# FUNDAMENTALS OF HIGH SCHOOL MATHEMATICS

RUGG – CLARK

*RUGG–CLARK MATHEMATICS TEXTS*
By Harold O. Rugg and John R. Clark

# Fundamentals of High School Mathematics

## A Textbook Designed to Follow Arithmetic

BY

HAROLD O. RUGG
Associate Professor of Education
University of Chicago

AND

JOHN R. CLARK
Head of Department of Mathematics
Francis W. Parker School, Chicago

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# FOREWORD

## THE CONSTRUCTION OF COURSES OF STUDY IN HIGH SCHOOL MATHEMATICS

THE one-year course of study represented by this textbook is the outcome of five years of critical investigation of high-school mathematics. A recent monograph published by the writers[1] presents in detail the evidence developed by these years of investigation. It reveals striking weaknesses in the content and teaching of first-year algebra. It shows that the achievement of more than 22,000 pupils on carefully designed standardized tests is very unsatisfactory; that the present course of study is not organized primarily to provide an opportunity for training in problem-solving; and that much of the material included in the traditional course will never be used in actual life situations. The evidence is clear, therefore, that the present course of study in first-year algebra must be completely reconstructed.

The study of the historical development of the present first-year high school algebra shows that it has come down from its former position in the upper college curriculum with but two slight modifications of content and presentation. One has been revealed in the increased emphasis on the formal and manipulative aspects; the second in the slight improvements which have been made in methods of presentation through the use of more concrete devices. On the whole, however, the course has retained that emphasis upon rigorous, logical organization which characterizes the mathematical thinking of "mathematicians."

The need is evident, therefore, for a course of study which will avoid the two fundamental weaknesses pointed out above; namely, *first*, a course which will eliminate formal non-essentials, basing such elimination on scientific investigation; *second*, one which will tie together, in a psychologically and sequentially worked-out scheme, the fundamental mathematical notions and tool operations which are needed by adults to facilitate quanti-

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[1] *Scientific Method in the Reconstruction of Ninth-Grade Mathematics.* University of Chicago Press. 1918.

tative thinking. The course of study presented in this text is therefore unique in these particulars.

### PRINCIPLES OF TEXTBOOK DESIGN

Three important criteria have been used to determine the content and the organization of this experimental course of study: *first*, subject matter must be organized in terms of a real psychological analysis of "learning" in mathematics; *second*, those mathematical notions and devices which have the widest application must be emphasized roughly in proportion to their frequency of use; thus the material has been selected in terms of the social criterion; *third*, the course must be organized to provide the pupil with the maximum opportunity to do genuine thinking, real problem-solving, rather than to emphasize the drill or manipulative aspects which now commonly require most of the pupil's time.

**The first criterion.** To organize a course of study in terms of a real psychology of "learning" in mathematics shows us that new meanings, new concepts which are to be learned, must be acquired by the pupil in the same natural way in which human beings learn and acquire new meanings outside of a textbook. To be more specific, this means that the exact logical sequence of the mature mathematician, which begins with abstract definitions and statements of general principles and then proceeds to their applications and specific uses, must give way to a directly psychological method. "Definitions" and general principles must grow out of the pupil's concrete experience. He must begin with details, particulars, concrete elements, and finally arrive at a generalization or the statement of a general principle (seldom, however, at "definitions" in first-year algebra). Contrast the introductory lesson in algebra which explains that "Algebra is the science of general numbers" with the one which begins, "It saves time to use abbreviations or letters to represent numbers; for example, you have used C to stand for 100, M for 1000, etc." The authors have attempted to *visualize* *the pupil's mental background* in writing each page of the exposition of this text and in the selection and statement of problem material. This represents at least a first step toward the designing of textbooks on this psychological criterion.

**The second criterion.** The authors' investigations of the mathematical experience which pupils need to have to prepare them to meet adequately later quantitative situations have resulted in the *elimination* of a great deal of the content of the traditional book. The time usually given to the operations on polynomials, special products, factoring, complicated fractions, etc., can have no justification in terms of this *criterion of social worth* or utility. The application of this criterion demands a complete recognition of the graphic method of representing number as one of the three methods around which a course should be constructed, and that the formula, the equation, the properties of the more important space forms, and the principle of dependence or functionality should form the basic material of the course. Likewise three chapters of the text are devoted to a non-demonstrative study of the triangle. Pupils are shown that mathematics supplies notions and devices which people need to master in order to solve many practical problems. The use of scale drawings, angular measurement, the principle of similarity, and the simple trigonometric functions of a right triangle have infinitely more value either (1) from the standpoint of their use in other situations, (2) from their appropriateness and adaptability to the child's interests and abilities, or (3) from the criterion of thinking value, than does the excessive formalism and manipulation of symbols which they supplant in this course. To summarize, the authors have been concerned to incorporate here the most important mathematical notions which investigation shows that all pupils ought to know.

**The third criterion.** Experimental investigation of the possibility of developing powers of generalization shows that such are to be developed only by so complete an organization of courses as will provide a maximum of opportunity for problem-solving, for reflective thinking, for calling into play mental processes of analysis, comparison, and recognition of relations between the parts of problems. Unfortunately, algebra courses have been deprived of most of their "training" value. An emphasis upon formalism, drill, the routine practice in manipulation of meaningless symbols, and lack of genuine motive are typical examples of the way in which we have hampered teachers in the development of problem-solving abilities. The general practice of devoting 80 per cent of the problem-material to these formal drill examples, leaving only 20 per cent for the verbal problem, — which of all types provides most completely opportunity for "thinking," — has been radically modified in the course submitted in this book. The entire course has been organized around a central core of "problem-solving." Even the purely formal materials themselves have been so organized, wherever possible, as to provide an opportunity for real thinking and not mere habit formation.

## THE PRIMARY FUNCTION OF MATHEMATICAL INSTRUCTION

The writers' thesis in constructing this course of study is this: **The central element in human thinking is seeing relationships clearly. In the same way the primary function of a high-school course in mathematics is to give ability to recognize verbally stated relationships between magnitudes, to represent such relationships economically by means of symbols, and to determine such relationships.** To carry out this aim the course of study, therefore, should be organized in such a way as *to develop ability in the intelligent use of the equation, the formula, methods of graphic representation, and the properties of the more important space forms in the expression and determination of relationships.*

Thus we may summarize the chief characteristics of this text, which has been organized on this fundamental aim, as follows:

(1) A marked decrease in the emphasis upon formal manipulation. The whole course is aimed at providing an opportunity for problem-solving.

(2) Three methods of representing number are recognized: the tabular method, the graphic method, and the equational or formula method. Thus graphic representation is an integral part of the course and is not treated as an isolated operation.

(3) A vast amount of useless material has been omitted, — for instance, addition, subtraction, multiplication, and division of polynomials; the more complicated work with fractions, all but the simpler work with radicals, etc.

(4) All material shown by investigation to have social utility, and which is omitted from current courses in mathematics, has been included. For example: construction and evaluation of formulas, emphasis on "evaluation" as a most important operation, trigonometric functions of the right triangle, a very complete study of variation or dependence, etc.

(5) The exposition of the text develops so gradually in accordance with the writers' discoveries concerning "learning" that the average pupil can read any part of the discussion and then solve the problems unaided. Thus the text develops in accordance with the natural development of the pupil's method rather than that of the highly trained, logical mathematician. This is the outcome of an original analysis of the psychology of mathematics.

(6) The equation is emphasized throughout as the primary formal operation of the course, — not as an end in itself, but because it is the essential tool for stating and determining quantitative relationships.

## THE TEACHER'S TESTS FOR A TEXTBOOK

The authors would propose to teachers as tests of the value of material contained in this or in any other textbook the three following principles: (1) Is the type of subject matter presented here or in any other textbook organized in terms of the way children naturally learn; that is, has the psychological criterion been kept constantly and adequately in mind? (2) Is the pupil who takes this or any other course prompted to do real and genuine thinking? Does he have ample opportunity for practice in "problem-solving"? Is the subject matter of the course organized primarily around a core of problem-solving situations? (3) Does the kind of subject matter presented in this or in any other book sufficiently justify itself from the point of view of its use or importance either in later mathematics courses, in other school subjects, or in situations outside the school?

The application of these criteria in the construction and selection of school textbooks will go far to bring about the type of reconstruction for which the writers' investigations show there is a real demand.

HAROLD O. RUGG
JOHN R. CLARK

CHICAGO, ILLINOIS
*July* 25, 1918

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# FUNDAMENTALS OF HIGH SCHOOL MATHEMATICS

## CHAPTER I

## A NEW WAY TO REPRESENT NUMBERS

**Section 1. It saves time to use abbreviations and letters, instead of words, to represent numbers.** In order to save time in reading and writing numbers in your studies in arithmetic, you have already found it convenient to use certain **abbreviations** or **letters** to represent numbers. For example, instead of "*dozen*" you have used "*doz.*" to stand for 12; C to stand for 100; M for 1000; cwt. (hundredweight) for 100 lb., mo. (month) for 30 days, etc. It is necessary that we learn more about this **new way of representing numbers by letters** because we shall use it in all our later work in mathematics.

### EXERCISE 1

### PRACTICE IN USING ABBREVIATIONS AND LETTERS TO REPRESENT NUMBERS

1. How many eggs are 6 doz. eggs and 2 doz. eggs?
2. How many days in 3 mo. and $2\frac{1}{2}$ mo.?
3. Change 5 yr. and 2 mo. to mo.
4. If 1 ft. $=$ 12 in., change 4 ft. $+$ 5 in. to in.
5. If $R$ (ream) stands for 500 sheets of paper, how many sheets in $2R + 3R$?
6. Change 5 yd. $-$ 2 ft. $-$ 3 in. to in.
7. Using $d$ for 12, how many eggs in $2d + 3d$ eggs?

8. How many sheets of paper in $5R + 3R - 6R$ sheets?

9. Change $5m + 4m - 6m + m$ to smaller time units, that is, to "days," if $m = 30$ days.

10. Change $7y$ (yards) $+ 6f$ (feet) to smaller **units**, that is, to "inches."

11. How many cents in $4q + 6d$, if $q$ and $d$ stand for the **number of cents** in a quarter and dime respectively?

12. If $h$ (hour) equals $60m$ (minutes), and $m$ equals $60s$ (seconds), how many $s$ in $2h + 3m$?

In these examples, you have used **abbreviations** or **single letters** to represent numbers or known quantities. We make use of abbreviations or single letters instead of ___?___ because __________________________ (State the reason here). You need a great deal of practice in doing this. The next exercise will give more practice in representing numbers by letters in different kinds of examples.

## EXERCISE 2

### FURTHER PRACTICE IN USING LETTERS FOR NUMBERS

1. Change $10y + 4f$ to inches (smaller UNITS) if $y$ and $f$ stand for the number of inches in a yard and in a foot, respectively.

2. Express $2h + 5m$ in seconds.

3. What will $4R + 3R + R$ sheets of paper cost at $\frac{1}{2}$¢ per sheet?

4. At 4¢ each, what will $7d - 2d$ eggs cost?

5. The **length** of the rectangle in Fig. 1 is represented by the expression $3f$, and the **width** by the expression $2f$. What expression will represent the **perimeter**? How many inches in the **perimeter** if $f = 12$ inches?

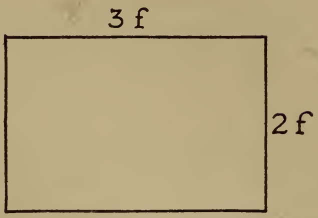

FIG. 1

6. Change $7p + 5p - 8p + 3p$ to ounces, if $p$ stands for the number of ounces in one pound.

7. The expression $14y + 8m + 5d$ represents the age of a pupil in an algebra class. Express this pupil's age in *days* or as *a certain number of d.*

8. Find the cost of $4\frac{1}{2}$ $T$ of coal at 30 cents per *cwt.*, using the **relation**, $T = 20$ *cwt.*

9. If $d = 4q$, find how many $q$ in $3d + 5d$.

10. If $y = 12m$, and $m = 30d$, how many $d$ in $2y + 4m$?

11. A boy earned 27 dollars in a month; his father earned $n$ dollars. How many dollars would both earn in 3 months, if $n$ stands for 60 dollars?

12. If $r = 3t$ and $t = 6s$, how many $s$ in $4r - 5t$?

13. Write the **relation** between $y$ and $m$ (*i.e.* year and month); between $f$ and $i$ (foot and inch); between $T$ and *cwt* (ton and hundredweight); between $h$, $m$, and $s$ (*i.e.* hour, minute, and second).

In this you *always* express "how many" of one *unit* equals a "certain number" of the other *unit.* For example, $1f = 12$ inches, or, more abbreviated,

$$f = 12\,i.$$

> Are *yards*, *feet*, and *inches* **related units?** Are *hours*, *minutes*, and *seconds* **related units?** Are *pounds* and *dollars* **related units?** Are *dollars* and *cents* **related units?**

**Section 2. Word statements about quantities may be much more briefly expressed by using a single letter to represent a quantity.** In the last section we saw that it saved time to use abbreviations or letters to represent quantities. Now we shall show that **entire word statements** about quantities may be expressed much more briefly by using a single letter to represent a number. To illustrate, consider next the four different ways of writing the statement of the same example.

**Illustrative example.**

| | |
|---|---|
| (*a*) The "word" method of stating the example. | (*a*) There is a certain number such that if you add 5 to it the result will be 18. What is the number? |
| (*b*) An abbreviated way to write it. | (*b*) What no. plus 5 equals 18? |
| (*c*) A more abbreviated way to write it. | (*c*) No. $+ 5 = 18$. |
| (*d*) The best way to write it. | (*d*) $n + 5 = 18$. |

It is clear that in all these cases the number is 13, and that it is most easily represented by the single letter $n$. Thus, the fourth method, $n + 5 = 18$, illustrates the very great saving that is obtained by the use of *single* letters for numbers. This method will be used throughout all our later work. This is one of the aims of mathematics: to help you solve problems by better methods than you knew in arithmetic.

### EXERCISE 3

Express the following word statements in the briefest possible way, using a single letter to represent the quantity you are trying to find.

**Illustrative example.** If a certain *unknown* number be increased by 7, the result will be 16. This may be most easily written:

$$n + 7 = 16,$$
$$n = 9.$$

---

1. There is a certain number such that if you add 12 to it, the result will be 27. What is the number?
2. If John had 7 more marbles, he would have 18. How many has he?
3. If the length of a rectangle were 5 inches less, it would be 21 inches long. What is its length?
4. A certain number increased by 12 gives as a sum 35. What is the number?
5. The sum of a certain number and 7 is 18. Find the number.
6. Three times a certain number is 21. Find the number.

**Explanation**: Again, to save time, we agree that 3 times $n$ (8 times $p$, or 12 times $x$, etc.) shall be written $3 \cdot n$ or, more briefly, $3\,n$. Understand, therefore, that whenever you meet expressions like $8\,p$, $12\,x$, $17\,y$, etc., they mean multiplication, even though no "times" sign (×) is printed. In the same way $\frac{2}{3}$ of a certain number is written $\frac{2}{3}n$ or $\frac{2\,n}{3}$; $\frac{1}{2}$ of a certain number, $\frac{n}{2}$ or $\frac{1}{2}n$.

**7.** Two thirds of a certain number is 10. Find the number.

**8.** Three fourths of a certain number is 15. Find the number.

**9.** The difference between a certain number and 5 is 9. What is the number?

**10.** The difference between a certain number and 12 is 13. What is the number?

**11.** The sum of 16 and a certain number is 29. Find the number.

**12.** Three fifths of a certain number is 27. What is the number?

**13.** The product of 11 and a certain number is 77. Find the number.

**14.** If your teacher had $50 less, he would have $15. How much has he?

**15.** The quotient of a number and 7 is 3. What is the number?

**16.** If the area of a rectangle were increased 12 sq. ft., it would contain 40 sq. ft. What is its area?

**17.** 13 exceeds a certain number by 4. What is the number?

> **Explanation: Does this mean that 13 is larger, or smaller, than the certain number? How do you determine how much larger one number is than another?**

**18.** Two thirds of the number of pupils in a class is 28. How large is the class?

**19.** Tom lacked $7 of having enough to buy a $50 Liberty Bond. How much did he have?

**20.** Three times a certain number plus twice the same number is 90. Find the number.

**21.** The difference between 20 and a certain number is 4. What is the number?

> **Explanation: It seems most consistent to interpret, "*the difference between two numbers,*" as meaning, "*the first number minus the second number.*"**

**22.** The number of pennies Harry has exceeds 30 by 7. How many has he?

**23.** The sum of two numbers is 40. One of them is 27. What is the other?

**24.** The difference between two numbers is 21. The larger is 60. What is the smaller?

**25.** The product of two numbers is 95. One is 5. What is the other?

**26.** The quotient of two numbers is 13. The divisor is 5. Find the dividend.

In these exercises you have been changing, or **translating**, from the language of **ordinary words** into **algebraic language**; you have been making *algebraic* statements out of word statements. The essential thing in this translation is **the representation of numbers by letters.** We should note carefully also that we have begun the practice of using a letter to stand for a number which is **unknown.** Of course in these simple examples there is only *one unknown* number and it is easy to see at a glance each time what it is.

### HOW TO REPRESENT TWO OR MORE UNKNOWN NUMBERS, WHEN THEY HAVE A DEFINITELY KNOWN RELATION TO EACH OTHER

**Section 3.** In the examples which you have worked in preceding lessons, you have had to represent **only one number** in each problem. To illustrate, in Example 9, Exercise 3, as in all the other examples solved thus far — "the difference between a certain number and 5 is 9."

