase accuracy, or the check may be obtained by measuring a base line and triangulating. Again, the angles required at a triangulation station may be measured directly and each repeated as many times as necessary, or they may be measured indirectly as described in § 51. If in either case the horizon is closed, an additional check is secured and a modification of the adjustment will be required.

Again, even if the angles at the station are directly observed, on entering the triangulation algebraic equations due to geometric conditions are found to exist between them and a least squares adjustment is necessary for the best results.

Such conditions are: that the sum of the angles in a triangle shall have a fixed value, that the length of a triangle side when computed from the same initial side through different triangles shall have the same value, etc.

The method of least squares thus has for its object the finding of the best or most probable values which can be obtained from a given series of observations for a set of unknown quantities depending upon physical measurement. It also has for its object the determination of the degree of confidence which can be placed in the results as determined from the agreement of the observations among themselves and from the propagation of error through a function or equation from observed to required quantities.

120. Classes of Errors.—Observations are subject to several classes of errors, as follows:

1st. *Constant errors*, or those which under the same circumstances, and in the measures of the same quantity, have the same value; or those in which the value can be made to depend upon the circumstances by some definite law. They are usually subdivided into: *theoretical*, such as refraction and curvature in leveling, whose effects, when their causes are once thoroughly understood, can be computed in advance, and hence they cease to exist as errors; *instrumental*, such as the line of collimation of a level not being horizontal when the bubble is in the center, which are discovered by an examination of the instruments, or of the observations made with them, and may be removed, when their causes are understood, either by a proper method of using the instruments or by subsequent computation; *personal*, such as always setting a target a little too high, and which depend upon the peculiarities of the observer. These latter are often the subject of special investigation under the name of "personal equation"; while not strictly constant, they are nearly so with trained observers.

These are frequently separated into two classes, using the term *systematic errors* for those which vary in accordance with more or less definite laws.

2d. *Mistakes or abnormal errors*, such as reading a circle a degree out of the way, the slipping of a clamp, the sighting at a wrong object, etc.

3d. *Accidental errors*, or the necessary *inaccuracies* which cannot be computed in advance from the circumstances of the observations and eliminated.

The limit of the first class is fixed by the limit of knowledge of instruments and of physical phenomena.

The limit of the second class can only be approximately fixed, as there are no means of distinguishing between inaccuracies and small mistakes.

In what follows the third class alone should be understood, unless otherwise stated.

The following may be assumed as axioms:

1. Small errors occur more frequently, or are more probable, than large ones.

2. Positive and negative errors of the same magnitude are equally probable, and in a large number of observations are equally frequent.

3. Very large errors do not occur.

121. Mean-square Error.—The square root of the average square of the errors is called the mean-square error. It is denoted by m.s.e. or ε, and is used in comparing the accuracy of different sets of observations.

Thus if $\Delta_1, \Delta_2, \ldots, \Delta_n$ are the *true* errors committed in a series of n equally good observations,

$$\varepsilon^2=\frac{\Delta_1^2+\Delta_2^2+\ldots+\Delta_n^2}{n}=\frac{[\Delta^2]}{n}^* \quad . \quad . \quad . \quad . \quad . \quad (120)$$

Ex. In Bessel's *Gradmessung in Ostpreussen* the excess over 180° plus the spherical excess, is given in seconds for the measured angles in 22 triangles as follows:

No	Δ	Δ^2	No.	Δ	Δ^2	No.	Δ	Δ^2	No.	Δ	Δ^2
				Forward,	6.249		Forward,	10.855		Forward,	24.411
1	+0.36	+0.130	7	+1.76	3.098	13	−1.36	1.850	19	+1.67	2.789
2	+0.93	0.865	8	+0.92	0.846	14	+1.86	3.460	20	−0.72	0.518
3	−0.51	0.260	9	+0.56	0.314	15	−0.42	0.176	21	−1.35	1.822
4	−1.46	2.132	10	0.00	0.000	16	+1.68	2.822	22	−0.98	0.960
5	−0.95	0.902	11	−0.59	0.348	17	+1.62	2.624			
6	−1.40	1.960	12	0.00	0.000	18	+1.62	2.624			
		6.249			10.855			24.411			30.500

* The square brackets are used to denote summation.

Hence

$$\text{The m.s.e., } \epsilon = \sqrt{30.500/22} = 1''.18.$$

If the triangle closures were larger the m.s.e. would be larger, so that ϵ increases with the inaccuracy of the observations.

122. Law of Propagation of Error.—(*a*) Let

$$x = \pm a_1 M_1 \pm a_2 M_2 \pm \ldots a_n M_n, \quad . \quad . \quad . \quad . \quad (121)$$

where $a_1, a_2, \ldots, a_n$ are constants unaffected by error, and $M_1, M_2, \ldots, M_n$ are observed independent quantities with m.s.e.'s $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$.

If $\Delta_1', \Delta_1'', \Delta_1''', \ldots, \Delta_2', \Delta_2'', \Delta_2''', \ldots, \Delta_n', \Delta_n'', \Delta_n'''$ are the errors for different observed values of $M_1, M_2, \ldots, M_n$, the errors in the corresponding values of x will be

$$\left.\begin{array}{l} \pm\Delta_x' = \pm a_1\Delta_1' \pm a_2\Delta_2' \pm \ldots a_n\Delta_n', \\ \pm\Delta_x'' = \pm a_1\Delta_1'' \pm a_2\Delta_2'' \pm \ldots a_n\Delta_n'', \\ \pm\Delta_x''' = \pm a_1\Delta_1''' \pm a_2\Delta_2''' \pm \ldots a_n\Delta_n'''. \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{array}\right\} \quad . \quad . \quad . \quad (a)$$

Squaring both members of each equation and adding,

$$[\Delta_x^2] = a_1^2[\Delta_1^2] + a_2^2[\Delta_2^2] \ldots + a_n^2[\Delta_n^2] \pm 2a_1a_2[\Delta_1\Delta_2] \ldots$$
$$\pm 2a_1a_n[\Delta_1\Delta_n] \pm 2a_2a_n[\Delta_2\Delta_n] \ldots \quad (b)$$

Positive and negative errors of the same magnitude being equally liable to occur, by Axiom 2, the products

$$\pm 2a_1a_2[\Delta_1\Delta_2] \ldots \pm 2a_1a_n[\Delta_1\Delta_n] \pm 2a_2a_n[\Delta_2\Delta_n] \ldots$$

will tend to foot up zero (approaching it more nearly the greater the number of observed values) and may be neglected.

Hence, dividing (*b*) by n, and remembering the definition of m.s.e.,

$$\epsilon_x^2 = a_1^2\epsilon_1^2 + a_2^2\epsilon_2^2 + \ldots + a_n^2\epsilon_n^2. \quad . \quad . \quad . \quad (122)$$

In the general case,

$$x = f(M_1, M_2, \ldots, M_n), \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

where f denotes any function.

If the different observed values be substituted for the true values of the observed quantities, we shall have

$$\left.\begin{array}{l} x\pm\Delta_x' = f(M_1\pm\Delta_1', M_2\pm\Delta_2', \ldots, M_n\pm\Delta_n'), \\ x\pm\Delta_x''=f(M_1\pm\Delta_1'', M_2\pm\Delta_2'', \ldots, M_n\pm\Delta_n''). \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}$$

Expanding the second members by Taylor's theorem, and supposing the observations accurate enough so that the squares, products, and higher powers of the Δ's may be neglected,*

$$\left.\begin{array}{l} \Delta_x' = \pm\frac{df}{dM_1}\Delta_1' \pm\frac{df}{dM_2}\Delta_2' \ldots \pm\frac{df}{dM_n}\Delta_n', \\ \Delta_x''= \pm\frac{df}{dM_1}\Delta_1''\pm\frac{df}{dM_2}\Delta_2''\ldots \pm\frac{df}{dM_n}\Delta_n''. \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}$$

These correspond to (*a*); hence, from (122),

$$\varepsilon_x^2 = \left(\frac{df}{dM_1}\varepsilon_1\right)^2 + \left(\frac{df}{dM_2}\varepsilon_2\right)^2 + \ldots + \left(\frac{df}{dM_n}\varepsilon_n\right)^2. \quad . \quad (124)$$

Ex. 1. Find the m.s.e. in the length of a city block 500 ft. long measured with a 100-ft. tape having a m.s.e. in its length of 0.01 ft. *Ans.* 0.05 ft.

---

* If $u=f(x, y)$, and x be increased by h,

$$f(x+h, y)=u+\frac{du}{dx}\cdot\frac{h}{1}+\frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2}+\ldots,$$

where $\frac{du}{dx}=f'(y)$, which we may place $=z$.

If y be increased by k,

$$z'=f'(y+k)=z+\frac{dz}{dy}\cdot\frac{k}{1}+\frac{d^2z}{dy^2}\cdot\frac{k^2}{1.2}+\ldots=\frac{du}{dx}+\frac{d\frac{du}{dx}}{dy}k+\ldots$$

Hence

$$f(x+h, y+k)=f(x, y+k)+\frac{du}{dx}h+\frac{d\frac{du}{dx}}{dy}hk+\frac{d^2u}{dx^2}\cdot\frac{h^2}{2}+\ldots$$

But

$$f(x, y+k)=f(x, y)+\frac{du}{dy}\cdot\frac{k}{1}+\frac{d^2u}{dy^2}\cdot\frac{k^2}{1.2}+\ldots$$

Substituting,

$$f(x+h, y+k)=f(x, y)+\frac{du}{dy}k+\frac{du}{dx}h\ldots$$

Similarly we may extend to three or more variables, as assumed above.

Ex. 2. Find the m.s.e. in the length of a city block 500 ft. long measured with five 100-ft. tapes, each with a m.s.e. of 0.01 ft. *Ans.* 0.02 ft.

Ex. 3. In the triangle *ABC*, Fig. 89, *AC* or $M_1=1060$ ft. with $\epsilon_1=0.1$ ft.; the angle *A* or $M_2=50°$ with $\epsilon_2=10''$ (in arc $=10 \sin 1''$); the angle *B* or $M_3=64°$ with $\epsilon_3=10''$. Find the m.s.e. for *BC* or *X*.

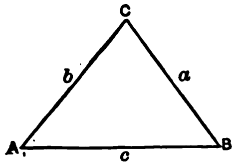

FIG. 89.

Solution.—Eq. (123) reduces to

$$x=f(M_1, M_2, M_3)=M_1\frac{\sin M_2}{\sin M_3}=903.44 \text{ ft.}$$

$$\frac{df}{dM_1}=\frac{\sin M_2}{\sin M_3}=\frac{x}{M_1}=.85,$$

$$\frac{df}{dM_2}=\frac{M_1 \cos M_2}{\sin M_3}=x \cot M_2=758,$$

$$\frac{df}{dM_3}=-M_1\frac{\sin M_2 \cos M_3}{\sin^2 M_3}=-x \cot M_3=-441.$$

Substituting in (124),

$$\begin{aligned}\epsilon_x^2&=\frac{x^2}{M_1^2}\epsilon_1^2+x^2\cot^2 M_2\epsilon_2^2+x^2\cot^2 M_3\epsilon_3^2\\&=(0.85\times0.1)^2+(758\times10\times0.000\,004\,85)^2+(441\times10\times0.000\,004\,85)^2\\&=0.0072+0.0013+0.0005=0.009,\end{aligned}$$

$$\epsilon_x=0.095 \text{ ft.}$$

Ex. 4. The weight of a piece of paving-brick in air, $M_1=1.67$ oz. The weight of the same piece in water, $M_2=0.96$ oz. Find the specific gravity and its m.s.e. for $\epsilon_1=\epsilon_2=0.01$ oz.

$$\text{Specific gravity, } x=\frac{M_1}{M_1-M_2}.$$

Ex. 5. The diameter of a steel test-piece which broke under a load of 30 000 lbs. is 0.6846 inch ±0.0012. What is the unit strength of the steel and its m.s.e.?

Ex. 6. A line 1km. long was measured first with a 5m line measure bar, whose m.s.e. was 2.5μ. It was then measured with two end measure bars, whose m.s.e.'s were also 2.5μ. Find in each case the m.s.e. of the measured kilometer due to this source of error.

123. **Law of Propagation of Error.**—(*b*) In (*b*) of § 122 it may be noted that in summing the products of the Δ's one of the factors, Δ, may contain constant error, or otherwise differ from the accidental errors of observation included in Class 3, § 120, and the sum of the products will still approximate zero, so that the value of ϵ_x^2 will be given by (122) or (124).

Ex. If the m.s.e. in placing a 50-ft. tape length is 0.02 ft., and the m.s.e. in the length is 0.01 ft., find that in a 1500-ft. line measured with the tape, due to both causes.

From the first cause, by (122),

$$\epsilon_x'^2 = (0.02)^2 + (0.02)^2 + \ldots \text{ for } 1500/50 = 30 \text{ terms}$$

(each measurement giving an M). Or

$$\epsilon_x' = .02\sqrt{30} = 0.11 \text{ ft.}$$

From the second cause, by (122),

$$\epsilon_x'' = \frac{1500 \times 0.01}{50} = 0.30 \text{ ft.}$$

From both together,

$$\epsilon_x = \sqrt{(0.11)^2 + (0.30)^2} = 0.32 \text{ ft.}$$

Reducing the m.s.e. from the first source to one-half its value would only reduce the final m.s.e. to 0.30 ft., while reducing that from the second source to one-half would reduce the final m.s.e. to 0.19 ft.

It is thus seen that but little is gained in accuracy by reducing the errors from one source when larger ones from another remain unnoticed; the great gain comes from reducing the larger ones.

124. The Simple Arithmetic Mean.—When a number of equally good, direct, and independent observations are taken for the value of an unknown quantity, the arithmetic mean is always taken for the best or most probable value, there being no reason for giving more influence to one than to another of the observations.

Thus if the observed values are $M_1, M_2, \ldots, M_n$, the most probable value

$$x_0 = \frac{[M]}{n}. \quad \ldots \ldots \ldots \quad (125)$$

This can be written

$$x_0 = \frac{1}{n}M_1 + \frac{1}{n}M_2 + \frac{1}{n}M_3 \ldots,$$

so that if ϵ = m.s.e. for an observed M and ϵ_0 = the m.s.e. for x_0, we have, from (122),

$$\epsilon_0^2 = \left(\frac{\epsilon}{n}\right)^2 + \left(\frac{\epsilon}{n}\right)^2 + \ldots \text{ to } n \text{ terms} = n\left(\frac{\epsilon}{n}\right)^2,$$

or

$$\epsilon_0^2 = \frac{\epsilon^2}{n}, \quad \ldots \ldots \ldots \quad (126)$$

i.e., *the m.s.e. of the arithmetic mean decreases as the square root of the number of observations increases.*

The difference between the arithmetic mean and the different observed values are called residuals.

If they are denoted by v_1, v_2, . . . , and the error of the arithmetic mean by δ, we shall have:

Residuals,

$$v_1 = x_0 - M_1, \quad v_2 = x_0 - M_2 \ldots$$

True errors,

$$\Delta_1 = (x_0 \pm \delta) - M_1, \quad \Delta_2 = (x_0 \pm \delta) - M_2, \ldots,$$

or

$$\Delta_1 = v_1 \pm \delta, \quad \Delta_2 = v_2 \pm \delta \ \ldots$$

Squaring and adding,

$$[\Delta^2] = [v^2] \pm 2\delta[v] + n\delta^2.$$

Eq. (125) can be written

$$(x_0 - M_1) + (x_0 - M_2) + (x_0 - M_3) \ldots = 0,$$

or

$$[v] = 0. \quad \ldots \quad (127)$$

Substituting and dividing by n,

$$\varepsilon^2 = \frac{[v^2]}{n} + \delta^2.$$

The most probable value of δ, the error of x_0, is usually assumed to be the m.s.e. of the mean itself, or $\delta = \varepsilon_0 = \varepsilon/\sqrt{n}$. Substituting,

$$\varepsilon^2 = \frac{[v^2]}{n} + \frac{\varepsilon^2}{n},$$

$$\varepsilon^2 = \frac{[v^2]}{n-1}. \quad \ldots \quad (128)$$

From (126),

$$\varepsilon_0{}^2 = \frac{[v^2]}{n(n-1)}. \quad \ldots \quad (129)$$

Ex. 1. The following values are given in *Pri. Tri. U. S. Lake Survey*, p. 895, for the observed difference in longitude between Detroit and Cambridge:

				$-v$	$+v$	v^2
June 21	0^h	47^m	$41^s.154$	.040		.0016
" 22			41 .171	.057		.0032
" 23			41 .138	.024		.0006
" 24			41 .110		.004	.0000
" 29			40 .995		.119	.0142
Totals....................			5 .568	.121	.123	.0196
Mean x_0=	0^h	47^m	$41^s.114$			

Each v is found by subtracting its M from x_0. If the mean were exact, $[v]=0$ instead of $+0.002$. In any case $[v]=nx_0-[M]$.

$$\varepsilon_0=\sqrt{\frac{[v^2]}{n(n-1)}}=\sqrt{\frac{0.0196}{20}}=0^s.031.$$

Ex. 2. From the following determinations of the length of an iron bar, find the most probable value of the length and its m.s.e.

$645^{mm}.67$	$645^{mm}.73$	
.69	.79	
.78	.72	
.81	.68	
.65	.70	*Ans.* 645.72 ± 0.02.

125. The Weighted Arithmetic Mean.—An observation is said to have the weight w, when its m.s.e. is equal to that of the mean of w observations of weight unity. If then ε' is the m.s.e. of an observation of weight unity, and ε_1, ε_2, . . . are the m.s.e.'s for weights w_1, w_2, . . . , we have from (126)

$$\varepsilon_1^2=\frac{\varepsilon'^2}{w_1}, \quad \varepsilon_2^2=\frac{\varepsilon'^2}{w_2},$$

or

$$\frac{w_1}{w_2}=\frac{\varepsilon_2^2}{\varepsilon_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

i.e., *the weights are inversely as the squares of the m.s.e.'s.*

If the different values of a quantity, M_1, M_2, M_3, . . . , have the weights w_1, w_2, w_3, . . . , each value being supposed to be the mean of w values of weight unity, the sum of the original values can be found by multiplying each mean by its number of observations and adding; the average can then be found by dividing by the total number; i.e., the arithmetic mean

$$x_0=\frac{(M_1w_1+M_2w_2+M_3w_3\ .\ .\ .)}{(w_1+w_2+w_3\ .\ .\ .)}=\frac{[Mw]}{[w]}. \quad . \quad . \quad (131)$$

The m.s.e. of the mean

$$\varepsilon_0=\frac{\varepsilon'}{\sqrt{[w]}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

Eq. (131) can be written

$$(x_0-M_1)w_1+(x_0-M_2)w_2+(x_0-M_3)w_3\ldots=0;$$

that is,

$$[vw]=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

As in § 124,

$$\varDelta_1=v_1\pm\delta, \quad \varDelta_2=v_2\pm\delta, \quad \varDelta_3=v_2\pm\delta.\ldots$$

If each equation be squared, then multiplied by its corresponding w, and added,

$$[w\varDelta^2]=[wv^2]\pm2\delta[wv]+\delta^2[w]. \quad . \quad . \quad . \quad . \quad (a)$$

The observations with weights w give errors $\varDelta$; the corresponding errors for weight unity would most probably be $\varDelta\sqrt{w}$, from the relation (130) between weights and m.s.e.'s. Therefore $[w\varDelta^2]$ is the sum of the squares of the errors for weight unity, $=n\varepsilon'^2$ by (120).

By § 124,

$$\delta=\varepsilon_0, \quad =\frac{\varepsilon'}{\sqrt{[w]}} \text{ by (132).}$$

Substituting in (a),

$$n\varepsilon'^2=[wv^2]+\varepsilon'^2,$$

or

$$\varepsilon'^2=\frac{[wv^2]}{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

$$\varepsilon_0{}^2=\frac{[wv^2]}{[w](n-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

Ex. 1. The following values are given in *Pri. Tri. U. S. Lake Survey*, p. 895, for the observed difference in longitude between Detroit, Mich., and Cambridge, Mass.

			w	wM	v	wv	wv^2
May 13	0^h 47^m	$41^s.163$	0.5	0.581	−.117	−.059	.00684
23		40 .966	0.5	0.483	+.080	+.040	.00320
24		41 .038	1.0	1.038	+.008	+.008	.00006
26		41 .030	1.0	1.030	+.016	+.016	.00026
June 4		41 .084	1.0	1.084	−.038	−.038	.00144
11		41 .012	1.0	1.012	+.034	+.034	.00116
Totals			5.0	5.228		+.001	.01296

Mean $= 0^h\ 47^m\ 41^s.046$.

$$\epsilon_0 = \sqrt{0.01296/25} = 0^s.023.$$

The mean is exact only to the third decimal place, but $[wv] = [w]x_0 - [wM]$ checking x_0.

Ex. 2. From the following measurements of a base line deduce the most probable value of the line and its m.s.e. Employ also the method of control of § 126.

3456′.378 ± 0.002
.369 ± 0.008
.365 ± 0.010
.372 ± 0.006
3456 .374 ± 0.005

126. Controls.—*Simple Arithmetic Mean.* Since

$$v_1 = x_0 - M_1,\ v_2 = x_0 - M_2,\ v_3 = x_0 - M_3, \ldots, \text{ and } nx_0 = [M],$$

$$[v^2] = nx_0^2 - 2x_0[M] + [M^2],$$

or

$$[v^2] = [M^2] - \frac{[M]^2}{n}. \quad \ldots \ldots \quad (136)$$

Also, from (127),

$$[v] = 0.$$

Weighted Arithmetic Mean. Since

$$v_1 = x_0 - M_1, \quad v_2 = x_0 - M_2, \quad v_3 = x_0 - M_3, \ldots, \quad \text{and by (131)},$$
$$[w]x_0 = [wM], \quad [wv^2] = x_0^2[w] - 2x_0[wM] + [wM^2],$$

or

$$[wv^2] = [wM^2] - \frac{[wM]^2}{[w]}. \quad \ldots \ldots \quad (137)$$

Also from (133),

$$[vw] = 0.$$

It may be noted that the left-hand places as far as they agree may be left off from the values of M, or any constant subtracted, whenever it will simplify the numerical computation for (136) or (137).

Ex. 1. In Ex. 1 of § 124 we have for the different values of M, subtracting 41 from each: 0.154, 0.171, 0.138, 0.110, −0.005.

Squaring and adding

$$[M^2] = 0.0843.$$

Adding and squaring,

$$\frac{[M]^2}{n}=0.0647.$$

$$[M^2]-\frac{[M]^2}{n}=0.0196,$$

checking $[v^2]$.

The mean x_0, when multiplied by 5 is 0.002 greater than $[M]$, so that $[v]$ should $= +0.002$ instead of 0.

Ex. 2. In Ex. 1 of § 125, subtracting 40 from each M,

$$[wM^2]=5.48041$$
$$\frac{[wM]^2}{[w]}=5.46744$$

$$0.01297, \text{ checking } [wv^2].$$

127. Closeness of Computation.—If the most probable value x as computed by a rigorous method have the errors $\Delta_1, \Delta_2, \Delta_3, \ldots,$ the value $x\pm c$, computed by an approximate method, will have the errors $\Delta_1\pm c, \Delta_2\pm c, \Delta_3\pm c, \ldots.$ Hence

$$\varepsilon^2_{x\pm c}=\frac{(\Delta_1\pm c)^2+(\Delta_2\pm c)^2+\ldots}{n}$$
$$=\frac{[\Delta^2]}{n}+c^2$$
$$=\varepsilon_x{}^2+c^2,$$

or

$$\varepsilon_{x\pm c}=\varepsilon_x\left(1+\frac{c^2}{2\varepsilon_x{}^2}\right) \text{ (approximately).}$$

If we allow the difference between $\varepsilon_{x\pm c}$ and ε_x to be $0.01\varepsilon_x$, i.e., allow the m.s.e. to be increased 1% by inaccuracy in computation, which would appear safe, then

$$0.01=c^2/2\varepsilon_x{}^2,$$

or

$$c=0.14\varepsilon_x; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

i.e., the error of computation can be 14% of the m.s.e. without sensibly increasing the inaccuracy of the result.

Ex. 1. In a 7-place log table the error in the last place will vary from 0 to 0.5, all values within these limits occurring with equal frequency. The m.s.e. for this method of distribution of error is $a/\sqrt{3}$, where $a=$the greatest error. This would give m.s.e.$=0.5/\sqrt{3}=0.29$ in the 7th place. An interpolated value, expressed as $M_1 + (M_2 - M_1)m$ (where M_1 and M_2 are the adjacent tabular quantities and m the percentage interval between the corresponding numbers), would have the following m.s.e.'s for different values of m, the 7th place only being retained in the interpolation (*Annals of Mathematics*, II, pp. 54–59; or *Smithsonian Geographical Tables*, p. lxxxvi).

m.	1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10
m.s.e.	.29	.41	.35	.38	.37	.38	.38	.39	.39	.39

The average m.s.e. will thus be well within 0.4, or $c = 0.4$. In geodetic work a m.s.e. of 0.3 second is about the minimum value for horizontal angles. A triangulation will be most exact, or the test most severe, when the angles of each triangle $= 60°$. The change in log sin 60° for a change of 1″ is 12.2 in the 7th place, so that the m.s.e. due to inaccuracy of measurement $= 0.3 \times 12.2 = 3.7 = \epsilon_x$. Substituting, $c/\epsilon_x = 0.4/3.7 = 11\%$, instead of the 14% allowed by (138).

Again a m.s.e. of 1/1 000 000 is excellent base line work. The log of 1 000 000 is changed 4.3 in the 7th place by a change of unity in the number, so that $c/\epsilon_x = 0.4/4.3 = 9\%$, instead of the 14 allowed by (138).

Seven-place logs are thus ample for the best geodetic work.

For six-place logs,

$$\begin{aligned} c &= 0.4 \text{ with units in the 6th place;} \\ \epsilon_x &= c/0.14 = 2.86 \text{ in log units} \\ &= 2.86/1.22 = 2''.3 \text{ in angle} \\ &= 2.86/4.3 = 7 \text{ in 1 000 000 in distance.} \end{aligned}$$

This is ample for the best city work.

Five-place logs are ample for

23″ in angle,
7 in 100 000 in distance,

or for the best railroad, or ordinary first-class field work.

Four-place logs are ample for

230″, or say 4′, in angle,
7 in 10 000 in distance,

or for the best chain and compass work, and much of the stadia work.

With suitable tables, like Vega's 7-place, Bremiker's 6-place, and Gauss' 5-place, Encke says the times required for the same computation are as 3, 2, 1, respectively. He also says 4 places are sufficient for minutes and 1/4 000 in sides, 5 places for 5″ and 1/40 000, 6 places for 1/2″, and 7 places for 1/20″, limits not as conservative as the above.

Ex. 2. In Ex. 1 of § 125 find the number of decimal places to use in expressing the value of x_0.

128. Independent Observations upon Independent Quantities.—In the general case of indirect observations let the observation equations be of the form

$$\left.\begin{array}{l} f_1(X, Y, Z, \ldots)-M_1=0, \text{ wt. } w_1, \\ f_2(X, Y, Z, \ldots)-M_2=0, \text{ wt. } w_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot, \end{array}\right\} \quad . \quad . \quad . \quad (139)$$

in which the number n of the observed quantities $M_1, M_2, \ldots$ is greater than m, that of the required ones, $X, Y, Z, \ldots$.

The observations being imperfect, no set of values can be found for the unknowns which will not leave residuals, so that (139) would be more correctly written

$$\left.\begin{array}{l} f_1(X, Y, Z, \ldots)-M_1=v_1, \\ f_2(X, Y, Z, \ldots)-M_2=v_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot, \end{array}\right\} \quad . \quad . \quad . \quad . \quad (140)$$

which are sometimes called error or *residual* equations.

We first find approximate values, by partial solution or otherwise, for $X, Y, Z, \ldots$, so that $X=X_0+x$, $Y=Y_0+y, \ldots$, ($x, y, \ldots$ being so small that terms containing the squares, products, and higher powers may be neglected without sensible error), then expand by Taylor's theorem, as in § 122. Thus

$$f_1(X_0, Y_0, Z_0, \ldots)-M_1+\frac{df_1}{dX_0}x+\frac{df_1}{dY_0}y+\frac{df_1}{dZ_0}z+\ldots=v_1,$$

or

Similarly

$$\left.\begin{array}{l} a_1x+b_1y+c_1z+\ldots+l_1=v_1, \\ a_2x+b_2y+c_2z+\ldots+l_2=v_2, \\ a_3x+b_3y+c_3z+\ldots+l_3=v_3, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot, \end{array}\right\} \quad . \quad . \quad . \quad . \quad (141)$$

where

$$a=\frac{df}{dX_0},\ b=\frac{df}{dY_0},\ c=\frac{df}{dZ_0}, \ldots, = \text{constants}.$$

$$l=f(X_0, Y_0, \ldots)-M,$$

the index following the equation.

As will be shown later, the most probable values for the corrections x, y, z, . . . will be those which will make $[wv^2]$ = minimum. Hence, since x, y, z, . . . are independent,

$$\frac{d[wv^2]}{dx}=0,\quad \frac{d[wv^2]}{dy}=0,\quad \frac{d[wv^2]}{dz}=0,\ldots,$$

or

$$\left.\begin{array}{l} w_1v_1\dfrac{dv_1}{dx}+w_2v_2\dfrac{dv_2}{dx}+\ldots=0, \\ w_1v_1\dfrac{dv_1}{dy}+w_2v_2\dfrac{dv_2}{dy}+\ldots=0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Substituting the values of v from (141),

$$\left.\begin{array}{l} [waa]x+[wab]y+[wac]z+\ldots+[wal]=0 \\ [wab]x+[wbb]y+[wbc]z+\ldots+[wbl]=0 \\ [wac]x+[wbc]y+[wcc]z+\ldots+[wcl]=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} \quad . \quad . \quad (142)$$

These are called normal equations, or better, *final* equations.

They can be more briefly written by substituting in (*a*) the values of the differential coefficients from (141):

$$[wva]=0,\quad [wvb]=0,\quad [wvc]=0,\ \ldots . \quad . \quad . \quad . \quad (143)$$

If the weights are equal or unity, w will disappear as a factor, giving

$$\left.\begin{array}{l} [aa]x+[ab]y+[ac]z+\ldots+[al]=0 \\ [ab]x+[bb]y+[bc]z+\ldots+[bl]=0 \\ [ac]x+[bc]y+[cc]z+\ldots+[cl]=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} \quad . \quad . \quad . \quad (144)$$

The solution of (142), (143), or (144) will give definite values for x, y, z, . . . , which applied to the approximate values X_0, Y_0, Z_0, . . . will give the most probable ones which can be found from the given equations or observations.

Linear equations can be arranged in the form of (141) without approximate values whenever it will lessen the numerical work, the loss of higher powers occurring in the reduction to linear form, and not in the later work. Usually, however, the advantage of solving for the small corrections rather than the large unknowns will more than compensate for the preliminary work of substituting approximate values.