Only ONE number has to be represented. But in most of the examples that you will meet in mathematics you will have to represent two or more numbers which have a definitely known RELATION to each other. For example, consider this problem:

> **Suppose Tom has 5 times as many marbles as John has. How many do they both have?**

It is clear that there are TWO numbers to be represented; namely, the number that Tom has and the number that John has. Furthermore, since there is a definite RELATION between these two numbers, that is, one is 5 times the other, it is important to see that *each* can be represented by the use of the *same letter*.

If you **let** *n* stand for the number John has, what **must** represent the number Tom has? Since the example states that Tom has 5 times as many as John, **then** Tom must have $5n$ marbles. In the same way, together they have the sum of the two; namely, $n + 5n$, or $6n$.

The best way to state this, however, *in algebraic language*, is to use a set form like the following:

**Let $n =$ the number John has.**
**Then $5n =$ the number Tom has,**
**and $5n + n$, or $6n =$ the number both have.**

The next exercises will show how two or more unknown numbers may be represented by using the same letter, if the numbers have a **definite relation** to each other.

## EXERCISE 4

**1.** Harry has four times as many dollars as James has. If you let $n$ stand for the *number* of dollars James has, what will stand for the *number* Harry has? for the number they together have?

**2.** The number of inches in a rectangle is 7 times the number in its width. If $n$ stands for the *number* of inches in its width, what will represent the number in its length? in its perimeter?

**3.** An agent sold three times as many books on Wednesday as he sold on Tuesday. *Represent* the number sold each day. State algebraically that he sold 28 books during both days.

**4.** There are twice as many boys as girls in a certain algebra class. If there are $n$ girls, how many boys are there? How many pupils?

State algebraically that there were 36 pupils in the class. Find the number of boys.

5. On a certain day Fred sold half as many papers as his older brother. How can you represent the number each sold? the number both sold?

6. During a certain vacation period there were three times as many cloudy days as clear days. Express the number of each kind of days, and the total number of days. If the vacation consisted of 60 days, how many days of each kind were there?

7. A rectangle is three times as long as it is wide. If it is $x$ feet wide, how long is it? What is its perimeter?

8. If one side of a square is $s$ inches long, what is the perimeter of the square? State algebraically that the perimeter is 108 inches.

9. The sum of three numbers is 60. The first is three times the third, and the second is twice the third. If $n$ represents the third number, what will represent the first? the second? their sum? State algebraically that the sum is 60, and then find each number. Why do you think it was advisable to represent the **third** number by $n$?

10. John sold five times as many papers as Eugene. If $n$ represents the number Eugene sold, what will represent the number John sold? What expression will represent the difference in the number sold? Make a statement showing that this expression is 30.

**11.** A farmer sold four times as many dollars' worth of wheat as of corn. If he received $x$ dollars for the corn, what will represent the amount he received for both?

**12.** A has $n$ dollars. B has three times as many as A, and C has as many as both A and B. What will represent the number of dollars all three together have?

**13.** A horse, carriage, and harness cost $500. The carriage cost three times as much as the harness, and the horse twice as much as the carriage. If you let $n$ represent the number of dollars the harness cost, what will represent the cost of the carriage? of the horse? of all together? Make an algebraic statement showing that all three cost $500. Can you now find the cost of each?

**14.** A man had 400 acres of corn and wheat, there being 7 times as much corn as wheat. Show how the number of acres of each could be represented by some letter. Make an algebraic statement showing that he had 400 acres of both.

**15.** The rectangle shown in Fig. 2 is three times as long as wide. State algebraically that the perimeter is 64 in. What are its dimensions?

FIG. 2

In the problems just studied you have been considering two or more numbers which had a definite **RELATION** to each other and each of which *had* to be represented by using the *same letter*. For example, you had to note that one number was always a certain number of times another one, or was a certain part of another one. In each problem you had to decide which of the unknown numbers you would represent by that letter. In general it is best to represent the SMALLER of the unknown numbers by **n** or by **p** or by ANY letter. *The other numbers must then be represented by using* **the same letter** which you selected to represent the first one.

## EXERCISE 5

Write out the solution of each of the following. Be sure to use the complete form illustrated below, in solving each example.

**Illustrative example. The larger of two numbers is 7 times the smaller. Find each if their sum is 32.**

$$
\begin{aligned}
\textbf{Let } s &= \textbf{smaller no.}\\
\textbf{Then } 7s &= \textbf{larger no.}\\
\textbf{and } 7s + s &= 32,\\
\textbf{or } 8s &= 32,\\
\textbf{or } s &= 4,\\
\textbf{and } 7s &= 28.
\end{aligned}
$$

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1. William and Mary tended a garden, from which they cleared $72. What did each receive if it was agreed that William should get three times as much as Mary?
2. The perimeter of a rectangle is 48 inches. Find the dimensions if the length is 5 times the width.
3. The sum of three numbers is 60. The second is twice the first, and the third equals the sum of the first and second. Find each.
4. Divide $48 between two boys so that one shall get three times as much as the other.

**5.** Twice a certain number exceeds 19 by 5. Find the number.

**6.** The product of a certain number and 5 is 35. Find the number.

**7.** A man is twice as old as his son. The sum of their ages is 90 years. Find the age of each.

**8.** The sum of three numbers is 120. The second is twice the first and the third is three times the first. Find each.

**9.** The perimeter of a certain square is 144 inches. Find the length of each side.

**10.** The perimeter of a rectangle is 160 inches. It is three times as long as it is wide. Find its dimensions.

**11.** William is three times as old as his brother. The sum of their ages is 36 years. How old is each?

**12.** One number is five times another. Their difference is 16. Find each.

**13.** The sum of three numbers is 14. The second is twice the first, and the third is twice the second. Find each number.

**14.** One number is eight times another. Their difference is 63. Find each.

**15.** A rectangle (Fig. 3) which is formed by placing two equal squares together has a perimeter of 150 feet. Find the side of each square, and the area of the rectangle.

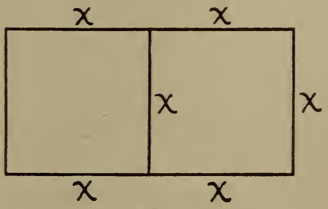

FIG. 3

**16.** Three men, A, B, and C, own 960 acres of land. B owns three times as many acres as A, and C owns half as many as A and B together. How many acres has each?

**17.** John sold half as many thrift stamps as Harry sold; Tom sold as many as both the other boys together. Find how many each sold, if all sold 144 thrift stamps.

**18.** Divide $21 among three boys, so that the first boy gets twice as much as the second, and the second boy gets twice as much as the third boy.

**Section 4. The most important thing in mathematics: the EQUATION.** In all the examples in Exercise 5 you have *translated* word sentences into algebraic sentences. These algebraic sentences are always called EQUATIONS.

They are called EQUATIONS because they show that one number expression is equal to another number expression. For example, you have stated that $n + 5 = 18$. This statement merely expresses equality between the number expression $n + 5$, on the left side of the $=$ sign, and the number expression 18, on the right side.

Furthermore, you have been *finding the value* of the unknown number in each of these EQUATIONS. From now on, instead of saying "*find the value of the unknown in an equation*," we shall say: "SOLVE THE EQUATION." For example, if you SOLVE THE EQUATION

$$7s + s = 32,$$

you "find the value" of $s$; namely, $s = 4$.

### EXERCISE 6

SOLVE the following EQUATIONS:

1. $p + 3 = 8$. This might be written: What no. $+ 3 = 8$, or $? + 3 = 8$.
2. $x - 5 = 10$. This might be written: What no. $- 5 = 10$, or $? - 5 = 10$.
3. $2n = 25$. This might be written: 2 times $? = 25$.
4. $5a = 275$. This might be written: 5 times $? = 275$.
5. $\frac{1}{2}x = 7$. This might be written: $\frac{1}{2}$ times $? = 7$.
6. $\frac{2}{3}c = 12$. This might be written: $\frac{2}{3}$ times $? = 12$.

> **It is always helpful to think of an equation as asking a question. Thus, $5a + 1 = 16$ should be thought of as the question: 5 times what number plus one gives 16?**

7. $2b + 1 = 21$
8. $5d - 3 = 27$
9. $4x = 13$
10. $12 = 3p$
11. $5 + n = 11$
12. $6 - n = 2$
13. $2p + 3p = 35$
14. $3x + 1 = 16$
15. $7b - 2 = 12$
16. $12s = 27$
17. $16 = 5y - 1$
18. $6t + 3t = 27$
19. $13 = 5y$
20. $21 = 5x + 1$
21. $4x = 17$

**Section 5. Translation from algebraic expressions into word expressions.** In the previous work you have translated from word statements into algebraic expressions. It is also very helpful to translate the algebraic expressions back into word expressions. For example, $n + 4 = 13$ is the same as the word statement "*the sum of a certain number and* 4 *is* 13." In the same way, the algebraic statement $4y = 26$ should be translated as follows:

"the product of a certain number and 4 is 26," or
"four times a certain number equals 26."

The next exercise will give practice in this important process, *i.e.* translating from algebraic statements into word statements.

### EXERCISE 7

Translate each of the following algebraic statements into word statements:

1. $y + 4 = 20$
2. $2b + 1 = 31$
3. $13 = 2 + y$
4. $2a + 3a = 55$
5. $\frac{1}{2}n + 1 = 12$
6. $n + 3n = 24$
7. $\frac{n}{5} + 1 = 8$
8. $a + b = 10$
9. $5x = 18$
10. $bh = 20$
11. $h = 60m$
12. $c - 4 = 12$
13. $a - b = 7$
14. $\frac{n}{d} = 4$
15. $\frac{1}{2}n - \frac{1}{5}n = 3$

---

### SUMMARY OF CHAPTER I

After studying this chapter you should have clearly in mind:

1. It saves time to *represent numbers by letters.*
2. Worded problems may be translated into *algebraic statements.*
3. *Equations* are statements that two numbers or two algebraic expressions are equal.
4. *Solving equations* means finding the value of the unknown number or letter in the equation.
5. *Algebraic expressions* may be *translated into word expressions.*

# CHAPTER II

## HOW TO CONSTRUCT AND EVALUATE FORMULAS

**Section 6. Further need for abbreviated language: Shorthand rules of computation.** In this chapter we shall study *abbreviated* or *shorthand* rules for solving problems. People who have found it necessary to compute over and over again the *areas* or *perimeters* of such figures as rectangles, triangles, circles, etc., have found it very convenient to abbreviate the rules for solving these problems into a kind of *shorthand expression* which can be more easily written or spoken than the long rules. For example, suppose you wanted to make a complete statement, either in writing or orally, concerning how to find the area of the rectangle which is represented by Fig. 4. You might express it, as you did in arithmetic, as follows:

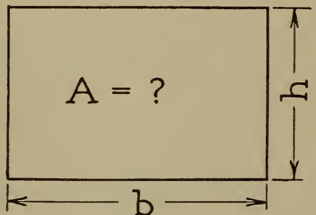

FIG. 4

(1) **The number of square units in the area of a rectangle is the number of units in its base times the number of units in its height.**

This long word rule can be greatly shortened by using abbreviations or suggestive letters to represent the number of units in each of its dimensions. Thus, a shorter way of expressing this rule is:

(2) $\text{Area} = \text{base} \times \text{height}.$

A third and still more abbreviated way of expressing it is:

(3) $A = b \times h,$

in which $A$, $b$, and $h$ mean, respectively, the **number of** UNITS in the area, base, and height. And finally, remembering that $b \times h$ is usually written as $bh$, the entire statement becomes:

(4) $A = bh.$

This last statement tells us everything that the first statement did, and requires much less time to read or to write. Such algebraic expressions are called FORMULAS.

**Section 7. What is a formula?** From the previous illustration we see that a *formula* is a shorthand, abbreviated rule for computing. We must remember, however, that the *formula* $A = bh$ is, at the same time, *an equation.* Since it is an equation that is *frequently* used, and which always appears in that particular *form*, we have come to call it a FORMULA.

## I. COMPUTATION OF AREAS AND PERIMETERS BY FORMULA

### EXERCISE 8

### COMPUTATION OF THE AREA OF RECTANGLES BY THE FORMULA

1. **Illustrative example.** Find the area of a rectangle (Fig. 5) in which $b = 10$ and $h = 7.5$, using the *formula*

$$A = bh.$$

Solution: (1) $A = bh.$
(2) $A = 10 \times 7.5.$
(3) $A = 75.$

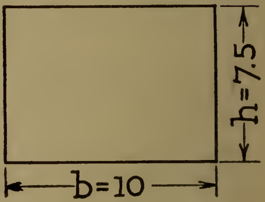

FIG. 5

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2. Find $A$ when $b = 8.25$ and $h = 4$.
3. Find $A$ when $h = 4.5$ and $b = 12$.
4. Find $b$ when $A = 50$ and $h = 5$.
5. What is $A$ if $b = 6.5$ and $h = 5.4$?
6. What is $h$ if $A = 40$ and $b = 6\frac{2}{3}$?
7. Find $A$ if $h = 2.5$ and $b = 6.4$.
8. What is $b$ if $A = 450$ and $h = 22.5$?

**9.** If $A = 200$ and $b = 7.5$, what does $h$ equal?

**10.** If $A = 625$ and $h = 50$, what does $b$ equal?

**11.** What is $A$ if $b = 40$ and $h$ is twice as large as $b$?

**12.** Find $A$ if $h = 16.2$ and $b = \frac{1}{2}$ of $h$.

**13.** $b = 12$ and $h = \frac{2}{3}b$. What is $A$?

**14.** Find $A$ if $h = 20$ and $h + b = 32$.

**Section 8. Perimeters of rectangles.** In Section 7 we saw that it was convenient to use a *formula* for the **area** of a rectangle. In the same way it is helpful to have a *formula* for the **perimeter** of any rectangle.

Since the perimeter of any rectangle is the sum of the bases and altitudes, the shortest way to express this is:

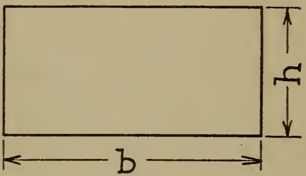


FIG. 6

(1) Perimeter = 2 × base plus 2 × height,
or by the *formula*

(2) $P = 2b + 2h$.

## EXERCISE 9

### COMPUTATION OF THE PERIMETERS OF RECTANGLES BY THE FORMULA

**1. Illustrative example.** Find the perimeter if the base is 13 and the height is 9; or, more briefly, find $P$ if $b = 13$ and $h = 9$.

Solution: (1) $P = 2b + 2h$.
(2) $P = 2 \cdot 13 + 2 \cdot 9$.
(3) $P = 26 + 18 = 44$.

---

**2.** Find $P$ if $h = 10.5$ and $b = 9$.

**3.** Find $P$ if $h = 18.4$ and $b = 12.8$.

**4.** What is $h$ if $P = 40$ and $b = 10$?

5. What is $b$ if $P = 60$ and $h = 14$?

6. Find $h$ if $P = 18.4$ and $b = 4.6$.

7. If $P = 110$ and $h = 22.5$, what is $b$?

8. What is $P$ if $h = 18$ and $b = 2\,h$?

9. $P = 100$. Find $b$ and $h$ if $b = h$.

10. What is $h$ if $P = 120$ and $b = \frac{1}{5}P$?

**Section 9. The formula for the area of any triangle.** What is the area of this triangle if its base is 12 ft. and its height is 8 ft.? How do you find the area of any triangle if you know its base and altitude?

Show that the most economical way to state this rule, or RELATION, between the area, the base, and the height, is by the formula $A = \frac{bh}{2}$.

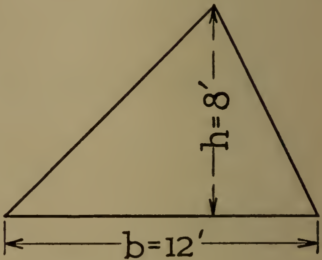


FIG. 7

The examples in the following exercise will give you practice in using this important *formula*.

### EXERCISE 10

#### COMPUTATION OF THE AREA OF TRIANGLES BY THE FORMULA

1. **Illustrative example.**

**Find the value of $A$ if $b = 22$ and $h = 12$.**

Solution: (1) $A = \frac{bh}{2}$.

(2) $A = \frac{22 \times 12}{2} = \frac{264}{2} = 132.$

---

**Write your work in a neat, systematic form.**

2. Find the value of $A$ if $b = 18$ and $h = 6\frac{1}{2}$.

3. What is the value of $A$ if $b = 12.5$ and $h = 20$?

4. If $h = 16.8$ and $b = 28$, what does $A$ equal?
5. What is the value of $b$ if $A = 300$ and $h = 50$?
6. Find $h$ if $A = 240$ and $b = 20$.
7. Determine $b$ if $A = 100$ and $h = 15$.
8. What is $A$ if $h = 6.25$ and $b = 10.5$?
9. Find the value of $A$ if $b = 22$ and $h = \frac{6}{11} b$.
10. What is $b$ if $A = 120$ and $h = \frac{1}{5} A$?
11. $h = 20$ and $b = \frac{7}{5} h$. What does $A$ equal?
12. Can you find $b$ and $h$ if $A = 256$ and $b = 2h$?
13. Two triangles have equal bases, 10 in. each, but the height or altitude of one is twice that of the other. Are their areas equal? Show this by an illustration.
14. What change occurs to $A$ if $b$ is fixed in value, but if $h$ gets larger? What is the RELATION between $A$ and $h$ if $b$ is fixed?

**Section 10. The formula for the circumference of any circle.** You will recall from arithmetic the following statement for the circumference of a circle:

> "The circumference of a circle is obtained by multiplying twice the radius by 3.1416."

With our new method of using letters instead of words or numbers, this is much more briefly expressed by the *formula*

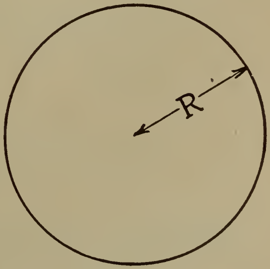


FIG. 8

$$C = 2\pi R.$$

$\pi$ is a symbol used to represent the number 3.1416. Use this value for $\pi$ in the problems which follow.

## EXERCISE 11

### COMPUTATION OF THE CIRCUMFERENCE OF CIRCLES BY THE FORMULA

1. What is the value of $C$ if $R = 12$?
2. What does $C$ equal if $R = 5\frac{1}{2}$?
3. Find the value of $R$ if $C = 31.416$.
4. Determine $R$ when $C = 100$.
5. Find $C$ if the diameter of the circle is 16.4.
6. What is the value of $C$ if $R = \frac{22}{5}$?
7. A Boy Scout wishes to make a circular hoop from a piece of wire 14 ft. long. Determine the radius of the largest possible hoop he can make.
8. The radius of one circle is 5, and the radius of another is twice as great. Find the circumference of each, and note whether one circumference is twice the other.
9. Think of a circle of some particular radius. Then imagine that the radius begins to increase. What happens to the circumference? Is there any particular connection or *relation* between $C$ and $R$?

## II. "EVALUATION": HOW TO FIND THE NUMERICAL VALUE OF AN ALGEBRAIC EXPRESSION

**Section 11.** In the examples which you have just solved we have used the long expression "What is the value of" or "Find the value of" in referring to the particular letter which was to be found. Instead of these long expressions we shall now use the single word EVALUATE. It means exactly the same thing as the longer expression. Thus to **evaluate** an algebraic expression means to find its numerical value, exactly as in the previous examples. This is done by "putting in" or by **substituting** numerical values for the letters. A few examples will make this clear.

### EXERCISE 12

#### EVALUATION OF COMMONLY USED FORMULAS

1. Evaluate $A = \frac{bh}{2}$ if $b = 10$ and $h = 14.6$.
2. Evaluate, or find the value of, $P$ in the expression $P = 2b + 2h$ if $b = 26$ and $h = 12.4$.
3. Evaluate $C = 2\pi R$ if $R = 14$.
4. Evaluate $V = lwh$ if $l = 10$, $w = 6\frac{1}{2}$, and $h = 5$.
5. Find the value of $i$ in the formula $i = prt$ if $p = \$640$, $r = \frac{5}{100}$, and $t = 4$.
6. Evaluate $A = \pi R^2$ if $R = 6$.
7. Evaluate $c = \frac{E}{R}$ if $E = 110$ and $R = 10.5$.
8. What is $h$ in the algebraic expression $P = 2b + 2h$ if $P = 80$ and $b = 12.8$?

## III. THE USE OF EXPONENTS TO INDICATE MULTIPLICATION

**Section 12. Need of short ways to indicate multiplication.** A very large part of our work in mathematics is that of **finding numerical values**. In many of our problems, therefore, we shall need short ways of *indicating* multiplication. For example, in arithmetic, the multiplication of $5 \times 5$ is sometimes written as $5^2$; or the multiplication of

$6 \times 6 \times 6$ as $6^3$. In *algebra*, to save time, this notation, or method of indicating multiplication, *is always used*. Thus, instead of writing $b \times b$ or $n \times n \times n$ we will write $b^2$ or $n^3$. This little number that is placed to the right of and above another number tells how many times that number is to be used as a factor. These numbers are called EXPONENTS. Numbers with exponents are read as follows:

> **$3a^2$ means 3 times $a$ times $a$, and is read "3 $a$ *square*." This does NOT mean 3 $a$ times 3 $a$. The exponent affects only the $a$.**
>
> **$5b^3$ means 5 times $b$ times $b$ times $b$, and is read "5 $b$ *cube*." This does NOT mean 5 $b$ times 5 $b$ times 5 $b$. The exponent affects only the $b$.**

Here, as well as throughout all later mathematical work, you will need to be able to evaluate algebraic expressions which involve exponents. For example, the *area* of the rectangle shown here is the expression $3W^2$, which is obtained by multiplying $3W$ by $W$. Now the *numerical value* of this area depends upon *the value of W;* that is, if $W$ is 4, then the area is $3 \cdot 4 \cdot 4$, or 48; but if $W$ is 2, then the area is $3 \cdot 2 \cdot 2$, or 12. In the same way the *volume* of the rectangular box in Fig. 10 is represented by the expression $2x^3$, or $2x \cdot x \cdot x$. Again, you see that the numerical value of the volume *depends upon the value of x*. Thus, if $x$ is 5, the volume is obtained by *evaluating* the expression $2x^3$, which gives $2 \cdot 5 \cdot 5 \cdot 5$, or 250. The next exercise gives practice in evaluating algebraic expressions containing exponents.

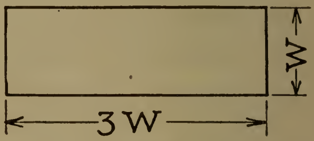


FIG. 9

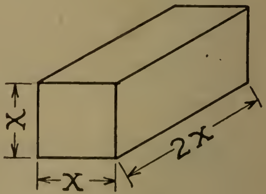


FIG. 10

## EXERCISE 13

### PRACTICE IN EVALUATION

1. **Illustrative example.**

**Evaluate $2\,ab^2 + 3\,a^2b + ac$, if $a = 4$, $b = 3$, and $c = 1$.**

**Solution:** $2 \cdot 4 \cdot 3 \cdot 3 + 3 \cdot 4 \cdot 4 \cdot 3 + 4 \cdot 1 = 72 + 144 + 4$
$= 220.$

**Note that the numbers are substituted for, or put in place of, the letters.**

---

Using the values of $a$, $b$, and $c$ given in Example 1, evaluate each of the following expressions:

2. $a^2 + b^2 + c^2$
3. $3\,abc$
4. $a^2b + ab^2$
5. $ac^2 + cb^2 + ba^2$
6. $a^3 + b^3 + c^3$
7. $\frac{a}{b} + \frac{b}{a} + \frac{c}{b}$
8. $\frac{a - b + c}{a + b - c}$
9. $a^2bc^2$
10. $\frac{4}{a} + \frac{3}{b} + \frac{1}{c}$
11. $\frac{a^3 - b^3 - c^3}{2\,a^2}$
12. $a^b + b^a + c^a$

13. The formula $d = 16\,t^2$ tells how far an object will fall in any number of seconds. Find how far a body will fall in 1 sec. of time, that is, when $t = 1$. Do you believe it? How could you test it?

14. Using the formula in Example 13, find how far an object will fall in 2 sec. of time, that is, when $t = 2$. How could you test the truth of this?

15. The horsepower of an automobile is given by the following formula: $\frac{DN^2}{2.5}$, in which $D$ represents the diameter of the piston, and $N$ the

number of cylinders. What is the horsepower of a Ford, which has 4 cylinders, and in which $D = 3\frac{1}{2}$ in.?

## IV. THE CONSTRUCTION OF FORMULAS

**Section 13.** It is very important to be able to make a formula for any computation that must be performed over and over again. For example, we often have to find the area of a square. Instead of saying or writing each time "the area of a square is equal to the square of the number of units in one of its sides," it saves time to use the formula $A = s^2$, in which $A$ = area and $s$ = one of the sides. *This formula tells all that the word rule says and requires much less effort.* To give practice in this kind of work, construct a formula for each of the examples in the following exercise.

### EXERCISE 14

**1.** (*a*) Find the volume of a rectangular box whose dimensions are 12, 8, and 6 inches.

(*b*) Make a formula for the volume of any box.

**2.** (*a*) What is the area of a circle whose radius is 9 in.?

(*b*) Write the formula for the area of any circle.

**3.** (*a*) How many square inches in the entire surface of a cube whose edge is 8 inches?

(*b*) Give a formula for the area of the entire surface of any cube.

**4.** (*a*) Find the interest on $400 for 2 years at 6%.

(*b*) Make a formula for the interest on any principal for any rate and for any time.

5. (*a*) How many cubic inches in a block $2'$ by $3'$ by $4'$?
   (*b*) Make a formula for the number of cubic inches in any rectangular solid whose dimensions are expressed in feet.
6. Make a formula for, or an equation which tells, the cost of any number of pounds of beans at 12 cents per pound.
7. What equation or formula will represent the area of any rectangle whose base is 5 inches, but whose height is unknown? Evaluate your formula for $h = 3.4$.
8. An automobilist travels 20 miles per hour. What formula or equation will represent the distance he travels in $t$ hours? Evaluate this formula: $t = 5$ hr. 20 min.

### REVIEW EXERCISE 15

1. What does an equation express? Is $7 + 4 = 6 + 6$ an equation?
2. Does $2n + 1 = 21$, if $n = 9$, make an equation? if $n = 10$?
3. The formula for the perimeter of a rectangle, $p = 2b + 2b$, contains three *unknown* numbers. How many of them must be *known* in order to use this formula to solve an example?
4. Read each of the following equations as questions, and find the value of the unknown number:

(*a*) $4y + 3 = 21$
(*b*) $20 = 6 + 2x$
(*c*) $5c + 2 = 42$
(*d*) $3y - 5 = 16$
(*e*) $x + x = 36$
(*f*) $b + b + 1 = 23$

5. Three times a certain number, plus 2, equals 38. Find the number.
6. Donald saved twice as much money as his older brother. Express in algebraic language that both together saved $96. How much did each save?
7. Evaluate the formula $V = l^3$ ($V =$ the volume of a cube), if $l = 4\frac{1}{2}$.
8. The first of three numbers is twice the second, and the third is twice the first. Find each number if their *sum* is 105.
9. Construct a formula for the cost of any number of eggs at 30 cents per dozen.
10. What is the difference in meaning between $10n$ and $n + 10$? Does $4w$ mean the same as $4 + w$?

---

## SUMMARY

The most important principles and methods which we have learned in this chapter are the following:

1. **A formula is merely a shorthand rule of computation.**
2. **Formulas are "evaluated" or "solved" by substituting numbers for the letters in the formula.**
3. **Exponents are used as short methods of indicating multiplication. An exponent of a number tells how many times that number is taken as a factor.**
4. **We should construct a formula for any kind of problem which we have to solve frequently.**

# CHAPTER III

## HOW TO USE THE EQUATION

**Section 14. The importance of the equation.** Nothing else in mathematics is as important as the *equation*, and the power to use it well. It is a *tool* which people use in stating and solving problems in which an unknown quantity must be found. In the last chapter we saw that the *formula*, or *equation*, was used to find unknown quantities, sometimes the area, sometimes the perimeter, etc. The fact that the *equation* is used as a means of solving such a large number of problems is the reason we shall study it very thoroughly in this chapter.

**Section 15. The equation expresses balance of numerical values.** The equation is used in mathematics for the same purpose that the weighing "scale" is used by clerks; that is, to help in finding some value which is unknown. The scale **represents balance of weights**; similarly the equation **represents balance of numerical values.** To understand clearly the principles which are applied in dealing

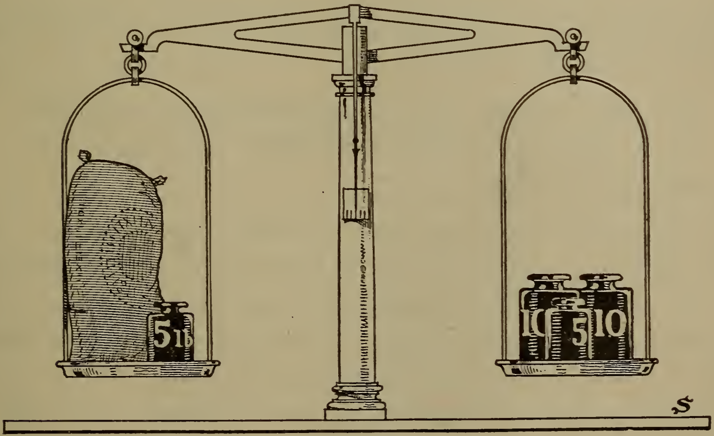


FIG. 11

with equations, we should consider the scale, as represented in Fig. 11. In this case a bag of flour of unknown weight, together with a 5-pound weight, balances weights which total 25 pounds on the other side of the scale.

Now, if $n$ represents the number of pounds of flour, it is clear that the *equation*

$$n + 5 = 25$$

represents a *balance of numerical values*. Obviously, $n$ is 20, for the clerk would take 5 pounds of weight from each side, and still keep a *balance of weights*.

This principle, namely, that the same weight may be taken from each side without destroying the balance of weights, can be applied to the equation

$$n + 5 = 25.$$

That is, we may *subtract 5 from each side* of the equation, giving another equation,

$$n = 20.$$

This suggests an important principle that may be used in solving equations; namely, —

> **The same number may be subtracted from each side of the equation without destroying the equality, or balance, of values.**

If you take something from one side of the scale, or of the equation, what MUST you do to the other side? **Why?**

The fact that the equation expresses the idea of balance makes it easy to reason about it, and find out all the things that can be done without changing the balance or equality. The next exercise suggests this kind of study of the equation.

## EXERCISE 16

By *thinking of the equation as a balance*, you should be able to complete the following statements. Fill in the blanks with the proper words.

1. Any number may be subtracted from one side of an equation if __?__ is __?__ from the other side.
2. Any number may be added to one side of an equation if __?__ is __?__ to the other.
3. One side of the equation may be multiplied by any number, if the other side is __?__ by the __?__.
4. One side of an equation may be divided by any number, if the other side is __?__ by the __?__.

These are very important principles, and are used in solving any equation. They are generally called AXIOMS. They must be understood and mastered. The examples of the next exercise have been planned to help you learn how to apply them.

## EXERCISE 17

In each of the following examples you can use one of the four principles stated above to explain what has been done, or to state the reason for doing it. Thus, if $2x = 8$, then $x = 4$, because of the principle:

"One side of an equation may be divided by a number if the other side is divided by the same number."

For each example, you are to state the principle which permits or justifies the conclusion.

1. If $4b = 22$, then what is done to each side to give $b = 5\frac{1}{2}$?
2. If $\frac{1}{2}y = 7$, then to get $y = 14$, what do you do to each side?
3. If $x + 4 = 13$, then to get $x = 9$, what do you do to each side?
4. If $5c = 32.5$, then what is done to each side to give $c = 6.5$?
5. If $6a = 12$, then what is done to each side to give $3a = 6$?
6. If $y - 4 = 7$, then what is done to each side to give $y = 11$?
7. If $x = 2$ and $y = 3$, then why does $x + y = 5$?
8. If $b = 3$ and $c = 10$, then why does $bc = 30$?
9. If $x = 12$ and $y = 4$, then why does $\frac{x}{y} = 3$?
10. If $a - 1 = 9$, then why does $a = 10$?
11. If $b = 2h$, then what is done to each side to give $3b = 6h$?
12. If $4x + 3 = 23$, then what is done to each side to give $4x = 20$?
13. If $5y - 3 = 27$, then what is done to each side to give $5y = 30$?
14. If $x + 7 = 19$, then to make $x = 12$, what is done to each side?
15. If $2c - 4 = 8$, then to make $2c = 12$, what is done to each side?
16. If $3b + 1 = 22$, then what is done to each side to give $b = 7$?

**17.** If $5b + 2 = 47$, why does $5b = 45$? Then why does $b = 9$?

**18.** If $6x + 2 = x + 22$, then why does $5x + 2 = 22$? and why does $5x = 20$? and why does $x = 4$?

**19.** If you know that $4w + 3 = w + 27$, then why does $4w = w + 24$?

and why does $3w = 24$?
and why does $w = 8$?

These examples are given to emphasize the fact that there are certain changes that can be made on both sides of an equation, without destroying the balance or equality. It should be clear that there must be some axiom or principle to justify *every* change that is made.

### EXERCISE 18

Find the value of the unknown number in each of the following equations, *telling exactly what you do to each side of the equation.*

**1.** $x + 5 = 13$
**2.** $3a = 17$
**3.** $26 = 4y$
**4.** $2b + 1 = 19$
**5.** $y - 5 = 12$
**6.** $\frac{1}{2}a = 4.5$
**7.** $\frac{a}{3} = 5$
**8.** $2x - 3 = 17$
**9.** $15 = x + 7$
**10.** $2x + 3x = 35$
**11.** $5y + 4y + y = 30$
**12.** $\frac{15}{b} = 3$
**13.** $5c - 2 = 38$
**14.** $27 = 6x - 3$
**15.** $4b + 7 = 47$
**16.** $4y - y = 21$
**17.** $5x + 1 = 23$
**18.** $2\frac{1}{2}x = 15$
**19.** $\frac{2}{3}a = 18$
**20.** $\frac{16}{y} = 8$

| | |
|---|---|
| **21.** $2b + 3b = 42$ | **24.** $18 + x = 13 + 10$ |
| **22.** $c - 4 = 13$ | **25.** $7 + 2x = 23 + 10$ |
| **23.** $2b - 1 = 18$ | **26.** $\frac{1}{2}x + \frac{1}{4}x = 18$ |

## HOW TO CHECK THE ACCURACY OF THE SOLUTION OF AN EQUATION

**Section 16. When is an equation solved?** We have already noted that *an equation is solved when the* **numerical value** *of the unknown number is found.* Thus, the equation $4a + 3 = 29$ is SOLVED when the numerical value of $a$ is found. This leads to another very important question; that is: How can you be certain your solution is correct? In other words, how can you *test* or *check* the accuracy of your work?