129. Control, Normal Equations.—If in (141) we place

$$\left.\begin{aligned} a_1+b_1+c_1+\ldots l_1&=s_1,\\ a_2+b_2+c_2+\ldots l_2&=s_2,\\ \ldots\ldots&\ldots, \end{aligned}\right\}$$

and treat s similarly to l, i.e., multiply each by its wa, and add the products, each by its wb and add, etc., the terms of the first members will be the coefficients of the normal equations and the second members check terms for them, as below:

$$\left.\begin{aligned} [waa]+[wab]+[wac]+\ldots+[wal]&=[was]\\ [wab]+[wbb]+[wbc]+\ldots+[wbl]&=[wbs]\\ [wac]+[wbc]+[wcc]+\ldots+[wcl]&=[wcs]\\ \ldots\ldots\ldots\ldots&\ldots \end{aligned}\right\} \quad . \quad . \quad (145)$$

Ex. 1. **Find the normal equations from the following observation equations:**

$$\begin{array}{lc} & w\\ 3x+2y-z-10=v_1 & 1\\ 2x+\ y+z-16=v_2 & 2\\ x+\ y+z-11=v_3 & 1\\ x-\ y-z+\ 6=v_4 & 3\\ x+\ y\ \ \ \ -\ 6=v_5 & 1 \end{array}$$

Since the equations are linear and the values of the unknowns small, the normal equations will be found directly without assuming approximate values and deriving corrections as in the general case.

TABLE FOR FORMING THE NORMAL EQUATIONS.

No.	w	a	b	c	l	s	waa	wab	wac
1	1	+3	+2	−1	−10	− 6	+ 9	+6	−3
2	2	+2	+1	+1	−16	−12	+ 8	+4	+4
3	1	+1	+1	+1	−11	− 8	+ 1	+1	+1
4	3	+1	−1	−1	+ 6	+ 5	+ 3	−3	−3
5	1	+1	+1	0	− 6	− 4	+ 1	+1	0
							+22	+9	−1

No.	wal	was	wbb	wbc	wbl	wbs	wcc	wcl	wcs
1	−30	−18	+ 4	−2	−20	−12	+1	+10	+ 6
2	−64	−43	+ 2	+2	−32	−24	+2	−32	−24
3	−11	− 8	+ 1	+1	−11	− 8	+1	−11	− 8
4	+18	+15	+ 3	+3	−18	−15	+3	−18	−15
5	− 6	− 4	+ 1	0	− 6	− 4	0	0	0
	−93	−63	+11	+4	−87	−63	+7	−51	−41

NORMAL EQUATIONS.

Substituting in (142),

	Check.
$22x+9y-z-93=0$	-63
$9x+11y+4z-87=0$	-63
$-x+4y+7z-51=0$	-41

From the solution of these equations given in the example of § 132,

$$x=+3.23,\quad y=+3.09,\quad z=+5.99.$$

These values are to hundredths. If approximate values had been assumed and corrections found, the same number of figures would give the unknowns to four or to five decimal places.

Ex. 2. Jordan, *Vermessungskunde*, Vol. I, p. 46, gives as the means of meteorological observations extending over twelve years barometer readings at nine stations, as follows:

1. Bruchsal,	$h=120^{m}.2$	$B=751^{mm}.18$	6. Heiden,	$h=492^{m}.4$	$B=718^{mm}.16$
2. Cannstatt,	225 .1	742 .37	7. Isnv,	708 .1	700 .48
3. Stuttgart,	270 .6	738 .50	8. Freuden,	733 .5	697 .64
4. Calw,	347 .6	731 .27	9. Schop,	768 .9	695 .23
5. Freidrich,	406 .7	726 .99			

Plotting these values with height h above sea level and barometer reading B as coordinates, the curve will be nearly or quite a straight line.

Assuming with Jordan a straight line, the equation would be of the form (the theoretical equation is a logarithmic one)

$$X+hY-B=0,$$

where X is the reading at sea level, Y the coefficient of increase with elevation, h the elevation known accurately enough to be considered a constant, and B the observed barometer reading.

$$f(X,Y)-M=v$$

becomes

$$X+hY-M=v.$$

Differentiating,

$$\frac{df}{dX}=1,\text{ i.e., } a_1=1,\ a_2=1,\ldots,$$

$$\frac{df}{dY}=h,\text{ i.e., } b_1=h_1,\ b_2=h_2,\ldots.$$

Assume $X_0=760^{mm}$, $Y_0=-0.08$,* and to equalize coefficients, put

$$\frac{h}{100}\cdot 100y=h'y'.$$

* X_0 and Y_0 may be found, for instance, by plotting the points and taking these values from the curve.

This will give

$$\left.\begin{array}{l} l_1=760-0.08\times120.2-751.18=-0.80, \\ l_2=760-0.08\times225.1-742.37=-0.38, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \, , \end{array}\right\}$$

TABLE FOR FORMING THE NORMAL EQUATIONS.

No.	a	b	l	s	b^2	bl	bs
1	1	1.20	−0.80	1.40	1.44	−0.96	1.68
2	1	2.25	−0.38	2.87	5.06	−0.85	6.46
3	1	2.71	−0.15	3.56	7.34	−0.41	9.65
4	1	3.48	+0.92	5.40	12.11	+3.20	18.79
5	1	4.07	+0.47	5.54	16.56	+1.91	22.55
6	1	4.92	+2.45	8.37	24.21	+12.05	41.18
7	1	7.08	+2.87	10.95	50.13	+20.32	77.53
8	1	7.34	+3.68	12.02	53.88	+27.01	88.23
9	1	7.69	+3.26	11.95	59.14	+25.07	91.90
	9	40.74	+12.32	62.06	229.87	+87.34	357.97

$$9x+\ 40.74y'+12.32=0. \qquad \text{Check}=\ 62.06$$
$$40.74x+229.87y'+87.34=0. \qquad \text{``} \ \ =357.97$$

Solving, $x=1.78$; $y'=-0.695$; $y=-0.00695$; $X=X_0+x=761.78$; $Y=Y_0+y=-0.08695$.

Substituting, the required equation becomes

$$B^{mm}=761.78^{mm}-0.08695h^{m}.$$

The check terms do not check the entries in the a, b, and l columns, but they do check the numerical work of the table. The coefficients and absolute term of each equation foot up equal to the check term as required by (145).

Equalizing coefficients, as in this example, will reduce the numerical work of forming and solving the normal equations, if the coefficients differ greatly in magnitude.

Ex. 3. Given the following observation equations:

$$\begin{array}{rl} x+2y+3z&=24 \\ 3x+4y+\ z&=40 \\ 6x\qquad\ -\ z&=44 \\ 3y+4z&=21 \\ x+2y+\ z&=19 \end{array}$$

Derive the normal equations from which the most probable values of x, y, and z may be obtained, employing both the method of approximation and the direct method used in Ex. 1.

130. Mean Square Errors of the Unknowns.—If in solving (144) the elimination were fully carried out, each unknown would be finally expressed as a linear function of $l_1, l_2, \ldots$, and the m.s.e.'s of the latter being the same as those of $M_1, M_2, \ldots$, and known, those of the former would follow from § 122. To effect this elimination, indeterminate multipliers are used, i.e., the first equation of (144) is multiplied by Q', the second by Q'', . . . , and the products added. Then to find x, such values are given to $Q', Q'', \ldots$, that in the sum or final equation the cofficients of the unknowns, $x, y, z, \ldots$, shall be zero, except those of x, which shall be unity. This gives * for the assumed coefficients:

$$\left.\begin{array}{l} \text{Coeff. of } x = [aa]Q' + [ab]Q'' + [ac]Q''' + \ldots = 1, \\ \quad\text{“}\quad\text{“}\ \ y = [ab]Q' + [bb]Q'' + [bc]Q''' + \ldots = 0, \\ \quad\text{“}\quad\text{“}\ \ z = [ac]Q' + [bc]Q'' + [cc]Q''' + \ldots = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots , \end{array}\right\} \quad . \quad (a)$$

so that the sum equation reduces to

$$x+[al]Q'+[bl]Q''+[cl]Q'''+\ldots=0 \quad . \quad . \quad . \quad . \quad (b)$$

(i.e., the sum of (144) with absolute terms of $1, 0, 0, \ldots = x +$ const. terms $=0$).

The coefficients of the unknowns in (144) and (a) are the same. Hence if the values $x, y, z, \ldots$ are found from (144) in terms of $l_1, l_2, \ldots$, those of $Q', Q'', \ldots$ would result from them by putting $[al]=-1$, $[bl]=[cl]\ldots=0$. This is also evident from (b). We now wish to show that if $\varepsilon=$ m.s.e. of an observation of weight

* The method of indeterminate multipliers may be more readily followed as applied to ordinary algebraic equations; thus,

$$\begin{array}{ll} 6x+\ y-\ z=10. & \text{Multiply by } Q'. \\ 4x-\ y \qquad = \ 6. & \quad\text{“}\quad\ \text{“}\ \ Q''. \\ 3x+4y-4z=\ 4. & \quad\text{“}\quad\ \text{“}\ \ Q'''. \end{array}$$

Sum equation,

$$x(6Q'+4Q''+3Q''')+y(Q'-Q''+4Q''')-z(Q'+4Q''')=10Q'+6Q''+4Q'''.$$

Placing the coefficient of $x=1$ and that of y and of z each $=0$, gives by inspection $Q''=0$, $Q'''=-\frac{1}{4}Q'$, $(6-\frac{1}{4}\times3)Q'=1$, or $Q'=\frac{4}{21}$. The second member foots up $(10-\frac{1}{4}\times4)Q'=\frac{36}{21}$, $=$the value of x.

If y had been required its coefficient would have been placed $=1$ in the sum equation, while those of x and z would $=0$.

The method is frequently very convenient.

unity, and ε_x = m.s.e. of the value of x found from the normal equations, then

$$\varepsilon_x^2 = Q'\varepsilon^2.$$

In (b), x being a linear function of $l_1, l_2, \ldots$, we may place

$$x + \alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \ldots = 0, \quad \ldots \ldots \quad (c)$$

in which, by comparing coefficients with (b),

$$\left.\begin{array}{l} \alpha_1 = a_1 Q' + b_1 Q'' + c_1 Q''' + \ldots \\ \alpha_2 = a_2 Q' + b_2 Q'' + c_2 Q''' + \ldots \\ \alpha_3 = a_3 Q' + b_3 Q'' + c_3 Q''' + \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad \ldots \ldots \quad (d)$$

If each of these equations be multiplied by its a and added, each by its b, etc., then by (a),

$$[a\alpha] = 1, \ [b\alpha] = 0, \ [c\alpha] = 0, \ldots \quad \ldots \ldots \quad (e)$$

The number of these equations is m, the number of unknown quantities. Multiply each of (d) by its α and add, then by (e),

$$[\alpha^2] = Q'. \quad \ldots \ldots \quad (f)$$

From the value of x in (c) we have, by § 122,

$$\varepsilon_x^2 = \alpha_1^2 \varepsilon^2 + \alpha_2^2 \varepsilon^2 + \alpha_3^2 \varepsilon^2 + \ldots,$$

or

$$\varepsilon_x^2 = [\alpha^2]\varepsilon^2 = Q'\varepsilon^2. \quad \ldots \ldots \quad (146)$$

Hence to find the m.s.e. of x in terms of that of an observation write $-1, 0, 0, \ldots$ for the absolute terms of the normal equations and solve for x; the value thus found multiplied by the square of the m.s.e. of an observation will give the square of the m.s.e. required.

In the same way it may be shown that the m.s.e. of y can be found by using $0, -1, 0, \ldots$ for the absolute terms, etc.

If the observations have different weights, $w_1, w_2, w_3, \ldots$, the multiplication of each by its $\sqrt{w}$ will reduce the m.s.e.'s, $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$, to ε', the m.s.e. for weight unity by (130). All the observations now having the m.s.e. ε', (146) will apply to (142), or to the normal equations with weights, ε being replaced by ε'. (146) could also have been derived directly from (142).

131. **Mean Square Error of an Observation.**—The most probable values of the unknowns substituted back in (141) will give the residuals, $v', v'', \ldots$, while the true values, $x+dx$, $y+dy, \ldots$, if known, would give the true errors,

$$\left.\begin{array}{l} a_1(x+dx)+b_1(y+dy)+c_1(z+dz)+\ldots+l_1=\Delta_1, \\ a_2(x+dx)+b_2(y+dy)+c_2(z+dz)+\ldots+l_2=\Delta_2, \\ a_3(x+dx)+b_3(y+dy)+c_3(z+dz)+\ldots+l_3=\Delta_3, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} \quad . \quad (147)$$

and we should at once have

$$\varepsilon^2=[\Delta^2]/n,$$

n being the number of observations.

If the first equation be multiplied by a_1, the second by a_2, etc., then by b_1, b_2, etc., we will have, by (144),

$$\left.\begin{array}{l} [aa]dx + [ab]dy + [ac]dz + \ldots -[a\Delta] = 0, \\ [ab]dx + [bb]dy + [bc]dz + \ldots -[b\Delta] = 0, \\ [ac]dx + [bc]dy + [cc]dz + \ldots -[c\Delta] = 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\}$$

These being the same form as (144), the value of dx can be found from that of x, by substituting $-\Delta$ for l in (c), § 130, giving

$$dx-\alpha_1\Delta_1-\alpha_2\Delta_2-\alpha_3\Delta_3-\ldots=0. \quad . \quad . \quad . \quad . \quad (a)$$

If we multiply (147) by $v_1, v_2, v_3, \ldots$, respectively, the sum of the products will be, by (143),

$$[vl]=[v\Delta].$$

Similarly, from (141),

$$[vl]=[v^2].$$

Hence

$$[v\Delta]=[v^2]=[vl]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

Again, multiply (141) by $\Delta_1, \Delta_2, \Delta_3, \ldots$, respectively, and add,

$$[a\Delta]x+[b\Delta]y+[c\Delta]z+\ldots+[l\Delta]=[v\Delta]=[v^2].$$

Multiply (147) similarly,

$$[a\varDelta]x+[b\varDelta]y+[c\varDelta]z\ldots+[l\varDelta]+[a\varDelta]dx+[b\varDelta]dy+[c\varDelta]dz\ldots=[\varDelta^2].$$

From these two equations,

$$[\varDelta^2]=[v^2]+[a\varDelta]dx+[b\varDelta]dy+[c\varDelta]dz+\ldots \qquad (149)$$

The value of $[a\varDelta]dx$ can be found by multiplying

$$[a\varDelta]=a_1\varDelta_1+a_2\varDelta_2+a_3\varDelta_3+\ldots \text{ and } (a),$$
$$dx=\alpha_1\varDelta_1+\alpha_2\varDelta_2+\alpha_3\varDelta_3+\ldots,$$

giving, since the sum of the products, $\alpha_1 a_2\varDelta_1\varDelta_2, \ldots, \alpha_2 a_1\varDelta_1\varDelta_2, \ldots, \alpha_3 a_1\varDelta_1\varDelta_3, \ldots$, will approximate zero,

$$[a\varDelta]dx=\alpha_1 a_1\varDelta_1^2+\alpha_2 a_2\varDelta_2^2+\alpha_3 a_3\varDelta_3^2+\ldots$$

If we substitute the average value of $\varDelta^2$, which is ε^2, for $\varDelta_1^2$, $\varDelta_2^2, \ldots$, this reduces to (e), § 130,

$$[a\varDelta]dx=\varepsilon^2.$$

Similarly, the mean value of the other terms, $[b\varDelta]dy$, $[c\varDelta]dz, \ldots$, will be ε^2. Substituting in (149),

$$[\varDelta^2]=n\varepsilon^2=[v^2]+m\varepsilon^2,$$

or

$$\varepsilon^2=\frac{[v^2]}{n-m}. \qquad (150)$$

If the observations have different weights, they can be reduced to the same weight by multiplying by $\sqrt{w}$, as in § 130, giving

$$\varepsilon'^2=\frac{[wv^2]}{n-m}. \qquad (151)$$

Having the m.s.e. of an observation from (150) or (151), that for each of the required quantities can be found from § 130.

132. Solution of Normal Equations.—The ordinary algebraic methods are convenient with few unknowns, but when the number is great, say four or more, the method of substitution, due to Gauss, will generally be found preferable. It can be made very mechanical and the work checked step by step.

NORMAL EQUATIONS.

$$\left.\begin{array}{ll} & \text{Check.} \\ [aa]x + [ab]y + [ac]z + \ldots + [al] = 0, & [as] \\ [ab]x + [bb]y + [bc]z + \ldots + [bl] = 0, & [bs] \\ [ac]x + [bc]y + [cc]z + \ldots + [cl] = 0. & [cs] \\ \cdots\cdots\cdots\cdots & \end{array}\right\} \quad (a)$$

From the first equation,

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z - \ldots - \frac{[al]}{[aa]}. \qquad \frac{[as]}{[aa]}$$

Substituting in the other equations of (a),

$$\left.\begin{array}{ll} [bb\,.\,1]y + [bc\,.\,1]z + \ldots + [bl\,.\,1] = 0, & [bs\,.\,1] \\ [bc\,.\,1]y + [cc\,.\,1]z + \ldots + [cl\,.\,1] = 0, & [cs\,.\,1] \\ \cdots\cdots\cdots\cdots & \end{array}\right\} \quad . \quad (b)$$

where

$$[bb\,.\,1] = [bb] - \frac{[ab]}{[aa]}[ab],$$

$$[bc\,.\,1] = [bc] - \frac{[ab]}{[aa]}[ac],$$

$$\cdots\cdots\cdots$$

$$[bl\,.\,1] = [bl] - \frac{[ab]}{[aa]}[al],$$

$$[bs\,.\,1] = [bs] - \frac{[ab]}{[aa]}[as],$$

$$[cc\,.\,1] = [cc] - \frac{[ac]}{[aa]}[ac],$$

$$\cdots\cdots\cdots$$

$$[cl\,.\,1] = [cl] - \frac{[ac]}{[aa]}[al],$$

$$[cs\,.\,1] = [cs] - \frac{[ac]}{[aa]}[as].$$

From the first of (b),

$$y = -\frac{[bc\,.\,1]}{[bb\,.\,1]}z - \ldots - \frac{[bl\,.\,1]}{[bb\,.\,1]}. \qquad \overset{\text{Check.}}{\frac{[bs\,.\,1]}{[bb\,.\,1]}}$$

Substituting,

$$\left.\begin{array}{ll}[cc\,.\,2]z + \ldots + [cl\,.\,2] = 0, & [cs\,.\,2] \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \end{array}\right\} \quad . \quad . \quad . \quad (c)$$

where

$$[cc\,.\,2] = [cc\,.\,1] - \frac{[bc\,.\,1]}{[bb\,.\,1]}[bc\,.\,1],$$

$$\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$[cl\,.\,2] = [cl\,.\,1] - \frac{[bc\,.\,1]}{[bb\,.\,1]}[bl\,.\,1],$$

$$[cs\,.\,2] = [cs\,.\,1] - \frac{[bs\,.\,1]}{[bb\,.\,1]}[bs\,.\,1].$$

The value of z from (c) substituted in (b) will give y; the values of y and z in (a) will give x.

The check terms carried through as indicated will check the equations (b) and (c). The value of y can be checked from the second of (b), while $x, y, z, \ldots$, can be checked by satisfying all of (a).

Bringing down the first equation of each group, we have the

Derived Normal Equations.

$$\left.\begin{array}{rl}[aa]x + [ab]y + [ac]z + \ldots + [al] = 0, & \overset{\text{Check.}}{[as]}, \\ [bb\,.\,1]y + [bc\,.\,1]z + \ldots + [bl\,.\,1] = 0, & [bs\,.\,1], \\ [cc\,.\,2]z + \ldots + [cl\,.\,2] = 0, & [cs\,.\,2]. \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \end{array}\right\} (152)$$

Experience has shown the form of solution on page 229 to be convenient in carrying out this elimination. This form is illustrated numerically in the solution of the normal equations of Ex. 1, § 129, as shown in solution A on page 230.

Solution A may be modified as shown in solution B. The derived normal equations I, VI, and XI of the first solution are equations I, II, and III of the second. The procedure to find equation III, for instance, illustrates the method of obtaining any derived equation. In the first place, the normal equations are written without the duplication of any numbers, and in the

FORM OF SOLUTION.

No.	x	y	z	Ab. Term.	Check.	Remarks.
I	$[aa]$	$[ab]$	$[ac]$	$[al]$	$[as]$	
II	$[ab]$	$[bb]$	$[bc]$	$[bl]$	$[bs]$	
III	$[ac]$	$[bc]$	$[cc]$	$[cl]$	$[cs]$	

Solution.

IV	1	$\frac{[ab]}{[aa]}$	$\frac{[ac]}{[aa]}$	$\frac{[al]}{[aa]}$	$\frac{[as]}{[aa]}$	$\mathrm{I}\times\frac{1}{[aa]}$
V		$[ab]\frac{[ab]}{[aa]}$	$[ab]\frac{[ac]}{[aa]}$	$[ab]\frac{[al]}{[aa]}$	$[ab]\frac{[as]}{[aa]}$	IV $\times [ab]$
II		$[bb]$	$[bc]$	$[bl]$	$[bs]$	II
VI		$[bb.1]$	$[bc.1]$	$[bl.1]$	$[bs.1]$	II–V
VII		$[ac]\frac{[ab]}{[aa]}$	$[ac]\frac{[ac]}{[aa]}$	$[ac]\frac{[al]}{[aa]}$	$[ac]\frac{[as]}{[aa]}$	IV $\times [ac]$
III		$[bc]$	$[cc]$	$[cl]$	$[cs]$	III
VIII		$[bc.1]$	$[cc.1]$	$[cl.1]$	$[cs.1]$	III–VII
IX		1	$\frac{[bc.1]}{[bb.1]}$	$\frac{[bl.1]}{[bb.1]}$	$\frac{[bs.1]}{[bb.1]}$	$\mathrm{VI}\times\frac{1}{[bb.1]}$
X			$[bc.1]\frac{[bc.1]}{[bb.1]}$	$[bc.1]\frac{[bl.1]}{[bb.1]}$	$[bc.1]\frac{[bs.1]}{[bb.1]}$	IX $\times [bc.1]$
VIII			$[cc.1]$	$[cl.1]$	$[cs.1]$	
XI			$[cc.2]$	$[cl.2]$	$[cs.2]$	VIII–X
XII			1	$\frac{[cl.2]}{[cc.2]}$	$\frac{[cs.2]}{[cc.2]}$	$\mathrm{XI}\times\frac{1}{[cc.2]}$

solution all sets of numbers which it is known will sum up to zero are omitted. In finding equation III, which has z for its first unknown, I is multiplied by minus the quotient of the coefficients of its z- and x-terms. Likewise II is multiplied by minus the quotient of its z- and y-terms To these equations is added the normal equation which in the tabulation at the beginning of the solution apparently begins with a z-term. In making the addition the x- and y-terms which must sum up to zero are omitted.

SOLUTION A.

No.	x	y	z	Ab. Term.	Check.	Remarks.
I	+22.	+ 9.	−1.	−93.	−63.	
II	+ 9.	+11.	+4.	−87.	−63.	
III	− 1.	+ 4.	+7.	−51.	−41.	
IV	1.	+ 0.409	−0.045	−4.227	− 2.864	$I\times\frac{1}{22}$
V		+ 3.682	−0.409	−38.045	−25.773	IV×9
II		+11.	+4.	−87.	−63.	
VI		+ 7.318	+4.409	−48.955	−37.227	II−V
VII		− 0.409	+0.045	+ 4.227	+ 2.864	IV×(−1)
III		+ 4.	+7.	−51.	−41.	
VIII		+ 4.409	+6.955	−55.227	−43.864	III−VII
IX		+ 1.	+0.603	− 6.689	− 5.087	$VI\times\frac{1}{7.318}$
X			+2.656	−29.495	−22.429	IX×4.409
VIII			+6.955	−55.227	−43.864	
XI			+4.299	−25.732	−21.435	VIII−X
XII			+1.	− 5.986	− 4.986	$XI\times\frac{1}{4.299}$

Whence $z= +5.99$, and by substitution in equations VIII and III we have $y= +3.09$ and $x= +3.23$.

SOLUTION B.

No.	x	y	z	Const.	Check.	Remarks.
I	+22.	+ 9.	−1.	−93.	−63.	
2		+11.	+4.	−87.	−63.	
3			+7.	−51.	−41.	
4		− 3.682	+0.409	+38.045	+25.773	$I\times\left(-\frac{9}{22}\right)$
2		+11.	+4.	−87.	−63.	
II		+ 7.318	+4.409	−48.955	−37.227	4+2
5			−0.045	− 4.227	− 2.864	$I\times\left(+\frac{1}{22}\right)$
6			−2.656	+29.495	+22.429	$II\times\left(-\frac{4.409}{7.318}\right)$
3			+7.	−51.	−41.	
III			+4.299	−25.732	−21.435	5+6+3
			+1.	− 5.986	− 4.986	

Further modifications of Gauss' solution along lines similar to the above have been developed in the method of Mr. M. H. Doolittle, which is employed in the Coast Survey, and is advantageous when a very large number of equations are to be solved. See Coast Survey *Report*, 1878, p. 115, and Wright and Hayford, *Adjustment of Observations*, § 91.

133. Indirect Observations, Problem.—A problem in astronomy is taken to illustrate the method of reducing a set of time transits for clock error, azimuth error, and collimation error. The observed time of transit t_1 requires: *

Correction for azimuth error, x,

$$=x \sin (\phi-\delta) \sec \delta = xa. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Correction for inclination of telescope axis, i,

$$=i \cos (\phi-\delta) \sec \delta = iI. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Correction for collimation error, y,

$$=y \sec \delta = yb, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

where ϕ = the latitude of the place and δ the declination of the star.

The true clock face time of transit will thus be

$$t=t_1+ax+iI+by.$$

If t_2 = true time of transit (computed from the right ascension of the star),

Clock correction,

$$\Delta t = t_2 - t,$$

or

$$\Delta t = t_2 - (t_1+ax+iI+by). \quad . \quad . \quad . \quad . \quad . \quad (d)$$

If the clock correction at the time t_0 is Δt_0 and the rate r,

$$\begin{aligned}\Delta t &= \Delta t_0 + (t-t_0)r \\ &= \Delta t_0' + z + (t-t_0)r,\end{aligned}$$

where z is a correction to the assumed value $\Delta t_0'$.

* Campbell's *Astronomy*, § 92 et seq.

Substituting in (*d*),

$$\Delta t_0' + (t-t_0)r + t_1 - t_2 + iI + ax + by + z = 0.$$

The observation equation thus becomes

$$ax + by + z + l = v, \quad \dots \dots \dots \quad (e)$$

where

$$l = \Delta t_0' + (t-t_0)r + t_1 - t_2 + iI. \quad \dots \dots \quad (f)$$

Each observed transit gives an equation (*e*) in which a and b can be computed from (*a*) and (*c*). l can be computed from the transit data and clock rate after assuming $\Delta t_0'$, while the most probable values of the corrections x, y, z are required.

The following data were obtained by the class in astronomy at the University, Oct. 2, 1895, from the transits of ten time stars over the five wires of the transit instrument used without reversal. Required the clock correction at 7 P.M., the azimuth correction. and the collimation correction, with their m.s.e.'s.

t_1-t_2	iI	a	b
$7^m\ 52^s.33$	$+0^s.32$	−0.07	+1.41
7 51 .68	+0 .14	+0.68	+1.00
7 51 .70	+0 .17	+0.52	+1.02
7 48 .38	−0 .34	+2.51	−2.67
7 53 .46	+0 .42	−0.73	+2.13
7 51 .84	+0 .14	+0.75	+1.01
7 51 .69	+0 .18	+0.53	+1.02
7 51 .33	+0 .23	+0.68	+1.00
7 51 .55	+0 .16	+0.81	+1.02
7 52 .43	+0 .33	+0.09	+1.27

The clock rate is small. Assume $r=0$, $\Delta t_0' = -7^m\ 52^s$ at 7 P.M. Substituting in (*f*),

$$l_1 = -7^m\ 52^s + 0 + 7^m\ 52^s.33 + 0^s.32 = +0^s.65,$$
$$l_2 = -7\ \ 52\ + 0 + 7\ \ 51\ .68 + 0\ .14 = -0\ .18,$$

. . . .

(See table, page 233.)

NORMAL EQUATIONS.

$9.541x - 4.226y + 5.77z - 12.098 = 0$	− 1.013 check
$-4.226x + 21.407y + 8.21z + 15.257 = 0$	+40.648 "
$5.77x + 8.21y + 10.00z - 1.86 = 0$	+22.12 "

Since the m.s.e. is required for x, y, and z, the Q' of § 130 will be necessary for x, for y, and for z. To find Q_x the absolute terms

TABLE FOR FORMING THE NORMAL EQUATIONS.

$a=ac$	$b=bc$	$c=cc$	$l=cl$	$s=cs$	aa
−0.07	+1.41	1	+0.65	+ 2.99	0.005
+0.68	+1.00	1	−0.18	+ 2.50	0.463
+0.52	+1.02	1	−0.13	+ 2.41	0.270
+2.51	−2.67	1	−3.96	− 3.12	6.300
−0.73	+2.13	1	+1.88	+ 4.28	0.533
+0.75	+1.01	1	−0.02	+ 2.74	0.563
+0.53	+1.02	1	−0.13	+ 2.42	0.281
+0.68	+1.00	1	−0.44	+ 2.24	0.462
+0.81	+1.02	1	−0.29	+ 2.54	0.656
+0.09	+1.27	1	+0.76	+ 3.12	0.008
+5.77	+8.21	10	−1.86	+22.12	+9.541

ab	al	as	bb	bl	bs
−0.099	− 0.045	−0.209	1.988	+ 0.917	+ 4.216
+0.680	− 0.122	+1.700	1.000	− 0.180	+ 2.500
+0.531	− 0.068	+1.253	1.040	− 0.133	+ 2.458
−6.702	− 9.940	−7.831	7.129	+10.573	+ 8.330
−1.555	− 1.372	−3.125	4.537	+ 4.004	+ 9.116
+0.758	− 0.015	+2.055	1.020	− 0.020	+ 2.767
+0.541	− 0.070	+1.283	1.040	− 0.133	+ 2.468
+0.680	− 0.299	+1.523	1.000	− 0.440	+ 2.240
+0.826	− 0.235	+2.057	1.040	− 0.296	+ 2.591
+0.114	+ 0.068	+0.281	1.613	+ 0.965	+ 3.962
−4.226	−12.098	−1.013	+21.407	+15.257	+40.648

are placed equal to zero except for the first or x equation, for which it is unity and the corresponding value of x found. This is done by adding a Q_x column and carrying it through the same as the absolute term column. (See table, page 234.) Similarly a Q_y and a Q_z column are added. The check terms are changed to include the new absolute terms.

Substituting the value of z in V will give $y=-0.5058$.

Substituting the value of z' (-0.9516) from the Q_y column will give the corresponding value of y, or $Q_y=+0.58$. The value of z'' (-1.4646) from the Q_x column will give the corresponding value of $y'=+0.8290$. The values of z and y substituted in I will give $x=+1.045$. The values of z' and y' will give $Q_x=+1.36$.

Collecting results, $x=+1.04$, $y=-0.51$, $z=0.00$, $Q_x=+1.36$, $Q_y=+0.58$, $Q_z=+1.73$.

SOLUTION OF NORMAL EQUATIONS.

No.	x	y	z	l	Q_x	Q_y	Q_z	Check.	Remarks.
I	+9.541	− 4.226	+ 5.77	−12.098	+1			−0.013	
II	−4.226	+21.407	+ 8.21	+15.257		+1		+41.648	
III	+5.77	+ 8.21	+10.	− 1.86			+1	+23.12	

Solution.