For example, suppose that in solving the equation

$$4a + 3 = 29$$

one member of your class obtains 8 for the value of $a$. Is his result correct? There is only one way to be sure. That is to *substitute* or "put in" 8 in place of $a$ in the equation, to see whether the numerical value of the left side equals the numerical value of the right side. In other words, does

$$4 \cdot 8 + 3 = 29?$$

Clearly, not. Therefore, the solution is incorrect; it does not CHECK. Then what is the correct value of $a$? Some of you doubtless think it is $6\frac{1}{2}$. Let us *test* or *check* by substituting $6\frac{1}{2}$ for $a$, to see if the numerical value of one side of the equation will equal the numerical value of the other side. Does

$$4 \cdot 6\tfrac{1}{2} + 3 = 29?$$

Yes. Then **the equation is solved**, or, to use the more general term, **the equation is** SATISFIED when $a = 6\frac{1}{2}$.

Summing up, then, an equation is SOLVED **when a value of the unknown is found which** SATISFIES **the equation**; that is, one which makes the numerical value of one side equal to the numerical value of the other side. The *solution* of the equation *is checked* by substituting for the unknown number the value which we *think* it has. If, as the result of the substitution, we get a balance of values, then we know that the equation has been solved correctly.

We have already had practice in substituting numerical values for letters. In Chapter II, we called this EVALUATION. Thus you see that each time you *check* the solution of an equation, you are *evaluating* the *original equation.*

## EXERCISE 19

### PRACTICE IN CHECKING THE SOLUTION OF EQUATIONS

1. The pupils in a class tried to solve the equation
$$6a - 3 = 39.$$
A few decided that $a = 7$, while the others insisted that $a = 6$. Which group was right? Show how they could have checked or tested their result. Why, do you think, some pupils got 6 for the value of $a$?

2. Does $x = 5$ in the equation $12x - 7 = 10x + 3$? In other words, does $x = 5$ **satisfy** this equation?

3. Would you give full credit on an examination to a pupil who said that $y = 4\frac{1}{2}$ would **satisfy** the equation $8y - 4 = 6y + 3$? Justify your answer.

4. Show whether the equation $b^2 + 5b = 24$ is **satisfied** or solved if $b = 3$; if $b = 2$.

5. Do you agree that the value of $x$ is 6 in the equation $10x - 4 = 58$? Justify your answer.

6. Is the equation $\frac{b+10}{3}+6=\frac{3}{4}b+8$ **satisfied** when $b=8$?

7. Does $x=24$ **satisfy** the equation

$$\tfrac{1}{2}x+\tfrac{1}{3}x+\tfrac{3}{8}x=29?$$

8. State in words how the solution of an equation is tested or checked.

9. What is the value of learning to check very carefully every kind of work you do?

## EXERCISE 20

Solve each of these equations. *Write out your work for each one* in the complete form illustrated in the first example. *Check each* one so that you can be absolutely certain that your work is correct.

1. **Illustrative example.**

$$6b-4=24.$$

**(1) By adding 4 to each side, we get**

$$6b=28.$$

**(2) By dividing each side by 6, we get**

$$b=4\tfrac{2}{3}.$$

**(3) Checking,**

$$6\cdot 4\tfrac{2}{3}-4=24.$$

$$28-4=24.$$

---

2. $5c-2=38$
3. $6b+3=45$
4. $7x=x+30$
5. $10a-3a=17\frac{1}{2}$
6. $22=5x+2$
7. $2\frac{1}{2}y+1=26$
8. $b+5b=20+b$
9. $4y=13+y$
10. $3x+2x+6x=66$
11. $5c+3=78$
12. $\frac{1}{2}y=8\frac{1}{2}$
13. $\frac{n+1}{5}=4$
14. $10b+3=7b+15$
15. $12x-2=5x+26$
16. $13y=2y+3y+4y+8$

## HOW TO GET RID OF FRACTIONS IN AN EQUATION

**Section 17. The use of the most convenient multiplier.** In many equations that you have solved already it has been necessary to multiply each side of the equation by some number. For example, in $\frac{1}{2}x = 10$, it is necessary to multiply each side by 2, which gives $x = 20$. Or, if you wanted to solve the equation $\frac{1}{5}x = 3$, it is necessary to multiply each side by 5, giving $x = 15$.

But, suppose you had an equation

$$\tfrac{1}{2}x + \tfrac{1}{5}x = 14,$$

would you get rid of *both* fractions by multiplying each side by 2? Would you get rid of both fractions by multiplying each side by 5? Here, as in all equations of this kind, you have to find some number which is a multiple of the different denominators. For this reason, in this example, 10 is the *most convenient number* by which to multiply each term in the equation.

**Illustrative example.**

Multiplying each side of (1) by 10, we get

$$10 \cdot \tfrac{1}{2}x + 10 \cdot \tfrac{1}{5}x = 10 \cdot 14,$$

or

$$5x + 2x = 140,$$

or

$$7x = 140,$$

or

$$x = 20.$$

Checking, by substituting the value of $x$ in the original equation, gives

$$\tfrac{1}{2} \cdot 20 + \tfrac{1}{5} \cdot 20 = 14,$$
$$10 + 4 = 14.$$

The study of this example shows that we can get rid of fractions in an equation if we multiply each side by the lowest common multiple of the denominators. We shall call this the MOST CONVENIENT MULTIPLIER.

## EXERCISE 21

### PRACTICE IN SOLVING FRACTIONAL EQUATIONS

*Solve and check* each example.

1. **Illustrative example.** $\frac{2}{3}n + \frac{1}{4}n = 22$. What is $n$?

(1) Multiplying each side by the *most convenient multiplier*, 12, gives

$$12 \cdot \tfrac{2}{3}n + 12 \cdot \tfrac{1}{4}n = 12 \cdot 22,$$

or

$$8n + 3n = 264.$$

(2) By adding $8n$ and $3n$, we get

$$11n = 264.$$

(3) By dividing each side by 11, we get

$$n = 24.$$

(4) Checking, by substituting the value of $n$ (24), in the original equation,

$$\tfrac{2}{3} \cdot 24 + \tfrac{1}{4} \cdot 24 = 22.$$
$$16 + 6 = 22.$$

---

2. What is the value of $x$ in the equation

$$\tfrac{1}{2}x + \tfrac{1}{5}x = 14?$$

3. A man spent $\frac{1}{5}$ of his income for rent and $\frac{1}{6}$ for groceries. Using $n$ to represent his income, make an equation which will state that he spent \$660 for rent and groceries. Solve the equation.

4. The dimensions of a rectangle are indicated on Fig. 12. What equation will state that the perimeter is 36 in.? Solve the equation for $l$.

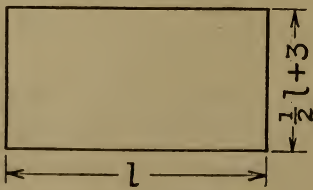


FIG. 12

5. $\frac{4}{5}b + b = \frac{1}{2}b + 13$. What is $b$?

6. If three fourths of a single number be diminished by one half of the number, the remainder is 10. Find the number.

7. $\frac{2}{3}x + \frac{3}{4}x - 1 = 3$.

8. Three boys together had 65 cents. Tom had half as much as Harry, and Bill had two thirds as much as Harry. Translate this into an equation, and solve.

9. $\frac{n}{2} + 2 + \frac{n}{3} = 4 + \frac{n}{2}$. What does $n$ equal?

10. One half of a certain number increased by four fifths of the same number gives 52 as a result. Find the number.

11. Harry made two thirds as much money last year selling the Saturday Evening Post as John made; Edward made three fourths as much as John. How much did each boy earn if all together earned $145?

12. $x + \frac{2}{3}x - 6 = 24$.

13. The sum of the third, fourth, and sixth parts of a number is 18. Find the number.

**Section 18. How word problems are solved by equations.** It is important to note the principal steps involved in solving word problems. Let us take, as an illustration, Example No. 8 in Exercise 21.

> **Three boys together had 65 cents. Tom had half as much as Harry, and Bill had two thirds as much as Harry. How much had each?**

1. The first important step in solving a word problem is to get in mind very clearly **what is known** and to recognize **what is to be found out.** In all problems some things are known and some things are to be determined. Thus, in this problem, we know how much money all the boys have together; and we also know that Tom has half as much as Harry; furthermore, we know that Bill has two thirds as much as Harry. That is, we see that the statement of the amount that Tom and Bill each has DEPENDS UPON the statement of the amount that Harry has.

2. But we do not know how much Harry has. Then, as in all word problems, we represent by some letter, such as $n$, the number of dollars Harry has. In other words, the *second step* is to get clearly in mind what quantities are unknown, and **to represent one of them by some letter.**

3. Next, **all the parts** or conditions of the problem **must be expressed by using the** SAME **letter.** Thus, if Harry has $n$ dollars, the number that Tom and Bill each has must be represented by using the same letter $n$, and NOT some other letter. That is, the word statement must be *translated* into an algebraic statement. It is always necessary, and usually difficult, to see that there must be a balance, an equality, between the parts of the problem. Thus, we must see that Harry's money, $n$, plus Tom's money, $\frac{1}{2}n$, plus Bill's money, $\frac{2}{3}n$,

must *balance*, or equal, 65 cents. This gives the complete algebraic statement:

$$n + \tfrac{1}{2}n + \tfrac{2}{3}n = 65.$$

4. The equation which we have obtained must be *solved.* A value of the unknown must be found which will **satisfy** the equation. In this case $n$ proves to be 30.

5. Finally, the **accuracy** of the result must be **tested** by substituting the obtained value of $n$ in the original word statement of the problem, to see if the statement holds true.

### EXERCISE 22

Translate into algebraic language, and solve each of the following word statements. *Check each one.*

1. Six more than twice a certain number is equal to 12. Find the number.
2. Four times a certain number is equal to 35 diminished by the number. What is the number?
3. I am thinking of some number. If I treble it, and add 11, my result will be 32. What number have I in mind?
4. If fourteen times a certain number is diminished by 2, the result will be 40. Find the number.
5. What is the value of $y$ in the equation
$$4y + \tfrac{1}{3}y + 2 = 28?$$
6. If seven times a certain number is decreased by 8, the result is the same as if twice the number were increased by 32. Find the number.

**7.** An algebra cost 12 cents more than a reader. Find the cost of each if both cost $1.64.

**8.** The sum of the ages of a father and his son is 57 years. What is the age of each if the father is 29 years older than the son?

**9.** The length of a school desk top exceeds its width by 10 inches; and the perimeter of the top is 84 inches. What are its dimensions?

**10.** Divide $93 between A, B, and C, so that A gets twice as much as C, and B gets $10 more than C.

**11.** A farmer sold a certain number of hogs at $20 each, and twice as many sheep at $14 each. How many of each did he sell if he received $576 for all?

**12.** Should a teacher give James full credit for the solution of the equation

$$4\tfrac{1}{2}x - 7 = 3x + 5$$

if he obtained $x = 8\frac{1}{2}$? Justify your answer.

**13.** Make a drawing of a rectangle whose perimeter is represented by the expression $6y + 20$, writing the dimensions on the drawing.

**14.** The length of a rectangle is 5 inches more than twice its width; its perimeter is 46 inches. What are its dimensions?

**15.** A school garden was $3\frac{1}{2}$ times as long as wide. To walk around it required 31 steps (27 in. each). Tell *how* to find its width, but do not actually find it.

## REVIEW EXERCISE 23

**1.** The formula $h^2 = a^2 + b^2$ is used in finding a side of a right triangle. Evaluate it if the base is 13 and the altitude is 5.

**2.** Make a formula for the number of revolutions made by the front wheel of a Ford car in going a mile, if the radius of the wheel is 14 inches.

**3.** In what sense does the equation $7b - 5 = b + 25$ ask a question?

**4.** Give one illustration of the advantage of using letters for quantities.

**5.** What are the four fundamental principles or axioms which are used in solving equations?

**6.** Does $x = 4\frac{1}{2}$ satisfy the equation $x^2 - 3x = 6$?

**7.** Using $m$, $s$, and $d$ for minuend, subtrahend, and difference, respectively, what equation or equations can you make from them?

**8.** An autoist travels at an average rate of 24 mi. per hour. What distance will he cover in 2 hr.? in 5 hr.? in 10 hr.? Make an equation or formula for the distance he will travel in $t$ hr.

**9.** Write a formula for the cost of any number of pounds of bacon at 30 cents per pound.

**10.** Draw rectangles with bases of 2 inches each. What formula will represent the area of any such rectangle if $h$ represents the height?

**11.** How do you get rid of fractions in an equation? What is the *most convenient multiplier* in any particular equation?

12. *When* is an equation *solved?*
13. If $M$, $m$, and $p$ stand for the multiplicand, multiplier, and product, respectively, what formulas can you make from them?
14. If $n$ is a whole number, what is the whole number next larger than $n$? the whole number next smaller than $n$?
15. In getting rid of fractions in the equation

$$\tfrac{1}{2}x + \tfrac{2}{3}x = 14,$$

show that 6 is a *more convenient multiplier* than 12, 24, 18, 30, etc.

### SUMMARY

From your study of this chapter, the following principles and methods should be kept clearly in mind:

1. **Equations express *balance* of value.**
2. **If any change is made on one side of the equation, the *same* change must be made on the other side.**
3. **An equation is solved when a value of the unknown is found which *satisfies* the equation.**
4. **The accuracy of your solution is *checked* by evaluating the equation for the value of the unknown.**
5. **You can get rid of fractions in an equation by multiplying each side by the lowest common multiple of the denominators; that is, by the *most convenient multiplier*.**

# CHAPTER IV

## HOW TO REPRESENT THE RELATIONSHIP BETWEEN QUANTITIES WHICH CHANGE TOGETHER

**Section 19. The chief aim of mathematics.** As we go about our daily work, **we commonly deal with quantities which change together.** For example, the cost of a railroad ticket *changes* as the number of miles you travel *changes;* that is, the *cost* and the *distance change together.* Or, the *distance* traveled by an autoist, if he goes at the rate of, say, 20 miles per hour, *changes* as the number of hours which he travels *changes;* that is, the *distance* and the *time change together.* As a third illustration, suppose you wanted to make a trip of 100 miles. We know that the time required will change with or be determined by the way in which the rate changes; that is, the *time* and the *rate change together.*

The fact that we are always dealing with situations of this kind makes it necessary for us to know how to *represent and determine these quantities which change together*, or which are RELATED in some definite way. Mathematics shows us how to describe or express them. In fact, it is the **chief aim of mathematics** to help you to see how quantities are related to each other and to help you to determine their values.

**Section 20. The three methods of representing relationship.** People have used three different methods for doing this:

I. THE TABULAR METHOD
II. THE GRAPHIC METHOD
III. THE EQUATIONAL OR FORMULA METHOD

This chapter will show how quantities, which are so *related* that they *change together*, can be represented by these methods. (*It should be pointed out that it is not always possible to use the* EQUATIONAL OR FORMULA METHOD.)

To illustrate, suppose you wanted to tell some one about the temperature in Chicago on a certain July day. There are *two* ways to do this; first, you might make a *table*, like the following, which would tell the temperature at each hour during the day.

TABLE 1

AN ILLUSTRATION TO SHOW THE WAY TO TABULATE TEMPERATURE AT DIFFERENT HOURS OF THE DAY

| | A.M. | | | | | | | P.M. | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Hour | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 1 | 2 | 3 | 4 | 5 | 6 |
| Temperature | 70 | 72 | 72 | 76 | 80 | 83 | 85 | 90 | 96 | 94 | 86 | 76 | 77 |

This method, which we shall call the TABULAR METHOD, shows the way the temperature changes at different hours of the day. To understand the table, however, requires much more effort on the part of the reader than is required to understand the second method, which is shown below. This pictorial or GRAPHIC METHOD shows all that the *tabular method* shows, and has the advantage of being more easily interpreted.

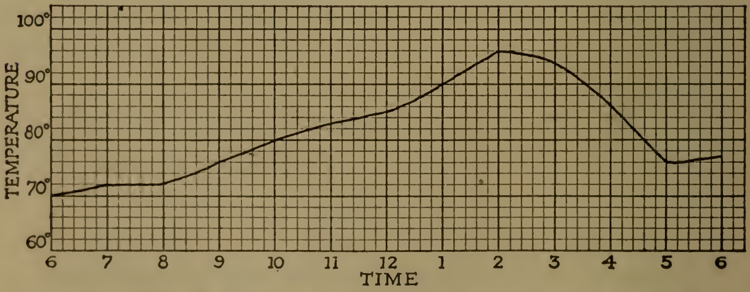


FIG. 13. **Graphic representation of temperatures at various hours of the day.**

Note that the horizontal line, or *scale*, shows the hours, or the time; each large space represents one hour. The vertical line, or *scale*, shows the temperature; on *this scale* each large space represents 10°, or each small space represents 2°. Suppose we wanted to read from the graph what the temperature was at 11 o'clock. We find it by looking along the *time line*, or *time axis*, until we come to the point marked 11 o'clock. We then look up, or down, to the line of the graph. In this case we have to go up to a point 11½ small spaces above the 11 o'clock point. By looking back, to the left, to the vertical or temperature scale, we see that any point on this horizontal line stands for 83°. Hence the graph shows that at 11 o'clock the temperature was 83°.

The following questions will help you compare the **graphic** and **tabular** methods of representing the relation between two numbers.

### EXERCISE 24

In order to answer each of these questions, refer to the data of Table 1 and Fig. 13.

1. Find, both from the table and from the graph, the highest temperature.
2. What was the lowest temperature? Which shows this the more easily, the table or the graph?
3. Between what hours did the temperature change the most rapidly?
4. About what do you think the temperature was at 9.30 A.M.?
5. Between what hours did the temperature change the least?
6. What might explain the rapid fall in temperature between 4 P.M. and 5 P.M.?

After answering the questions, are you not convinced that the *graphic* method gives the information which the reader may desire much more quickly and easily than the *tabular* method? The fact that this is true has brought about a very wide use of graphic methods in all kinds of business and industry. Nearly every newspaper and magazine contains "*graphs*" of some kind. Your teacher will be glad to have you bring to class any graphs you may find in the newspapers or magazines.

### EXERCISE 25

1. An east-bound train, running at 40 miles per hour, left Chicago at 8 A.M. Show from the graph, Fig. 14, how far the train was from Chicago at 10 A.M.; at 11 A.M.; at 11.30 A.M.; at 2 P.M. At what time was the train 100 miles from Chicago? 200 miles?

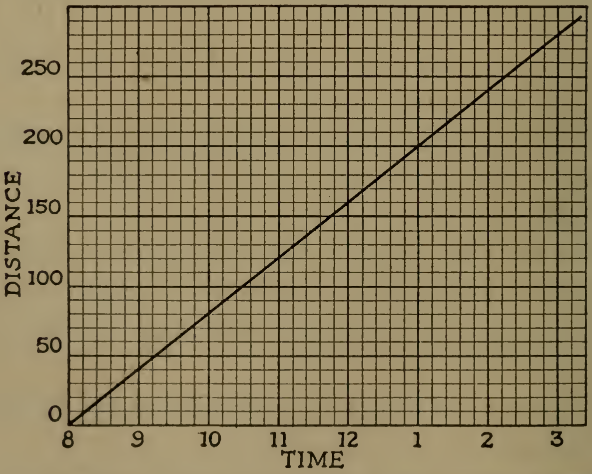


FIG. 14. **The line shows relationship between time spent and distance traveled.**

2. In Fig. 14 how many miles does each small space represent? How many hours does each large space equal?

3. In a newspaper the following graph, Fig. 15, was printed. It gives the prices of wheat, per bushel, from August 5 to August 10. What was the price on August 5? On August 7? On August 9?

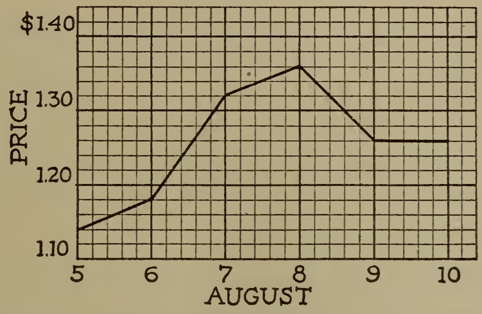


FIG. 15. **Graphic representation of prices of wheat on various days of August, 1916.**

4. When was the price the highest? the lowest? Between what dates did the price change most? change least?

5. Each large space on the vertical scale represents how many cents? What is measured along the horizontal scale? What is the unit used on this scale?

## SUMMARY OF IMPORTANT ASPECTS OF GRAPHIC REPRESENTATION

**Section 21.** In the study of the previous examples, the following important aspects of graphic representation should be noted:

1. Graphs always show the relation between two changing quantities; for example, they showed the relation between the number of miles traveled and the time required.
2. Two rectangular *axes* are drawn. One of the *changing quantities* is measured on the horizontal axis; the other *changing quantity* is measured on the vertical axis.
3. These axes, or reference lines, are SCALES, marked off in a series of UNITS. Thus, as in our illustrative examples, the horizontal axis may be a *time scale*, marked off into *units* of one hour each, and the vertical axis may be a *distance scale*, marked off into units of *one mile*, or *fifty miles*, each.
4. In making a graph one must choose units very carefully in order to be able to get all the information on the graph, and yet make it stand out as clearly as possible.

## EXERCISE 26

**1.** The table below gives the earnings of a book agent for the latter part of July, 1915. Show the same thing graphically.

| Date | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
|---|---|---|---|---|---|---|---|---|---|---|
| Earnings-$ | 2.00 | 3.50 | 4.00 | 5.00 | 7.00 | 4.50 | 3.00 | 8.00 | 7.50 | 9.00 |

SUGGESTION. Represent time on the horizontal scale, and earnings on vertical scale.

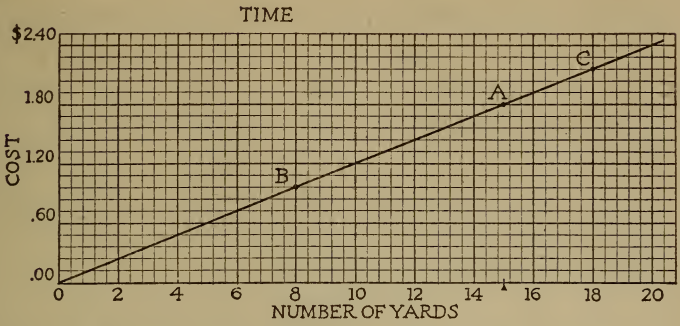


FIG. 16. **The line shows the relationship between the number of yards of cloth purchased and the total cost.**

**2.** Figure 16 is a price graph which shows the cost of *any* number of yards of cloth at 12 cents per yard. From it we can find the cost of *any number* of yards. For example, the cost of 15 yd. is found by finding the point on the horizontal scale which stands for 15 yd., then by finding the point on the cost line directly above this point. This appears on the cost line as point $A$. Now, to find the cost of 15 yd. we find the point on the cost axis ($OY$) horizontally opposite the point $A$ which already stands for 15 yd. The cost proves to be $1.80. Thus we see that point $A$ stands both for 15 yd. and for $1.80. In the same way the point $B$ shows that 8 yd. on the horizontal scale *corresponds* to 96¢ on the vertical scale. The point $C$ shows that 18 yd. on the horizontal scale corresponds to $2.26 on the vertical scale.

**3.** Make a *formula* for the cost of any number of yards of cloth at 12¢ per yard.

NOTE THAT THE GRAPH AND THE FORMULA TELL EXACTLY THE SAME THING. The graph tells the relation between the cost and the number of yards purchased more clearly because it presents it to the eye as a picture. To tell from the graph the cost of any particular number of yards requires only a glance; to tell from the formula or equation

$$C = .12\,n$$

requires that we substitute some particular value of $n$ in the equation and then that we find the value of $C$.

**4.** Draw a graph showing the price of any number of pounds of beans at 9 cents a pound. From it find the cost of $5\frac{1}{2}$ pounds; of 12 pounds.

**5.** Now, write a *formula* which represents the cost of any number of pounds at 9 ¢ a pound. Note that *the graph and the formula tell the same thing.*

**6.** Draw a graph for the cost of a railroad ticket at 3 ¢ a mile.

**7.** If $c = .03\,m$ is used as the equation for the cost of any railroad ticket at 3 ¢ a mile, show that by letting $m$ have particular values, such as 2, 3, 7, 10, etc., we get values for $c$, from which we can make the graph.

**8.** A number of rectangles have the same base, 5 in. Write an equation for the area of any rectangle which has a 5-inch base. (Use $h$ for the altitude, or height.)

**9.** Draw a graph for the area of any rectangle whose base is 5 in. by using the equation you got in Example 8. (Let $h$ have particular values, such as 2, 3, 4, 7, 10, and find the corresponding area, in each case.)

**10.** A west-bound train leaves Chicago at 7 A.M., going 30 miles per hour. Show graphically its progress until 4 P.M.

**11.** Using $d = 30\,t$ for the equation of the train in Example 10, show that the graph could have been made from the results obtained by letting $t$ have particular values.

**12.** The movement of a train is described by the equation $d = 25\,t$. Draw a graph showing the same thing.

**13.** A boy joined a club which charged an initiation fee of 25 cents. His dues were 10 cents each month. Draw a graph to show how much he had spent at the end of any number of months.

**14.** What formula or equation will represent the same thing as the graph in Example 13?

### VARIABLES AND CONSTANTS

**Section 22.** In all the examples which you have just solved graphically there have been **changing or varying quantities**; for example, in the graph of the motion of a train, the *distance and the time vary* as the train moves along its trip; or in any cost graph the *cost varies* (that is, increases and decreases) *as the number of articles varies.*

But in these examples, some of the quantities do not change or vary. To illustrate: the *rate* of the train (as in Example 10, 30 miles per hour) remains fixed, or **constant**, as the train moves along; and the price per unit of any article (for example, cloth at 12¢ per yard) remains fixed or constant in any particular example.

Thus, in any problem we may have two kinds of quantities: *first*, those that change or **vary**; and *second*, those that remain fixed or **constant**. We call them, respectively, VARIABLES AND CONSTANTS. For example, in the formula for the area of any rectangle whose base is 4 units, $A = 4\,h$, it is clear that $A$ and $h$ are *variables*, and that the base, 4, remains *constant*. In other words, if $h$ is 2, then $A$ is 8; if $h$ is 3, then $A$ is 12; if $h$ is 7, then $A$ is 28, etc. Thus, $h$ can change, but as it changes, $A$ also changes, since $A$ is always 4 times as large as $h$. Hence, 4 is the "constant" in the equation, and $A$ and $h$ are the "variables." Note that there is a definite **relation** between $A$ and $h$. $A$ is always 4 **times** $h$.

## EXERCISE 27

Determine the **variables** and the **constants** in each of the following examples. Give reasons for each decision that you make.

**1.** $c = 2\,\pi R$

**2.** $d = 40\,t$

**3.** $A = \frac{bh}{2}$

**4.** $x = y + 4$

**5.** $c = 10\,m + 25$

**6.** $A = s^2$

**7.** $P = 2\,b + 2\,h$

**8.** $c = 10\,m + 50$

## GRAPHS SHOW THE RELATION BETWEEN TWO VARIABLES

**Section 23.** A cost graph, such as Fig. 16, really shows the **relation** between the number of units (lb., doz., or yd., etc.) purchased and the total price paid. A graph of the movement of a train (*e.g.* Fig. 14) which runs at a constant rate shows the **relation** between the number of hours (the time) and the number of miles traveled (the distance). Saying that these graphs show the **relation** between the numbers represented by them means that if we read a particular value of the time, such as 2 hr. or 5 hr., we can find the number of miles which *corresponds* to that number of hours. Thus, graphs show the **relation** between two variables; that is, they show the **values** of one variable **which correspond** respectively to the values of another **related** variable.

*A formula also shows the relation* or connection between the two variables. For example, the formula for the area of any rectangle with a 3-inch base, which is $A = 3h$, shows that *the value of A must always be three times the value of h*, or, in other words, the area is always three times the height. At first it is more difficult to understand the formula than the graph, but as you advance in mathematics the formula will become more important and significant.

Thus, as we stated at the beginning of the chapter, there are three methods of showing the relationship or connection between the kinds of variables we have studied:

I. THE FORMULA METHOD
II. THE TABULAR METHOD
III. THE GRAPHIC METHOD

Let us illustrate one example by each of these methods.

A man walks at the rate of 6 miles per hour. Show the relation or connection between the distance he walks and the number of hours he walks.

I. **Formula method:**

$$D = 6h.$$

II. **Tabular method:**

TABLE 2

| | | | | | | |
|---|---|---|---|---|---|---|
| If the no. of hours is | 1 | 2 | 5 | 8 | 10 | 12 |
| then the distance is | 6 | 12 | 30 | 48 | 60 | 72 |

III. **Graphic method:**

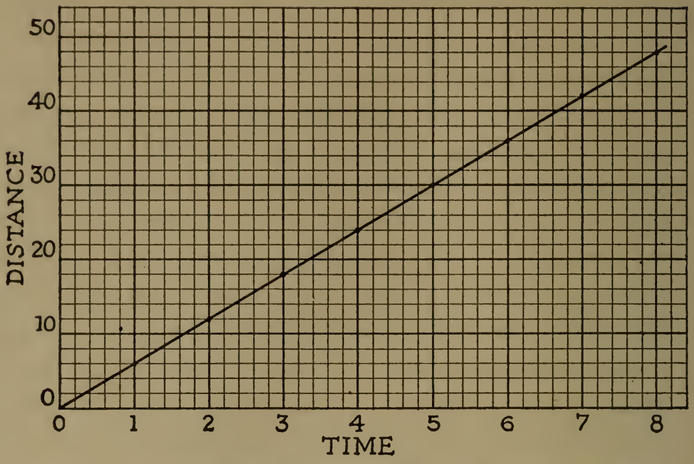


FIG. 17. **The line shows relationship between time spent and distance traveled.**

## EXERCISE 28

### PRACTICE IN REPRESENTING THE RELATION BETWEEN VARIABLES

Show by three methods the relation between the variables in the following:

1. The area of a rectangle whose base is 8 in. and its height.
2. The cost of belonging to a club which charges an initiation fee of 50¢, and 10¢ per month for dues.
3. A freight train leaves Chicago at 10 A.M., at the rate of 25 miles per hour; at 1 P.M. a passenger train leaves Chicago, running in the same direction, at the rate of 40 miles per hour. Show graphically at what time the passenger train will overtake the freight train. See Fig. 18 for solu-

tion. How does the graph show that one train will overtake the other? If $t$ represents the

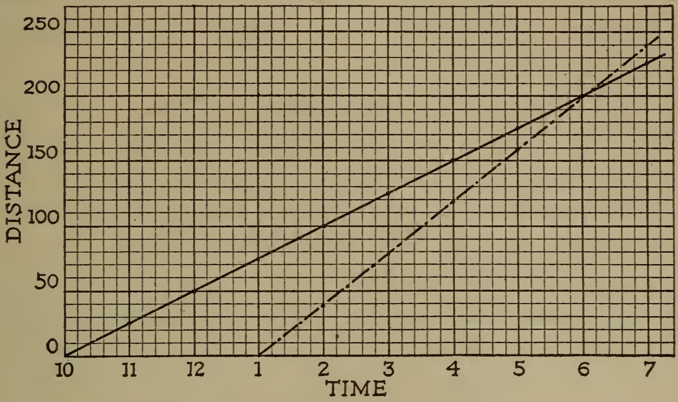


FIG. 18. **The lines show relationship between the time spent and the distance traveled by each train. The point of intersection indicates the time at which they will meet and how far each travels.**

time of the freight train, what formula will represent the distance it travels? What will represent the time the passenger train travels? What formula will represent its distance?

4. A slow train left Cleveland at 6 A.M., running uniformly at the rate of 30 miles per hour. At 10 A.M. a faster train left Cleveland, running in the same direction, at the rate of 40 miles per hour. Show graphically at what time the faster train will overtake the slower one.

5. A freight train left St. Louis at 7 P.M., running 30 miles per hour. At 11.30 P.M. an express train started in the same direction. Show graphically at what time it will overtake the freight train, if it runs 45 miles per hour.

**Section 24. Two different kinds of graphs.** We should distinguish between the *two kinds* of examples which we have graphed. The first kind includes all those for which no formula or equation can be made. Recall the first illustrative example in this chapter: *the relation between the time of day and the temperature.* Clearly, no formula can be made which will always show the relation between the two variables in this kind of example. Thus, there are only two ways to show or represent this kind of relation: (1) **the tabular method,** (2) **the graphic method.**

The second kind of example which we have been graphing is illustrated by any of those examples *for which we made a formula.* For example, we have such illustrations as: the *graph showing the relation between the distance traveled* by a train running at 30 miles per hour *and the time the train travels.* This belongs to the second kind of graph, because we can make a formula for the *relation* between its variables. The formula is:

$$d = 30\,t.$$

Thus, there are **three ways** to show the relation between these variables: (1) the *formula* or *algebraic* method, (2) the *tabular* method, and (3) the *graphic* method.

In mathematics, we say that the second kind of graph, for which an equation can always be made, states ALGEBRAIC LAWS, or MATHEMATICAL LAWS, **because there is always a definite relation between the variables.** The first kind of graph, for which no definite *law* or *equation* can be made, is sometimes called a STATISTICAL GRAPH. It is this kind that is most frequently seen in newspapers and magazines. In mathematics, however, the other kind, that which states "laws," is nearly always used.

The next exercise will give practice in making both kinds of graphs. It is important to tell whether the information to be graphed (generally called the *data*) can be expressed by an algebraic law or formula.

### EXERCISE 29

1. The following table shows the average heights of boys of different ages. Construct a graph showing this information or data.

| Age in years | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| Height in feet | 1.6 | 2.6 | 3.0 | 3.5 | 4.0 | 4.8 | 5.2 | 5.5 | 5.6 | 5.7 |

Represent ages on the horizontal scale.

2. When does the average boy grow the most rapidly? the most slowly?
3. Is there an algebraic "law," or formula, which shows the relation between these two variables, *age* and *height?*
4. Mr. Smith joined a lodge which charged $25 initiation fee, and dues of $2 per month. Show graphically the **relation** between the *cost* of belonging and the *time* one belongs.
5. Is there an algebraic "law," or formula, which shows the **relation** between the variables, *cost* and *time?*
6. The information or data of the following table represent the area of a square of varying sides:

| If the side is | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| then the area is | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |

Show this **relation** between the area of the square and its side graphically, using the vertical scale to measure *areas* and the horizontal scale to measure *sides*.