No.	x	y	z	l	Q_x	Q_y	Q_z	Check.	Remarks.
IV	1	− 0.4429	+ 0.6048	− 1.2680	+0.1048			− 0.0014	$I\times\frac{1}{9.541}$
		− 1.872	+ 2.556	− 5.359	+0.442			− 0.006	IV×4.226
II		+21.407	+ 8.21	+15.257		+1		+41.648	
V		+19.535	+10.766	+ 9.898	+0.442	+1		+41.642	
		+ 2.555	− 3.489	+ 7.316	−0.605			+ 0.008	IV×(−5.77)
III		+ 8.21	+10.	− 1.86			+1	+23.12	
VI		+10.765	+ 6.511	+ 5.456	−0.605		+1	+23.128	
VII		1	+ 0.5511	+ 0.5067	+0.0226	+0.0512		+ 2.1315	$V\times\frac{1}{19.535}$
			− 5.932	− 5.455	−0.243	−0.551		−22.948	VII×(−10.765)
VI			+ 6.511	+ 5.456	−0.605		+1	+23.128	
VIII			+ 0.579	+ 0.001	−0.848	−0.551	+1	+ 0.180	
			1	+ 0.0017	−1.4646	−0.9516	+1.7271		$VIII\times\frac{1}{0.579}$

$z = -0.0017, \qquad Q_z = +1.73.$

Substituting back in the observation equations (as inferred from the table for forming the normal equations):

$-v$	$+v$	v^2	$-v$	$+v$	v^2
0.14	....	0.020	....	0.24	0.058
....	0.02	0.000	0.10	....	0.010
0.11	....	0.012	0.24	....	0.058
....	0.01	0.000	....	0.03	0.001
....	0.03	0.001	....	0.20	0.040
0.25	0.06	0.033	0.34	0.47	0.167
			0.25	0.06	0.033
			0.59	0.53	0.200

$[v] = -0.59 + 0.53 = -0.06$ instead of 0.

$$\epsilon^2 = \frac{[v^2]}{n-m} = \frac{0.200}{10-3} = 0.029,$$

$$\epsilon_x = \sqrt{Q_x \epsilon^2} = \sqrt{1.36 \times 0.029} = 0.20,$$

$$\epsilon_y = \sqrt{0.58 \times 0.029} = 0.13, \qquad \epsilon_z = \sqrt{1.73 \times 0.029} = 0.22.$$

Hence at 7 P.M., Oct. 2, 1895:

Clock correction	$= -7^m\ 52^s$	$\pm 0^s.22$,
Azimuth correction	$= +$	$1^s.04 \pm 0^s.20$,
Collimation correction	$= -$	$0^s.51 \pm 0^s.13$.

134. Methods of Solution.—The most common method of performing the arithmetic work for a large number of equations is with a computing machine. The Thomas, Grant, Burckhardt, and Brunsviga computing machines are in use; while the Thatcher slide rule giving five places will usually be accurate enough. The Crelle multiplication table will give about as good speed as a machine if the number of figures required in the product is so small that little or no interpolation is required in the table.

The forms already given are convenient for these methods.

Logarithms are also used in forming the products for the normal equations and in their solution. A table is formed of the log coefficients or log factors, while the log of the constant factor (or its arithmetic complement if of a constant divisor) is written at the bottom of a slip of paper and carried along just above the log coefficients and added, forming the log products. From these the products are found and recorded.

A table of squares is also used for forming the products, and it is considered preferable to a log table. Thus,

$$ab = \tfrac{1}{2}\{(a+b)^2 - a^2 - b^2\},$$

and therefore

$$[ab] = \tfrac{1}{2}\{[(a+b)^2] - [aa] - [bb]\}. \quad . \quad . \quad . \quad . \qquad (153)$$

Hence form the square sums

$$\begin{array}{llllll} [aa], & [(a+b)^2], & [(a+c)^2], & \ldots & [(a+l)^2], & [(a+s)^2], \\ & [bb], & [(b+c)^2], & \ldots & [(b+l)^2], & [(b+s)^2], \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & [ll], & [(l+s)^2], \end{array}$$

and perform the necessary subtractions.*

The squares aa, bb, ... can be taken directly from the table and added. For the sum squares, write a_1 on a slip of paper and move it over b_1, c_1, ..., a_2 for b_2, c_2, ..., forming the sums a_1+b_1, a_1+c_1, ...; a_2+b_2, a_2+c_2, ... Take out the squares and sum them. Similarly for b_1, b_2, ... Substitute in (153).

In solving the normal equations it is considered preferable to multiply by a reciprocal of a number rather than to divide, and to change the sign of the reciprocal and add rather than subtract. The operations are thus reduced to two, multiplication and addition.

Attention is called to the equality of the side coefficients both in the normal equations and in the derived normal equations, as shown in § 132, so that in the solution it is unnecessary to compute these equal terms more than once.

The desirability of equalizing coefficients was noted in Ex. 2, § 129. During the solution closer results will be obtained by dividing by the larger coefficients and multiplying by the smaller, rather than the reverse, but the equality of coefficients in equations to be added should be kept in mind so that the same number of decimal places will represent about the same degree of precision in each.

In substituting back after one or more unknowns have been found, always take an equation in which the coefficient of the required quantity is large and the coefficients of the known quantities are small in order to reduce the cumulative effects due to omitted decimal places.

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* See Barlow's *Tables of Squares, Cubes, etc.*

135. Independent Observations upon Dependent Quantities. —In the observation equations of § 128 care is taken not to introduce more unknown quantities than are independent. Thus if x and y represent two adjacent angles, $X+Y$ and not Z would be used in expressing their sum in an observation equation. The last angle in closing the horizon would be expressed as 360° less the sum of the others, rather than by another letter.

But there are many cases where it is more convenient to consider each observed quantity as an unknown, giving direct observations, and then by the method of indeterminate multipliers eliminate an unknown for each rigid condition connecting them. This leaves the remaining unknowns independent so that § 128 will apply, but it leads to special formulas. These conditions or equations are called rigid because they must be exactly satisfied, in distinction from observation equations which can be only approximately satisfied.

Let the m' rigorous equations be

$$\left.\begin{array}{l} f_1(V_1,\ V_2,\ V_3,\ldots)=0, \\ f_2(V_1,\ V_2,\ V_3,\ldots)=0, \\ \ldots\ldots\ldots \end{array}\right\} \quad \ldots\ldots \quad (154)$$

where V_1, V_2, V_3, ... are the most probable values of the unknowns.

For each V substitute the observed value M plus a correction v, i.e., $V_1=M_1+v_1$, $V_2=M_2+v_2$, $V_3=M_3+v_3$, ... ; expand by Taylor's theorem as in § 122, and put

$$\left.\begin{array}{llll} \dfrac{df_1}{dM_1}=a_1, & \dfrac{df_2}{dM_1}=b_1, & \dfrac{df_3}{dM_1}=c_1, & \ldots \\ \dfrac{df_1}{dM_2}=a_2, & \dfrac{df_2}{dM_2}=b_2, & \dfrac{df_3}{dM_2}=c_2, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}\right\}$$

$$f_1(M_1,\ M_2,\ M_3,\ \ldots)=q_1, \quad f_2(M_1,\ M_2,\ M_3,\ \ldots)=q_2,\ \ldots$$

This gives, by substitution in (154),

$$\left.\begin{array}{l} a_1v_1+a_2v_2+a_3v_3+\ \ldots\ +q_1=0, \\ b_1v_1+b_2v_2+b_3v_3+\ \ldots\ +q_2=0, \\ c_1v_1+c_2v_2+c_3v_3+\ \ldots\ +q_3=0. \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots \quad (155)$$

These equations must be rigorously satisfied by v_1, v_2, v_3, ...

The observation equations are

$$V_1-M_1=v_1, \quad V_2-M_2=v_2, \quad \ldots,$$

or $$(V_1-M_1)\sqrt{w_1}=v_1\sqrt{w_1}, \quad (V_2-M_2)\sqrt{w_2}=v_2\sqrt{w_2}. \ldots$$

By Chapter X the most probable corrections, v_1, v_2, ..., will be those which make

$$w_1v_1^2+w_2v_2^2+w_3v_3^2+\ldots \text{ a minimum,}$$

or $$w_1v_1dv_1+w_2v_2dv_2+w_3v_3dv_3+\ldots=0. \quad . \quad . \quad . \quad (156)$$

This minimum is conditioned by (155).

Differentiating,

$$\left.\begin{array}{l} a_1dv_1+a_2dv_2+a_3dv_3+\ldots=0, \\ b_1dv_1+b_2dv_2+b_3dv_3+\ldots=0, \\ c_1dv_1+c_2dv_2+c_3dv_3+\ldots=0, \\ \ldots\ldots\ldots\ldots, \end{array}\right\}$$

which must be satisfied at the same time with (156).

The number of these equations is m'; the number of terms in (156) is m, and as $m>m'$, we can find the values of m' differentials in terms of the $m-m'$ others and substitute in (156). The remaining differentials being independent, their coefficients will separately equal zero. This elimination is effected by indeterminate multipliers, i.e., the first equation is multiplied by A, the second by B, etc., and (156) by -1; the products are then added and A, B, C, ... given such values that m' coefficients of dv shall equal zero. The other $(m-m')$ differentials being independent, their coefficients must $= 0$.

Placing each of these coefficients equal zero gives

$$\left.\begin{array}{l} \text{Coefficient of } dv_1=Aa_1+Bb_1+Cc_1+\ldots-w_1v_1=0, \\ \quad\text{"} \qquad \text{"} \quad dv_2=Aa_2+Bb_2+Cc_2+\ldots-w_2v_2=0, \\ \quad\text{"} \qquad \text{"} \quad dv_3=Aa_3+Bb_3+Cc_3+\ldots-w_3v_3=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}. \quad (157)$$

Multiply the first by $\frac{a_1}{w_1}$, the second by $\frac{a_2}{w_2}$, ..., then multiply the same equations by $\frac{b_1}{w_1}, \frac{b_2}{w_2}, \ldots, \frac{c_1}{w_1}, \frac{c_2}{w_2}, \ldots$, and add the products in each case. This will give by comparison with (155) m' normal equations containing m' unknowns.

$$\left.\begin{aligned}
\left[\frac{aa}{w}\right]A+\left[\frac{ab}{w}\right]B+\left[\frac{ac}{w}\right]C+\ldots+q_1=0,\\
\left[\frac{ab}{w}\right]A+\left[\frac{bb}{w}\right]B+\left[\frac{bc}{w}\right]C+\ldots+q_2=0,\\
\left[\frac{ac}{w}\right]A+\left[\frac{bc}{w}\right]B+\left[\frac{cc}{w}\right]C+\ldots+q_3=0,\\
\ldots\ldots\ldots\ldots\ldots,
\end{aligned}\right\} \quad . \quad (158)$$

in which for equal weights $w=1$.

A, B, C, ... are called correlatives of the equations of condition. Their values from (158) substituted in (157) will give

$$\left.\begin{aligned}
v_1=(a_1A+b_1B+c_1C+\ldots)\frac{1}{w_1},\\
v_2=(a_2A+b_2B+c_2C+\ldots)\frac{1}{w_2},\\
v_3=(a_3A+b_3B+c_3C+\ldots)\frac{1}{w_3},\\
\ldots\ldots\ldots\ldots,
\end{aligned}\right\} \quad . \quad . \quad . \quad (159)$$

from which

$$V_1=M_1+v_1,\quad V_2=M_2+v_2,\quad V_3=M_3+v_3,\ldots.$$

Since there were m observations and m observed quantities, while m' quantities have been eliminated, the difference between the number of observations and that of the unknowns is m', so that (150) and (151) become

$$\varepsilon^2=\frac{[v^2]}{m'},\quad \varepsilon'^2=\frac{[wv^2]}{m'}. \quad . \quad . \quad . \quad . \quad . \quad (160)$$

136. Control.—If in (157) we place

$$\left.\begin{aligned}
a_1+b_1+c_1+\ldots=s_1,\\
a_2+b_2+c_2+\ldots=s_2,\\
a_3+b_3+c_3+\ldots=s_3,\\
\ldots\ldots\ldots,
\end{aligned}\right\}$$

and treat s the same as one of the other terms in deriving (158), the following checks will result. It should be noted that they do not contain the absolute terms as in (145).

$$\left.\begin{array}{l}\left[\frac{aa}{w}\right]+\left[\frac{ab}{w}\right]+\left[\frac{ac}{w}\right]\ldots=\left[\frac{as}{w}\right]\\ \left[\frac{ab}{w}\right]+\left[\frac{bb}{w}\right]+\left[\frac{bc}{w}\right]\ldots=\left[\frac{bs}{w}\right]\\ \left[\frac{ac}{w}\right]+\left[\frac{bc}{w}\right]+\left[\frac{cc}{w}\right]\ldots=\left[\frac{cs}{w}\right]\\ \cdots\cdots\cdots\cdots\end{array}\right\}\quad . \quad . \quad . \qquad (161)$$

To check $[wv^2]$ multiply (159) by $\sqrt{w}$, square, and add.

$$\begin{aligned}[wv^2]=\left[\frac{aa}{w}\right]A^2+2\left[\frac{ab}{w}\right]AB+2\left[\frac{ac}{w}\right]AC+\ldots\\ \left[\frac{bb}{w}\right]B^2+2\left[\frac{bc}{w}\right]BC+\ldots\\ +\left[\frac{cc}{w}\right]C^2.+\ldots\\ \cdots\cdots\end{aligned}$$

Substituting the values of q from (158),

$$[wv^2]=-Aq_1-Bq_2-Cq_3-\ldots \quad . \quad . \quad . \quad . \qquad (162)$$

Similarly for independent observations upon independent quantities, § 128, multiply (141) by $\sqrt{w}$, square, and add.

$$\begin{aligned}[wv^2]=[waa]x^2+2[wab]xy+2[wac]xz\ldots+2[wal]x,\\ [wbb]y^2+2[wbc]yz\ldots+2[wbl]y,\\ [wc^2]z^2\ldots+2[wcl]z,\\ \cdots\cdots\cdots\\ +[wll].\end{aligned}$$

Substituting the values of $[wal]$, $[wbl]$, . . . , from (142),

$$[wv^2]=x[wal]+y[wbl]+z[wcl]\ldots+[wll]. \quad . \qquad (163)$$

Applying (163) to the problem of § 133,

$$[v^2]=-12.582-7.781+20.560=0.197,$$

nearly checking $[v^2]$ as found at the end of the paragraph.

137. Example.—In the Coast Survey *Report*, 1880, App. 6, are given the following differences of longitude: *

Year.	Observed Differences.					Cor.
1851	Cambridge–Bangor....	0^h	9^m	$23^s.080$	± 0.043	v_1
1857	Bangor–Calais........		6	00 .316	± 0.015	v_2
1866	Calais–Hts. Content...		55	37 .973	± 0.066	v_3
1866	Hts. Cont.–Foilh......	2	51	56 .356	± 0.029	v_4
1866	Foilhommer.-Green....		41	33 .336	± 0.049	v_5
1872	Brest–Greenwich......		17	57 .598	± 0.022	v_6
1872	Brest–Paris...........		27	18 .512	± 0.027	v_7
1872	Greenwich–Paris......		9	21 .000	± 0.038	v_8
1872	St. Pierre–Brest.......	3	26	44 .810	± 0.027	v_9
1872	Camb.–St. Pierre......		59	48 .608	± 0.021	v_{10}
1869–70	Camb.–Duxbury......		1	50 .191	± 0.022	v_{11}
1870	Duxbury–Brest.......	4	24	43 .276	± 0.047	v_{12}
1867–72	Washington–Cambridge		23	41 .041	± 0.018	v_{13}
1872	Washington–St. Pierre.	1	23	29 .553	± 0.027	v_{14}

Number of conditions (154) or (155) † $= l - p + 1 = 14 - 11 + 1 = 4$ ($l =$ No. of observed differences of longitude, $p =$ No. of stations).

A glance at Fig. 90 shows a triangle, Brest, Greenwich, Paris; a polygon, Cambridge, Bangor, Calais, H. Content, Foilhommer.,

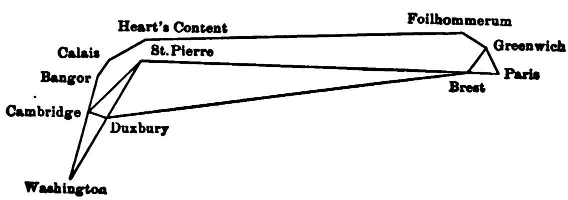

FIG. 90.

Greenwich, Brest, St. Pierre; a polygon, St. Pierre, Brest, Duxbury, Cambridge; and a triangle, Cambridge, St. Pierre, Washington; each giving a check or rigid condition.

* The Pacific arcs from San Francisco to Manila were observed in 1903–4, completing the circuit of the earth. (Coast Survey *Report*, 1904, p. 263.)

† With l lines and p points, l must equal p for a closed figure and this will give one check. All extra lines after the figure is closed must be check lines. Hence

$$\text{No. of checks} = l - p + 1.$$

These give linear equations. Substituting in (155),

$$-v_6+v_7-v_8-0.086=0;$$
$$-v_1-v_2-v_3-v_4-v_5+v_6+v_9+v_{10}-0.045=0;$$
$$-v_9-v_{10}+v_{11}+v_{12}+0.049=0;$$
$$v_{10}+v_{13}-v_{14}+0.096=0;$$

with weights inversely as the squares of the uncertainties.

TABLE FOR NORMAL EQUATIONS.

v	$\sqrt{\frac{1}{w}}$	$\frac{1}{w'}=\frac{100}{w}$	a	b	c	d	s	$\frac{bb}{w'}$	$\frac{bs}{w'}$
v_1	0.043	0.18	......	−1	......	......	−1	+0.18	+0.18
v_2	0.015	0.02	......	−1	......	......	−1	+0.02	+0.02
v_3	0.066	0.44	......	−1	......	......	−1	+0.44	+0.44
v_4	0.029	0.08	......	−1	......	......	−1	+0.08	+0.08
v_5	0.049	0.24	......	−1	......	......	−1	+0.24	+0.24
v_6	0.022	0.05	−1	+1	......	......	0	+0.05	....
v_7	0.027	0.07	+1	......	......	......	+1	....	....
v_8	0.038	0.14	−1	......	......	......	−1	....	....
v_9	0.027	0.07	......	+1	−1	......	0	+0.07	....
v_{10}	0.021	0.04	......	+1	−1	+1	+1	+0.04	+0.04
v_{11}	0.022	0.05	......	......	+1	......	+1	....	....
v_{12}	0.047	0.22	......	......	+1	......	+1	....	....
v_{13}	0.018	0.03	......	......	......	+1	+1	....	....
v_{14}	0.027	0.07	......	......	......	−1	−1	....	....
								+1.12	+1.00

NORMAL EQUATIONS.

					Check.
$+0.26A$	$-0.05B$			$-0.086=0,$	$+0.21$
$-0.05A$	$+1.12B$	$-0.11C$	$+0.04D$	$-0.045=0,$	$+1.00$
	$-0.11B$	$+0.38C$	$-0.04D$	$+0.049=0,$	$+0.23$
	$+0.04B$	$-0.04C$	$+0.14D$	$+0.096=0,$	$+0.14$

From which $A=+0.342$; $B=+0.063$; $C=-0.191$; $D=-0.763$.
Substituting in (159),

$v_1=.18(-.063) \quad =-0^s.011$	$v_8=.14(-.342) \quad =-0^s.048$
$v_2=.02(-.063) \quad =-0\ .001$	$v_9=.07(.063+.191) \quad =+0\ .018$
$v_3=.44(-.063) \quad =-0\ .028$	$v_{10}=.04(.063+.191-.763)=-0\ .020$
$v_4=.08(-.063) \quad \doteq -0\ .005$	$v_{11}=.05(-.191) \quad =-0\ .010$
$v_5=.24(-.063) \quad =-0\ .015$	$v_{12}=.22(-.191) \quad =-0\ .042$
$v_6=.05(-.342+.063)=-0\ .014$	$v_{13}=.03(-.763) \quad =-0\ .023$
$v_7=.07(.342) \quad =+0\ .024$	$v_{14}=.07(.763) \quad =+0\ .053$

Adding each v to the corresponding observed value will give the most probable value for the difference in longitude between two adjacent points, while the same difference will be found between any two distant points by any circuit.

If we square each v, multiply by w', and add,

$$[w'v^2]=0.1148.$$

Computing by (162),

$$[w'v^2]=0.1149.$$

By (160),

$$\varepsilon'=\sqrt{\frac{0.1149}{4}}=0^s.170.$$

The ε for each observed value can be found by multiplying ε' by $\sqrt{\frac{1}{w'}}=10\sqrt{\frac{1}{w}}$.

Thus for the first

$$\varepsilon_1=0.170\times10\times0.043=0^s.07.$$

Ex. 2. The following differences of level were determined between points along a small lake; points A, D, and F being at the mean surface of the lake:

A to B	$+12.61$
B " C	$+24.07$
C " D	-36.38
C " E	-22.30
E " F	-14.55

Find the most probable values of the elevations of B, C, and E above mean lake level.

Ex. 3. The angles A, B, and C of a spherical triangle are measured with the weights w_1, w_2, and w_3; required their adjusted values.

138. M.S.E of a Function of the Required Quantities for Indirect Observations.—Let the function be

$$\begin{aligned}F&=f(X, Y, Z, \ldots)\\&=f(X_0+x, Y_0+y, Z_0+z, \ldots)\\&=f(X_0, Y_0, Z_0, \ldots)+\frac{df}{dX_0}x+\frac{df}{dY_0}y+\frac{df}{dZ_0}z+\ldots\end{aligned}$$

expanded to the linear form.

This may be written

$$F=H+G_1x+G_2y+G_3z+\ldots$$

Since x, y, z, ... are not independent, being connected by the normal equations, they had best be expressed in terms of the observed quantities. As in § 130, (c),

$$x = -[\alpha l], \quad y = -[\beta l], \quad z = -[\gamma l], \ldots, \quad . \quad . \quad . \quad . \quad (a)$$

where α, β, γ, ... are functions of a, b, c, ... of the normal equations.

Substituting,

$$F = H - (G_1\alpha_1 + G_2\beta_1 + \ldots)l_1 - (G_1\alpha_2 + G_2\beta_2 + \ldots)l_2 - \ldots$$

Hence, by § 122,

$$\varepsilon_F^2 = (G_1\alpha_1 + G_2\beta_1 + \ldots)^2\varepsilon_{l_1}^2 + (G_1\alpha_2 + G_2\beta_2 + \ldots)^2\varepsilon_{l_2}^2 + \ldots \quad (164)$$

Reducing the observations to weight unity by multiplying by $\sqrt{w} = \varepsilon'/\varepsilon$ (see § 125),

$$\frac{\varepsilon_F^2}{\varepsilon'^2} = (G_1\alpha_1 + G_2\beta_1 + \ldots)^2 + (G_1\alpha_2 + G_2\beta_2 + \ldots)^2 + \ldots,$$

or

$$\begin{aligned}\frac{1}{w_F} = \frac{\varepsilon_F^2}{\varepsilon'^2} = G_1^2[\alpha\alpha] + 2G_1G_2[\alpha\beta] + \ldots \\ + \quad G_2^2[\beta\beta] + \ldots \\ + \ldots \quad . \quad . \quad (165)\end{aligned}$$

To find the values of $[\alpha\alpha]$, $[\alpha\beta]$, ..., assume, as in § 130,

$$x = -[\alpha l], \quad y = -[\beta l], \quad z = -[\gamma l], \quad \ldots$$

Substituting in the normal equations,

$$\left.\begin{array}{l}[aa][\alpha l] + [ab][\beta l] + \ldots - [al] = 0, \\ [ab][\alpha l] + [bb][\beta l] + \ldots - [bl] = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}$$

Arranging according to l_1, l_2, ...,

$$\left.\begin{array}{l}([aa]\alpha_1 + [ab]\beta_1 + \ldots - a_1)l_1 \\ \qquad + ([aa]\alpha_2 + [ab]\beta_2 + \ldots - a_2)l_2 + \ldots = 0. \\ ([ab]\alpha_1 + [bb]\beta_1 + \ldots - b_1)l_1 \\ \qquad + ([ab]\alpha_2 + [bb]\beta_2 + \ldots - b_2)l_2 + \ldots = 0. \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}$$

The undetermined quantities α_1, α_2, ... may be so determined that the coefficients of l_1, l_2, ... shall each equal zero, i.e.,

$$\left.\begin{array}{l}[aa]\alpha_1 + [ab]\beta_1 + \ldots - a_1 = 0,\\ [aa]\alpha_2 + [ab]\beta_2 + \ldots - a_2 = 0,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ [ab]\alpha_1 + [bb]\beta_1 + \ldots - b_1 = 0,\\ [ab]\alpha_2 + [bb]\beta_2 + \ldots - b_2 = 0,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Multiply the first equation of the first set by a_1, the second by $a_2, \ldots$, and add; then since the equation must hold whatever the number of terms $[aa]$, $[ab], \ldots$, their coefficients must each equal zero, giving

$$[a\alpha]=1, \quad [a\beta]=0, \quad [a\gamma]=0, \ldots \quad . \quad . \quad . \quad . \quad . \quad (166)$$

Again, multiply the second set by b_1, $b_2, \ldots$, the third by c_1, $c_2, \ldots$, and add as above.

$$\left.\begin{array}{l}[b\alpha]=0, \quad [b\beta]=1, \quad [b\gamma]=0, \ldots\\ [c\alpha=0, \quad [c\beta]=0, \quad [c\gamma]=1, \ldots\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{array}\right\} \quad . \quad . \quad . \quad (167)$$

Again, multiply each set by α_1, $\alpha_2, \ldots$, each by β_1, $\beta_2, \ldots$, each by γ_1, $\gamma_2, \ldots$, and add.

$$\left.\begin{array}{l}[aa][\alpha\alpha]+[ab][\alpha\beta]+ \ldots =[a\alpha]=1,\\ [ab][\alpha\alpha]+[bb][\alpha\beta]+ \ldots =[b\alpha]=0,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ [aa][\alpha\beta] +[ab][\beta\beta] + \ldots =[a\beta]=0,\\ [ab][\alpha\beta] +[bb][\beta\beta] + \ldots =[b\beta]=1,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{array}\right\} \quad . \quad . \quad . \quad (168)$$

The coefficients of these equations are the same as for the normal equations, so that the values of $[\alpha\alpha]$, $[\alpha\beta], \ldots$ of (165) can be found from them by substituting $-1, 0, 0, \ldots; \; 0, -1, 0, \ldots; \ldots$ for the absolute terms and carrying out the solution as in the Ex. of § 133, the quantities being the same as Q_x, Q_y, Q_z of that example.

Ex. 1. In the Ex. of § 133 it is required to find the m.s.e. of the angle between the line of collimation and the meridian.

$$F=f(X,Y)=X+Y.$$

$$\frac{df}{dX_0}=G_1=1, \qquad \frac{df}{dY_0}=G_2=1.$$

Substituting in (165),

$$\frac{\varepsilon_F^2}{\varepsilon'^2}=[\alpha\alpha]+2[\alpha\beta]+[\beta\beta]$$

$$=Q_x+2y'+Q_z$$

of § 133, where y' is the value of y used in finding Q_x and coming from the absolute terms $-1, 0, 0$.

$$\therefore\ \frac{\varepsilon_F^2}{\varepsilon'^2}=1.36+2\times0.83+0.58=3.60,$$

$$\varepsilon_F=\sqrt{\varepsilon'^2\times3.60}=0.32.$$

Ex. 2. *Show that the average value of the ratio of the weight of the observed value of a quantity to that of its adjusted value is the ratio of the number of independent unknowns to the number of observed quantities* (m/n).

The m.s.e. of an adjusted value M_1+v_1 is the same as that for $-l_1+v_1$. But

$$-l_1+v_1=a_1x+b_1y+\ldots$$

Placing $\qquad -[\alpha l]$ for x, $\quad -[\beta l]$ for $y, \ldots,$

$$-l_1+v_1=-(a_1\alpha_1+b_1\beta_1+\ldots)l_1-(a_1\alpha_2+b_1\beta_2+\ldots)l_2+\ldots$$

$$\varepsilon^2{}_{(-l_1+v_1)}=(a_1\alpha_1+b_1\beta_1+\ldots)^2\varepsilon_1{}^2+(a_1\alpha_2+b_1\beta_2+\ldots)^2\varepsilon_2{}^2+\ldots$$

$$=a_1([\alpha\alpha\varepsilon^2]a_1+[\alpha\beta\varepsilon^2]b_1+\ldots)+b_1([\alpha\beta\varepsilon^2]a_1+[\beta\beta\varepsilon^2]b_1+\ldots),$$

$$\ldots\ldots\ldots\ldots$$

The normal equations with weights are (eq. 142)

$$\left.\begin{array}{l}[waa]x+[wab]y+\ \ldots\ +[wal]=0,\\ [wab]x+[wbb]y+\ \ldots\ +[wbl]=0,\\ \ldots\ldots\ldots\ldots\end{array}\right\}\quad\ldots\ldots\quad(c)$$

If these had been used in deriving (*b*), each term would contain w. In deriving (166) (167), and (168), $\frac{1}{w}$ would be used as a part of the multiplier, giving for (168)

$$\left.\begin{array}{l}[waa]\left[\frac{\alpha\alpha}{w}\right]+[wab]\left[\frac{\alpha\beta}{w}\right]+\ \ldots\ =1,\\ [wab]\left[\frac{\alpha\alpha}{w}\right]+[wbb]\left[\frac{\alpha\beta}{w}\right]+\ \ldots\ =0,\\ \ldots\ldots\ldots\ldots\end{array}\right\}\quad\ldots\ldots\quad(d)$$

Solving (*c*) for $x, y, \ldots$, and (*d*) for $\left[\frac{\alpha\alpha}{w}\right], \left[\frac{\alpha\beta}{w}\right], \ldots$, we find

$$\left.\begin{array}{l}-x=[\alpha l]=\left[\frac{\alpha\alpha}{w}\right][wal]+\left[\frac{\alpha\beta}{w}\right][wbl]+\ldots\\ -y=[\beta l]=\left[\frac{\alpha\beta}{w}\right][wal]+\left[\frac{\beta\beta}{w}\right][wbl]+\ldots\\ \ldots\ldots\ldots\ldots\end{array}\right\}$$

Equating the coefficients of l_1 in the two members of the equations,

$$\left.\begin{array}{l}\left[\frac{\alpha\alpha}{w}\right]a_1+\left[\frac{\alpha\beta}{w}\right]b_1+\ldots=\frac{\alpha_1}{w_1},\\ \left[\frac{\alpha\beta}{w}\right]a_1+\left[\frac{\beta\beta}{w}\right]b_1+\ldots=\frac{\beta_1}{w_1},\\ \ldots\ldots\ldots\ldots\end{array}\right\} \quad \ldots \ldots \quad d)$$

Substituting in the value of $\varepsilon^2_{(-l_1+v_1)}=\frac{1}{W_1}$,

$$\frac{w_1}{W_1}=(a_1\alpha_1+b_1\beta_1+\ldots).$$

Similarly for $\frac{w_2}{W_2}$, $\frac{w_3}{W_3}$, . . .

Adding $\left[\frac{w}{W}\right]=[a\alpha]+[b\beta]+\ldots$

$=m$ by (166) and (167).

Hence the average value of $\left[\frac{w}{W}\right]=\frac{m}{n}$, as assumed.

139. M.S.E of a Function of Dependent Quantities.—Let

$$\begin{aligned}F&=f_1(V_1, V_2,\ldots)\\ &=f_1(M_1, M_2,\ldots)+\frac{df_1}{dM_1}v_1+\frac{df_1}{dM_2}v_2+\ldots \quad \ldots \quad (169)\\ &=H_1+g_1v_1+g_2v_2+\ldots, \quad \ldots \quad (a)\end{aligned}$$

where the v's are connected by (155), i.e.,

$$\left.\begin{array}{l}a_1v_1+a_2v_2+\ldots+q_1=0,\\ b_1v_1+b_2v_2+\ldots+q_2=0,\\ \ldots\ldots\ldots,\end{array}\right\} \quad \ldots \quad (170)$$

with $[wv^2]=$ a minimum.

Multiply the first equation of (170) by k_1, the second by k_2, etc., then add to (a), giving

$$\begin{aligned}F=H_1&+(g_1+a_1k_1+b_1k_2+\ldots)v_1+\\ &+(g_2+a_2k_1+b_2k_2+\ldots)v_2+\ldots+q_1k_1+q_2k_2+\ldots \quad (b)\end{aligned}$$

If now proper values be given to the correlatives k_1, k_2, . . . , we can treat v_1, v_2, . . . as if independent as in § 135, giving, § 122,

$$\left.\begin{aligned}\varepsilon_F^2 &= (g_1+a_1k_1+b_1k_2+\ldots)^2\varepsilon_1^2\\ &\quad+(g_2+a_2k_1+b_2k_2+\ldots)^2\varepsilon_2^2+\ldots\\ &\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot,\end{aligned}\right\}\quad .\ .\quad (171)$$

or, using weights,

$$\left.\begin{aligned}\frac{1}{w_F} &= \frac{1}{w_1}(g_1+a_1k_1+b_1k_2+\ldots)^2\\ &\quad+\frac{1}{w_2}(g_2+a_2k_1+b_2k_2+\ldots)^2+\ldots.\\ &\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad,\end{aligned}\right\}\quad .\ .\ .\quad (172)$$

If the most probable values of the v's are substituted in the value of F, this function will have its most probable value. Hence by the principle of Least Squares, to be proved in the following chapter, ε_F^2 will be a minimum. This condition will determine $k_1, k_2, \ldots$ by differentiating (171) with respect to them as independent variables. Hence

$$\frac{d\varepsilon_F^2}{dk_1}=0,\quad \frac{d\varepsilon_F^2}{dk_2}=0,\quad \ldots,$$

giving

$$\left.\begin{aligned}&[\varepsilon^2aa]k_1+[\varepsilon^2ab]k_2\ldots+[\varepsilon^2ag]=0,\\ &[\varepsilon^2ab]k_1+[\varepsilon^2bb]k_2\ldots+[\varepsilon^2bg]=0,\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}\right\}\quad .\ .\ .\quad (173)$$

or, using weights,

$$\left.\begin{aligned}&\left[\frac{aa}{w}\right]k_1+\left[\frac{ab}{w}\right]k_2+\ldots\left[\frac{ag}{w}\right]=0,\\ &\left[\frac{ab}{w}\right]k_1+\left[\frac{bb}{w}\right]k_2+\ldots\left[\frac{bg}{w}\right]=0.\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}\right\}\quad .\ .\quad (174)$$

These equations have the same coefficients as the normal equations (158), so that the values of k can be easily found by adding a column of absolute terms in the solution as in § 134.

Ex. 1. Find the m.s.e. in a triangle side due to the m.s.e.'s of the measured angles.

The function equation (169) is

$$F=f_1(V_1,\ V_2)=b\,\frac{\sin\,(M_1+v_1)}{\sin\,(M_2+v_2)}=a.$$

Differentiating,

$$\frac{df_1}{dM_1}=g_1=a \cot M_1, \qquad \frac{df_1}{dM_2}=g_2=-a \cot M_2.$$

The rigorous equation to be satisfied in closing the triangle is

$$V_1+V_2+V_3-(180+s)=0,$$

giving $a_1=a_2=a_3=1$, and for (170)

$$v_1+v_2+v_3+q_1=0.$$

There being only one condition, we have only one correlative, k_1. Substituting in (173),

$$k_1=-\frac{[\varepsilon^2 g]}{[\varepsilon^2]}=\frac{-\varepsilon_1^2 a \cot M_1+\varepsilon_2^2 a \cot M_2}{\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2}.$$

Substituting in (171),

$$\varepsilon_F^2=a^2 \sin^2 1''\left\{\left(1-\frac{\varepsilon_1^2}{[\varepsilon^2]}\right) \cot M_1+\frac{\varepsilon_2^2}{[\varepsilon^2]} \cot M_2\right\}^2\varepsilon_1^2$$

$$+a^2 \sin^2 1''\left\{\frac{\varepsilon_1^2}{[\varepsilon^2]} \cot M_1+\left(1-\frac{\varepsilon_2^2}{[\varepsilon^2]}\right) \cot M_2\right\}^2\varepsilon_2^2$$

$$+a^2 \sin^2 1''\left(\frac{\varepsilon_1^2}{[\varepsilon^2]} \cot M_1-\frac{\varepsilon_2^2}{[\varepsilon^2]} \cot M_2\right)^2\varepsilon_3^2.$$

Making use of the value of $[\varepsilon^2]$ $(=\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2)$ and reducing,

$$\varepsilon_F^2=a^2 \sin^2 1''\left\{\left(\varepsilon_1^2-\frac{\varepsilon_1^4}{[\varepsilon^2]}\right) \cot^2 M_1+\left(\varepsilon_2^2-\frac{\varepsilon_2^4}{[\varepsilon^2]}\right) \cot^2 M_2+\frac{2\varepsilon_1^2\varepsilon_2^2}{[\varepsilon^2]}\cot M_1 \cot M_2\right\}$$

If $\varepsilon_1=\varepsilon_2=\varepsilon_3$,

$$\varepsilon_F^2=\tfrac{2}{3}a^2 \sin^2 1''(\cot^2 M_1+\cot^2 M_2+\cot M_1 \cot M_2)\varepsilon^2.$$

If the triangle is equilateral,

$$\varepsilon_F^2=\tfrac{2}{3}a^2 \sin^2 1''\varepsilon^2.$$

If the base has the m.s.e. ε_b, ε_F^2 would be increased (§ 122) by $\frac{a^2}{b^2}\varepsilon_b^2$.

Ex. 2. Find the m.s.e. of the adjusted angles of a triangle ε_a in terms of those of the measured ones ε. *Ans.* $\varepsilon_a^2=\tfrac{2}{3}\varepsilon^2$.

CHAPTER X.

THEORY.

140. Principles of Probability.—The mathematical probability of the occurrence of an event is defined as the ratio of the number of ways it may happen to the total number of ways in which it may either happen or fail; each being supposed independent and equally liable to occur.

Thus if an urn contain *a* white balls, *b* black, and *c* red ones, the probability in a single draw

$$\left.\begin{array}{ll}
\text{Of drawing a white ball} & =\dfrac{a}{a+b+c}. \\[2ex]
\text{Of failing to draw a white ball} & =\dfrac{b+c}{a+b+c}. \\[2ex]
\text{The sum of the two probabilities} & =\dfrac{a+b+c}{a+b+c}=1. \\[2ex]
\text{Of drawing either a white or a black ball} & =\dfrac{a+b}{a+b+c}. \\[2ex]
\text{Of drawing a black, white, or red ball} & =\dfrac{a+b+c}{a+b+c}=1. \\[2ex]
\text{Of drawing a green ball} & =\dfrac{0}{a+b+c}=0.
\end{array}\right\} \quad (175)$$

We thus see that the probability is an abstract number which varies with the degree of confidence which can be placed in the occurrence of an event, zero denoting impossibility and unity certainty; that the probability of occurrence plus that of failure must always equal unity; and that the probability of the occurrence of an event which can happen in several independent ways is the sum of the separate probabilities.

If a second urn contain a' white balls, b' black, and c' red ones, the number of possible combinations or cases in a single draw from each urn

$$=(a+b+c)(a'+b'+c'),$$

while the number of favorable cases for two white balls

$$=aa'.$$

Hence in two successive draws, one from each urn,

$$\left.\begin{array}{c}\text{Probability of drawing}\\ \text{two white balls}\end{array}\right\}=\frac{aa'}{(a+b+c)(a'+b'+c')}. \quad . \quad . \quad . \quad (176)$$

This, by (175), equals the product of the separate probabilities. The same could be proved for any number of events.

We thus see that the probability of a compound event, produced by the occurrence of several simple and independent events, equals the product of the separate probabilities.

141. Probability Curve.—In connection with the accidental errors of observation, the following axioms, derived from experience, were stated in § 120.

1. Small errors occur more frequently, or are more probable, than large ones.
2. Positive and negative errors of the same magnitude are equally probable, and in a large number of observations are equally frequent.
3. Very large errors do not occur.

From the first axiom it may be assumed that the probability p, of an error Δ, is some function of the error, or

$$p=f(\Delta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Practically there is a limit to the graduation and use of instruments by which Δ can have only definite numerical values differing by the finest reading, $d\Delta$, so that the probability of an error Δ is the probability that the error lies between Δ and $\Delta+d\Delta$, a value which will vary with $d\Delta$; hence (a) would be more correctly written

$$p=f(\Delta)d\Delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$$

Mathematically we have to treat Δ as a continuous variable.

Taking p as a continuous function of Δ, (177) represents a curve of the general form shown in Fig. 91, for, by the first axiom above, small values of Δ must have the largest probabilities, p; by the second, the curve must be symmetrical about the axis of P; and by the third, p must be zero for all values of Δ greater than a

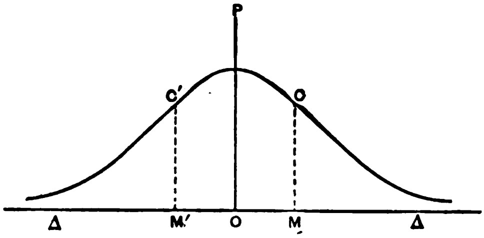

FIG. 91.

given limit $\pm l$, an impossibility except for $l=\infty$, although it can be closely approximated.

142. Form of $f(\Delta)$.—Observations may be direct or indirect, i.e., the observed quantities may be the required ones or they may be functions of them. As the first is but a special case of the second, only the latter need be considered.

Let us take the observation equations of § 128,

$$\left.\begin{array}{l} f'(X. Y, Z, \ldots)-M_1=v_1, \\ f''(X, Y, Z, \ldots)-M_2=v_2, \\ \ldots\ldots\ldots\ldots, \end{array}\right\} \quad \ldots . \quad (178)$$

there being n equations or observed values M, and m unknowns with $n>m$.

The probability of the occurrence of a given series of errors, $\Delta_1, \Delta_2, \ldots$, in $M_1, M_2, \ldots$ will be, by (176) and (177),

$$p=f(\Delta_1)d\Delta_1 f(\Delta_2)d\Delta_2 \ldots \quad \ldots . \quad (179)$$

But the true values of $X, Y, \ldots$ are unknown; and since Δ_1, $\Delta_2, \ldots$ are found from them by substituting in (178), their true values are also unknown. The most probable values of the errors, which, if the number of observations is great, may be taken as the true ones, and hence also of the unknowns, will be those which make p a maximum; or, since $\log p$ varies with p, and the unknowns are independent except as connected by the observations themselves, the derivatives of $\log p$ with reference to $X, Y, \ldots$ must equal zero.

This gives, since

$$\log p = \log f(\Delta_1) + \log f(\Delta_2) + \log d\Delta_1 + \log d\Delta_2 + \ldots,$$

$$\left.\begin{aligned} f'(\Delta_1)\frac{d\Delta_1}{dX} + f'(\Delta_2)\frac{d\Delta_2}{dX} \ldots &= 0, \\ f'(\Delta_1)\frac{d\Delta_1}{dY} + f'(\Delta_2)\frac{d\Delta_2}{dY} \ldots &= 0, \\ \ldots\ldots\ldots\ldots&, \end{aligned}\right\} \quad \ldots \quad (180)$$

in which

$$f'(\Delta) = \frac{df(\Delta)}{f(\Delta)d\Delta}. \quad \ldots \quad (181)$$

The number of these equations being the same as that of the unknowns, they will serve to determine them when $f'(\Delta)$ is known. Furthermore, $f(\Delta)$ and $f'(\Delta)$ being general, they must hold whatever the number of unknowns.

When the number is one, and the unknown is directly observed, the errors will be

$$\Delta_1 = X - M_1, \quad \Delta_2 = X - M_2, \quad \ldots,$$

from which

$$\frac{d\Delta_1}{dX} = \frac{d\Delta_2}{dX} = \ldots = 1,$$

and (180) reduces to

$$f'(\Delta_1) + f'(\Delta_2) + \ldots = 0,$$

or

$$\frac{f'(\Delta_1)}{\Delta_1}\Delta_1 + \frac{f'(\Delta_2)}{\Delta_2}\Delta_2 + \ldots = 0. \quad \ldots \quad (182)$$

It is usually assumed that the arithmetic mean is the best or most probable value that can be found for a single quantity from a set of direct observations all equally good. Making this assumption, and also that the number of observations is great, it may be called the true value, or

$$X = (M_1 + M_2 + M_3 + \ldots)\frac{1}{n};$$

transposing,

$$(X - M_1) + (X - M_2) + (X - M_3) + \ldots = 0,$$

i.e.,

$$\Delta_1 + \Delta_2 + \Delta_3 + \ldots = 0.$$

Comparing this with (182), and remembering that each equation must hold whatever the value of n,

$$\frac{f'(\Delta)}{\Delta} = \text{a constant} = k.$$

Hence, from (181),

$$\frac{df(\Delta)}{f(\Delta)} = k\Delta d\Delta.$$

Integrating,

$$\log f(\Delta) = \tfrac{1}{2}k\Delta^2 + \log C.$$

Passing to numbers,

$$f(\Delta) = Ce^{\frac{k\Delta^2}{2}},$$

in which e is the base of the Naperian system of logarithms.

Since as $f(\Delta)$ increases Δ diminishes, k must be essentially negative. As its value is unknown we may replace it by another unknown constant, i.e., place $k = -1/\varepsilon^2$, giving

$$f(\Delta) = Ce^{-\frac{\Delta^2}{2\varepsilon^2}},$$

whence

$$p = f(\Delta)d\Delta = Ce^{-\frac{\Delta^2}{2\varepsilon^2}}d\Delta. \quad . \quad . \quad . \quad (183)$$

143. Constant *C*.—In deriving (177), the probability of an error between Δ and $\Delta + d\Delta$, it was assumed that p increased directly with $d\Delta$, which would be true for small intervals. For larger intervals the probability varies with Δ, so that the sum of the separate probabilities would have to be taken, giving

$$p_a^b = \int_a^b f(\Delta)d\Delta = C\int_a^b e^{-\frac{\Delta^2}{2\varepsilon^2}}d\Delta. \quad . \quad . \quad . \quad . \quad (a)$$

Since all errors are included between $+\infty$ and $-\infty$, the probability of an error between these limits $=1$, and of an error between 0 and ∞ (plus and minus errors being equally probable) $=\frac{1}{2}$. Hence

$$\tfrac{1}{2} = C\int_0^\infty e^{-\frac{\Delta^2}{2\varepsilon^2}}d\Delta.$$

If we place the exponent of e, or

$$\frac{\Delta^2}{2\varepsilon^2}=t^2, \quad d\Delta=\varepsilon\sqrt{2}dt.$$

For $\Delta=\infty$, t will also $=\infty$, so that the limits will be the same. Substituting,

$$\tfrac{1}{2}=C\varepsilon\sqrt{2}\int_0^\infty e^{-t^2}dt.$$

Since the definite integral is independent of the variable, we may also put

$$\tfrac{1}{2}=C\varepsilon\sqrt{2}\int_0^\infty e^{-u^2}du,$$

giving

$$\tfrac{1}{8}=C^2\varepsilon^2\int_0^\infty\int_0^\infty e^{-t^2-u^2}dtdu. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

To integrate, take a surface of revolution generated by a curve with equation $z=e^{-t^2}$ in the ZT plane, or $z=e^{-u^2}$ in the ZU plane. Its equation will be

$$z=e^{-t^2-u^2}, \quad \text{or} \quad z=e^{-r^2}.$$

Its differential volume above the plane TU, as found by dividing into elementary prisms, whose base is $dt\cdot du$, will be

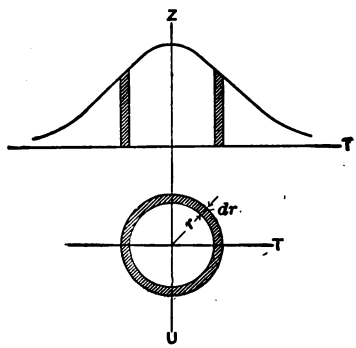

FIG. 92.

$$dV=zdtdu=e^{-t^2-u^2}dtdu,$$

giving

$$V=4\int_0^\infty\int_0^\infty e^{-t^2-u^2}dtdu. \quad . \quad . \quad . \quad . \quad . \quad (c)$$

(Note that the integral represents only one-fourth of the volume.)

Its differential volume, as found by dividing the plane TU into elementary rings of area $=2\pi r dr$, and erecting hollow cylinders of heights z, will be

$$dV=2\pi rzdr=2\pi re^{-r^2}dr.$$

Integrating,*

$$V=\pi\int_0^\infty e^{-r^2}2rdr,$$

or

$$V=-\pi(e^{-r^2})_0^\infty=\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

By (c) it is seen that the required integral $=\frac{1}{4}V$, which by (d)

$=\frac{\pi}{4}$. Substituting this value in (b),

$$\tfrac{1}{8}=\tfrac{1}{4}C^2\varepsilon^2\pi, \quad \text{or} \quad C=\frac{1}{\varepsilon\sqrt{2\pi}}.$$

Hence (183) becomes

$$p=f(\Delta)d\Delta=\frac{d\Delta}{\varepsilon\sqrt{2\pi}}e^{-\frac{\Delta^2}{2\varepsilon^2}}. \quad . \quad . \quad . \quad . \quad (184)$$

144. Value of Probability Integral by Series.—Substituting the value of C in § 143 (a) with the limits changed to $-a$ and $+a$,

$$p_{-a}^{+a}=\frac{1}{\varepsilon\sqrt{2\pi}}\int_{-a}^{+a}e^{-\frac{\Delta^2}{2\varepsilon^2}}d\Delta,$$

or, with $\frac{\Delta^2}{2\varepsilon^2}=t^2$ and $d\Delta=\varepsilon\sqrt{2}dt$ as in § 143, with the limits changed to $-t=-\frac{\Delta}{\varepsilon\sqrt{2}}$ and $+t=+\frac{\Delta}{\varepsilon\sqrt{2}}$,

$$p_{-t}^{+t}=(p)_t=\frac{1}{\sqrt{\pi}}\int_{-t}^{+t}e^{-t^2}dt=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt.$$

Expanding e^x by Maclaurin's theorem,

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

---

* $d(a^x)=a^x la dx$, where la is the Naperian log. Hence for the base e,

$$d(e^x)=e^x dx.$$

Hence

$$e^{-t^2}=1-\frac{t^2}{1!}+\frac{t^4}{2!}-\frac{t^6}{3!}+\dots$$

Substituting,

$$(p)_t=\frac{2}{\sqrt{\pi}}\left(t-\frac{t^3}{3\times1!}+\frac{t^5}{5\times2!}-\frac{t^7}{7\times3!}+\frac{t^9}{9\times4!}-\dots\right), \quad . \quad (185)$$

which converges rapidly for small values of t.

For large values of t a more rapidly converging series may be obtained by transforming according to the foot-note upon the previous page and integrating by parts,* thus:

$$\int e^{-t^2}dt=\int\left(-\frac{1}{2t}\right)de^{-t^2}=-\frac{1}{2t}e^{-t^2}-\frac{1}{2}\int e^{-t^2}\frac{dt}{t^2},$$

or, by continued integration by parts,

$$=-\frac{1}{2t}e^{-t^2}+\frac{1}{2^2t^3}e^{-t^2}+\frac{1\,.\,3}{2^2}\int e^{-t^2}\frac{dt}{t^4}.$$

Hence

$$\int_t^\infty e^{-t^2}dt=\frac{e^{-t^2}}{2t}\left(1-\frac{1}{2t^2}+\frac{1\,.\,3}{(2t^2)^2}-\frac{1\,.\,3\,.\,5}{(2t^2)^3}+\dots\right),$$

the terms for the limit ∞ being all zero.

But

$$\int_0^t e^{-t^2}dt=\int_0^\infty e^{-t^2}dt-\int_t^\infty e^{-t^2}dt$$

$$=\frac{\sqrt{\pi}}{2}-\int_t^\infty e^{-t^2}dt.$$

Substituting,

$$(p)_t=1-\frac{e^{-t^2}}{t\sqrt{\pi}}\left(1-\frac{1}{2t^2}+\frac{1\,.\,3}{(2t^2)^2}-\frac{1\,.\,3\,.\,5}{(2t^2)^3}+\dots\right). \quad . \quad (186)$$

From (185) and (186) Table XIII has been constructed, from which $(p)_t$ can be found for any value of t or $\frac{\Delta}{\varepsilon\sqrt{2}}$.

$$*\int xdy=xy-\int ydx.$$

In a given set of observations errors of different magnitude should occur in proportion to their probabilities as found from Table XIII. This gives a method of testing theory by practice, as below in the eighteen independently observed values for the angle Mednicken-Fuchsberg at station Trenk, given in *Gradmessung in Ostpreussen*, p. 78.

Angle.	$-v$	$+v$	v^2	Angle.	$-v$	$+v$	v^2
83° 30′ 36″.25	1.38		1.90	For'd 49″.36	8.50	7.84	40.74
7 .50	2.63		6.92				
6 .00	1.13		1.28	83° 30′ 33″.16		1.71	2.92
4 .77		0.10	0.01	4 .57		0.30	0.09
3 .75		1.12	1.25	4 .75		0.12	0.01
0 .25		4.62	21.34	6 .50	1.63		2.66
3 .70		1.17	1.37	5 .00	0.13		0.02
6 .14	1.27		1.61	4 .75		0.12	0.01
4 .04		0.83	0.69	4 .25		0.62	0.38
6 .96	2.09		4.37	5 .25	0.38		0.14
Sums 49″.36	8.50	7.84	40.74	87″.59	10.64	10.71	46.97

Mean = 83° 30′ 34″.87; $[v^2] = 46.97$; ε is found (by a method shown later) = 1.66.

For probability of error $\overline{\overline{<}} 1''$,

$$t = \frac{\Delta}{\varepsilon\sqrt{2}} = \frac{1}{1.66\sqrt{2}} = 0.426.$$

Hence $(p)_t$ from Table XIII = 45%.

Number of errors $\overline{\overline{<}} 1'' = n(p)_t = 0.45 \times 18 = 8.$

Similarly as below:

No. of Errors.	t	p	np	
			Theory.	Actual.
<0″.5	.5×0.426=0.213	24%	4.3	6
<1	1.0×0.426=0.426	45	8.1	8
<2	2.0×0.426=0.852	77	13.8	14
<3	3.0×0.426=1.278	93	16.8	17
<4	4.0×0.426=1.704	99	17.8	17
>4	. . . . . .	0.01	0.2	1

With a larger number of observations a closer agreement would be expected.

145. Degree of Precision.—It should be noted that the value of p in § 144 for a given value of Δ depends not on Δ but on $t=\dfrac{\Delta}{\varepsilon\sqrt{2}}$, so that in two sets of observations the probability of an error less than δ in the first will be equal that of an error less than δ' in the second, if $\delta/\varepsilon=\delta'/\varepsilon'$; e.g., if $\varepsilon'=2\varepsilon$, the probability of an error less than δ in the first will be the same as that of one less than $\delta'/2$ in the second, or the probability of an error less than, say, 1″ in the first will be as great as that of one less than 2″ in the second, or the *degree of precision* of the second is said to be only one-half as great as that of the first.

The degree of precision is then inversely as ε, and observations can be reduced to the same degree of precision, and their errors directly compared by dividing them by their corresponding ε's.

These quotients must in fact be abstract numbers, since $\dfrac{\Delta^2}{2\varepsilon^2}$ is the exponent of e in (184).

146. Constant ε. —In a large number of observations errors of different values will appear in proportion to their probabilities (as found to be nearly the case for a small number of observations in § 144, so that in n observations, or errors, there should be by (184)

$$n\frac{d\Delta'}{\varepsilon\sqrt{2\pi}}e^{-\frac{\Delta^2}{2\varepsilon^2}} \text{ errors of value } \Delta',$$

$$n\frac{d\Delta''}{\varepsilon\sqrt{2\pi}}e^{-\frac{\Delta^2}{2\varepsilon^2}} \quad \text{“} \quad \text{“} \quad \text{“} \quad \Delta'',$$

. . . .

Squaring each error and adding, we have for the total the sum of a series of terms of the form

$$\Delta^2 n\frac{d\Delta}{\varepsilon\sqrt{2\pi}}e^{-\frac{\Delta^2}{2\varepsilon^2}}.$$

Placing $\dfrac{\Delta^2}{2\varepsilon^2}=t^2$ and $d\Delta=\varepsilon\sqrt{2}dt$ as before, with the limits of $\pm\infty$ for Δ,

$$\text{Average square}=\frac{[\Delta^2]}{n}=\frac{2\varepsilon^2}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}t^2dt$$

$$=\frac{4\varepsilon^2}{\sqrt{\pi}}\int_0^{\infty}e^{-t^2}t^2dt.$$

Integrating by parts,

$$\int_0^\infty \left(\frac{-t}{2}\right) de^{-t^2} = -\left(\frac{t}{2}e^{-t^2}\right)_0^\infty + \frac{1}{2}\int_0^\infty e^{-t^2}dt$$

$$= 0 + \frac{\sqrt{\pi}}{4}, \text{ by § 143.}$$

Substituting,

$$\frac{[\Delta^2]}{n} = \varepsilon^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

which by comparison with (120) shows that the constant ε of § 142 is the m.s.e. of § 121.

147. Average Error.—As in § 146, we have the mean value of the errors taken without regard to sign:

$$\eta = \frac{[\pm\Delta]}{n} = \frac{2\varepsilon\sqrt{2}}{\sqrt{\pi}}\int_0^\infty e^{-t^2}tdt$$

$$= \frac{2\varepsilon\sqrt{2}}{\sqrt{\pi}}\left(-\frac{1}{2}e^{-t^2}\right)_0^\infty,$$

whence

$$\eta = \frac{\varepsilon\sqrt{2}}{\sqrt{\pi}} = 0.7979\varepsilon,$$

or

$$\varepsilon = 1.2533\eta = \frac{1}{n}1.2533[\pm\Delta] \quad . \quad . \quad . \quad . \quad (188)$$

148. Probable Error r. —If a series of errors are arranged in order of magnitude, the central one is called the probable error. There thus being as many errors with less values as with greater, the probability that any error taken at random will be less than r will be the same as that it is greater, and each equals one-half. Its value is found by placing $(p)_t = \frac{1}{2}$ in § 144, and solving for

$$t\left(=\frac{\Delta}{\varepsilon\sqrt{2}}\right), \text{ by Table XIII}$$

giving

$$\frac{\Delta'}{\varepsilon\sqrt{2}}, \quad \text{or} \quad \frac{r}{\varepsilon\sqrt{2}} = 0.4769,$$

from which

$$r=0.6745\varepsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (189)$$

The p.e. and m.s.e. are both used in expressing the precision of observations. The p.e. is almost exclusively used in this country. It should be remembered that this is not a limit for the actual error, which will be greater as often as less.

149. Graphic Representation.—If in (184) $\varepsilon=\frac{1}{\sqrt{2}}$, which reduces Δ to t,

$$\frac{p}{dt}=f(t)=\frac{e^{-t^2}}{\sqrt{\pi}},$$

from which the curve $f(t)$ of Fig. 93 can be plotted by assuming

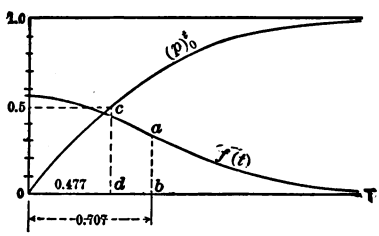

FIG. 93.

values for t and solving for $f(t)$, as below. Its general form was shown in Fig. 91.

t	$f(t)$	t	$f(t)$	t	$f(t)$
0.0	0.564	0.6	0.394	1.5	0.079
0.2	0.542	0.8	0.297	2.0	0.010
0.4	0.481	1.0	0.208	3.0	0.000

Since p/dt is an ordinate, p, the probability of an error t, will be an area $=f(t)dt$; while $(p)_t$ of § 144, the probability of an error between 0 and t, will be the area from 0 to t below the $f(t)$ curve. Laying these values of $(p)_t$ off as ordinates for given values of t by Table XIII, we have the curve $(p)_0^t$.

If $\Delta=\varepsilon$, $t'=1/\sqrt{2}=0.707$, corresponding to the m.s.e. If $\Delta=0.6745\varepsilon$, $t''=0.6745/\sqrt{2}=0.477$, corresponding to the p.e. These ordinates are laid off at ab and cd. The latter, cd, will bisect the area between the $f(t)$ curve and the axes, and cut the $(p)_0^t$ curve at the height 0.5, from the definition of p.e.

The former, ab, will give the point of inflection of the $f(t)$ curve, for, placing the second differential coefficient equal zero,

$$\frac{d^2f(t)}{dt^2}=\frac{4t^2}{\sqrt{\pi}}e^{-t^2}-\frac{2}{\sqrt{\pi}}e^{-t^2}=0,$$

or

$$t=1/\sqrt{2}=t', \text{ as above.}$$

150. Principle of Least Squares.—In § 142 we saw that with n unknowns dependent upon observation, their most probable values were those which made

$$p=f(\Delta_1)d\Delta_1\cdot f(\Delta_2)d\Delta_2\cdot f(\Delta_3)d\Delta_3 \ldots$$

a maximum, or substituting the values of $f(\Delta_1), f(\Delta_2), \ldots$ from (184),

$$p=\frac{d\Delta_1 d\Delta_2 d\Delta_3 \ldots}{\varepsilon_1\varepsilon_2\varepsilon_3 \ldots}\,\frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\left[\frac{\Delta^2}{2\varepsilon^2}\right]},$$

a maximum which, since $d\Delta_1$, $d\Delta_2, \ldots$, ε_1, $\varepsilon_2, \ldots$ are constants, or are known from the observations, will be a maximum when

$$\frac{1}{2}\left[\frac{\Delta^2}{\varepsilon^2}\right]$$

is a minimum; i.e., each error being divided by its m.s.e., or reduced to a standard degree of precision, § 145, the most probable values of the unknowns will be those which make the sum of the squares of the quotients a minimum. Hence the name *Least Squares.*

If the degrees of precision are equal, ε can be factored out, leaving $[\Delta^2]$ a minimum. When $[\Delta^2]$ is a minimum $[v^2]$ will also be a minimum.

For, § 124,

$$n\varepsilon^2, \quad \text{or} \quad [\Delta^2]=[v^2]+n\delta^2.$$

But

$$n\delta^2=\varepsilon^2=\frac{[\Delta^2]}{n}.$$

Substituting,

$$[v^2]=\frac{n-1}{n}[\Delta^2]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (190)$$

Hence we may also say that each residual being divided by its m.s.e., or reduced to a standard degree of precision, the most probable values of the unknowns will be those which make the sum of the squares a minimum.