7. Is there an algebraic "law," or formula, which shows the **relation** between the variables here?

**Section 25. Summary of chapter.** This chapter should make clear the following truths:

1. **Important facts about quantities are more easily read and interpreted if they are represented *graphically*.**
2. **Graphs always show the relation between two varying quantities.**
3. **Two *scales*, a horizontal scale and a vertical scale, at right angles to each other, are required in order to mark off or measure the values of the varying quantities. These scales must be divided into convenient units.**
4. **The information or data must be tabulated in order to show it graphically.**
5. **There are three fundamental methods of describing the relationship between related variables:**
   **a. The *Formula*, or Algebraic Method, of stating "Law";**
   **b. The *Tabular* Method of expressing "Law";**
   **c. The *Graphic* Method of expressing "Law."**

EXERCISE 30

1. If $n$ represents a boy's present age, state in words what the expression $n + 7 = 22$ means.

2. Give a formula for the base of a rectangle when the area and height are known.
3. Represent the number of cubic yards in a box-shaped excavation when the dimensions are expressed in feet.
4. If $m$, $s$, and $d$ represent the minuend, subtrahend, and difference respectively, what formula will show the relation between these numbers?
5. Show by a formula the relation between the product, $p$, multiplicand, $M$, and multiplier, $m$.
6. Give the meaning of the formula $i = prt$.
7. Divide each side of the formula $V = lwh$ by $lw$ and tell what the resulting formula means.
8. Give a formula for the volume of a cube whose edge is $s$.
9. Evaluate the above formula when $s = 3.2$.
10. Translate into words the formula $d = rt$.
11. Divide each side of the formula $d = rt$ by $r$ and tell what the resulting formula *means*.
12. In the formula $c = np$, $n$ represents the number of articles bought, $p$ represents the price of each, and $c$ represents the total cost. Translate it into a word statement.
13. Divide each side of the formula $c = np$ by $n$, and tell what the resulting formula means.
14. Does $x = 4$ satisfy the equation $x^2 + 6x = 40$?
15. Solve the equation $10y + 7 = 52 + 4y$.
16. Is the equation $x + y + 3 = 20$ satisfied if $x = 8$ and $y = 9$? Can you find any other values of $x$ and $y$ which will satisfy this equation?

**17.** Solve each of the following equations, thinking of each example as asking a question:

(*a*) $\frac{10}{x} = 2$

(*b*) $.5\,y = 17$

(*c*) $.4\,p = 80$

(*d*) $\frac{34}{x} = 4.25$

(*e*) $\frac{22\,y}{y} = 5.5$

(*f*) $1.5\,b = 45$

(*g*) $\frac{x}{.42} = 60$

(*h*) $\frac{h}{.34} = 85$

Tell what you do to each side of the equation; that is, tell whether you add, subtract, multiply, or divide, on each side.

### REVIEW EXERCISE 31

**1.** Write in algebraic language: The volume of a sphere is four thirds the cube of the radius times $\pi$.

**2.** The weights of a baby boy who weighed 8 lb. at birth are given for each month of his first year by the table:

| Month | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Weight | $9\frac{1}{4}$ | $11\frac{1}{4}$ | $12\frac{3}{4}$ | $14\frac{1}{4}$ | $15\frac{1}{2}$ | $16\frac{1}{2}$ | 18 | 19 | $19\frac{1}{2}$ | 20 | 21 | 22 |

Represent this graphically.

**3.** Which would you rather have, $4\,x + 5\,y$ dollars, or $5\,x + 4\,y$ dollars, if $x = \$\,20$ and $y = \$\,16$?

**4.** If $V$ is the volume of a cone, $b$ the area of its base, and $h$ its height, then

$$V = \tfrac{1}{3}\,bh.$$

Write this formula in words.

**5.** Nurses keep temperature records of fever patients. For one patient the following degrees of fever were noted:

2, 5.4, 4, 6.1, 4.5, 6.5, 5.3, 6.7, 4.5, 6.5, 5.9, 6.2, 6.9, 5, 6.4, 4.7, 5.8, 7.6.

These readings were taken an hour apart. Show graphically this patient's successive temperatures.

**6.** If you know that

$$10a - 7 = 4a + 35,$$

then what do you do to each side to get

$$10a = 4a + 42?$$

How do you get $6a = 42$?
Why does $a = 7$?
Prove that $a = 7$.

**7.** Get the hourly temperature for 24 hours from your daily paper, and construct a graph to represent the changes in temperature.

**8.** The sum of two numbers is 18 and their difference is 4. What are the numbers?

**9.** Show that $2a$ and $a^2$ are unequal by choosing some particular value for $a$, such as $a = 6$. Do you think there is any possible value for $a$ which would make $2a = a^2$?

**10.** What product is obtained by using 7 as a factor twice? by using $2a$ as a factor three times?

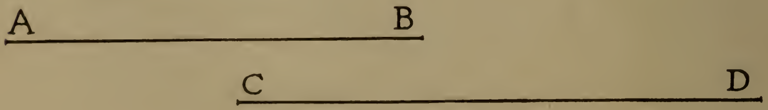
# CHAPTER V

## HOW TO FIND UNKNOWN DISTANCES BY MEANS OF SCALE DRAWINGS: THE FIRST METHOD

**Section 26. We need to know how to find unknown distances.** The methods of mathematics are really all planned to help us find *unknown* values. The equation, which we have studied so carefully, is the best *algebraic* tool with which to do that. Many times, however, in practical life work the *unknown values* that we need to know are *distances.* For example, the surveyor may need to know the distance across a river and may not be able actually to measure it. Or, he may need to know the distance between two points, with some other intervening object between which prevents him from measuring it directly. Now, mathematics has given us **three ways** to find such an unknown distance. In Chapters V, VI, and VII we shall discuss these methods.

The *first method* is to **make a scale drawing**, which will include in some way the unknown distance. Next, therefore, we shall study *how to determine unknown distances by means of scale drawings.* Before we take up that particular subject, however, we must study *how to measure the lines and angles* which make up scale drawings.

**Section 27. The measurement of lines.** We are already familiar with certain methods of measuring distances. For example, we have measured the length of lines, such as the distance from $A$ to $B$ or from $C$ to $D$. If we use a metric *scale*, in which the units are *centimeters*, the distance from



*A* to *B*, which is read "*line AB*," is 5.08 centimeters long, and the line *CD* is 6.35 centimeters long. If we use a foot-rule in which the units are inches, the distance between *A* and *B*, or the line *AB*, is 2 inches, and line *CD* is 2.5 inches. Note here that the distances or lengths that we obtain for these lines depend upon the kind of *scale*, or kind of *unit*, that is used in measuring.

## THE MEASUREMENT OF ANGLES

**Section 28. An angle is determined by one line turning about another.** In order to construct scale drawings, we must know how to measure angles. Let us think of an angle as being formed by one line turning, or rotating about a fixed point on some fixed or stationary line. The line *OY* turns or rotates about point *O*. For example, in Fig. 19, think of *AX* as a fixed, or stationary, line. (It is easiest always to take this line as *horizontal.*) Think also of another line, say *OY*, as turning, or rotating, about some point on the fixed line *AX*, say point *O*. As the line *OY* rotates about the point *O*, it constantly forms a larger and larger ANGLE with the fixed line *AX*. (The symbol for angle is ∠.) The point *O*, about which the line turns, is always the point at which the two sides of the angle meet, and is called the *vertex of the angle.*

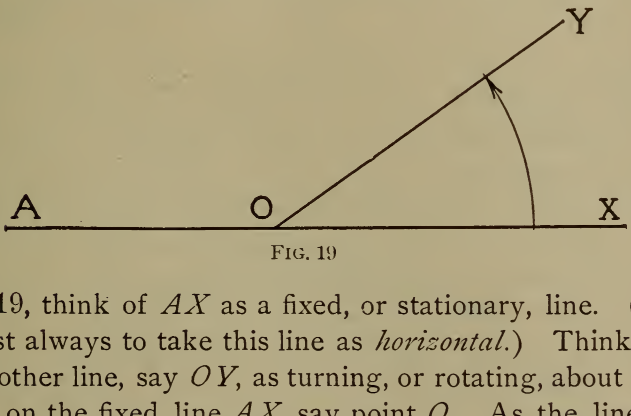


FIG. 19

The arrow is drawn to indicate that **the line *OY* is turning, or rotating, about the point *O*.**

**Section 29. The unit of angular measurement.** Just as we have **units** and **scales** for measuring straight lines, so we have **units** and **scales** for measuring angles. Evidently the **unit** with which we must measure the size of the angle is one that will measure the amount that the line has rotated about the fixed point. Figure 20 shows that we can think of the rotating line as turning clear around until it occupies its original position again. That is, any point *P* on the line *OX* has turned through a complete *circle* in rotating about *O* and returning to its original position.

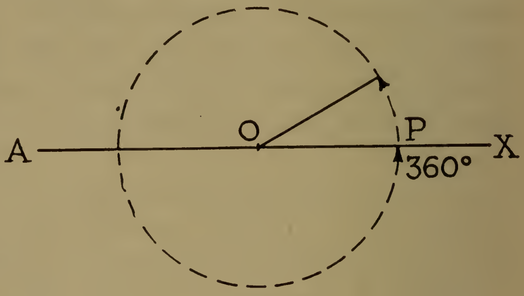


FIG. 20

This suggests that the **unit** with which we measure angles *will be some definite fraction of the circle.* For a long time people have agreed that the circle be divided into **360 units** and that each one of these *units of angular measure* be called a DEGREE. The symbol used for degree is a small ° placed at the right above the number. For example: 45° is read "45 degrees." Thus, Figs. 21, 22, and 23 illustrate angles of different sizes or of different numbers of degrees.

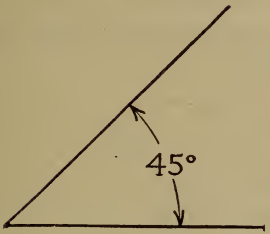


FIG. 21

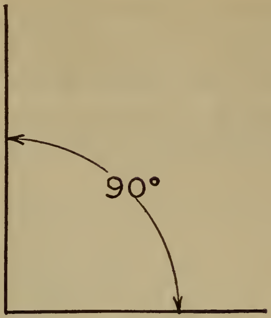


FIG. 22

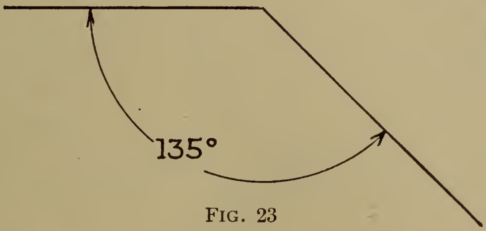


FIG. 23

**Section 30. The PROTRACTOR: How to measure angles.** Just as we use foot rules, yardsticks, meter sticks, etc., to measure straight-line distances, so we have an instrument called a PROTRACTOR to measure angular distances. Figure 24 shows that the circular edge of the

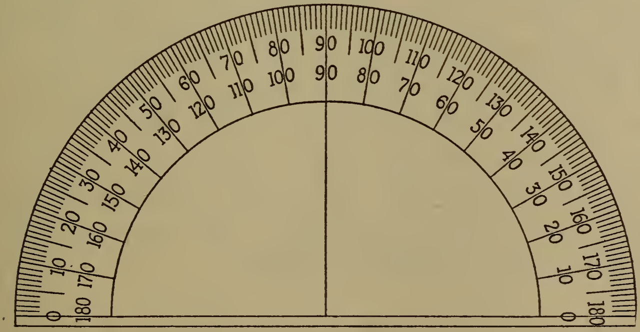


FIG. 24. **A protractor for constructing and measuring angles.**

protractor is marked off (*i.e.* is "graduated") into degrees. Note from the figure that the protractor is divided into 180 equal parts (half of the total number of angular units in the circle), called *degrees.* Sometimes the whole circle is used and marked off to give 360°.

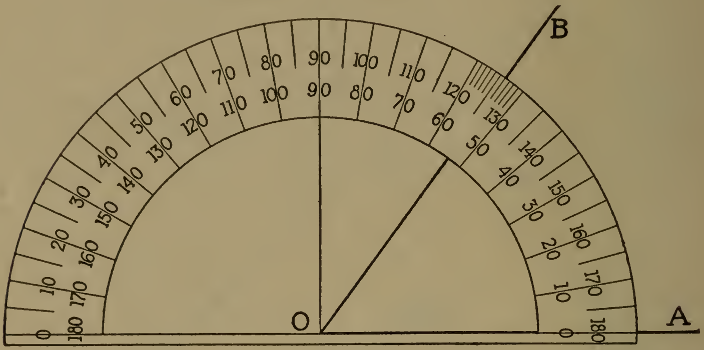


FIG. 25

The next figure, Fig. 25, shows *how to measure an angle* with a protractor. First, lay the straight edge of the protractor so that it will fall exactly upon one of the two lines that form the angle, and with the center of the protractor exactly upon the VERTEX, *O*, of the angle. Then the other side of the angle, *OB*, for example, will appear to cut across the circular edge of the protractor. Now count the number of degrees from the point where the curved edge of the protractor touches *OA* to the point where it crosses the line *OB*. Hence, in Fig. 25, the angle *AOB* contains 54°. It is very important for us to be able *to read angles accurately.* The next exercise will give you practice in reading angles.

## EXERCISE 32

### PRACTICE IN MEASURING ANGLES

**1.** Measure each of these angles with a protractor.

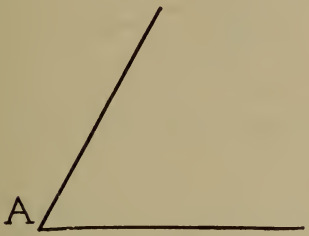


FIG. 26

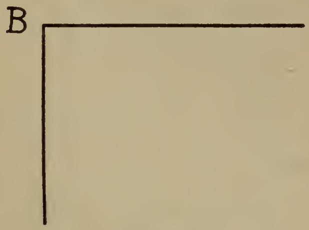


FIG. 27

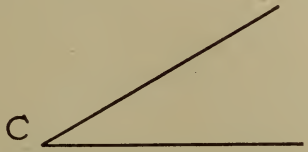


FIG. 28

**2.** Compare angle *A* and angle *C*. Which has the longer sides? What effect has the length of a side of an angle upon the size of the angle?

**3.** Measure each angle of triangle *ABC*. From the results of your measurement, what is the *sum* of all three angles of *this* triangle?

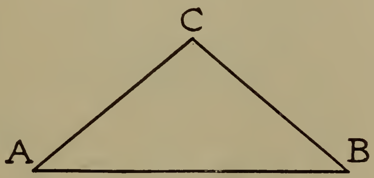


FIG. 29

4. How large is $\angle x$? How many degrees in $\angle y$? in $\angle z$? How many degrees in the *sum* of the angles of *this* triangle, $XYZ$?

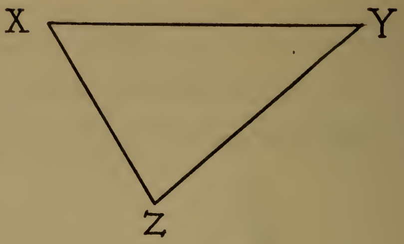


FIG. 30

5. Draw with the protractor an angle of 30°; 45°; 60°; 100°.

6. At each end of a line 6 cm. long draw angles of 50°. Produce these lines until they meet, and measure the angle formed by them. How many degrees in it? Compare the lengths of the lines you drew. How many degrees does the *sum* of the three angles of *this* triangle make?

7. Draw a triangle such as triangle $ABC$, so that $AB = 4$ inches, angle $A = 60°$ and $AC = 3$ inches. Then find the number of degrees in angle $B$ and angle $C$.

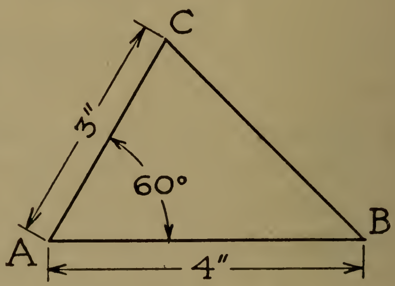


FIG. 31

8. Construct triangle $ABC$ so that $AC = 5$ cm., angle $C = 40°$, and $CB = 5$ cm. Compare angle $A$ with angle $B$. How many degrees in each?

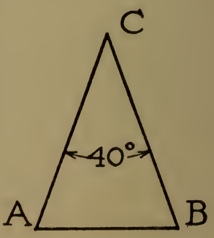


FIG. 32

**Explanation**: A triangle having two sides equal, such as *AC* and *CB*, is an *isosceles triangle*. It is proved in geometry that the angles opposite these equal sides are always equal; for example, angle $A$ = angle $B$. How many degrees ought there to be in either angle $A$ or angle $B$?

**Section 31. How to describe an angle.** An angle is *described* by using three letters, *i.e.* the letter which represents the vertex is written between the two letters at the ends of the sides. Thus, ∠1, in Fig. 33, is read as *angle AOB* or *angle BOA*, and is written ∠ *AOB* or ∠ *BOA*. In the same way, ∠2 is read *angle BOC* or *angle COB*, and is written ∠ *BOC* or ∠ *COB*.

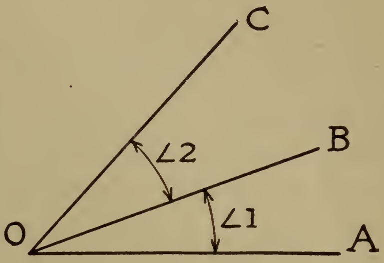


FIG. 33

## EXERCISE 33

### PRACTICE IN READING ANGLES

1. Why would it not be clear to read ∠2 as ∠ *O*?
2. Read the angle formed by lines *OA* and *OB*.
3. Read the angle formed by lines *OB* and *OC*.
4. Determine the number of degrees in ∠ *AOB*, in Fig. 34, *without using the protractor*.

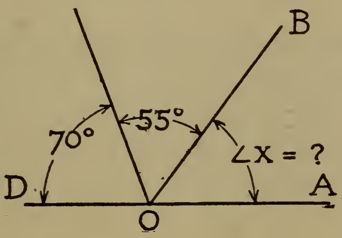


FIG. 34

5. If in Fig. 35 you know that angle *ABC* is 40° and that ∠ *BCA* is 90°, could you find ∠ *CAB* without measuring it? How? How large is it?

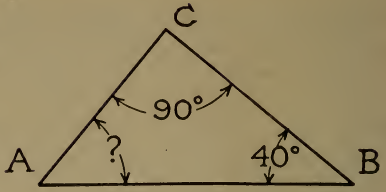


FIG. 35

**Section 32. We must be able to find unknown distances which cannot be measured directly.** The preceding section took up only examples in which the distances, linear and angular, could be measured directly, by means of instruments. There are many instances, however, in which the lengths of the lines and the sizes of the angles cannot be measured directly. For example, consider the case of finding the distance across a river, or the height of a tree, which we mentioned at the beginning of the chapter. In cases like this we need *indirect* methods of measuring. Mathematics makes it possible for us to determine the lengths of such lines **by measuring the lengths of other lines and the sizes of angles that are related to them.** This leads us to the main topic of this chapter.

## HOW TO FIND UNKNOWN DISTANCES BY MEANS OF SCALE DRAWINGS

**Section 33. How to draw distances to scale.** One of the methods that you will use commonly in indirect *measurement* is that of drawing distances "to scale." So much use is made of mechanical drawings that we need to be very proficient in making them and in reading them. Let us take a simple illustration of the drawing of distances "to scale" and of measuring distances on scale drawings.

**Illustrative example.** A man starts at a given point and walks 2.5 miles east, then 2.5 miles north. How far is he from his starting point?

*First*, set point *O*, in Fig. 36, as his starting point. *East* is measured to the *right* of point *O* and *north above* point *O*.

*Second*, to represent distances "to scale," we need to *select a unit of distance on the scale which will represent a unit of distance in the example.* Let us take, for example, three quarters of an inch on the drawing to represent each mile which the man actually walks. This is indicated on the scale drawing (Fig. 36) by writing "Scale = $\frac{3}{4}$ in. to 1 mi." It is very important to select the scale unit carefully and always to indicate the scale that has been used on the drawing.

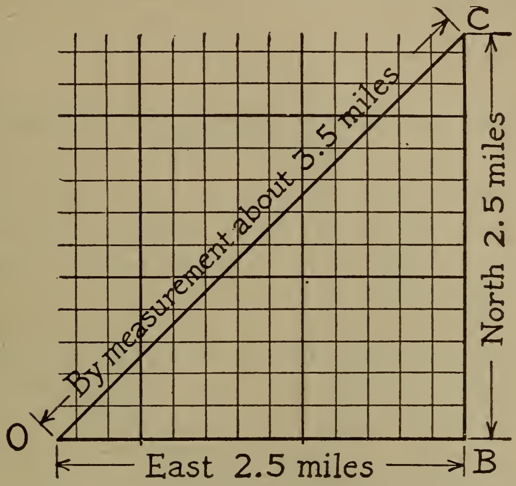


FIG. 36

*Third*, to represent the man's path, we lay off *OB* horizontally to the right of *O*, 2.5 miles (on the drawing this amounts to $1\frac{7}{8}$ inches) and *BC* vertically, 2.25 miles. Then, by using the cross-section paper as a scale, *we can measure at once the distance*, *OC*, that the man is from his starting point. The distance is 2.63 inches on the figure, or 3.54 miles actually.

This work illustrates by a very simple example how we make *scale drawings.* Mechanical drawings made "to scale" are used very commonly by such workers as architects, carpenters, machinists, and engineers.

## EXERCISE 34

### PRACTICE IN FINDING UNKNOWN DISTANCES BY THE CONSTRUCTION OF SCALE DRAWINGS

1. Draw to the scale 1 cm. to 2 ft. a floor plan of a room 28 ft. by 20 ft. By measuring the distance diagonally across the plan, compute the diagonal of the room.

2. Draw a plan of a baseball diamond 90 ft. square and find the distance from first base to third base. Use 1 cm. to represent 20 ft.

3. Two bicyclists start from the same point. One rides 12 miles north and then 8 miles east; the other rides 10 miles south and then 6 miles west. How far apart will they be? Use the scale 1 cm. to 2 mi.

4. Draw to the scale 1 cm. to 4 ft. a plan of the end of a garage such as in Fig. 37. Find the height from the floor to the top of the roof.

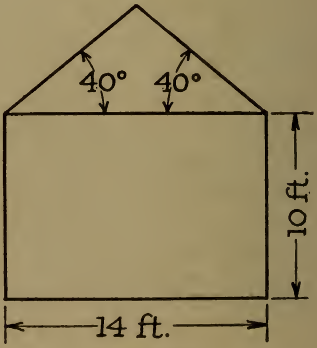


FIG. 37

**5.** In Fig. 38 *AB* is 200 ft., angle *A* is 60°, and angle *B* is 50°. Draw to the scale 1 cm. to 50 ft. and determine the length of *AC* and *BC*.

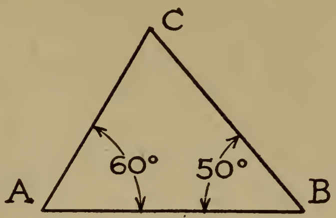


FIG. 38

**6.** A surveyor sometimes finds it necessary to measure the distance across a swamp, such as

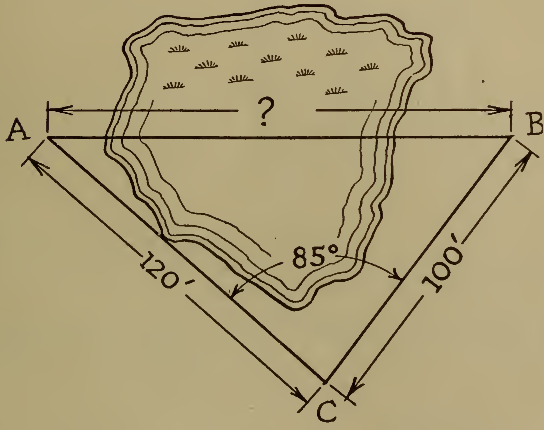


FIG. 39

*AB* in Fig. 39. He measures from a stake *A* to a stake *C*, 120 ft. From *C* to *B* he finds it is 100 ft. Find, by a scale drawing, the distance *AB* across the swamp, if angle *C* is 85°.

**7.** How could a surveyor find the distance from *A* to *B*, if there were some obstacle in the way

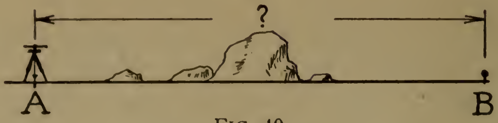


FIG. 40

preventing his measuring directly the distance *AB*?

8. Find by a scale drawing the distance *AC* across

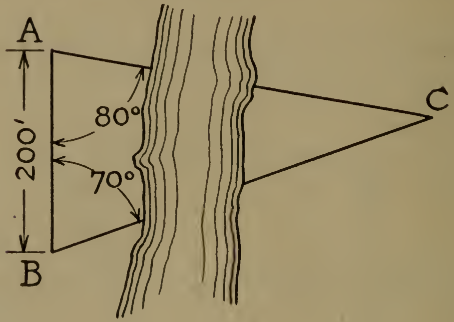


FIG. 41

the river, if it is known that angle $A = 80°$, $AB = 200$ ft., and angle $B = 70°$.

9. **Illustrative example.** **A boy wishes to determine the height of a flagpole. A scale drawing will aid him in**

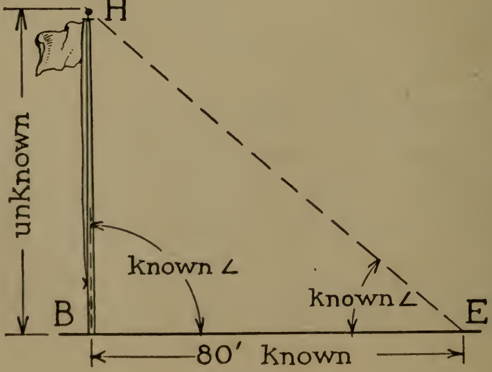


FIG. 42

doing this. For example, he can measure a line of any length on the ground out from the base of the flagpole. Suppose he takes a line 80 feet long. Then he sets at *E* an instrument called a *transit*, with which he can read the angle between the horizontal base line and the *line of sight* from *E*, where he stands, to *H*, the top of the pole. He knows also that the angle *B* is a right angle. So he knows the length of the line *EB*, the size of the angle *E* and the angle *B*. He constructs a scale drawing to represent the known lengths and the known angles. From this drawing he is able to "scale" or measure the height of the flagpole.

---

The angle *BEH* or angle *E* between the horizontal and the line of sight in this example is called the ANGLE OF ELEVATION. If the boy had taken a longer base line, what would have been true of the size of the angle of elevation with respect to what it was before?

**10.** If the angle of elevation in Fig. 42 is 40° when the observer is 80 ft. from the foot of the pole, find its height.

**11.** A flagpole 50 ft. high casts a shadow 60 ft. long on level ground. What is the angle of elevation of the sun? If the length of a shadow cast by this pole increases, what conclusion can be drawn concerning the angle of elevation?

**12.** The angle of elevation of the top of a tree is 42° when the observer stands 30 yd. from the tree. How high is the tree? If the distance from the observer to the tree decreases, what change in the angle of elevation follows?

**13.** On the top of a church is a tall spire. At a point $P$ on level ground 75 ft. from the point $D$ directly beneath the spire, the angle of elevation of the top of the spire is 38°. At a point $Q$ 150 ft. from $D$, in line with $P$ and $D$, the angle of elevation of the top of the spire is 28°. Find the distance from $Q$ to the top of the spire.

**14.** An anchored balloonist from a height $HT$, Fig. 43, of 2500 yd., observes the enemy at $D$.

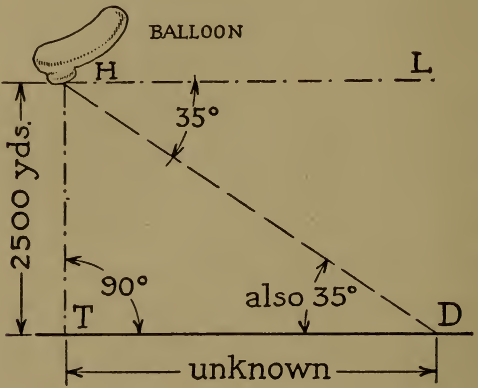


FIG. 43

He wishes to compute the distance $DT$, on level ground. To do so, he measures the angle which is formed by the horizontal line $HL$ and the line of sight $HD$. This angle is called the ANGLE OF DEPRESSION, and has the same number of degrees as the angle of elevation. (Can you see that this is true?) He next finds angle $THD$ by subtracting the angle of depression from 90°, angle $THL$. He then knows enough about the triangle *DTH* to make a scale drawing of it. Find *DT* if the angle of depression is 35°.

**15.** From the top of a lighthouse 80 ft. high the angle of depression of a ship is 35°. How far is the ship from the base of the lighthouse? Compare your result with that obtained by the other members of the class.

**16.** From the top of a cliff 120 ft. above the surface of the water the angle of depression of a boat is 20°. How far is it from the top of the cliff to the boat?

**17.** A searchlight on the top of a building is 180 ft. above the street level. Through how many degrees from the horizontal must its beam of light be depressed so that it may shine directly on an object 400 ft. down the street from the base of the building? How does your result compare with that obtained by other pupils?

**18.** From the tenth story of a building the angle of depression of an object on the street level is 30°; from the eighteenth story the angle of depression of the same object is 50°. Find the distance from the object to the base of the building if the second observation point is 80 ft. above the first observation point.

**19.** An observer is 200 ft. from the ground. The angle of depression of a point *A* is 24°, of a point *B* 42°, and of a point *C* 15°. Which point is closest to the observer? farthest from the observer?

**20.** In Fig. 44 measure (1) the angle of elevation of point *D* from point *E*; (2) the angle of depression of point *E* from point *D*. Compare these angles. Note that line *DH* is parallel to *EC*, and that these angles are formed by line *DE* cutting (or intersecting) these two parallels. In geometry it is proved that such angles are always equal.

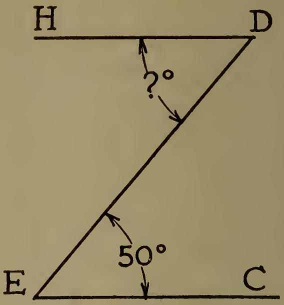
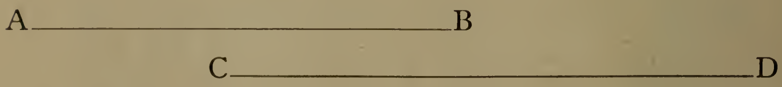


FIG. 44

**21.** Is the scale drawing a very *accurate* method of determining unknown distances? Do many of the pupils in the class get the same answer for any particular example? Why?

## A NEW WAY TO COMPARE TWO QUANTITIES; NAMELY, TO FIND THEIR **RATIO**

**Section 34.** We have been comparing two quantities by finding how much larger or smaller one quantity was than another quantity. For example, if the line *AB* is 4 units long and the line *CD* is 6 units long, we should have said, in describing the comparative lengths, that the line *AB* was 2 units shorter than line *CD*, or that line *CD* was 2 units longer than the line *AB*.



Another method, however, of comparing quantities is used very extensively in mathematics. It is the method of dividing one quantity by the other, or finding the quotient of the two quantities. Thus, to compare the line $AB$ with the line $CD$ we divide $AB$ by $CD$, which gives:

$$\frac{AB}{CD}=\frac{4}{6}.$$

This result is read "the quotient, or RATIO, of $AB$ to $CD$ is equal to four sixths." This process is described as finding the RATIO of the two lines. The *ratio* of two numbers means, then, the quotient which results from dividing one of the numbers by the other. Thus, the ratio of 5 to 10 is $\frac{1}{2}$, and is written $\frac{5}{10}=\frac{1}{2}$. The ratio of 10 to 5 is 2, and is written $\frac{10}{5}=2$. In the same way the ratio of 1 in. to 1 ft. is $\frac{1}{12}$; the ratio of $\frac{1}{2}$ to $\frac{3}{4}$ is $\frac{2}{3}$; and the ratio of $4x$ to $7x$ is $\frac{4x}{7x}$ or $\frac{4}{7}$.

## EXERCISE 35

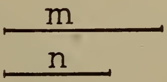
### PRACTICE IN DEALING WITH RATIOS OF NUMBERS

**1.** What is the ratio (in lowest terms) of 10 to 12? of 20 to 24? of 15 to 8? of 25 to 30? Show that $5x$ and $6x$ represent all pairs of numbers whose ratio is $\frac{5}{6}$.

**2.** Give several pairs of numbers having the ratio $\frac{3}{4}$. Show that $3x$ and $4x$ represent all pairs of numbers having this ratio.

**3.** The ratio of two numbers, $a$ and $b$, is $\frac{5}{4}$. What is $b$ when $a$ is 40?

**4.** The ratio of two lines, $m$ and $n$, is $\frac{3}{2}$. Find the length of $m$ if $n$ is 18 in.



**5.** Divide 40 into two parts whose ratio is $\frac{6}{4}$.

**6.** A father and his son agreed to divide the profits from their garden in the ratio of $\frac{7}{3}$. Find each one's share if the total profits were $210.

7. The ratio of $\angle A$ to $\angle B$ is $\frac{5}{3}$. Find $\angle B$ when $\angle A$ is 80°.

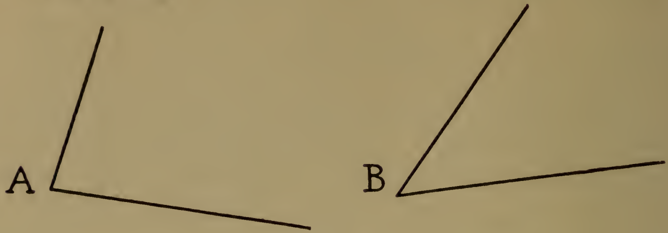


FIG. 45

8. The ratio of the areas of the two squares, $S_1$ and $S_2$, is $\frac{4}{1}$. Find the area of each if the sum of their areas is 45 sq. in.

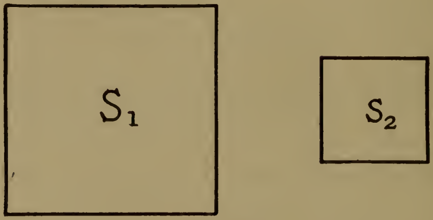


FIG. 46

9. Divide an angle of 90° into two angles having the ratio of 4 to 5.

10. Measure each angle, Fig. 47, with a protractor. Find their ratio.

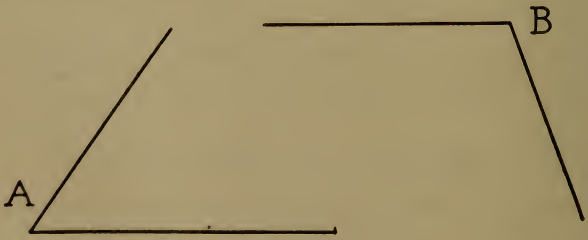


FIG. 47

**11.** In the two triangles $ABC$ and $XYZ$, what is the ratio of $\angle C$ to $\angle Z$? Of $\angle A$ to $\angle X$? Of $\angle B$ to $\angle Y$? Do these triangles have the **same shape**? Do all triangles have the **same shape**?