We may also note that, since it was assumed as an axiom that the arithmetic mean of a number of equally good observations is the most probable value, the arithmetic mean must make the sum of the squares of the residuals a minimum.

To test this, take some other value of the unknown as $x_0+\delta$. The residuals will be $v_1'=v_1+\delta$, $v_2'=v_2+\delta$.

Squaring and adding,

$$[v'^2]=[v^2]+2\delta[v]+n\delta^2,$$

which, since $[v]=0$ and $n\delta^2$ is positive, will always be greater than $[v^2]$.

151. Relation between Average, Mean Square, and Probable Errors.—To find the average error of § 147 in terms of the residuals v, with one unknown, directly observed, we have from (190),

$$[v^2]=\frac{n-1}{n}[\Delta^2].$$

Hence it may be concluded that on the average

$$\frac{\Delta^2}{v^2}=\frac{n}{n-1}, \quad \text{and} \quad \frac{\Delta}{v}=\sqrt{\frac{n}{n-1}},$$

or, if v and Δ are added without regard to sign,

$$[\pm\Delta]=\sqrt{\frac{n}{n-1}}[\pm v].$$

Hence the average error,

$$\left.\begin{aligned}\eta&=\frac{[\pm\Delta]}{n}=\frac{[\pm v]}{\sqrt{n(n-1)}}\\ \eta_0&=\frac{\eta}{\sqrt{n}}=\frac{[\pm v]}{n\sqrt{n-1}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (191)$$

Substituting these values of η and η_0 in (188),

$$\left.\begin{aligned} \varepsilon &= 1.2533 \frac{[\pm v]}{\sqrt{n(n-1)}}, \\ \varepsilon_0 &= 1.2533 \frac{[\pm v]}{n\sqrt{n-1}}, \end{aligned}\right\} \quad \ldots \quad (192)$$

which are known as Peter's formulas.

From (128) and (129),

$$\varepsilon = \sqrt{\frac{[v^2]}{n-1}}, \qquad \varepsilon_0 = \sqrt{\frac{[v^2]}{n(n-1)}},$$

From (189)

$$r = 0.6745\sqrt{\frac{[v^2]}{n-1}}, \qquad r_0 = 0.6745\sqrt{\frac{[v^2]}{n(n-1)}}, \quad . \quad (193)$$

which are known as Bessel's formulas.

For the general case of indirect observations we have (150)

$$\varepsilon^2 = \frac{[v^2]}{n-m}, \quad \text{while} \quad \varepsilon^2 = \frac{[\varDelta^2]}{n}.$$

Hence as above,

$$\frac{\varDelta}{v} = \sqrt{\frac{n}{n-m}}, \quad \text{or} \quad \frac{[\pm \varDelta]}{[\pm v]} = \sqrt{\frac{n}{n-m}},$$

$$\eta = \frac{[\pm \varDelta]}{n},$$

or

$$\left.\begin{aligned} \eta &= \frac{[\pm v]}{\sqrt{n(n-m)}}, \\ \varepsilon &= 1.2533 \frac{[\pm v]}{\sqrt{n(n-m)}}, \\ r &= 0.8454 \frac{[\pm v]}{\sqrt{n(n-m)}}, \end{aligned}\right\} \quad \ldots \ldots \quad (194)$$

which are known as Lueroth's formulas.

For the case of direct observations upon dependent quantities m' takes the place of $n-m$ as in (160), giving

$$\left.\begin{aligned} &\varepsilon = 1.2533\frac{[\pm v]}{\sqrt{nm'}}, \quad r = 0.8454\frac{[\pm v]}{\sqrt{nm'}}, \\ &(150) \text{ and } (160) \text{ being} \\ &\varepsilon = \sqrt{\frac{[v^2]}{n-m}} \quad \text{and} \quad \varepsilon = \sqrt{\frac{[v^2]}{m'}}. \end{aligned}\right\} \quad . \quad . \quad (195)$$

The values of ε and r derived from the first powers of the residuals are often used because they are slightly more easy to compute; they are not, however, as accurate as those derived from the second powers. Weights are readily introduced if desired.

152. Limit of Accuracy.—In deriving the preceding formulas it has been assumed: (*a*) That the number of observations is great. (*b*) That Δ can be regarded as a continuous variable. (*c*) That all constant errors have been eliminated. With but few observations, (*a*) and (*b*) are only partially satisfied, still if (*c*) is satisfied, the computed m.s.e. will, on the average, be the true one, although in an individual case it may be somewhat in error.

But as constant error is usually present, the computed m.s.e. may be very misleading, unless the circumstances under which the observations were taken, or the reputation of the observer, are known. Again, when the number of observations is great, an increase in the number does not reduce the m.s.e. as rapidly as theory would indicate ($\varepsilon_0 = \varepsilon/\sqrt{n}$), and finally there is in every species of observations an ultimate limit of accuracy beyond which no mass of accumulated observations can ever penetrate.

For example, Chauvenet (*Astronomy*, Vol. I, p. 370) states that the ultimate limit of accuracy in finding longitude from moon culminations with a sextant is about one second of time, and that this result can be reached by taking advantage of the available clear nights for some two to three years. Professor Rogers found the p.e. of a single complete determination of the declination of a star with the meridian circle to be $0''.36$ and of the right ascension of an equatorial star $0^s.026$. He says: "If therefore the p.e. can be taken as a measure of the accuracy of the observations, there ought to be no difficulty in obtaining from a moderate number of observations the right ascension within $0^s.02$ and the declination within $0''.2$. Yet it is doubtful, after continuous observations in all parts of the world for more than a century, if there is a single star in the heavens whose absolute coordinates are known within these limits."

The reason is that the observations are not arranged so that constant error is eliminated, but only the accidental errors.

Even the theoretical gain in accuracy by increasing the number of observations is small after a fairly large number have been taken, so that if it is possible to improve in instruments or in methods, it is a waste of time to push the repetition of observations to anything like the ultimate limit of accuracy. The improvement in method is mainly in the arrangement of the observations so as to eliminate constant error. A study of the methods of observing which have been developed and of the instruments employed, as described in Part I, is of interest in this connection.

153. Rejection of Doubtful Observations.—This is one of the most difficult points in connection with the adjustment of observations. An observer is at liberty to arrange the observations and choose the conditions under which he will observe as his experience and best judgment may dictate. Having begun the observations, if he finds the conditions unfavorable he is at liberty to stop, reject the work already done, and begin again under more favorable auspices. When it comes to individual results in a set, if there is reason to suspect that an observation is poor before obtaining the result, a note should be made to that effect and a line drawn through the value. If the only reason for suspecting it is because it differs from the others, the young observer should hesitate about rejection unless the discrepancy is so great that a mistake is certain. The attitude of an observer should be that of perfect honesty and fairness, directing his effort each time to obtaining the best possible value of the quantity sought without being biased by the preceding results, and without regard to them except to know in a general way that no mistakes are being made.

Having the different results together, and being familiar with the circumstances under which the observations were made, the observer can decide which, if any, he will leave out in making up the mean.

The computer in revising the work usually assumes the right to revise the rejection of observations. For this purpose he, if not the observer, will usually require a criterion. Several have been proposed. Peirce's is perhaps in most common use, but the following, based upon Table XIII, has able advocates and is simple.

If $\Delta=3\varepsilon$, $t=3/\sqrt{2}=2.12$, giving, by Table XIII, $(p)_t=0.997$; i.e., only 3 errors in 1000 should exceed 3 times the m.s.e. On this account the criterion calls for rejecting errors greater than 3ε in limited series of observations. Many object to any criterion and leave the matter to the judgment of the observer, or to the computer in cases where more data are obtained by subsequent observations or by an advance in theoretical knowledge.

See on this subject Wright and Hayford's *Adjustment of Observations*, pp. 87–93.

CHAPTER XI.

APPLICATION TO TRIANGULATION.

154. Triangulation.—In adjusting the horizontal angles of a triangulation there are two classes of errors or discrepancies which are to be considered, one in the adjustment of the observed angles or directions at a station, the other in their adjustment in the

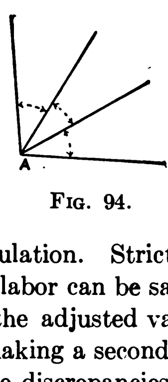

FIG. 94.

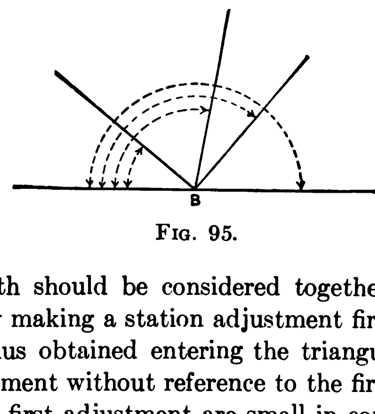

FIG. 95.

triangulation. Strictly, both should be considered together, but much labor can be saved by making a station adjustment first and with the adjusted values thus obtained entering the triangulation and making a second adjustment without reference to the first.

The discrepancies in the first adjustment are small in comparison with those in the second, as the effects of eccentricity, phase, etc., are not wholly eliminated by the method of observing, but enter more or less as constant errors which appear in the second adjustment. On this account there is but little loss in accuracy and the method of separation is usually adopted to save labor.

155. Station Adjustment.—The station adjustment can be avoided by measuring the angles at a station independently and without checks. This can be done by measuring the angles between adjacent stations, as in Fig. 94, or by measuring the angle from a reference line around to the right to each station, as in Fig. 95. In the latter case each measured angle would correspond to a bearing or direction for the line to its right. They are sometimes used as directions in the figure adjustment and sometimes the differences are taken and treated as angles. In the first case (Fig. 94) a considerable gain in accuracy can be secured by closing the horizon. This would reduce the station adjustment to simply dividing the discrepancy equally among the angles if of equal weight, or inversely as the weights if the weights are unequal. If instead of closing the horizon the sum angle is measured the discrepancy would be divided equally among the angles, including the sum, if of equal weight, or inversely as the weights if unequal.

If the signals are tied together as in Fig. 96, either with a direction instrument or a repeating instrument, as described in §§51 and 52, a station adjustment will be required. If treated as angles the angle between each two adjacent lines would be considered an unknown, X, Y, Z, . . . , while each observed value would furnish an observation equation. Thus in Fig. 96 there would be four unknowns, with four observation equations from the angles swung from 1–6, three from 2–6, two from 3–6, and one from 4–6.

The first would be

$$
\begin{aligned}
X - M_1 &= v_1, \\
X + Y - M_2 &= v_2, \\
X + Y + Z - M_3 &= v_3, \\
X + Y + Z + U - M_4 &= v_4.
\end{aligned}
$$

The next

$$
\begin{aligned}
Y - M_5 &= v_5, \\
Y + Z - M_6 &= v_6, \\
\ldots\ldots
\end{aligned}
$$

This is a simple case under § 128.

If treated as directions there would be five unknowns and the observation equations would be written:

$$
\begin{aligned}
- X_1 + Y_1 - M_1 &= v_1, \\
- X_1 + Z_1 - M_2 &= v_2, \\
- X_1 + U_1 - M_3 &= v_3, \\
- X_1 + W_1 - M_4 &= v_4
\end{aligned}
$$

for the first, etc. for the others.

This method would be used for the observations of both §§ 51 and 52.

For the method of direction observations described in § 50 denote the required directions of the signals or the angles which they make with the reference line by $Y, Z, U, \ldots$, and the angle between the zero of the circle for each position and the reference line by $X', X'', X''', \ldots$. Then if the readings of the circle on station 1 are $M_1', M_1'', \ldots$, on station 2, M_2', $M_2'', \ldots$, the observation equations will be;

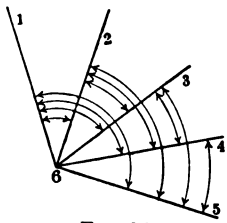

FIG. 96.

For the first position,

$$X' - M_1' = v_1'$$
$$X' + Y - M_2' = v_2'$$
$$\cdots\cdots\cdots$$

For the second,

$$X'' \qquad - M_1'' = v_1''$$
$$X'' + Y - M_2'' = v_2''$$
$$\cdots\cdots\cdots$$

The solution would be as given in § 128.

It is stated in the 1903 Coast Survey *Report*, p. 846, that for the method of direction observations as described in § 54, no local adjustments of directions were made on the ninety-eighth meridian work because their method of observing has rendered them unnecessary. With the repeating instrument it will be remembered the angles between adjacent lines are measured and the horizon closed, giving a very simple station adjustment.

156. Weighting.—In the station adjustment the observed values for the observation equations are the means of quite a number of observations, so that their p.e.'s can be found from the residuals for the separate observations by (193). The reciprocals of the squares will give the weights for the observation equations.

The p.e.'s for the computed angles can be found in solving the normal equations as in the problems of § 133. If the angles are observed independently the p.e.'s can be found as for the station adjustment above.

Having the adjusted angles or directions, the next step will be to make up the triangles and compute the triangle errors. Usually more triangles can be made up than are necessary to express all of the angle checks, and as using this extra number will result in a waste of labor, it is important to find the exact number required.

The number of lines required to connect p points with a closed figure is p, and this will give one check upon the observed angles. Every additional line will give an additional check, so that with l lines and p points

$$\text{No. of angle checks} = l - p + 1. \quad . \quad . \quad . \quad . \quad (196)$$

This will usually give the number of required triangles. In exceptional cases a sufficient number of triangles to satisfy (196) cannot be found, and polygons will have to be used instead. When there is an excess in the total number of triangles the best-shaped ones should usually be taken.*

The triangle errors can then be computed by comparing the sums of the angles in each with $(180° + s)$.† Squaring these, adding and dividing by the number of triangles will give the average square, or ε^2 for a triangle. Dividing by three will give the average ε^2 for an angle, or by six the average ε^2 for a direction.‡ Comparing this value with the average ε^2 found for the adjusted angles or directions at the station, it will usually be found greater. The reason is that the former include only the observing errors, while the latter include both the observing and triangle errors, or those due to eccentricity of signal and instrument, lateral refraction, one-sided illumination, etc. Subtracting the former from the latter will give the ε^2 due to triangle error, which must be regarded as constant. Adding this to the ε^2 due to observing error for each angle we have the total for each angle; the weights for the triangle adjustment will be proportional to the reciprocals of these.

In case more than three of the adjusted angles are required to form a triangle, the sum of the squares of the triangle errors should be divided by the total number of angles used for the average ε^2 for an angle; while in forming the sum of the ε^2 for the adjusted angles at a station, each should be repeated as many times as the angle is used in different triangles and the total number of ε's used as a divisor in obtaining the average. Polygons can be included

* For a discussion of this subject see Wright and Hayford's *Adjustment of Observations*, § 154.

† s = spherical excess.

‡ An angle is made up of the difference of two directions, just as in compass work it is made up of the difference of two bearings. Counting clockwise the left-hand direction would be negative and the right-hand positive.

with the triangles in following out this method for angles or directions if there are not triangles enough to satisfy (196).

The effect of the triangle error is to make the weights more nearly equal; if it is to be neglected, nearly as good results will be obtained by neglecting weights as by taking them from the ε^2 of the adjusted angles and with less labor. It is good practice to arrange the observations so as to avoid station adjustment, except that due to closing the horizon when a repeating instrument is used, to measure the angles with about equal accuracy, and then to neglect weights in the figure adjustment.

157. **Figure Adjustment.**—The geometrical conditions to be satisfied in the triangulation are:

(*a*) A fixed sum for the angles in a figure.

For a triangle

$$S=180°+s.$$

For a polygon

$$S=180°l+s-360°,$$

where S = the sum of the angles, and l = the number of sides in the polygon.

(*b*) The same length for a triangle side when computed through different systems from the same base or starting line.

(*a*) gives rise to *angle equations;* the number is given by (196). (*b*) gives rise to *side* equations; the number will be found in § 160 Both come under the case of independent observations upon dependent quantities developed in § 135.

PENTAGON WITH INTERIOR POINT. (FIG. 97).

ANGLE EQUATIONS.

There are five triangle checks besides the station condition that the angles about the central point f must remain equal to 360°.

Substituting in (155),

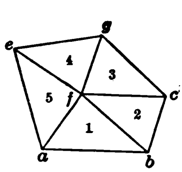

FIG. 97

$$(a_1)+(f_1)+(b_1)+q_1=0,$$

$$(b_2)+(f_2)+(c_2)+q_2=0,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\ ,$$

$$(f_1)+(f_2)+(f_3)+(f_4)+(f_5)=0,$$

where q_1, q_2, . . . are the triangle errors, and (a_1), (b_1) . . . are the corrections to the angles, or the v's of § 135.

Side Equations.

The triangles which give a side equation, or a check upon the length of a side, will usually have one vertex in common, called a pole, while the sides radiating from it will each be common to two triangles.

In making up the check equation, the two radiating sides of each triangle are written as a fraction, beginning with any one and taking the adjacent ones in order in either direction around to the first again, the denominator of the first becomes the numerator of the second, etc., until finally the last denominator will be the same as the first numerator, giving unity for the continued product. Each fraction can be replaced by the ratio of the sines of the opposite angles in the same triangle, giving the required check on the angles.

Thus the five triangles have a common vertex at f, giving

$$\frac{af}{bf}\cdot\frac{bf}{cf}\cdot\frac{cf}{gf}\cdot\frac{gf}{ef}\cdot\frac{ef}{af}=1,$$

or

$$\frac{\sin b_1 \sin c_2 \sin g_3 \sin e_4 \sin a_5}{\sin a_1 \sin b_2 \sin c_3 \sin g_4 \sin e_5}=1.$$

Taking logs,

$$\log \sin b_1 - \log \sin a_1 + \log \sin c_2 - \log \sin b_2 + \log \sin g_3 \\ - \log \sin c_3 + \log \sin e_4 - \log \sin g_4 + \log \sin a_5 - \log \sin e_5 = 0.$$

The differential coefficients of § 135 become

$$\frac{d\ (\log \sin V)}{dV} = \text{ratio of change in log sin to change in arc} = \frac{d}{\sin 1''},$$

where d = tabular difference of log sin for 1″ (minus for $V > 90°$).

Substituting, (155) becomes

$$d_1(b_1) - d_2(a_1) + d_3(c_2) - d_4(b_2) + d_5(g_3) - d_6(c_3) \\ + d_7(e_4) - d_8(g_4) + d_9(a_5) - d_{10}(e_5) + q_7 = 0,$$

where q_7 = the value of the log sin equation when the observed angles are substituted.

For convenience the decimal point is moved either six or seven places to the right for d and q.

158. Adjustment of a Quadrilateral. Method of Angles.—To illustrate the application of the method of least squares to the adjustment of a figure, the following example, taken from the triangulation of Seneca Lake in 1882, will be worked out. The angles were observed independently and the weights of each angle determined from the observations by the method of § 156. The spherical excess was inappreciable, since the distances were comparatively small.

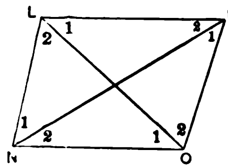

Fig. 98.

The observed angles in the three triangles of the figure which were used for angle equations were as follows:

Triangle *ONL*.

					$\frac{1}{w}$
O_1 =	49°	13′	05″	.4	0.6
N_{1+2} =	70	27	53	.0	1.5
L_2 =	60	19	06	.5	0.6
	180	00	04	.9	

Triangle *LMO*.

					$\frac{1}{w}$
L_1 =	48°	53′	29″	.3	0.4
M_{1+2} =	60	41	56	.9	0.5
O_2 =	70	24	31	.0	0.6
	179	59	57	.2	

Triangle *NLM*.

					$\frac{1}{w}$
N_1 =	41°	21′	25″	.8	1.1
L_2 =	60	19	06	.5	0.6
L_1 =	48	53	29	.3	0.4
M_2 =	29	25	52	.7	1.0
	179	59	54	.3	

These three triangles thus afford three angle equations of condition, in which $q' = +04''.9$, $q'' = -02''.8$, and $q''' = -5''.7$. By taking a pole at L, the following side equation is obtained:

$$\log \sin O_1 + \log \sin M_{1+2} + \log \sin N_1 - \log \sin N_{1+2} - \log \sin O_2 - \log \sin M_2 = 0.$$

The absolute term q^{IV} is found from the following tabulation.

	Log sine (+).	d for 1″.		Log sine (−).	d for 1″.
O_1	9.879 2119	18.1	N_{1+2}	9.974 2517	7.5
M_{1+2}	9.940 5473	11.8	O_2	9.974 1006	7.5
N_1	9.820 0378	23.9	M_2	9.691 4172	37.4
	9.639 7970			9.639 7695	
	695				
	$q^{\text{IV}} = +275$				

The values of q for the several condition equations having now been determined, the normal equations may be determined as below. First, however, the condition equations will be stated in tabular form, thus:

v	a	b	c	d*	s	$\frac{1}{w}$
(O_1)	+1			+18.1	+19.1	0.6
(N_{1+2})	+1			− 7.5	− 6.5	1.5
(L_2)	+1		+1		+ 2.0	0.6
(L_1)		+1	+1		+ 2.0	0.4
(M_{1+2})		+1		+11.8	+12.8	0.5
(O_2)		+1		− 7.5	− 6.5	0.6
(N_1)			+1	+23.9	+24.9	1.1
(M_2)			+1	−37.4	−36.4	1.0

By equation (158) we can now form four normal equations of the form:

$$\left[\frac{aa}{w}\right]A + \left[\frac{ab}{w}\right]B + \left[\frac{ac}{w}\right]C + \left[\frac{ad}{w}\right]D + q' = 0.$$

$$\left[\frac{ab}{w}\right]A + \left[\frac{bb}{w}\right]B + \left[\frac{bc}{w}\right]C + \left[\frac{bd}{w}\right]D + q'' = 0.$$

. . . .

The quantities A, B, C, and D are the correlates of the corrections and are equal in number to the number of conditions. The formation of the normal equations is facilitated by the use of the following table:

* Note that the coefficients of the side equation have their units taken in the seventh place of decimals.

v	$\frac{aa}{w}$	$\frac{ab}{w}$	$\frac{ac}{w}$	$\frac{ad}{w}$	$\frac{as}{w}$	$\frac{bb}{w}$	$\frac{bc}{w}$
O_1	+0.6			+10.86	+11.46		
N_{1+2}	+1.5			−11.25	− 9.75		
L_2	+0.6		+0.6		+ 1.2		
L_1						+0.4	+0.4
M_{1+2}						+0.5	
O_2						+0.6	
N_1							
M_2							
	+2.7		+0.6	− 0.39	+ 2.91	+1.5	+0.4

v	$\frac{bd}{w}$	$\frac{bs}{w}$	$\frac{cc}{w}$	$\frac{cd}{w}$	$\frac{cs}{w}$	$\frac{dd}{w}$	$\frac{ds}{w}$
O_1						+ 196.57	+ 207.43
M_{1+2}						+ 84.37	+ 73.12
L_2			+0.6		+ 1.2		
L_1		+0.8	+0.4		+ 0.8		
M_{1+2}	+5.9	+6.4				+ 69.62	+ 75.52
O_2	−4.5	−3.9				+ 33.76	+ 29.25
N_1			+1.1	+26.29	+27.39	+ 628.33	+ 654.62
M_2			+1.0	−37.4	−36.4	+1398.76	+1361.36
	+1.4	+3.3	+3.1	−11.11	− 7.01	+2411.41	+2401.30

The normal equations are now written, employing the coefficients and check sums of the table. The checks only affect the coefficients of the equations.

NORMAL EQUATIONS.

$$
\begin{array}{lr}
 & \text{Check.} \\
+2.7A \qquad\quad + 0.6C - 0.39D + 4.9 = 0 & -\ 2.91 \\
\qquad +1.5B + 0.4C + 1.4D - 2.8 = 0 & -\ 3.3 \\
+0.6A + 0.4B + 3.1C - 11.11D - 5.7 = 0 & +\ 7.01 \\
-0.39A + 1.4B - 11.11C + 2411.4D + 275.0 = 0 & -2401.3
\end{array}
$$

These equations are more readily solved with a smaller coefficient for D in the fourth one. If D_1 be taken equal to $10D$ we should have the following equations:

$$
\begin{array}{lr}
+2.7A \qquad\quad + 0.6C - 0.04D_1 + 4.9 = 0 & -\ 8.16 \\
\qquad +1.5B + 0.4C + 0.14D_1 - 2.8 = 0 & +\ 0.76 \\
+0.6A + 0.4B + 3.1C - 1.11D_1 - 5.7 = 0 & +\ 2.71 \\
-0.04A + 0.14B - 1.11C + 24.11D_1 + 27.5 = 0 & -\ 50.60
\end{array}
$$

The figures in the check column now include the values of the constants, and are to be used in checking the solution, which is to be performed by one of the methods of Chapter IX.

The solution of the normal equations gives the values

$$A=-2.20,\quad B=+1.52,\quad C=1.68,\quad D_1=-1.08,\quad D=-0.108.$$

Applying now equations (159), we have for triangle ONL:

Correction to O_1 $=0.6\{-2.20+18.1(-0.108)\}=-2''.5$
" " $N_{1+2}=1.5\{-2.20-\ 7.5(-0.108)\}=-2\ .1$
" " L_2 $=0.6\{-2.20+\ 1.68\}$ $=-0\ .3$

$-4\ .9=-q'$

In like manner for triangles LMO and NLM,

Corr. to L_1 $=+1''.3$	Corr. to N_1 $=-1.0$
" " $M_{1+2}=+0\ .1$	" " M_2 $=+5.7$
" " O_2 $=+1\ .4$	" " $L_1+L_2=+1.0$
$+2\ .8=-q''$	$+5.7=-q'''$

We now substitute these corrections in the side condition equation:

For O_1,	$-2.5\times18.1=-45.2$	For N_{1+2},	$-2.1\times\ 7.5=-\ 15.7$
" M_{1+2},	$+0.1\times11.8=+\ 1.2$	" O_2,	$+1.4\times\ 7.5=+\ 10.5$
" N_1,	$-1.0\times23.9=-23.9$	" M_2,	$+5.7\times37.4=+213.2$
	$-\ 67.9$		$+208.0$
	-208.0		
	-275.9, which should equal $-q^{IV}$.		

The work of obtaining the corrections is now checked, and the adjusted angles are obtained by adding these corrections to the observed angles.

159. Adjustment of a Quadrilateral, Method of Directions.— To illustrate the application of the method of directions to the adjustment of a figure, the angles of the previous example are expressed as differences of direction. Weights are assumed for the directions illustrated in Fig. 99, which correspond very nearly to the weights of the angles in the preceding case.

In this example each direction has a correction applied to it, and consequently we have a case of twelve unknown quantities subject to the same four conditions as before. As might be supposed, the method is peculiarly adapted to the adjustment of observations made with a direction instrument.

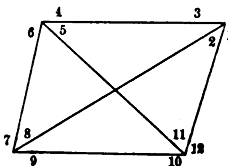

Fig. 99.

In the present example the steps followed are substantially the same as in the preceding one, so that no further explanation is necessary. The normal equations are stated and solved by the method of § 134, and the corrections when obtained are checked by substitution in the condition equations.

TRIANGLES.

$-(10)+(11)$	49° 13′ 05″.4
$-(7)+(9)$	70 27 53 .0
$-(5)+(6)$	60 19 06 .5
	180 00 04 .9
$-(4)+(5)$	48 53 29 .3
$-(1)+(3)$	60 41 56 .9
$-(11)+(12)$	70 24 31 .0
	179 59 57 .2
$-(4)+(6)$	109 12 35 .8
$-(2)+(3)$	29 25 52 .7
$-(7)+(8)$	41 21 25 .8
	179 59 54 .3

SIDE EQUATION.

		Log sin (+).	d for 1″.
$-(10)+(11)$	49° 13′ 05″.4	9.879 2119	+18.1
$-(1)+(3)$	60 41 56 .9	9.940 5473	+11.8
$-(7)+(8)$	41 21 25 .8	9.820 0378	+23.9
		9.639 7970	

		Log sin (−).	d for 1″.
$-(7)+(9)$	70° 27′ 53″.0	9.974 2517	+ 7.5
$-(11)+(12)$	70 24 31 .0	9.974 1006	+ 7.5
$-(2)+(3)$	29 25 52 .7	9.691 4172	+37.4
		9.639 7695	

9.639 7970
695

+275 (*Take unit in sixth place.*)

TABLE FOR FORMATION OF NORMAL EQUATIONS.

Direction.	$\frac{1}{w}$	a	b	c	d	s	$\frac{a^2}{w}$	$\frac{ab}{w}$	$\frac{ac}{w}$	$\frac{ad}{w}$	$\frac{as}{w}$
(1)	0.2		−1		−1.18	−2.18					
(2)	0.7			−1	+3.74	+2.74					
(3)	0.3		+1	+1	−2.56	−0.56					
(4)	0.1		−1	−1		−2.00					
(5)	0.3	−1	+1			0	0.3	−0.3			
(6)	0.3	+1		+1		+2.00	0.3		+0.3		+0.60
(7)	0.7	−1		−1	−1.64	−3.64	0.7		+0.7	+1.15	+2.55
(8)	0.4			+1	+2.39	+3.39					
(9)	0.8	+1			−0.75	+0.25	0.8			−0.60	+0.20
(10)	0.3	−1			−1.81	−2.81	0.3			+0.54	+0.84
(11)	0.3	+1	−1		+2.56	+2.56	0.3	−0.3		+0.77	+0.77
(12)	0.3		+1		−0.75	+0.25					
S'ms							+2.70	−0.60	+1.00	+1.86	+4.96

Direction.	$\frac{b^2}{w}$	$\frac{bc}{w}$	$\frac{bd}{w}$	$\frac{bs}{w}$	$\frac{c^2}{w}$	$\frac{cd}{w}$	$\frac{cs}{w}$	$\frac{d^2}{w}$	$\frac{ds}{w}$
(1)	0.2		+0.24	+0.44				+ 0.28	+ 0.51
(2)					0.7	−2.62	−1.92	+ 9.79	+ 7.17
(3)	0.3	+0.3	−0.77	−0.17	0.3	−0.77	−0.17	+ 1.97	+ 0.43
(4)	0.1	+0.1		+0.20	0.1		+0.20		
(5)	0.3								
(6)					0.3		+0.60		
(7)					0.7	+1.15	+2.55	+ 1.88	+ 4.18
(8)					0.4	+0.96	+1.36	+ 2.28	+ 3.24
(9)								+ 0.45	− 0.15
(10)								+ 0.98	+ 1.53
(11)	0.3		−0.77	−0.77				+ 1.97	+ 1.97
(12)	0.3		−0.22	+0.08				+ 0.17	− 0.06
S'ms	+1.50	+0.40	−1.52	−0.22	+2.50	−1.28	+2.62	+19.77	+18.82

NORMAL EQUATIONS.

	A	*B*	*C*	*D*	Const.	Check.	*S*	
I	+2.70	−0.60	+1.00	+ 1.86	+ 4.90	+ 9.86	+ 4.96	
		+1.50	+0.40	− 1.52	− 2.80	− 3.02	− 0.22	
			+2.50	− 1.28	− 5.70	− 3.08	+ 2.62	
				+19.77	+27.50	+46.33	+18.83	
		+1.50	+0.40	− 1.52	− 2.80	− 3.02		(Factors)
		−0.13	+0.22	+ 0.41	+ 1.09	+ 2.19		(+0.222)
II		+1.37	+0.62	− 1.11	− 1.71	− 0.83		

	C	D	Const.	Check.	
	+2.50	− 1.28	− 5.70	− 3.08	(Factors)
	−0.37	− 0.69	− 1.82	− 3.66	(−0.371)
	−0.28	+ 0.50	+ 0.77	+ 0.38	(−0.453)
III	+1.85	− 1.47	− 6.75	− 6.37	
		+19.77	+27.50	+46.33	
		− 1.28	− 3.38	− 6.79	(−0.689)
		− 0.90	− 1.38	− 0.67	(+0.810)
		− 1.17	− 5.37	− 5.06	(+0.795)
IV		+16.42	+17.37	+33.81	

$$D=\frac{-17.37}{+16.42}=-1.058;$$

$$C=\frac{+6.75+1.47D}{+1.85}=\frac{+5.195}{+1.85}=+2.808;$$

$$B=\frac{+1.71+1.11D-0.62C}{+1.37}=\frac{-1.205}{+1.37}=-0.880;$$

$$A=\frac{-4.90-1.86D-C+0.60B}{+2.70}=\frac{-6.268}{+2.70}=-2.321.$$

TABLE FOR CORRECTIONS.

	(1)	(2)	(3)	(4)	(5)	(6)
$\frac{1}{w}$	0.2	0.7	0.3	0.1	0.3	0.3
A					+2.321	−2.321
B	+0.880		−0.880	+0.880	−0.880	
C		−2.808	+2.808	−2.808		+2.808
D	+1.249	−3.957	+2.709			
Sums	+2.129	−6.765	+4.637	−1.928	+1.441	+0.487
	+0.426	−4.736	+1.391	−0.193	+0.432	+0.146
Corr'n	+0.43	−4.74	+1.39	−0.19	+0.43	+0.14

	(7)	(8)	(9)	(10)	(11)	(12)
$\frac{1}{w}$	0.7	0.4	0.8	0.3	0.3	0.3
A	+2.321		−2.321	+2.321	−2.321	
B					+0.880	−0.880
C	−2.808	+2.808				
D	+1.735	−2.528	+0.793	+1.915	−2.709	+0.793
Sums	+1.248	+0.280	−1.528	+4.236	−4.150	−0.087
	+0.874	+0.112	−1.222	+1.271	−1.245	−0.026
Corr'n	+0.87	+0.11	−1.22	+1.27	−1.25	−0.03

$$\text{Sums}\times\frac{1}{w}=\text{Correction to direction.}$$

TESTS OF CORRECTIONS.

(*a*) $-0.43+0.14^{*}-0.87-1.22-1.27-1.25=-4.90.$
(*b*) $-0.43+1.39\ \ +0.19+0.43+1.25-0.03=+2.80.$
(*c*) $+4.74+1.39\ \ +0.19+0.14-0.87+0.11=+5.70.$
(*d*) $-1.18(+0.43)+3.74(-4.74)-2.56(+1.39)-1.64(+0.87)+2.39(+0.11)$
$-0.75(-1.22)-1.81(+1.27)+2.56(-1.25)-0.75(-0.03)$
$=-0.51-17.73-3.56-1.43+0.26+0.92-2.30-3.20+0.02=-27.5.$

160. Number and Formation of the Side Equations.—When in any system of triangles the first two points are determined by the length of the line joining them, the determination of any additional point will require two sides or two directions, so that in any system of p points we have to determine $p-2$ points, which requires $2(p-2)$ directions, or by adding the first $2p-3$.

Hence in a system of l sides and p points,

$$\text{No. of side equations}=l-2p+3, \quad . \quad . \quad . \quad (197)$$

where each side requires to be observed over from one end only.

Stations between which side equations exist form systems about a central point or pole including it in a triangle or polygon. Frequently the pole falls outside, which makes no difference in the solution. In either case there is one characteristic property, i.e., at every station three lines meet, save one, where $p-1$ meet, there being p stations. Complications arise from systems within systems. It involves less work to take the pole where the least angles have been observed in cases which permit of choice

In a completed quadrilateral where the angles are measured independently, it is best to take the pole at the vertex of the three triangles giving the angle equations; in other cases where the adjacent angles are used giving four to a triangle the pole is conveniently taken at the intersection of the two diagonals.

The number of angle equations was found in (196), each side requiring to be sighted over in both directions.

In a chain of triangles where two bases have been measured, both being regarded perfect, the absolute term of the side equation becomes the ratio of the bases instead of unity. If the discrepancy in length is not all thrown into the angles the p.e. of the computed base can be found by § 162. The discrepancy can then be divided

* Alterations are made in last place in order to close the equations. They are necessitated by omitting the third places.

in the ratio of the p.e. squared, and the portion thrown into the angles used as the absolute term in the equation as above.

It is best in making up the equations in any difficult case to start with a triangle or even with a side of the net, add a point and draw all the lines connecting it with the first points, then apply the formulas for the number of angle and side equations and write them out. All equations added by the lines radiating from the point must involve the point. Add a second point, draw the radiating lines, apply the formulas, and add the new equations as before. In this way it is comparatively easy to distribute the equations among the proper stations and avoid an excess in one group, which would mean an excess of labor, or a deficiency in another, which would mean an imperfect adjustment.

161. Adjustment of Secondary to Primary Work.—The primary work having been adjusted by itself, the entire discrepancy would be thrown into the secondary. This would be accomplished

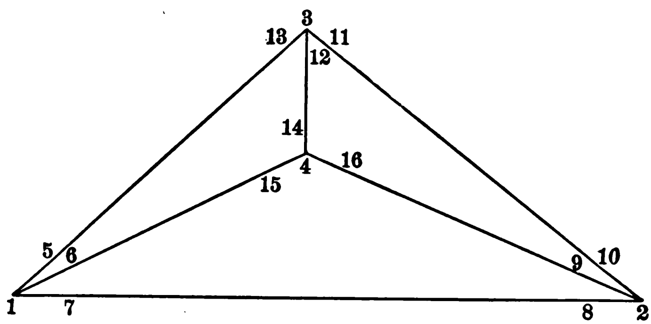

FIG. 100.

by placing the correction to the adjusted or perfect angle, or its v, equal zero, so that the term containing it would disappear from (155).

Thus in Fig. 100 we have given the angles of the primary triangle 1–2–3, and those of the secondary triangles 1–2–4, 2–3–4, 3–4–1, derived as differences of direction.

Angle equations:

$$\begin{array}{ll} \text{Tri. } 1\text{–}2\text{–}4, & -(6)-(16)+(15)+(9)+q_1=0. \\ 2\text{–}4\text{–}3, & -(9)-(14)+(16)+(12)+q_2=0. \\ 1\text{–}3\text{–}4, & +(6)-(12)-(15)+(14)+q_3=0. \end{array}$$

Side equation, pole at station 4:

$$\frac{1\text{–}4}{2\text{–}4}\cdot\frac{2\text{–}4}{3\text{–}4}\cdot\frac{3\text{–}4}{1\text{–}4}=1,$$

or

$$\frac{\sin 1\text{–}2\text{–}4}{\sin 4\text{–}1\text{–}2}\cdot\frac{\sin 2\text{–}3\text{–}4}{\sin 4\text{–}2\text{–}3}\cdot\frac{\sin 4\text{–}1\text{–}3}{\sin 4\text{–}3\text{–}1}=1.$$

Expressing the corrections as in § 157,

$$d_1(9)+d_2(6)+d_3(12)+d_4(9)+d_5(6)+d_6(12)+q_4=0$$

From these the corrections can be derived as usual.

If a secondary chain connects at each end with a primary side, and in many other cases, the checks due to the connection are often brought in as a side equation, azimuth equation, latitude equation, and longitude equation, thus making the computed side of the same length as, parallel to, and coinciding with, the primary side.

162. Probable Error of Any Side.—In Ex. 1 of § 139 it was found that

$$\epsilon_{a_1}{}^2=\tfrac{2}{3}a_1{}^2\sin^2 1''(\cot^2 A_1+\cot^2 B_1+\cot A_1\cot B_1)\epsilon^2+\epsilon_b{}^2\frac{a_1{}^2}{b^2}.$$

Similarly for the next side, using p.e. instead of m.s.e.:

$$r_{a_2}{}^2=\tfrac{2}{3}a_2{}^2\sin^2 1''\,(\cot^2 A_2+\cot^2 B_2+\cot A_2\cot B_2)r^2+r_{a_1}{}^2\frac{a_2{}^2}{a_1{}^2}.$$

Substituting,

$$r_a{}^2=\tfrac{2}{3}a^2\sin^2 1''[\cot^2 A+\cot^2 B+\cot A\cot B_2]r^2+r_b{}^2\frac{a^2}{b^2}.\quad . \quad . \quad (198)$$

This will give the p.e. of any side in a chain of triangles due to the p.e. of the base and the p.e. of the intervening angles. If, as is quite customary, the base is considered exact the last term will disappear. A and B are the distance angles used in computing the side a, and r is the average p.e. for the adjusted angles. If more than one chain of triangles can be used the one giving the smallest p.e. should be taken.

If in Ex. 1 of § 139 the log equation for a had been used, the differential coefficients $a\cot M_1$ and $-a\cot M_2$ would have been $d_A/\sin 1''$ and $-d_B/\sin 1''$, as found in § 157.

Substituting in (198) and calling the base exact,

$$r^2_{\log a}=\tfrac{2}{3}r^2[d_A{}^2+d_B{}^2+d_Ad_B],\quad . \quad . \quad . \quad . \quad . \quad (199)$$

where d_A and d_B are the tabular log sin differences for 1″ for the angles A and B.

If the average p.e. r_1 for an adjusted direction is used (199) becomes

$$r^2_{\log a} = \tfrac{4}{3} r_1^2 [d_A^2 + d_B^2 + d_A d_B]. \quad . \quad . \quad . \quad . \quad (200)$$

To find the p.e. of the adjusted directions requires considerable extra labor in connection with the solution of the normal equations, as shown in the problem of § 133. On this account it is more convenient to determine the average value of the adjusted p.e. from that of the observed from the relation developed in § 138. The average adjusted p.e. squared equals the average observed p.e. squared multiplied by the ratio of the number of independent unknowns to the number of observed quantities; i.e., if d = average p.e. for the observed direction, N_d = number of directions observed and N_c = number of conditions or checks to be satisfied.

$$r_1^2 = d^2 \frac{N_d - N_c}{N_d}.$$

Substituting,

$$r^2_{\log a} = \tfrac{4}{3} d^2 \frac{N_d - N_c}{N_d} [d_A^2 + d_B^2 + d_A d_B], \quad . \quad . \quad . \quad (201)$$

as assumed in (1) of § 9.

163. Approximate Adjustment for Azimuth.—An azimuth equation may be due to joining new work at two points with a common side at each to a triangulation which has already been adjusted, or it may be formed in adjusting between astronomically observed azimuths. Strictly the azimuth equation should be included with the others in the figure adjustment, but much labor is saved, and often sufficient accuracy attained, by considering it separately after the first adjustment has been made.

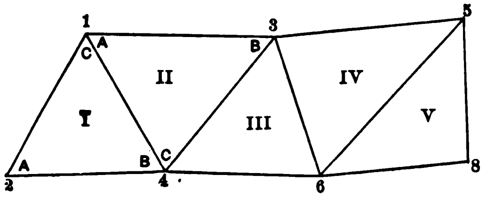

FIG. 101.

In Fig. 101 the distance angles used in computing 6–8 from 1–2 are marked A and B. The azimuth should be computed through the C angles. Counting azimuth clockwise as usual, the azimuth of 1–4 would be found from that of 1–2 by subtracting C_1. The azimuth of 4–1 would be found from that of 1–4 by (85). The azimuth of 4–3 can be found from that of 4–1 by adding C_2, etc. If q_z = computed azimuth of 6–8 less the observed or direct value, we shall have, since the angles have already been adjusted to close the triangles,

AZIMUTH EQUATION.

$$-(C_1)+(C_2)-(C_3)\ldots+q_z=0.$$

ANGLE EQUATIONS.

$$(A_1)+(B_1)+(C_1)=0$$
$$(A_2)+(B_2)+(C_2)=0$$
$$\cdots\cdots\cdots$$

for n triangles.

Forming the normal equations as usual,

$$\begin{aligned} nA-B+C-D\ldots+q_z&=0\\ -A+3B\qquad&=0\\ +A+3C\qquad&=0\\ \cdots\cdots\cdots& \end{aligned}$$

Finding the values of the correlatives B, C, ..., in terms of A and substituting in the first equation,

$$nA-\tfrac{1}{3}A-\tfrac{1}{3}A-\ldots+q_z=0,$$

or,

$$nA-n\tfrac{1}{3}A\qquad+q_z=0.$$

Solving,

$$A=-\frac{3q_z}{2n},\ B=-\frac{q_z}{2n},\ C=+\frac{q_z}{2n},\ D=-\frac{q_z}{2n},\ldots$$

Corrections (159):

$$(A_1)=-\frac{q_z}{2n},\ (B_1)=-\frac{q_z}{2n},\ (C_1)=+\frac{q_z}{n},$$

$$(A_2)=+\frac{q_z}{2n},\ (B_2)=+\frac{q_z}{2n},\ (C_2)=-\frac{q_z}{n},$$

$$\cdots\cdots\cdots\cdots,$$

i.e., divide the excess of the computed over the observed azimuth by the number of the triangles and apply one-half of this quantity to each of the angles used in computing distance through the chain, and the total quantity, with the sign changed, to the third angle, the latter being so applied each time as to reduce the discrepancy.

164. Approximate Adjustment Between Bases.—This equation should be added with the others in the figure adjustment, but frequently it is omitted until the other adjustment has been made in order to see how closely the bases will check, or the check base may not have been measured until the figure adjustment has been completed. In such cases the base adjustment can be made separately as below.

BASE EQUATION.

$$[d_A(A)-d_B(B)]+q_b=0,$$

where, as in Fig. 101, the A angles are opposite the required sides and the B angles opposite the known ones in passing from the first to the second base, d_A and d_B are the differences of the log sines of the angles for 1″, and q_b is the discrepancy in the logs of the bases when the observed values are substituted.

ANGLE EQUATIONS.

$$(A_1)+(B_1)+(C_1)=0,$$
$$(A_2)+(B_2)+(C_2)=0,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

for n triangles.

NORMAL EQUATIONS.

$$([d_A{}^2]+[d_B{}^2])A+(d_{A_1}-d_{B_1})B+(d_{A_2}-d_{B_2})C+\,.\,.+q_b=0.$$
$$(d_{A_1}-d_{B_1})A+3B \qquad =0.$$
$$(d_{A_2}-d_{B_2})A+3C \qquad =0.$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

From the second equation,

$$B=-\tfrac{1}{3}(d_{A_1}-d_{B_1})A.$$

From the third equation,

$$C=-\tfrac{1}{3}(d_{A_2}-d_{B_2})A.$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Substituting in the first,

$$A=-\frac{3q_b}{2\,[d_A{}^2+d_Ad_B+d_B{}^2]}.$$

Substituting in (159),

$$(A_1)=(2d_{A_1}+d_{B_1})\frac{A}{3}.$$

$$(B_1)=-(d_{A_1}+2d_{B_1})\frac{A}{3}.$$

$$(C_1)=-(d_{A_1}-d_{B_1})\frac{A}{3}.$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

The corrections to the C angles will tend to foot up zero, the differences for 1″ for the A and B angles averaging about equal in a triangulation. The disturbance in the azimuth adjustment will thus be small.

By calling the C corrections zero ($d_A = d_B$) the angle equations become

$$(A_1)+(B_1)=0,$$
$$(A_2)+(B_2)=0, \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

NORMAL EQUATIONS.

$$([d_A^2]+[d_B^2])A+(d_{A_1}-d_{B_1})B+(d_{A_2}-d_{B_2})C+\ldots+q_b=0,$$
$$(d_{A_1}-d_{B_1})A+2B \qquad =0,$$
$$(d_{A_2}-d_{B_2})A+2C \qquad =0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot ,$$

from which

$$B=-(d_{A_1}-d_{B_1})\frac{A}{2}, \quad C=-(d_{A_2}-d_{B_2})\frac{A}{2}, \ldots$$

$$A=-\frac{2q_b}{[d_A^2+2d_Ad_B+d_B^2]}.$$

Substituting in (159),

$$(A_1)=(d_{A_1}+d_{B_1})\frac{A}{2},$$

$$(B_1)=-(d_{A_1}+d_{B_1})\frac{A}{2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

These corrections when applied will not disturb the azimuth adjustment, so that the length and direction of any line will be the same computed from either end of the chain.

165. Adjustment for Latitude and Longitude.—In joining new work to old adjusted work at two points, as in filling in secondary triangulations, the junction side computed through the new work must be parallel to the old (azimuth equation), must have the same length (base line equation), and must coincide in position at one end, which is best effected by a latitude and longitude equation. This last can be introduced in the figure adjustment, but the discrepancy in good work will be so small that the equation can be omitted in the first adjustment, and the error in latitude and longitude distributed as in a land survey without serious loss of accuracy. Each station can then be reduced to center by § 59, making the figure consistent throughout.

For the development of the latitude and longitude equations in form for use with the others in forming the normal equations, reference is made to Wright and Hayford's *Adjustment of Observations*, §§ 184, 185.

166. Adjustment of Levels.—The three methods described in §§ 80 and 81 for finding the difference in level trigonometrically are from non-simultaneous readings at the two stations, from simultaneous readings, and from readings at one of the stations only. The corresponding approximate formulas are:

$$h_1 = s \tan \tfrac{1}{2}(z_2 - z_1) + (m_2 - m_1)\frac{s^2}{2R_a},$$

$$h_2 = s \tan \tfrac{1}{2}(z_2 - z_1),$$

$$h_3 = s \cot z_1 + (1 - 2m_1)\frac{s^2}{2R_a},$$

where s = horizontal distance; z_1, z_2 = observed zenith distances; m_1, m_2 = coefficients of refraction; R_a = radius of curvature of the arc joining the two stations.

The p.e. for each result can be found as in § 122, remembering that s is well known and that z is nearly 90°.

Thus

$$r_{h_1}^2 = \tfrac{1}{2}s^2 \sin^2 1'' r_z^2 + \frac{s^4 r_m^2}{2R_a^2}. \quad . \quad . \quad . \quad . \quad (202)$$

$$r_{h_2}^2 = \tfrac{1}{2}s^2 \sin^2 1'' r_z^2 + \frac{s^4 r_m^2}{2R_a^2}. \quad . \quad . \quad . \quad . \quad (203)$$

$$r_{h_3}^2 = s^2 \sin^2 1'' r_z^2 + \frac{s^4 r_m^2}{R_a^2}. \quad . \quad . \quad . \quad . \quad (204)$$

It should be remembered that in the third method there will usually be a determination from the other end of the line. The mean of the two will give the same value for r_h^2 as the others. It is usually assumed that placing $m_1 = m_2$ in the second method does away with the uncertainty due to refraction. If this assumption is accepted the last term in (203) will disappear, leaving $r_{h_2}^2$ the smallest of the three.

In adjusting a triangulation net, the algebraic sum of the differences in height in going around a triangle must equal zero, corresponding to the angle equations and giving for the number the same as for the number of angle equations, $l-p+1$.

There will usually be enough reciprocal observations so that the value of m can be computed for the lines observed at each station, assigning weights to each reciprocal set by Bessel's empirical formula,

$$\frac{n_1 n_2 \sqrt{s}}{n_1+n_2},$$

where n_1, n_2 are the numbers of observations for z_1, z_2.

The weights to be given to the differences in height would be the reciprocals of the r_h^2 found above.

In adjusting a net of level circuits run as precise levels the same method can be used as for trigonometric levels. The difficulty is in assigning weights. In closing circuits the discrepancies are usually considerably greater than indicated from the discrepancies in rerunning short sections between temporary benchmarks, thus indicating systematic error.

Accidental errors tend to accumulate as the square root of the lengths, giving weights inversely as lengths of circuit. Systematic errors tend to accumulate as the length, giving weights inversely as the squares of the lengths.

For examples of adjustment see App. 8, Coast Survey *Report* 1899, and App. 3 for 1903.

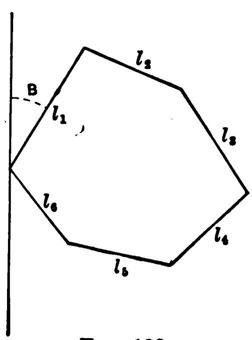

FIG. 102.

167. Adjustment of a Compass Survey.—For each side the length and bearing are directly measured, while the latitude and departure are computed.

The latitude equation is

$$L=l\cos B. \quad (a)$$

$$\therefore \frac{dL}{dl}=\cos B, \quad \frac{dL}{dB}=-l\sin B. \quad (b)$$

By § 122,

$$r_L^2=r_l^2\cos^2 B+r_B^2 l^2\sin^2 B. \quad (c)$$

For the departure

$$D=l\sin B. \quad (d)$$

$$\frac{dD}{dl}=\sin B, \quad \frac{dD}{dB}=l\cos B. \quad (e)$$

$$\therefore r_D^2=r_l^2\sin^2 B+r_B^2 l^2\cos^2 B. \quad (f)$$

If we assume, as was practically done by Dr. Bowditch, that

$$r_l^2 = r_B^2 \, l^2 = l \times \text{constant} = lC, \quad . \quad . \quad . \quad . \quad (205)$$

(*c*) and (*f*) will reduce to

$$r_L^2 = r_D^2 = lC, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (206)$$

i.e., the squares of the p.e.'s in latitude and in departure are each proportional to the lengths of the sides.

In § 150 it is shown that the most probable corrections will be those which make the sum of the squares of the quotients found by dividing each correction by its p.e. a minimum. Hence denoting the corrections in latitude for the different sides by $v_{L_1}, v_{L_2}, v_{L_3}, \ldots,$

$$\left[\frac{v_L^2}{l}\right] = \text{minimum}.$$

Differentiating,

$$dv_{L_1}\frac{v_{L_1}}{l_1} + dv_{L_2}\frac{v_{L_2}}{l_2} + \ldots = 0. \quad . \quad . \quad . \quad . \quad (207)$$

The sum of the corrections must equal the total error in latitude with its sign changed, $= -q_1$, a constant, i.e.,

$$v_{L_1} + v_{L_2} + v_{L_3} + \ldots = q_1.$$

Differentiating,

$$dv_{L_1} + dv_{L_2} + dv_{L_3} + \ldots = 0. \quad . \quad . \quad . \quad . \quad (208)$$

Comparing (207) and (208), and remembering that each must hold whatever the number of sides, or v's,

$$\frac{v_{L_1}}{l_1} = \frac{v_{L_2}}{l_2} = \frac{v_{L_3}}{l_3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

i.e., *the corrections in latitude are proportional to the lengths of the sides*, according to the Dr. Bowditch rule.

The same can be found for the corrections in departure, giving

$$\left.\begin{array}{r} v_{D_1} + v_{D_2} + v_{D_3} + \ldots = q_2 \\ \dfrac{v_{D_1}}{l_1} = \dfrac{v_{D_2}}{l_2} = \dfrac{v_{D_3}}{l_3} \ldots \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (210)$$

If the corrections are required for computing area, they can be applied directly to the values in the latitude and departure columns;

but if they are required for a geometrically consistent map or record the corresponding corrections must be found for the distances and bearings. This can be done by dividing the corrected departure by corrected latitude for the tangent of the corrected bearing, then dividing departure by sine and latitude by cosine for the corrected distance, giving weight to the value having the larger numerator and using the other as a check. This requires the use of as many decimal places as the original computation.

From the differential equations (*b*) and (*e*), the total correction to the side

$$\left.\begin{aligned} dl &= \frac{dL}{\cos B} + \frac{dD}{\sin B} \\ &= dL\frac{l}{L} + dD\frac{l}{D} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

The total correction to the bearing

$$dB = -\frac{dL}{D} + \frac{dD}{L},$$

or, in minutes, (212)

$$dB' = -\frac{dL}{D \sin 1'} + \frac{dD}{L \sin 1'}$$

Equations (211) and (212) are readily computed with a slide rule, or even by inspection from the coordinate sheet.

In equation (205) an uncertainty in chaining which would amount to 1 ft. in 500 would give

$$r_{500} = 1 = \sqrt{500C}, \quad \text{or} \quad \sqrt{C} = 0.0447.$$

$$\therefore \quad r_l = 0.0447\sqrt{l}, \quad r_B \text{ (in minutes)} = \frac{0.0447}{\sqrt{l} \sin 1'}.$$

Substituting for different distances:

Distance.	Uncertainty in Chaining.	Uncertainty in Bearing.
10 feet	0.14 feet	0° 48′
50	0.32	0 21
100	0.45	0 15
500	1.00	0 07
1000	1.41	0 05
2000	2.00	0 03

An examination of these results shows that the assumption of Dr. Bowditch is fairly reasonable, although it gives too great weight to the bearings of long lines, and perhaps too small to those of very short ones.

168. Adjustment of a Transit Survey.—In an ordinary transit survey no bearings are observed, but the horizontal angles between the lines are measured. In computing coordinates a meridian is observed or assumed and the bearings found from the angles. To express these bearings in terms of the measured angles in the adjustment equations, as should be done for accuracy, involves too much labor. To use them as observed quantities will give different bearings and different coordinates, depending upon the direction taken around the figure for each in case the angles do not "close."

In ordinary work the p.e. of an angle need not exceed 1 minute if care is taken in setting over the points and in plumbing the flag poles, using tacks on the stakes for all lines of less than 300 ft., swinging without delay from the back sight to the front sight, and lining in a "range" point to swing from for all lines of less than 50 ft. With these precautions the p.e. need not increase with the shortness of the line, as with the compass, for which it is a waste of time to guard against small errors of eccentricity in setting up or flagging.

On very rough ground, or in going through brush, where the flag pole is partly hidden, it may be difficult to keep the p.e. below 2 minutes, while for careful work the p.e. can be readily kept within ½ minute.

For good work the length of sight should be limited to about 1200 ft. Ordinarily it will be more difficult to measure distances to 1:500 than angles to minutes, while an acccuracy of 1:1000 is seldom reached except on level ground or in city work.

The accuracy of angle work is thus considerably greater than that of chaining, 1 minute in angle giving 0.15 ft. in 500 as compared with 1 ft. in chaining; or 0.5 minute, 0.15 ft. in 1000 as compared with the 1 ft. due to the more accurate chaining.

On this account it will be admissible to adjust the angles to close the figure (i.e., so that the sum of the interior angles shall equal twice as many right angles, less four, as the figure has sides) by distributing the error equally among the angles to the nearest ½ or ¼ minute if they all have been equally well measured, or concentrating the corrections somewhat upon the poorer angles if not equally well measured. The bearings or azimuths are then computed and assumed to be correct in the final adjustment. This leaves only the two conditions:

Sum of latitudes equal zero;

Sum of departures equal zero;

that is,

$$\left.\begin{array}{l} l_1 \cos B_1 + l_2 \cos B_2 + \ldots = 0, \\ l_1 \sin B_1 + l_2 \sin B_2 + \ldots = 0, \end{array}\right\} \quad . \quad . \quad . \quad (213)$$

where the total corrections are to be applied upon the basis of inaccuracy in chaining.

Denote the observed distances by M_1, M_2, M_3, ..., and the required corrections by v_1, v_2, v_3, ... The corrected distances will be

$$l_1 = M_1 + v_1, \quad l_2 = M_2 + v_2, \quad l_3 = M_3 + v_3 \ldots$$

Substituting in (213),

$$(M_1 + v_1) \cos B_1 + (M_2 + v_2) \cos B_2 + \ldots = 0,$$
$$(M_1 + v_1) \sin B_1 + (M_2 + v_2) \sin B_2 + \ldots = 0,$$

or

$$\left.\begin{array}{l} v_1 \cos B_1 + v_2 \cos B_2 + \ldots + q_1 = 0, \\ v_1 \sin B_1 + v_2 \sin B_2 + \ldots + q_2 = 0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (214)$$

where $q_1 = M_1 \cos B_1 + M_2 \cos B_2 + \ldots =$ error in latitude,
$q_2 = M_1 \sin B_1 + M_2 \sin B_2 + \ldots =$ error in departure.

For convenience change (214) to

$$\left.\begin{array}{l} v_1 \dfrac{L_1}{l_1} + v_2 \dfrac{L_2}{l_2} + \ldots + q_1 = 0, \\ v_1 \dfrac{D_1}{l_1} + v_2 \dfrac{D_2}{l_2} + \ldots + q_2 = 0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad (215)$$

where L_1, L_2, ..., D_1, D_2, ..., denote latitudes and departures.

If r^2 for chaining increase as l, or the weights inversely as l, the normal equations (158) become

$$\left[\frac{L^2}{l}\right] A + \left[\frac{LD}{l}\right] B + q_1 = 0.$$

$$\left[\frac{LD}{l}\right] A + \left[\frac{D^2}{l}\right] B + q_2 = 0.$$

Solving,

$$\left.\begin{aligned} A&=\frac{q_2\left[\frac{LD}{l}\right]-q_1\left[\frac{D^2}{l}\right]}{\left[\frac{D^2}{l}\right]\left[\frac{L^2}{l}\right]-\left[\frac{LD}{l}\right]^2},\\ B&=\frac{q_1\left[\frac{LD}{l}\right]-q_2\left[\frac{L^2}{l}\right]}{\left[\frac{D^2}{l}\right]\left[\frac{L^2}{l}\right]-\left[\frac{LD}{l}\right]^2}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (216)$$

(159) become

$$\left.\begin{aligned} v_1&=L_1A+D_1B\\ v_2&=L_2A+D_2B\\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217)$$

Adding,

$$[v]=A[L]+B[D]=0, \text{ nearly.}$$

Also,

$$\left.\begin{aligned} v_{L_1}&=v_1\frac{L_1}{l_1}=A\frac{L_1^2}{l_1}+B\frac{L_1D_1}{l_1},\\ v_{L_2}&=v_2\frac{L_2}{l_2}=A\frac{L_2^2}{l_2}+B\frac{L_2D_2}{l_2}.\\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}\right\}$$

$$\left.\begin{aligned} v_{D_1}&=v_1\frac{D_1}{l_1}=A\frac{D_1L_1}{l_1}+B\frac{D_1^2}{l_1},\\ v_{D_2}&=v_2\frac{D_2}{l_2}=A\frac{L_2D_2}{l_2}+B\frac{D_2^2}{l_2},\\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \end{aligned}\right\}$$

with $[v_L]=-q_1$, and $[v_D]=-q_2$.

If the inaccuracy in chaining increases directly with the distance (r varying as l), or the weights inversely as l^2, (158) become

$$\left.\begin{aligned} [L^2]A+[LD]B+q_1&=0,\\ [LD]A+[D^2]B+q_2&=0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (218)$$

with

$$\left.\begin{aligned} A&=\frac{q_2[LD]-q_1[D^2]}{[D^2][L^2]-[LD]^2},\\ B&=\frac{q_1[LD]-q_2[L^2]}{[D^2][L^2]-[LD]^2}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (219)$$

$$\left.\begin{array}{l} v_1 = L_1 l_1 A + D_1 l_1 B \\ v_2 = L_2 l_2 A + D_2 l_2 B. \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\}$$

In order to equalize coefficients so as to retain the same number of decimal places throughout, $100l$ is used in place of l in (216), making the values of A and B 100 times too great and requiring the values of v to be divided by 100.

If it is assumed that the error in chaining increases directly with the distance, (205) may be changed to

$$r_l = r_B l = l \times \text{constant} = lC, \quad \ldots \ldots \quad (220)$$

which changes

$$(206) \text{ to } r_L^2 = r_D^2 = l^2C, \quad \ldots \ldots \quad (221)$$

$$(207) \text{ to } \frac{dv_{L_1}v_{L_1}}{l_1^2} + \frac{dv_{L_2}v_{L_2}}{l_2^2} + \ldots = 0, \quad \ldots \quad (222)$$

$$\left.\begin{array}{l} (209) \text{ to } \dfrac{v_{L_1}}{l_1^2} = \dfrac{v_{L_2}}{l_2^2} = \dfrac{v_{L_3}}{l_3^2} \ldots, \\ (210) \text{ to } \dfrac{v_{D_1}}{l_1^2} = \dfrac{v_{D_2}}{l_2^2} = \dfrac{v_{D_3}}{l_3^2} \ldots, \end{array}\right\} \quad \ldots \ldots \quad (223)$$

i.e., the corrections in latitude and the corrections in departure are each proportional to the squares of the sides.

An examination of (220) shows that an error of 1:500 in distance will give

$$1/500 = r_B,$$

or

$$r_B' = \frac{1}{500 \times 0.00029} = 7',$$

or for 1/1000,

$$r_B' = 3'.5.$$

These ratios are more reasonable for transit work than those tabulated from (205), but it would require an accuracy of 1/10 000 in chaining, or the best grade of level-ground city work, to reduce the corresponding angle error to a value easily attained in ordinary transit work, unless the figure has a very large number of sides.

In this method the error of closure of the angles would first have to be distributed before computing the coordinates the same as by the first method.

Ex. The following field measurements were made with transit and tape: Sta. 1, 44° 38′.5*R*, 287.24 ft.; sta. 2, 8° 04′*R*, 451.75 ft.; sta. 3, 129° 17′.5*R*, 921.60 ft.; sta. 4, 89° 26′*R*, 212 ft.; sta. 5, 2° 35′.5*L*, 317.3 ft.; sta. 6 91° 09′.5*R*, 443.6 ft.

The deflections foot up 360°, requiring no adjustment for angle closure. The line 6–1 is nearly north and south and it is taken for the meridian.

In computing the coordinates, columns are added for $L^2/100l$, $D^2/100l$, $LD/100l$, made up with slide rule from the distances and coordinates as given below.

Station.	Bearing.	Distance.	Latitude, L. +	Latitude, L. −	Departure, D. +	Departure, D. −	$\frac{L^2}{100l}$	$\frac{D^2}{100l}$	$\frac{LD}{100l}$
1	N 44° 38′.5 E	287.24	204.37		201.83		1.45	1.42	1.44
2	N 52 42′.5 E	451.75	273.70		359.40		1.65	2.85	2.18
3	S 2 00 W	921.60		921.04		32.16	9.20	0.01	0.32
4	N 88 34 W	212.	5.30			211.93	0.00	2.11	−0.05
5	S 88 50 .5 W	317.3		6.41		317.24	0.01	3.17	0.06
6	North	443.6	443.60				4.44	0.00	0.00
	Totals.....	......	926.97	927.45	561.23	561.33	16.75	9.56	3.95
				926.97		561.23			
			$q_1 =$	−0.48	$q_2 =$	−0.10			

$$A = \frac{-0.10 \times 3.95 + 0.48 \times 9.56}{160.13 - 15.60} = +0.029.$$

$$B = \frac{-0.48 \times 3.95 + 0.10 \times 16.75}{160.13 - 15.60} = -0.001.$$

$v_1 = +0.05$	$v_{L_1} = +0.04$	$v_{D_1} = +0.04$
$v_2 = +0.08$	$v_{L_2} = +0.05$	$v_{D_2} = +0.06$
$v_3 = -0.26$	$v_{L_3} = +0.26$	$v_{D_3} = +0.01$
$v_4 = 0.00$	$v_{L_4} = 0.00$	$v_{D_4} = 0.00$
$v_5 = 0.00$	$v_{L_5} = 0.00$	$v_{D_5} = 0.00$
$v_6 = +0.13$	$v_{L_6} = +0.13$	$v_{D_6} = 0.00$
$[v] = 0.00$	$[v_L] = +0.48 = -q_1$	$[v_D] = +0.11 = -q_2$

If any line is regarded as perfect, as in connecting with a survey already adjusted, the corresponding correction is made zero and the corresponding $L^2/100l$, $D^2/100l$, and $LD/100l$ omitted in the summation for A and B.

TABLES.

TABLE I.

FORMULAS AND CONSTANTS.

$\sin^2 x+\cos^2 x=1$. 1]

$\tan x=\frac{1}{\cot x}=\frac{\sin x}{\cos x}=\sqrt{\sec^2 x-1}$. 2]

$\sin(x\pm y)=\sin x\cos y\pm\cos x\sin y$ 3]

$\cos(x\pm y)=\cos x\cos y\mp\sin x\sin y$. 4]

$\tan(x\pm y)=\frac{\tan x\pm\tan y}{1\mp\tan x\tan y}$. 5]

$\cos(180^\circ-y)=-\cos y$; $\sin(180^\circ+y)=-\sin y$. 6]

$\sin x=\tan x=x''\sin 1''=x''\text{ arc }1''$ (for small angles). 7]

$\sin x\pm\sin y=2\sin\frac{1}{2}(x\pm y)\cos\frac{1}{2}(x\mp y)$. 8]

$\cos(x+y)+\cos(x-y)=2\cos x\cos y$. 9]

$\sin 2x=2\sin x\cos x$. 10]

$2\cos^2\frac{1}{2}x=1+\cos x$. 11]

$2\sin^2\frac{1}{2}x=1-\cos x$. 12]

$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\ldots$ 13]

$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\ldots$ 14]

$\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\frac{1382x^{11}}{155925}+\ldots$ 15]

$\text{arc}\sin x=x+\frac{x^3}{6}+\frac{3x^5}{40}+\frac{5x^7}{112}+\frac{35x^9}{1152}+\frac{63x^{11}}{2816}+\ldots$ 16]

$\text{arc}\tan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\frac{x^{11}}{11}+\ldots$ 17]

(In the equations 13]–17] inclusive x is in π-measure. Should x be given in seconds, multiply by $\sin 1''$.)

PLANE OBLIQUE TRIANGLES.

$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$. 18]

$a^2=b^2+c^2-2bc\cos A$. 19]

$\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\tan\frac{1}{2}(A+B)$. 20]

Area of a triangle $=\frac{1}{2}bc\sin A$. 21]

ELLIPSE AND MERIDIAN SECTION.

$e=\frac{\sqrt{a^2-b^2}}{a}$. 22]

$c=\frac{a-b}{a}$. 23]

$r^2=1-e^2\sin^2\phi$. 24]

$V^2=1+e'^2\cos^2\phi=\frac{r^2}{1-e^2}$. 25]

TABLE I.—(*Continued*).

SPHERICAL OBLIQUE TRIANGLES.

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}. \qquad 26]$$

$$\cos a=\cos b \cos c+\sin b \sin c \cos A. \qquad 27]$$

$$\tan \tfrac{1}{2}(A+B)=\cot \frac{C}{2}\cdot\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}. \qquad 28]$$

$$\cot B=\frac{\sin c \cot b-\cos c \cos A}{\sin A}. \qquad 29]$$

Area of spherical triangle $=A=\frac{s}{90^\circ}\cdot\frac{1}{2}\pi r^2$ (s = spherical excess). . . . 30]

SPHERICAL RIGHT TRIANGLE.

$$\tan A \sin b=\tan a. \qquad 31]$$

BINOMIAL THEOREM.

$$(a+b)^m=a^m+ma^{m-1}b+\frac{m(m-1)}{2!}a^{m-2}b^2+. \qquad 32]$$

MACLAURIN'S THEOREM.

$$u'=f(x)=(u)_{x=0}+\left(\frac{du}{dx}\right)_{x=0}\frac{x}{1!}+\left(\frac{d^2u}{dx^2}\right)_{x=0}\frac{x^2}{2!}+\dots\left(\frac{d^nu}{dx^n}\right)_{x=0}\frac{x^n}{n!}. \qquad 33]$$

TAYLOR'S THEOREM.

$$u'=f(x+y)=u+\frac{du}{dx}\cdot\frac{y}{1!}+\frac{d^2u}{dx^2}\cdot\frac{y^2}{2!}+\dots\frac{d^nu}{dx^n}\cdot\frac{y^n}{n!}. \qquad 34]$$

RADIUS OF CURVATURE.

$$R=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}\Big/\frac{d^2y}{dx^2}. \qquad 35]$$

$$\log(1+x)=M\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\dots\right). \qquad 36]$$

$$\log(1-x)=-M\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\dots\right). \qquad 37]$$

A. R. Clarke of the English Ordnance Survey gives the following values for the ellipsoid of revolution as found from the various degree measurements. These values were adopted in 1875 by the U. S. Coast and Geodetic Survey, and those of the following tables which involve the ellipsoid are based upon these data.

Semi-major axis........	$a=6\,378\,206^m.4$	log 6.804 6985
Semi-minor axis........	$b=6\,356\,583\;.8$	" 6.803 2238
Square of eccentricity...	$e^2=0.006\,768\,658$	" 7.830 5026

One meter = 39.37 inches (by Act of Congress).

The following are approximate values ef the coefficients of expansion for 1° F., the unit being 0.000 001 of the length:

Glass	4.7	Zinc	16.1
Platinum	4.8	Nickel-iron (Coast Survey Precise Level)	2.2
Steel	6.2	Nickel-steel (Coast Survey Base Tapes)	0.17
Iron	6.5		
Brass	10.2		

Table II.

LOGARITHMS OF FACTORS *A*, *B*, *C*, *D*, *E*, *F*, FOR LATITUDE, LONGITUDE, AND AZIMUTH COMPUTATIONS. (See Chapter VII.)

The distance or triangle side should be used in meters.

Latitude.		Log *A*.	Log *B*.	Log *C*.	Log *D*.	Log *E*.	Log *F*.
0°	00′	8.5097266	8.5126761	−∞	−∞	5.6125	−∞
	20	265	760	9.1717	0.457	125	6.057
	40	264	756	9.4728	0.758	126	358
1	00	261	748	9.6489	0.934	128	534
	20	258	738	9.7738	1.059	132	658
	40	253	724	9.8708	1.156	136	755
2	00	248	708	9.95002	1.2347	140	834
	20	241	688	0.01703	1.3015	146	901
	40	234	666	0.07509	1.3594	153	6.959
3	00	225	641	0.12633	1.4104	160	7.010
	20	216	612	0.17217	1.4560	168	055
	40	206	581	0.21367	1.4971	178	096
4	00	194	547	0.25156	1.5347	188	133
	20	182	510	0.28644	1.5692	199	168
	40	168	470	0.31875	1.6011	210	200
5	00	154	427	0.34885	1.6308	223	229
	20	139	381	0.37703	1.6586	236	256
	40	122	332	0.40351	1.6846	251	282
6	00	105	280	0.42850	1.7090	266	306
	20	087	225	0.45216	1.7322	282	329
	40	068	167	0.47462	1.7541	299	351
7	00	047	107	0.49600	1.7749	316	371
	20	026	8.5126043	0.51641	1.7946	335	391
	40	8.5097004	8.5125977	0.53593	1.8135	354	409
8	00	8.5096981	907	0.55464	1.8315	374	427
	20	957	835	0.57260	1.8488	395	444
	40	932	760	0.58987	1.8653	416	461
9	00	906	682	0.60652	1.8812	439	476
	20	879	602	0.62257	1.8964	462	490
	40	851	518	0.63808	1.9111	486	505
10	00	822	432	0.65309	1.9253	511	518
	20	793	343	0.66762	1.9389	536	532
	40	762	251	0.68171	1.9521	563	544
11	00	730	156	0.69539	1.9648	590	556
	20	698	8.5125058	0.70867	1.9771	618	568
	40	665	8.5124958	0.72159	1.9890	646	580
12	00	630	855	0.73417	2.0006	675	591
	20	595	749	0.74642	2.0118	705	601
	40	559	641	0.75836	2.0226	736	611
13	00	8.5096522	8.5124530	0.77001	2.0331	5.6767	7.621

TABLE II—(*Continued*).

Latitude.	Log *A*.	Log *B*.	Log *C*.	Log *D*.	Log *E*.	Log *F*.
13° 00′	8.5096522	8.5124530	0.77001	2.0331	5.6767	7.621
20	6484	4416	0.78139	433	799	31
40	6445	4299	0.79251	533	832	40
14 00	6405	4180	0.80337	629	865	49
20	6365	4059	1401	723	900	58
40	6323	3934	2441	814	934	67
15 00	6281	3807	3461	903	5.6970	75
20	6238	3677	4460	2.0990	5.7006	83
40	6194	3545	5439	2.1074	042	91
16 00	6149	3411	6400	156	080	7.698
20	6103	3273	7343	236	118	7.705
40	6056	3133	8269	314	156	12
17 00	6009	2991	0.89178	390	196	19
20	5961	2846	0.90072	464	235	26
40	5912	2699	0952	536	276	32
18 00	5862	2550	1816	606	317	38
20	5811	2397	2667	675	358	44
40	5759	2243	3505	742	400	50
19 00	5707	2086	4330	808	443	56
20	5654	1927	5142	872	486	61
40	5600	1766	5943	934	530	67
20 00	5546	1602	6733	2.1996	574	72
20	5490	1435	7511	2.2055	619	77
40	5434	1267	8279	113	664	82
21 00	5377	1096	9037	170	711	87
20	5320	0924	0.99785	226	757	91
40	5261	0748	1.00524	280	804	7.796
22 00	5202	0571	1.01253	333	851	7.800
20	5142	0391	1974	385	899	04
40	5082	0210	2686	436	948	08
23 00	5020	0026	3390	485	5.7997	12
20	4959	9840	4086	534	5.8046	16
40	4898	9653	4775	581	096	19
24 00	4833	9463	5456	627	146	23
20	4769	9271	6130	672	197	26
40	4704	9077	6797	716	249	29
25 00	4639	8881	7457	759	300	32
20	4573	8684	8111	801	352	35
40	4507	8484	8758	842	405	38
26 00	4439	8283	1.09400	882	458	41
20	4372	8079	1.10036	922	512	44
40	4303	7874	1.10666	960	566	46
27 00	4234	7667	1290	2.2997	620	49
20	4165	7458	1909	2.3033	675	51
40	4094	7248	2523	069	730	53
28 00	8.5094024	8.5117036	1.13132	2.3104	5.8785	7.855

TABLE II—(*Continued*).

Latitude.	Log *A*.	Log *B*.	Log *C*.	Log *D*.	Log *E*.	Log *F*.
28° 00′	8.5094024	8.5117036	1.13132	2.3104	5.8785	7.855
20	3952	6822	3737	137	841	57
40	3881	6607	4337	170	898	59
29 00	3808	6389	4932	203	5.8955	61
20	3735	6171	5522	234	5.9012	63
40	3662	5950	6109	264	069	64
30 00	3588	5729	6692	294	127	66
20	3514	5505	7270	323	186	67
40	3439	5281	7845	351	245	69
31 00	3363	5054	8416	379	304	70
20	3287	4827	8983	405	363	71
40	3211	4598	1.19548	431	423	72
32 00	3134	4368	1.20108	456	484	73
20	3057	4136	0666	481	544	74
40	2980	3903	1220	504	605	75
33 00	2901	3669	1772	527	667	75
20	2823	3433	2321	550	729	76
40	2744	3197	2866	571	791	76
34 00	2665	2959	3409	592	853	77
20	2585	2720	3950	612	916	77
40	2505	2480	4488	632	5.9980	77
35 00	2425	2239	5024	651	6.0043	77
20	2344	1997	5557	669	107	77
40	2263	1754	6088	687	172	77
36 00	2182	1510	6617	704	237	77
20	2100	1265	7145	720	302	77
40	2018	1019	7670	735	367	77
37 00	1936	0772	8193	750	433	76
20	1853	0524	8715	765	499	76
40	1771	0276	9234	779	566	75
38 00	1687	8.5110027	1.29753	792	633	74
20	1604	8.5109777	1.30269	804	701	74
40	1521	9526	0785	816	768	73
39 00	1437	9275	1299	827	836	72
20	1353	9023	1811	838	905	71
40	1269	8771	2323	848	6.0974	70
40 00	1184	8517	2833	857	6.1043	69
20	1100	8264	3342	866	113	67
40	1015	8010	3850	874	183	66
41 00	0930	7755	4358	882	253	64
20	0845	7500	4864	889	324	63
40	0760	7245	5370	895	395	61
42 00	0675	6989	5875	901	467	60
20	0590	6733	6379	906	539	58
40	0504	6477	6883	910	612	56
43 00	8.5090419	8.5106220	1.37386	2.3914	6.1684	7.854

TABLE II—(*Continued*).

Latitude.		Log *A*.	Log *B*.	Log *C*.	Log *D*.	Log *E*.	Log *F*.
43°	00′	8.5090419	8.5106220	1.37386	2.3914	6.1684	7.854
	20	0333	5963	7889	18	758	52
	40	0247	5706	8392	21	831	50
44	00	0162	5449	8894	23	905	48
	20	8.5090076	5192	9396	25	6.1980	45
	40	8.5089990	4935	1.39898	26	6.2055	43
45	00	9904	4677	1.40400	26	130	40
	20	9818	4420	0902	26	206	38
	40	9733	4162	1404	25	283	35
46	00	9647	3905	1906	24	359	32
	20	9561	3648	2409	22	436	30
	40	9475	3391	2911	20	514	27
47	00	9390	3134	3414	17	592	24
	20	9304	2877	3917	13	671	21
	40	9219	2621	4421	09	750	17
48	00	9133	2364	4926	2.3904	830	14
	20	9048	2108	5431	2.3898	910	11
	40	8963	1853	5937	92	6.2990	07
49	00	8878	1598	6443	86	6.3071	04
	20	8793	1343	6950	78	153	7.800
	40	8708	1088	7459	71	235	7.796
50	00	8623	0835	7968	62	318	92
	20	8539	0581	8478	53	401	88
	40	8455	0328	8989	43	485	84
51	00	8371	8.5100076	1.49502	33	569	80
	20	8287	8.5099825	1.50016	22	654	76
	40	8203	9574	0531	2.3811	740	72
52	00	8120	9323	1048	2.3799	826	67
	20	8036	9074	1566	86	6.3912	63
	40	7953	8825	2086	73	6.4000	58
53	00	7871	8577	2608	59	088	53
	20	7788	8329	3131	44	176	48
	40	7706	8083	3656	29	265	43
54	00	7624	7838	4183	2.3713	355	38
	20	7543	7593	4712	2.3696	446	33
	40	7462	7349	5243	79	537	28
55	00	7381	7107	5777	61	629	23
	20	7300	6865	6312	42	721	17
	40	7220	6624	6850	23	815	11
56	00	7140	6385	7391	2.3603	6.4909	06
	20	7061	6147	7934	2.3583	6.5004	7.700
	40	6982	5909	8480	61	099	7.694
57	00	6903	5673	9028	39	196	88
	20	6825	5438	1.59580	2.3517	293	82
	40	6747	5205	1.60134	2.3493	391	75
58	00	8.5086669	8.5094972	1.60692	2.3469	6.5490	7.669

TABLE II—(*Concluded*).

Latitude.	Log *A*.	Log *B*.	Log *C*.	Log *D*.	Log *E*.	Log *F*.
58° 00′	8.5086669	8.5094972	1.60692	2.3469	6.5490	7.669
20	6592	4741	1253	444	590	62
40	6516	4512	1817	419	691	56
59 00	6440	4283	2384	392	792	49
20	6364	4056	2955	365	895	42
40	6289	3831	3530	337	6.5998	35
60 00	6214	3607	4109	309	6.6102	27
20	6140	3384	4691	279	208	20
40	6066	3163	5278	249	314	13
61 00	5993	2944	5869	218	422	7.605
20	5920	2726	6464	186	530	7.597
40	5848	2510	7063	154	640	89
62 00	5777	2295	7668	120	750	81
20	5706	2082	8277	086	862	73
40	5635	1871	8891	050	6.6975	64
63 00	5566	1661	1.69510	2.3014	6.7089	56
20	5496	1454	1.70135	2.2977	205	47
40	5428	1248	0765	939	322	38
64 00	5360	1043	1400	901	440	29
20	5292	0841	2042	861	559	20
40	5225	0641	2689	820	680	11
65 00	5159	0442	3343	778	802	7.501
20	5094	0245	4004	735	6.7926	7.491
40	5029	8.5090051	4670	692	6.8051	81
66 00	4964	8.5089858	5344	647	177	71
20	4901	9667	6025	601	306	61
40	4838	9478	6714	554	436	50
67 00	4776	9292	7410	506	567	40
20	4714	9107	8114	456	700	29
40	4653	8925	8826	406	835	18
68 00	4593	8745	1.79547	354	6.8972	7.406
20	4534	8566	1.80276	302	6.9111	7.395
40	4475	8390	1015	248	252	83
69 00	4417	8217	1763	192	395	71
20	4360	8045	2521	136	540	58
40	4304	7876	3289	078	687	46
70 00	4248	7709	4068	2.2018	836	33
20	4193	7544	4857	2.1957	6.9988	20
40	4139	7382	5658	2.1895	7.0142	7.307
71 00	4086	7222	6470	831	298	7.293
20	4033	7064	7295	766	457	79
40	3981	6908	8133	699	619	65
72 00	8.5083930	8.5086756	1.88984	2.1630	7.0784	7.250

Table III.

CORRECTIONS TO LONGITUDE FOR DIFFERENCE IN ARC AND SINE.

Log s(−).	Log Difference.	Log Δλ(+).	Log s(−).	Log Difference.	Log Δλ(+).
3.876	0.000 0001	2.385	4.871	0.000 0098	3.380
4.026	02	2.535	4.882	103	3.391
4.114	03	2.623	4.892	108	3.401
4.177	04	2.686	4.903	114	3.412
4.225	05	2.734	4.913	119	3.422
4.265	06	2.774	4.922	124	3.431
4.298	07	2.807	4.932	130	3.441
4.327	08	2.836	4.941	136	3.450
4.353	09	2.862	4.950	142	3.459
4.376	10	2.885	4.959	147	3.468
4.396	11	2.905	4.968	153	3.477
4.415	12	2.924	4.976	160	3.485
4.433	13	2.942	4.985	166	3.494
4.449	14	2.958	4.993	172	3.502
4.464	15	2.973	5.002	179	3.511
4.478	16	2.987	5.010	186	3.519
4.491	17	3.000	5.017	192	3.526
4.503	18	3.012	5.025	199	3.534
4.526	20	3.035	5.033	206	3.542
4.548	23	3.057	5.040	213	3.549
4.570	25	3.079	5.047	221	3.556
4.591	27	3.100	5.054	228	3.563
4.612	30	3.121	5.062	236	3.571
4.631	33	3.140	5.068	243	3.577
4.649	36	3.158	5.075	251	3.584
4.667	39	3.176	5.082	259	3.591
4.684	42	3.193	5.088	267	3.597
4.701	45	3.210	5.095	275	3.604
4.716	48	3.225	5.102	284	3.611
4.732	52	3.241	5.108	292	3.617
4.746	56	3.255	5.114	300	3.623
4.761	59	3.270	5.120	309	3.629
4.774	63	3.283	5.126	318	3.635
4.788	67	3.297	5.132	327	3.641
4.801	71	3.310	5.138	336	3.647
4.813	75	3.322	5.144	345	3.653
4.825	80	3.334	5.150	354	3.659
4.834	84	3.343	5.156	364	3.665
4.849	89	3.358	5.161	373	3.670
4.860	0.000 0094	3.369	5.167	0.000 0383	3.676

TABLE IV.

VALUES AT VARIOUS LATITUDES OF 1° OF PARALLEL, LOG N AND LOG R.

(Metric Units.)

Latitude.	1° of Parallel.	Log N.	Log R.	Latitude.	1° of Parallel	Log N.	Log R.
0° 00′	111 321	6.8046985	6.8017489	15° 00′	107 553	6.8047970	6.8020443
20	319	6986	7490	20	7 384	8013	0573
40	313	6987	7495	40	7 212	8057	0705
1 00	304	6990	7502	16 00	7 036	8102	0839
20	291	6993	7513	20	6 857	8148	0977
40	274	6998	7526	40	6 674	8195	1117
2 00	253	7003	7543	17 00	6 487	8242	1258
20	229	7010	7562	20	6 297	8290	1404
40	201	7017	7584	40	6 103	8339	1551
3 00	169	7025	7610	18 00	5 906	8389	1701
20	133	7035	7638	20	5 705	8440	1853
40	094	7046	7669	40	5 501	8491	2007
4 00	051	7057	7704	19 00	5 294	8544	2165
20	111 004	7069	7742	20	5 083	8597	2324
40	110 954	7083	7782	40	4 868	8651	2485
5 00	900	7097	7824	20 00	4 649	8705	2649
20	842	7112	7870	20	4 427	8761	2815
40	780	7129	7919	40	4 201	8817	2984
6 00	715	7146	7971	21 00	3 972	8874	3155
20	646	7164	8026	20	3 739	8931	3327
40	573	7183	8083	40	3 503	8989	3503
7 00	497	7203	8144	22 00	3 264	9049	3680
20	417	7225	8207	20	3 021	9108	3859
40	333	7247	8274	40	2 774	9169	4041
8 00	245	7270	8343	23 00	2 524	9231	4225
20	153	7294	8415	20	2 271	9292	4410
40	110 058	7319	8490	40	2 014	9353	4598
9 00	109 959	7345	8568	24 00	1 754	9418	4788
20	856	7372	8649	20	1 490	9482	4980
40	750	7400	8732	40	1 223	9547	5174
10 00	641	7428	8819	25 00	0 952	9612	5370
20	528	7458	8908	20	0 678	9678	5567
40	411	7489	9000	40	0 400	9744	5767
11 00	289	7520	9094	26 00	100 119	9812	5968
20	164	7553	9192	20	99 835	9879	6172
40	109 036	7586	9292	40	9 548	6.8049948	6377
12 00	108 904	7620	9395	27 00	9 257	6.8050017	6584
20	768	7656	9501	20	8 963	0086	6792
40	629	7692	9609	40	8 665	0156	7003
13 00	108 486	7729	9720	28 00	8 364	0227	7215
20	340	7767	9834	20	8 059	0298	7429
40	190	7806	6.8019951	40	7 752	0370	7644
14 00	108 036	7845	6.8020070	29 00	7 441	0443	7862
20	107 878	7886	0192	20	7 127	0515	8080
40	717	7928	0316	40	6 809	0589	8300
15 00	107 553	6.8047970	6.8020443	30 00	96 488	6.8050663	6.8028522

TABLE IV—(*Continued*).

Latitude.	1° of Parallel.	Log N.	Log R.	Latitude.	1° of Parallel.	Log N.	Log R.
30° 00′	96 488	6.8050663	6.8028522	45° 00′	78 849	6.8054347	6.8039574
20	6 164	0737	8745	20	8 390	4432	6.8039831
40	5 837	0812	8970	40	7 929	4518	6.8040088
31 00	5 506	0888	9197	46 00	7 466	4604	0346
20	5 172	0963	9424	20	6 999	4690	0603
40	4 835	1040	9653	40	6 530	4775	0860
32 00	4 495	1117	6.8029883	47 00	6 058	4861	1117
20	4 152	1194	6.8030115	20	5 584	4947	1374
40	3 805	1271	0348	40	5 107	5032	1630
33 00	3 455	1350	0582	48 00	4 628	5118	1887
20	3 102	1428	0817	20	4 146	5203	2142
40	2 746	1507	1054	40	3 661	5288	2398
34 00	2 387	1586	1292	49 00	3 174	5373	2653
20	2 025	1666	1531	20	2 685	5458	2908
40	1 659	1746	1771	40	2 193	5543	3162
35 00	1 290	1826	2012	50 00	1 698	5628	3416
20	0 918	1907	2254	20	1 201	5712	3669
40	0 544	1988	2497	40	0 702	5796	3922
36 00	90 166	2069	2741	51 00	70 200	5880	4175
20	89 785	2151	2986	20	69 696	5964	4426
40	9 401	2233	3232	40	9 189	6048	4677
37 00	9 014	2315	3479	52 00	8 680	6131	4928
20	8 624	2397	3726	20	8 169	6214	5177
40	8 231	2480	3975	40	7 656	6297	5426
38 00	7 835	2564	4224	53 00	7 140	6380	5674
20	7 436	2647	4474	20	6 622	6463	5921
40	7 034	2730	4724	40	6 101	6545	6167
39 00	6 629	2814	4976	54 00	5 578	6627	6413
20	6 221	2898	5228	20	5 053	6708	6657
40	5 810	2982	5480	40	4 526	6789	6901
40 00	5 396	3067	5734	55 00	3 996	6870	7144
20	4 979	3151	5987	20	3 464	6951	7385
40	4 560	3236	6241	40	2 931	7031	7626
41 00	4 137	3321	6496	56 00	2 395	7111	7866
20	3 712	3405	6750	20	1 856	7190	8104
40	3 284	3491	7006	40	1 316	7269	8341
42 00	2 853	3576	7262	57 00	0 774	7348	8578
20	2 420	3661	7518	20	60 230	7426	8812
40	1 983	3747	7774	40	59 683	7504	9046
43 00	1 543	3832	8031	58 00	9 135	7582	9279
20	1 101	3918	8287	20	8 584	7659	9509
40	0 656	4003	8544	40	8 032	7735	9739
44 00	80 208	4089	8802	59 00	7 478	7811	6.8049968
20	79 758	4175	9059	20	6 922	7887	6.8050194
40	9 305	4261	9316	40	6 363	7962	0419
45 00	78 849	6.8054347	6.8039574	60 00	55 802	6.8058037	6.8050644

TABLE IV—(*Concluded*).

Latitude.	1° of Parallel.	Log N.	Log R.	Latitude.	1° of Parallel.	Log N.	Log R.
60° 00′	55 802	6.8058037	6.8050644	75° 00′	28 903	6.8060742	6.8058759
20	5 240	8111	0866	20	8 276	0785	8887
40	4 676	8185	1087	40	7 647	0826	9012
61 00	4 110	8258	1307	76 00	7 017	0867	9135
20	3 542	8330	1524	20	6 386	0907	9255
40	2 972	8403	1741	40	5 755	0946	9372
62 00	2 400	8474	1956	77 00	5 123	0984	9487
20	1 827	8545	2169	20	4 490	1022	9598
40	1 252	8615	2380	40	3 855	1058	9707
63 00	0 675	8685	2590	78 00	3 220	1093	9814
20	50 096	8755	2797	20	2 584	1128	9.8059917
40	49 516	8823	3003	40	1 948	1162	9.8060018
64 00	48 934	8891	3208	79 00	21 311	1195	0118
20	8 350	8959	3409	20	20 673	1226	0212
40	7 765	9026	3610	40	20 034	1257	0304
65 00	7 178	9092	3809	80 00	19 394	1287	0394
20	6 589	9157	4005	20	18 754	1316	0481
40	5 998	9222	4200	40	18 113	1344	0565
66 00	5 406	9287	4393	81 00	17 472	1371	0646
20	4 813	9350	4583	20	16 830	1397	0725
40	8 218	9413	4772	40	16 188	1422	0800
67 00	3 622	9475	4959	82 00	15 545	1446	0873
20	3 024	9537	5143	20	14 901	1469	0943
40	2 424	9597	5325	40	14 257	1492	1009
68 00	1 823	9658	5506	83 00	13 612	1513	1074
20	1 221	9717	5684	20	12 967	1534	1134
40	0 617	9776	5860	40	12 321	1553	1192
69 00	40 012	9834	6034	84 00	11 675	1571	1248
20	39 405	9891	6205	20	11 029	1589	1300
40	8 797	6.8059947	6374	40	10 382	1605	1349
70 00	8 188	6.8060003	6542	85 00	9 735	1620	1395
20	7 578	0058	6707	20	9 087	1635	1439
40	6 966	0112	6869	40	8 439	1649	1479
71 00	6 353	0165	7029	86 00	7 792	1661	1517
20	5 738	0218	7186	20	7 144	1673	1551
40	5 123	0270	7342	40	6 495	1683	1583
72 00	4 506	0321	7495	87 00	5 846	1692	1611
20	3 888	0371	7645	20	5 197	1701	1637
40	3 269	0420	7793	40	4 548	1709	1659
73 00	2 648	0468	7938	88 00	3 898	1715	1679
20	2 027	0516	8081	20	3 248	1721	1695
40	1 405	0563	8222	40	2 599	1725	1709
74 00	30 781	0608	8361	89 00	1 949	1728	1719
20	30 156	0654	8496	20	1 300	1730	1727
40	29 530	0699	8629	40	650	1732	1731
75 00	28 903	6.8060742	6.8058759	90 00	0	6.8061733	6.8061733

Table V.

LENGTHS AT VARIOUS LATITUDES OF 1° OF MERIDIAN.

(Metric Units.)

Latitude.	1° of Meridian.	Latitude.	1° of Meridian.	Latitude.	1° of Meridian.
0° 00′	110 567.2	30° 00′	110 848.5	60° 00′	111 414.5
1	567.6	31	865.7	61	431.5
2	568.6	32	883.2	62	448.2
3	570.3	33	901.1	63	464.4
4	572.7	34	919.2	64	480.3
5	575.8	35	937.6	65	495.7
6	579.5	36	956.2	66	510.7
7	583.9	37	975.1	67	525.3
8	589.0	38	110 994.1	68	539.3
9	594.7	39	111 013.3	69	552.9
10	601.1	40	032.7	70	565.9
11	608.1	41	052.2	71	578.4
12	615.8	42	071.7	72	590.4
13	624.1	43	091.4	73	601.8
14	633.0	44	111.1	74	612.7
15	642.5	45	130.9	75	622.9
16	652.6	46	150.6	76	632.6
17	663.3	47	170.4	77	641.6
18	674.5	48	190.1	78	650.0
19	686.3	49	209.7	79	657.8
20	698.7	50	229.3	80	664.9
21	711.6	51	248.7	81	671.4
22	725.0	52	268.0	82	677.2
23	738.8	53	287.1	83	682.4
24	753.2	54	306.0	84	686.9
25	768.0	55	324.8	85	690.7
26	783.3	56	343.3	86	693.8
27	799.0	57	361.5	87	696.2
28	815.1	58	379.5	88	697.9
29	831.6	59	397.2	89	699.0
30 00	110 848.5	60 00	111 414.5	90 00	111 699.3

These lengths of 1° of meridian extend 0° 30′ north and south of the given latitude.

TABLE VI.
VALUES AT VARIOUS LATITUDES OF LOG Rα.
(Metric Units.)

Azimuth.	Latitude. 22°	24°	26°	28°	30°	32°	34°	36°
0°	6.80237	6.80248	6.80260	6.80272	6.80285	6.80299	6.80313	6.80327
5	239	250	262	274	287	300	314	329
10	244	255	267	279	292	305	319	333
15	254	264	276	288	300	313	326	340
20	266	277	288	299	311	324	337	350
25	282	292	302	313	325	337	349	362
30	300	309	319	330	340	352	364	376
35	320	329	338	348	358	369	380	391
40	341	350	358	367	377	386	397	407
45	364	371	379	387	396	405	414	424
50	386	392	399	407	415	423	432	441
55	407	413	420	426	434	441	449	457
60	427	432	438	445	451	458	465	472
65	445	450	455	461	467	473	480	486
70	461	465	470	475	481	486	492	498
75	473	478	482	487	492	497	502	508
80	483	487	491	495	500	505	510	515
85	489	492	496	501	505	510	514	519
90	490	494	498	502	507	511	516	521

	Latitude. 38°	40°	42°	44°	46°	48°	50°	52°
0°	6.80342	6.80357	6.80373	6.80388	6.80404	6.80419	6.80434	6.80449
5	344	359	374	389	404	420	435	450
10	348	363	378	393	408	423	438	453
15	355	369	384	399	413	428	442	457
20	364	378	392	406	420	434	448	462
25	375	388	402	415	429	442	456	469
30	388	401	413	426	439	452	465	477
35	402	414	426	438	450	462	474	486
40	418	429	440	451	462	474	485	496
45	434	444	454	464	475	485	495	505
50	450	459	468	478	487	496	506	515
55	465	474	482	490	499	508	516	524
60	480	487	495	502	510	518	526	533
65	493	500	507	514	520	528	534	541
70	504	510	517	523	529	536	542	548
75	513	519	525	530	536	542	548	554
80	520	525	531	536	542	547	553	558
85	524	529	534	540	545	550	555	560
90	526	531	536	541	546	551	556	561

TABLE VII.
VALUES OF LOGARITHM *m*.

Latitude.	Log *m*.	Latitude.	Log *m*.	Latitude.	Log *m*.	Latitude.	Log *m*.
0° 00′	1.40695	20° 00′	1.40626	40° 00′	1.40452	60° 00′	1.40253
2 00	694	22 00	612	42 00	431	62 00	235
4 00	692	24 00	597	44 00	411	64 00	219
6 00	688	26 00	582	46 00	390	66 00	203
8 00	683	28 00	565	48 00	369	68 00	188
10 00	677	30 00	548	50 00	349	70 00	174
12 00	669	32 00	530	52 00	329	72 00	1.40161
14 00	660	34 00	511	54 00	309		
16 00	650	36 00	491	56 00	290		
18 00	639	38 00	472	58 00	271		

TABLE VIII.
CORRECTIONS FOR INCLINATION FOR FOUR-METER BAR.

′	0°	1°	2°	3°	′	0°	1°	2°	3°
00	0.00000	0.00061	0.00244	0.00548	30	0.00015	0.00137	0.00381	0.00746
01	00	63	248	554	31	16	140	386	753
02	00	65	252	560	32	17	143	391	760
03	00	67	256	567	33	18	146	396	768
04	00	69	260	573	34	20	150	401	775
05	00	71	264	579	35	21	153	407	782
06	01	74	269	585	36	22	156	412	789
07	01	76	273	592	37	23	159	417	797
08	01	78	277	598	38	24	163	422	804
09	01	81	282	604	39	26	166	428	811
10	02	83	286	611	40	27	169	433	819
11	02	85	290	617	41	28	173	439	826
12	02	88	295	624	42	30	176	444	834
13	03	90	299	630	43	31	180	450	841
14	03	93	304	637	44	33	183	455	849
15	04	95	308	643	45	34	187	461	856
16	04	98	313	650	46	36	190	466	864
17	05	100	318	657	47	37	194	472	872
18	05	103	322	663	48	39	197	478	879
19	06	106	327	670	49	41	201	483	887
20	07	108	332	677	50	42	205	489	895
21	07	111	336	684	51	44	208	495	903
22	08	114	341	690	52	46	212	501	911
23	09	117	346	697	53	48	216	506	918
24	10	119	351	704	54	49	220	512	926
25	11	122	356	711	55	51	224	518	934
26	11	125	361	718	56	53	228	524	942
27	12	128	366	725	57	55	232	530	950
28	13	131	371	732	58	57	236	536	958
29	14	134	376	739	59	59	240	542	966
30	0.00015	0.00137	0.00381	0.00746	60	0.00061	0.00244	0.00548	0.00974

TABLE IX.
FOR PLOTTING MAPS IN POLYCONIC PROJECTION.
(Metric Units.)

Latitude.	Meridional Distance from Even-degree Parallel.	Abscissa of Developed Parallel for Longitude Interval.					
		10′	20′	30′	40′	50′	1° 00′
26° 00′	....	16 686	33 373	50 060	66 746	83 432	100 119
10	18464.1	663	326	49 989	652	314	99 977
20	36928.6	639	279	918	557	196	99 834
30	55393.6	615	231	846	461	83 076	9 691
40	73859.0	591	183	774	365	82 956	9 547
50	92324.8	567	134	701	268	835	9 402
27 00	....	543	086	628	171	713	9 256
10	18466.7	518	33 037	555	66 073	591	9 109
20	36933.9	494	32 987	481	65 974	468	8 961
30	55401.5	469	938	407	875	344	8 813
40	73869.6	444	888	332	776	220	8 664
50	92338.1	419	838	257	676	82 095	8 514
28 00	....	394	788	182	576	81 969	8 363
10	18469.4	369	737	106	474	842	8 211
20	36939.3	343	686	49 029	372	715	8 058
30	55409.6	318	635	48 953	270	587	7 905
40	73880.4	292	584	876	167	459	7 751
50	92351.6	266	532	798	65 064	330	7 596
29 00	....	240	480	720	64 960	81 200	7 440
10	18472.2	214	428	642	855	81 069	7 283
20	36944.8	188	375	563	750	80 938	7 125
30	55417.9	161	323	484	645	806	6 967
40	73891.5	135	270	404	539	673	6808
50	92365.5	108	216	324	432	540	6 648
30 00	....	081	163	244	325	406	6 487
10	18475.0	054	109	163	217	271	6 325
20	36950.4	027	055	082	109	136	6 163
30	55426.3	16 000	32 000	48 000	64 000	80 000	6 000
40	73902.7	15 973	31 946	47 918	63 891	864	5 836
50	92379.6	945	891	836	781	726	5 671
31 00	....	918	835	753	670	588	5 505
10	18477.9	890	780	670	559	449	5 339
20	36956.2	862	724	586	448	309	5 171
30	55435.0	834	668	502	336	169	5 003
40	73914.3	806	612	418	223	028	4 834
50	92394.1	778	555	333	63 110	887	4 664
32 00	....	15 749	31 498	47 247	62 996	78 745	94 494

Ordinates of Developed Parallel for Longitude Interval.

Lat.	10′	20′	30′	40′	50′	1° 00′
26° 00′	10.6	42.6	95.8	170.2	266.0	383.0
27	10.9	43.7	98.3	174.8	273.1	393.2
28	11.2	44.8	100.7	179.1	279.8	403.0
29	11.5	45.8	103.1	183.2	286.3	412.2
30	11.7	46.8	105.3	187.1	292.4	421.0
31	11.9	47.7	107.3	190.8	298.1	[illegible].3
32 00	12.1	48.6	109.3	194.2	303.5	[illegible].0

TABLE IX—(*Continued*).

Latitude.	Meridional Distance from Even-degree Parallel.	Abscissa of Developed Parallel for Longitude Interval. 10′	20′	30′	40′	50′	1° 00′
32° 00′	....	15 749	31 498	47 247	62 996	78 745	94 494
10	18480.8	721	441	162	882	602	4 323
20	36962.1	692	384	47 076	767	459	4 150
30	55443.8	663	326	46 989	652	315	3 977
40	73926.1	634	268	902	536	170	3 804
50	92408.8	605	210	815	420	78 025	3 629
33 00	....	576	152	727	303	77 879	3 454
10	18483.8	547	093	639	186	732	3 278
20	36968.0	517	31 034	551	62 068	585	3 101
30	55452.8	488	30 975	462	61 950	437	2 923
40	73938.0	458	915	373	831	288	2 745
50	92423.8	428	856	283	711	77 139	2 566
34 00	....	398	796	193	591	76 988	2 385
10	18486.8	368	735	103	470	838	2 205
20	36974.1	338	675	46 012	350	686	2 023
30	55461.9	307	614	45 921	228	534	1 841
40	73950.2	276	553	829	61 105	382	1 658
50	92439.0	246	492	737	60 983	228	1 474
35 00	....	215	430	645	860	76 074	1 289
10	18489.9	184	368	552	736	75 920	1 103
20	36980.2	153	306	459	611	764	0 917
30	55471.1	122	244	365	487	608	0 730
40	73962.5	091	181	271	362	452	0 542
50	92454.4	059	118	177	236	295	0 354
36 00	....	15 028	30 055	45 083	60 110	75 137	90 164
10	18493.0	14 996	29 992	44 988	59 983	74 979	89 974
20	36986.4	964	928	892	856	820	9 783
30	55480.4	932	864	796	728	660	9 592
40	73975.0	900	800	700	600	500	9 399
50	92470.0	868	736	604	471	339	9 206
37 00	....	836	671	507	342	177	9 012
10	18496.1	803	606	409	212	74 015	8 818
20	36992.7	771	541	312	59 082	73 852	8 622
30	55489.9	738	476	214	58 951	689	8 426
40	73987.6	705	410	115	820	525	8 229
50	92485.8	672	344	44 016	688	360	8 031
38 00	....	14 639	29 278	43 917	58 556	73 195	87 833

ORDINATES OF DEVELOPED PARALLEL FOR LONGITUDE INTERVAL.

Lat.	10′	20′	30′	40′	50′	1° 00′
32 00	12.1	48.6	109.3	194.2	303.5	437.0
33	12.3	49.4	111.0	197.4	308.4	444.2
34	12.5	50.1	112.7	200.4	313.1	450.8
35	12.7	50.8	114.2	203.1	317.3	456.9
36	12.8	51.4	115.6	205.6	321.2	462.5
37	13.0	51.9	116.9	207.8	324.6	467.5
38 00	13.1	52.4	118.0	209.8	327.7	471.9

TABLE IX—(*Continued*).

Latitude.	Meridional Distance from Even-degree Parallel.	Abscissa of Developed Parallel for Longitude Interval. 10′	20′	30′	40′	50′	1° 00′
38° 00′	....	14 639	29 278	43 917	58 556	73 195	87 833
10	18499.3	606	212	817	423	73 029	7 634
20	36999.1	573	145	717	290	72 862	7 434
30	55499.4	539	078	617	156	695	7 233
40	74000.3	505	29 011	516	58 022	527	7 031
50	92501.8	472	28 944	416	57 888	359	6 830
39 00	....	438	876	314	752	190	6 627
10	18502.5	404	808	212	616	72 020	6 423
20	37005.5	370	740	109	479	71 849	6 219
30	55509.1	336	672	43 007	343	679	6 014
40	74013.2	302	603	42 904	206	507	5 808
50	92517.8	267	534	801	57 068	335	5 602
40 00	....	233	465	698	56 930	71 162	5 394
10	18505.7	198	396	594	791	70 989	5 186
20	37012.0	163	326	489	652	815	4 978
30	55518.8	128	256	384	512	640	4 768
40	74026.1	093	186	279	372	466	4 558
50	92534.0	058	116	174	232	290	4 347
41 00	....	14 023	28 046	42 068	56 091	70 114	4 136
10	18509.0	13 988	27 975	41 962	55 950	69 937	3 923
20	37018.5	952	904	856	808	759	3 710
30	55528.5	916	833	749	665	581	3 497
40	74039.1	881	761	642	522	403	3 282
50	92550.3	845	689	534	379	223	3 067
42 00	....	809	617	426	235	69 043	2 851
10	18512.2	773	545	318	55 090	68 863	2 635
20	37025.0	737	473	209	54 946	682	2 417
30	55538.3	700	400	41 100	801	500	2 199
40	74052.2	664	327	40 991	655	318	1 981
50	92566.6	627	254	881	509	68135	1 761
43 00	....	591	181	771	362	67 952	1 541
10	18515.5	554	107	661	214	768	1 321
20	37031.6	517	27 034	550	54 067	583	1 099
30	55548.2	480	26 960	439	53 919	398	0 877
40	74065.3	443	885	328	770	213	0 654
50	92583.0	405	811	216	621	67 026	0 431
44 00	....	13 368	26 736	40 104	53 472	66 839	80 207

ORDINATES OF DEVELOPED PARALLEL FOR LONGITUDE INTERVAL.

Lat.	10′	20′	30′	40′	50′	1° 00′
38° 00′	13.1	52.4	118.0	209.8	327.7	471.9
39	13.2	52.9	118.9	211.5	330.4	475.8
40	13.3	53.2	119.8	212.9	332.6	479.0
41	13.4	53.5	120.4	214.1	334.5	481.7
42	13.4	53.8	120.9	215.0	336.0	483.8
43	13.5	53.9	121.3	215.7	337.0	485.3
44 00	13.5	54.0	121.5	216.1	337.7	486.2

TABLE IX—(*Concluded*).

Latitude.	Meridional Distance from Even-degree Parallel.	Abscissa of Developed Parallel for Longitude Interval.					
		10′	20′	30′	40′	50′	1° 00′
44° 00′	....	13 368	26 736	40 104	53 472	66 839	80 207
10	18518.8	331	661	39 992	322	652	79 982
20	37038.1	293	586	879	172	464	9 756
30	55558.0	255	511	766	53 021	276	9 530
40	74078.4	218	435	652	52 869	66 086	9 303
50	92599.5	180	359	538	718	65 896	9 075
45 00	....	142	283	424	566	706	8 847
10	18522.1	103	207	310	413	515	8 618
20	37044.7	065	130	195	260	324	8 388
30	55567.9	13 027	26 053	39 080	52 106	65 132	8 158
40	74091.7	12 988	25 976	38 964	51 952	64 940	7 927
50	92615.9	950	899	849	798	747	7 696
46 00	....	911	822	733	643	554	7 464
10	18525.4	872	744	616	488	360	7 231
20	37051.3	833	666	499	333	64 165	6 997
30	55577.8	794	588	382	176	63 969	6 763
40	74104.8	755	510	265	51 019	774	6 528
50	92632.4	716	431	147	50 862	578	6 292
47 00	....	676	353	38 029	705	381	6 056
10	18528.7	637	274	37 911	547	63 184	5 820
20	37057.9	597	195	792	389	62 986	5 582
30	55587.7	558	115	673	230	787	5 344
40	74118.0	518	25 036	553	50 071	588	5 105
50	92648.8	478	24 956	433	49 911	388	4 866
48 00	....	438	876	314	751	62 188	4 626
10	18531.9	398	796	194	591	61 988	4 385
20	37064.4	358	715	37 073	430	787	4 144
30	55597.5	317	634	36 952	268	585	3 902
40	74131.1	277	554	831	49 107	384	3 659
50	92665.2	236	473	709	48 945	61 181	3 416
49 00	....	196	391	587	782	60 977	3 172
10	18535.2	155	310	465	619	774	2 928
20	37071.0	114	228	342	456	570	2 683
30	55607.3	073	146	219	293	365	2 437
40	74144.2	032	24 064	36 096	48 128	60 160	2 191
50	92681.6	11 991	23 982	35 973	47 963	59 954	1 944
50 00	....	11 950	23 899	35 849	47 798	59 747	71 696

ORDINATES OF DEVELOPED PARALLEL FOR LONGITUDE INTERVAL.

Lat.	10′	20′	30′	40′	50′	1° 00′
44° 00′	13.5	54.0	121.5	216.1	337.7	486.2
45	13.5	54.1	121.6	216.2	337.9	486.5
46	13.5	54.0	121.6	216.1	337.7	486.3
47	13.5	53.9	121.4	215.7	337.1	485.4
48	13.4	53.8	121.0	215.1	336.1	484.0
49	13.4	53.5	120.5	214.2	334.7	481.9
50 00	13.3	53.3	119.8	213.0	332.8	479.3

Table X.

CORRECTIONS FOR RUN OF THE MICROMETER.

Corrections same sign as r for $m < 2' \, 30''$. Opposite signs for $m > 2' \, 30''$.

r	a=0'						a=1'						a=2'				r
	00″	10″	20″	30″	40″	50″	00″	10″	20″	30″	40″	50″	00″	10″	20″	30″	
°																	°
0.1	.05	.05	.04	.04	.04	.03	.03	.03	.02	.02	.02	.01	.01	.01	.00	.00	0.1
0.2	.10	.09	.09	.08	.07	.07	.06	.05	.05	.04	.03	.03	.02	.01	.01	.00	0.2
0.3	.15	.14	.13	.12	.11	.10	.09	.08	.07	.06	.05	.04	.03	.02	.01	.00	0.3
0.4	.20	.19	.17	.16	.15	.13	.12	.11	.09	.08	.07	.05	.04	.03	.01	.00	0.4
0.5	.25	.23	.22	.20	.18	.17	.15	.13	.12	.10	.08	.07	.05	.03	.02	.00	0.5
0.6	.30	.28	.26	.24	.22	.20	.18	.16	.14	.12	.10	.08	.06	.04	.02	.00	0.6
0.7	.35	.33	.30	.28	.26	.23	.21	.19	.16	.14	.12	.09	.07	.04	.02	.00	0.7
0.8	.40	.37	.35	.32	.29	.27	.24	.21	.19	.16	.13	.11	.08	.05	.03	.00	0.8
0.9	.45	.42	.39	.36	.33	.30	.27	.24	.21	.18	.15	.12	.09	.06	.03	.00	0.9
1.0	.50	.47	.43	.40	.37	.33	.30	.27	.23	.20	.17	.13	.10	.07	.03	.00	1.0
1.1	.55	.51	.48	.44	.40	.37	.33	.29	.26	.22	.18	.15	.11	.07	.04	.00	1.1
1.2	.60	.56	.52	.48	.44	.40	.36	.32	.28	.24	.20	.16	.12	.08	.04	.00	1.2
1.3	.65	.61	.56	.52	.48	.43	.39	.35	.30	.26	.22	.17	.13	.09	.04	.00	1.3
1.4	.70	.65	.60	.56	.51	.47	.42	.37	.33	.28	.23	.19	.14	.09	.05	.00	1.4
1.5	.75	.70	.65	.60	.55	.50	.45	.40	.35	.30	.25	.20	.15	.10	.05	.00	1.5
1.6	.80	.75	.69	.64	.59	.53	.48	.43	.37	.32	.27	.21	.16	.11	.05	.00	1.6
1.7	.85	.79	.74	.68	.62	.57	.51	.45	.40	.34	.28	.23	.17	.11	.06	.00	1.7
1.8	.90	.84	.78	.72	.66	.60	.54	.48	.42	.36	.30	.24	.18	.12	.06	.00	1.8
1.9	.95	.89	.82	.76	.70	.63	.57	.51	.44	.38	.32	.25	.19	.13	.06	.00	1.9
2.0	1.00	.93	.87	.80	.73	.67	.60	.53	.47	.40	.33	.27	.20	.13	.07	.00	2.0
2.1	1.05	.98	.91	.84	.77	.70	.63	.56	.49	.42	.35	.28	.21	.14	.07	.00	2.1
2.2	1.10	1.03	.95	.88	.81	.73	.66	.59	.51	.44	.37	.29	.22	.15	.07	.00	2.2
2.3	1.15	1.07	1.00	.92	.84	.77	.69	.61	.54	.46	.38	.31	.23	.15	.08	.00	2.3
2.4	1.20	1.12	1.04	.96	.88	.80	.72	.64	.56	.48	.40	.32	.24	.16	.08	.00	2.4
2.5	1.25	1.17	1.08	1.00	.92	.83	.75	.67	.58	.50	.42	.33	.25	.17	.08	.00	2.5
2.6	1.30	1.21	1.13	1.04	.95	.87	.78	.69	.61	.52	.43	.35	.26	.17	.09	.00	2.6
2.7	1.35	1.26	1.17	1.08	.99	.90	.81	.72	.63	.54	.45	.36	.27	.18	.09	.00	2.7
2.8	1.40	1.31	1.21	1.12	1.03	.93	.84	.75	.65	.56	.47	.37	.28	.19	.09	.00	2.8
2.9	1.45	1.35	1.26	1.16	1.06	.97	.87	.77	.68	.58	.48	.39	.29	.19	.10	.00	2.9
3.0	1.50	1.40	1.30	1.20	1.10	1.00	.90	.80	.70	.60	.50	.40	.30	.20	.10	.00	3.0
	60″	50″	40″	30′	20″	10″	60″	50″	40	30″	20″	10″	60″	50″	40″	30″	
r	a=4′						a=3′						a=2′				r

TABLE XI.

RECTANGULAR SPHERICAL COORDINATES.

VALUES OF THE CORRECTIVE TERM $\frac{m^2n}{2RN}$.

m in km.	n in km.									
	10	20	30	40	50	60	70	80	90	100
	m.	m.	m.	m.	m.	m.	m.	m.	m.	m.
10	0.01	0.02	0.04	0.05	0.06	0.07	0.09	0.10	0.11	0.12
20	0.05	0.10	0.15	0.20	0.25	0.29	0.34	0.39	0.44	0.49
30	0.11	0.22	0.33	0.44	0.55	0.66	0.77	0.88	0.99	1.11
40	0.20	0.39	0.59	0.79	0.99	1.19	1.39	1.58	1.78	1.97
50	0.31	0.61	0.92	1.23	1.54	1.85	2.16	2.47	2.77	3.08
60	0.45	0.89	1.34	1.78	2.22	2.66	3.10	3.55	3.99	4.43
70	0.60	1.21	1.81	2.41	3.02	3.62	4.22	4.82	5.43	6.03
80	0.79	1.57	2.36	3.15	3.94	4.73	5.52	6.30	7.09	7.88
90	0.99	1.99	2.99	3.99	4.99	5.98	6.98	7.98	8.97	9.97
100	1.23	2.46	3.69	4.92	6.16	7.39	8.62	9.85	11.08	12.31

VALUES OF THE CORRECTIVE TERM $\frac{mn}{RN \sin 1''}$.

m in km.	n in km.									
	10	20	30	40	50	60	70	80	90	100
	"	"	"	"	"	"	"	"	"	"
10	0.5	1.0	1.5	2.0	2.5	3.1	3.6	4.1	4.6	5.1
20	1.0	2.0	3.1	4.1	5.1	6.1	7.1	8.2	9.2	10.2
30	1.5	3.0	4.6	6.1	7.6	9.1	10.6	12.1	13.7	15.2
40	2.0	4.1	6.1	8.1	10.2	12.2	14.2	16.2	18.3	20.3
50	2.5	5.1	7.6	10.2	12.7	15.2	17.8	20.3	22.9	25.4
60	3.1	6.1	9.2	12.2	15.3	18.3	21.4	24.4	27.5	30.5
70	3.6	7.1	10.7	14.2	17.8	20.3	24.9	28.5	32.0	35.6
80	4.1	8.1	12.2	16.2	20.3	24.4	28.4	32 5	36.5	40.6
90	4.6	9.1	13.7	18.3	22.9	27.4	32.0	36.6	41.1	45.7
100	5.1	10.2	15.2	20.3	25.4	30.5	35.6	40.6	45.7	50.8

(These two tables are computed tor a latitude of 40°.)

TABLE XII.

FOR DETERMINING RELATIVE STRENGTH OF FIGURES IN TRIANGULATION.

	10°	12°	14°	16°	18°	20°	22°	24°	26°	28°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
10°	428	359																					
12	359	295	253																				
14	315	253	214	187																			
16	284	225	187	162	143																		
18	262	204	168	143	126	113																	
20	245	189	153	130	113	100	91																
22	232	177	142	119	103	91	81	74															
24	221	167	134	111	95	83	74	67	61														
26	213	160	126	104	89	77	68	61	56	51													
28	206	153	120	99	83	72	63	57	51	47	43												
30	199	148	115	94	79	68	59	53	48	43	40	33											
35	188	137	106	85	71	60	52	46	41	37	33	27	23										
40	179	129	99	79	65	54	47	41	36	32	29	23	19	16									
45	172	124	93	74	60	50	43	37	32	28	25	20	16	13	11								
50	167	119	89	70	57	47	39	34	29	26	23	18	14	11	9	8							
55	162	115	86	67	54	44	37	32	27	24	21	16	12	10	8	7	5						
60	159	112	83	64	51	42	35	30	25	22	19	14	11	9	7	5	4	4					
65	155	109	80	62	49	40	33	28	24	21	18	13	10	7	6	5	4	3	2				
70	152	106	78	60	48	38	32	27	23	19	17	12	9	7	5	4	3	2	2	1			
75	150	104	76	58	46	37	30	25	21	18	16	11	8	6	4	3	2	2	1	1	1		
80	147	102	74	57	45	36	29	24	20	17	15	10	7	5	4	3	2	1	1	1	0	0	
85	145	100	73	55	43	34	28	23	19	16	14	10	7	5	3	2	2	1	1	0	0	0	0

TABLE XII—(*Continued*).

	10°	12°	14°	16°	18°	20°	22°	24°	26°	28°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
90°	143	98	71	54	42	33	27	22	19	16	13	9	6	4	3	2	1	1	1	0	0	0	0
95	140	96	70	53	41	32	26	22	18	15	13	9	6	4	3	2	1	1	0	0	0	0	
100	138	95	68	51	40	31	25	21	17	14	12	8	6	4	3	2	1	1	0	0	0		
105	136	93	67	50	39	30	25	20	17	14	12	8	5	4	2	2	1	1	0	0			
110	134	91	65	49	38	30	24	19	16	13	11	7	5	3	2	2	1	1	1				
115	132	89	64	48	37	29	23	19	15	13	11	7	5	3	2	2	1	1					
120	129	88	62	46	36	28	22	18	15	12	10	7	5	3	2	2	1						
125	127	86	61	45	35	27	22	18	14	12	10	7	5	4	3	2							
130	125	84	59	44	34	26	21	17	14	12	10	7	5	4	3								
135	122	82	58	43	33	26	21	17	14	12	10	7	5	4									
140	119	80	56	42	32	25	20	17	14	12	10	8	6										
145	116	77	55	41	32	25	21	17	15	13	11	9											
150	112	75	54	40	32	26	21	18	16	15	13												
152	111	75	53	40	32	26	22	19	17	16													
154	110	74	53	41	33	27	23	21	19														
156	108	74	54	42	34	28	25	22															
158	107	74	54	43	35	30	27																
160	107	74	56	45	38	33																	
162	107	76	59	48	42																		
164	109	79	63	54																			
166	113	86	71																				
168	122	98																					
170	148																						

TABLE XIII.

PROBABILITY OF ERROR BETWEEN THE LIMITS 0 AND $\pm\frac{\Delta}{\epsilon\sqrt{2}}$.

t.	$(p)_t$.	t.	$(p)_t$.	t.	$(p)_t$.	t.	$(p)_t$.	t.	$(p)_t$.
0.00	0.0000	0.50	0.5205	1.00	0.8427	1.50	0.9661	2.00	0.9953
0.05	0.0564	0.55	0.5633	1.05	0.8624	1.55	0.9716	2.05	0.9963
0.10	0.1125	0.60	0.6039	1.10	0.8802	1.60	0.9763	2.10	0.9970
0.15	0.1680	0.65	0.6420	1.15	0.8961	1.65	0.9804	2.15	0.9976
0.20	0.2227	0.70	0.6778	1.20	0.9103	1.70	0.9838	2.20	0.9981
0.25	0.2763	0.75	0.7112	1.25	0.9229	1.75	0.9867	2.25	0.9985
0.30	0.3286	0.80	0.7421	1.30	0.9340	1.80	0.9891	2.30	0.9989
0.35	0.3794	0.85	0.7707	1.35	0.9438	1.85	0.9911	2.35	0.9991
0.40	0.4284	0.90	0.7969	1.40	0.9523	1.90	0.9928	2.40	0.9993
0.45	0.4755	0.95	0.8209	1.45	0.9597	1.95	0.9942	2.45	0.9995
0.50	0.5205	1.00	0.8427	1.50	0.9661	2.00	0.9953	2.50	0.9996

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