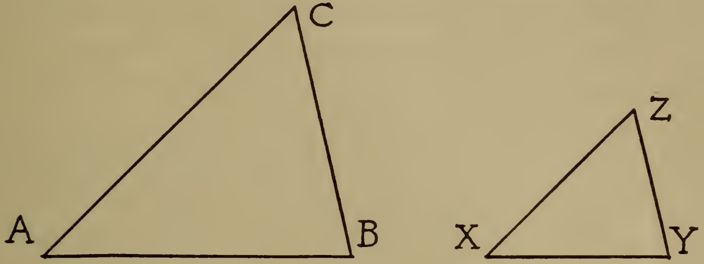


FIG. 48

**12.** Find the ratio of two lines if one is 2 feet long and the other 3 yards long.

**13.** What is the ratio of 3 pints to 4 quarts?

**14.** $\frac{2}{n} = \frac{5}{15}$. **15.** $\frac{4}{3} = \frac{n}{6}$. **16.** $\frac{5}{7} = \frac{10}{n}$.

**17.** Find the ratio of $AB$ to $CD$, Fig. 49, by measuring the length of each line. Express the result decimally.

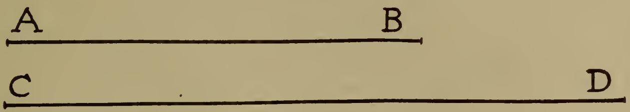


FIG. 49

**18.** A school baseball team won 7 of the 10 games it played. What was its standing in percentage?

Solution: The standing of the team in percentage means the ratio of the number of games won to the number played, *expressed decimally*. Thus, the standing of this team is expressed by the result obtained from changing the ratio, $\frac{7}{10}$, to a decimal. This gives .70 or .700.

**19.** What was the percentage or standing of a team which won 9 of its 12 games?

**20.** The winning team in one league won 14 of its 18 games; and the winning team in another league won 15 of the 19 games it played. Which had the higher percentage?

**21.** Express each of the following ratios as a decimal, correct to two places:

(*a*) $\frac{3}{5}$ (*c*) $\frac{5}{8}$ (*e*) $\frac{5}{16}$ (*g*) $\frac{21}{28}$ (*i*) $\frac{22}{81}$

(*b*) $\frac{2}{7}$ (*d*) $\frac{7}{11}$ (*f*) $\frac{4}{9}$ (*h*) $\frac{32}{45}$ (*j*) $\frac{16.2}{21}$

**22.** If 12 quarts of water are added to 25 gallons of alcohol, what is the ratio of the water to the entire mixture? Express decimally.

## REVIEW EXERCISE 36

**1.** A boy knows that *AB* is 100 ft. and that $\angle B = 40°$. From this information can he construct a scale drawing for the triangle? Give reasons for your answer.

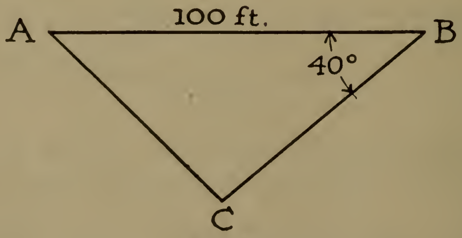


FIG. 50

**2.** What facts or data must be known about a triangle before you can make a scale drawing of it?

3. How does the scale of a drawing illustrate ratio?

4. A tree 90 ft. high casts a shadow 140 ft. long. Find from a scale drawing the angle of elevation of the sun.

5. The following table shows the number of feet required to stop an automobile running at various speeds.

| At a speed of (miles per hour) | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 50 |
|---|---|---|---|---|---|---|---|---|
| a car should stop (feet) | 9.2 | 20.8 | 37. | 58. | 83.3 | 104. | 148. | 231. |

Represent this graphically, measuring the speed on the horizontal axis.

6. Represent in the briefest way the product of five $x$'s; the sum of five $x$'s.

7. Does $a^2b = ab^2$ if $a$ is 4 and $b$ is 3?

8. Can you construct a triangle similar to Fig. 50 if you know that $AB$ is 200 ft. and $\angle B = 40°$? Why?

9. What must be known about a triangle before you can construct it accurately?

# CHAPTER VI

## A SECOND METHOD OF FINDING UNKNOWN DISTANCES: THE USE OF SIMILAR TRIANGLES

**Section 35. Scale drawings are very inaccurate.** In the last chapter we saw that scale drawings could be used to find unknown distances, either *linear* or *angular.* The results obtained, however, were very inaccurate. Seldom did many of you get the same answer for any example. Therefore *we need more accurate methods for determining unknown lines and unknown angles.* This chapter, and the next one, will show methods that depend less upon the accuracy or skill of the person who "scales" or measures. The first method, *based upon geometrical figures of exactly the same shape*, will be explained now.

**Section 36. What are similar figures?** You have already seen many objects or figures of exactly the same shape. A scale drawing has the same shape as the figure from which it was made; on a photographic plate the figure is the same in outline or shape as the original; the map of a state has the same shape or outline as the state itself.

Figures which have the same shape are said to be SIMILAR FIGURES. Which of the following figures are similar in shape?


FIG. 51


FIG. 53


FIG. 54


FIG. 55


FIG. 52


FIG. 56

**Similarity in shape** in geometrical figures is a very important principle that we are able to use in many ways in mathematics. Before this can be taken up, however, we need to be perfectly clear as to *what is meant by similarity of figures*. The next exercise will help to do this.

## EXERCISE 37

### PRACTICE WITH SIMILAR TRIANGLES

**1.** Draw a line $XY$ twice as long as $AB$, Fig. 57. At $X$ draw an angle equal to angle $A$. At $Y$ draw an angle equal to angle $B$. Produce the sides of these angles until they meet at $Z$. Measure the angle formed by these sides. How should it compare with angle $C$? Why?

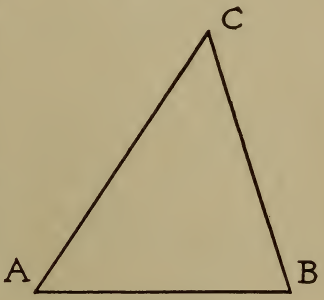


FIG. 57

**2.** (*a*) Angle $A$ *corresponds* to what angle in your triangle?

(*b*) Angle $B$ *corresponds* to what angle in your triangle?

(*c*) What is true, then, about the *corresponding angles* of the two triangles?

**3.** Measure the side in your triangle which *corresponds* to $AC$, and the side which *corresponds* to $BC$. What is the ratio of $AB$ to $XY$, or what is the value of $\frac{AB}{XY}$? of $\frac{AC}{XZ}$? of $\frac{BC}{YZ}$? What does this tell about *the ratios of corresponding sides?*

4. Construct a triangle larger than Fig. 58 but having its angles equal to the angles of Fig. 58. Is your triangle the same shape as Fig. 57? After careful measurement find the ratio of $AB$ to its *corresponding* side in your triangle. Then find the ratio of $AC$ to its *corresponding side.* Compare these ratios.

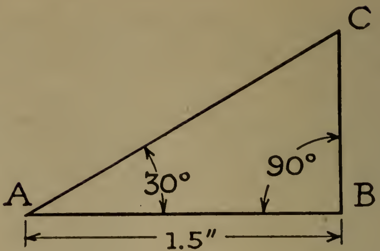


FIG. 58

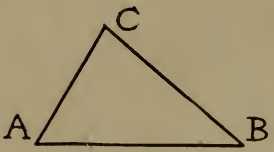


FIG. 59

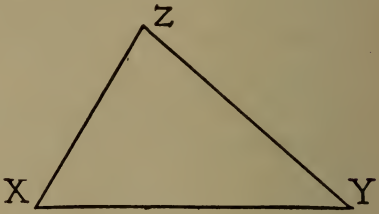


FIG. 60

5. In Figs. 59 and 60 it is known that $\frac{AC}{XZ}=\frac{2}{3}$; $\frac{BC}{YZ}=\frac{2}{3}$; and $\frac{AB}{XY}=\frac{2}{3}$. In other words, it is known that the ratios of *corresponding* sides are equal. Measure each angle of each triangle and compare each angle in Fig. 59 with its corresponding angle in Fig. 60. State carefully the conclusion to be drawn from this comparison. Are these triangles of the same shape?

**Section 37. Similar triangles.** The previous exercises illustrate two very important and widely used truths about similar triangles:

**(1) IF IT IS KNOWN THAT THE ANGLES OF ONE TRIANGLE ARE EQUAL RESPECTIVELY TO THE ANGLES OF ANOTHER TRIANGLE, IT FOLLOWS THAT THE RATIOS OF THE CORRESPONDING SIDES OF THE TRIANGLES ARE EQUAL.**

**(2) IF IT IS KNOWN THAT THE RATIOS OF THE CORRESPONDING SIDES OF TWO TRIANGLES ARE EQUAL, IT FOLLOWS THAT THE ANGLES OF ONE TRIANGLE ARE EQUAL RESPECTIVELY TO THE ANGLES OF THE OTHER TRIANGLE.**

Thus, to state that two triangles are **similar** is equivalent to stating that *the corresponding angles are equal*, and that *the ratios of the corresponding sides are equal.*

This principle or truth is used very much in mathematics. To illustrate, suppose we know that the angles of one triangle are equal *respectively* to the angles of another triangle; then we *also know* that the *ratios* of *corresponding sides* are *equal.* Hence, we can make an **equation** from these equal ratios and from this equation find important unknown distances. The next exercise will show how this is applied to finding the length of lines.

## EXERCISE 38

### ADDITIONAL PRACTICE WITH SIMILAR FIGURES

1. Figure 61 is a right triangle. Why? If angle $A$ is 30°, find angle $C$. If angle $A$ is one fourth of angle $C$, find the size of each angle.

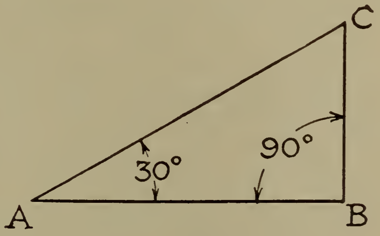


FIG. 61

2. In a right triangle one of the *acute* angles (that is, one of the angles smaller than a right angle) is 40°. Find the other acute angle.

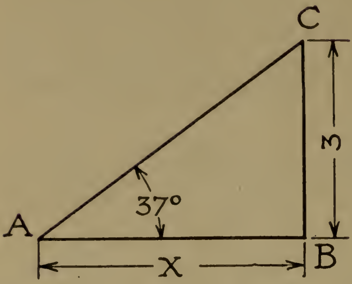


FIG. 62

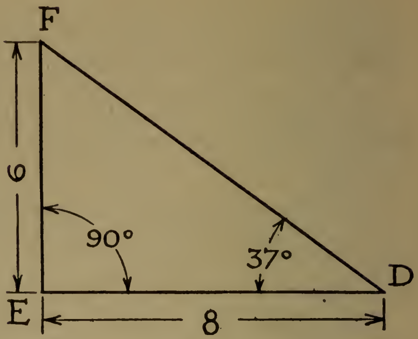


FIG. 63

3. Figures 62 and 63 are right triangles. If angle $A$ = angle $D$, are the triangles similar? Why?
4. In Figs. 62 and 63, if $EF = 6$, $ED = 8$, and $CB = 3$, what must $AB$ equal? To solve this problem we use the principle that the ratios of the corresponding sides of similar triangles are equal. This gives the equation $\frac{x}{8} = \frac{3}{6}$. What, therefore, is $AB$?
5. The sides of a triangular plat of ground are 150 ft., 100 ft., and 125 ft., respectively. The side of a scale drawing of this plat, corresponding to the 150-foot side, is 5 cm. Find the side of the scale drawing corresponding to the 100-foot side. Solve as in Example 4.
6. The sides of a triangle are 3, 4, and 5 cm. The shortest side of a similar triangle is 16 cm. Find the other sides of the second triangle.
7. A house is 36 ft. high and the garage is 16 ft. high. If the house is represented in a drawing as 18 in. high, how high should the drawing

of the garage be? What mathematical principle is used to show this?

**8.** Two rectangular gardens are the same shape, but of different size. The larger one is 72 ft. by 84 ft. If the length of the smaller one is 40 ft., what must be its width?

**9.** Two angles of one triangle are equal respectively to two angles of another triangle. Are the triangles similar? Why?

**10.** Line *AB* is parallel to line *CD*. Would they meet if produced, either to the right or to the left of the third line *MN*? Measure $\angle 1$ and $\angle 2$. These angles are called *corresponding angles* of parallel lines.

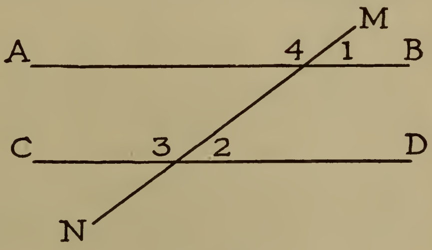


FIG. 64 *a*

**11.** In Fig. 64 measure the other pair of *corresponding angles*, $\angle 3$ and $\angle 4$. What do you find?

These two exercises illustrate a very important fact in mathematics; namely, that **the corresponding angles of parallel lines are always equal.** Later on this will be proved without measuring the angles; that is, without any possibility of error. You will make use of this fact without again measuring the angles.

**12.** In triangle *ABC*, *DE* is drawn parallel to *AB*. Does $\angle 1 = \angle 2$? Why? Is triangle *DEC similar* to triangle *ABC*? Why?

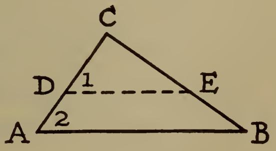


FIG. 64 *b*

**13.** In Example 12, $DC = 12$, $AC = 21$, and $CE = 10$. Show how $BC$ can be found, by using the principle that the ratios of corresponding sides of similar triangles are equal. What is the length of $BC$?

**14.** A boy wishes to measure the height of a tree. He notes that the tree, $AC$, its shadow, $AB$, and the sun's ray, $CB$, passing over the top of the tree, form a triangle. He measures the shadow and finds it 100 ft. long. At the same time a vertical stick 4 ft. high makes a shadow 10 ft. long. Why is the triangle formed by the stick, its shadow, and the sun's ray passing over the top of the stick similar to the other triangle? How can the boy find the height of the tree from the similar triangles? What is its height?

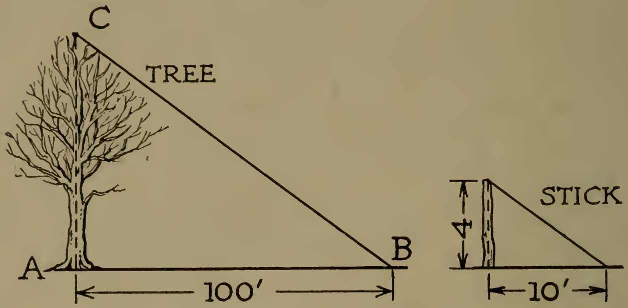


FIG. 65

**15.** A Boy Scout wagered he could find the distance between two trees, $A$ and $B$, on opposite sides of a river, without crossing it. Could he do it, and if so, how? If not, why not?

**16.** A crude way to measure the height of an object is by means of a mirror. Place a mirror horizontally on the ground at *M*, and stand at the point at which the image of the top of the object is just visible in the mirror. Show how, by measuring certain distances, this would enable one to compute the height of the object.

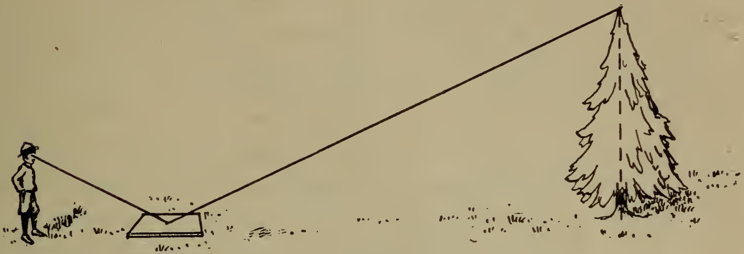


FIG. 66

**17.** In triangle *ABC*, *DE* is parallel to *CB*. Show that triangle *BED* is similar to triangle *ABC*. If $BC = 10$, $ED = 5$, and $AE = 8$, what is *AB*?

**18.** Show that in Fig. 67 the ratio of *DE* to *AE*, or $\frac{DE}{AE}$, *remains the same* even if *DE* is drawn in different positions (always parallel to *CB*).

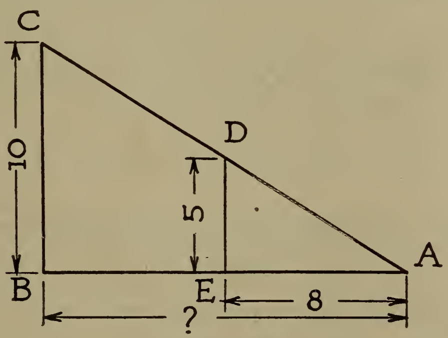


FIG. 67

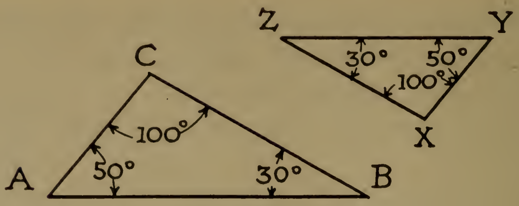


FIG. 68

**19.** Figure 68 shows two triangles, with the size of each angle indicated, which a teacher drew upon the blackboard for an examination. She asked the following questions about the two triangles:

(*a*) Are they similar triangles? Why?

(*b*) Does $\frac{AB}{ZY}=\frac{AC}{XY}$? Why?

(*c*) Does $\frac{AC}{XY}=\frac{BC}{ZY}$? Why?

(*d*) Does $\frac{ZY}{AB}=\frac{XZ}{BC}$? Why?

(*e*) Does the ratio of *any* two sides equal the ratio of *any other* two sides?

How would you have answered these questions?

**20.** The sides of a small triangle are 3, 4, and 6. Is it similar to a larger triangle whose sides are 15, 18, and 30? See Section 37.

---

### SUMMARY OF THE IMPORTANT POINTS OF THE CHAPTER

It is important to have clearly in mind the following important conclusions from the chapter:

1. If you know that the angles of one triangle are equal respectively to the angles of another triangle, then you know that *the ratios of the corresponding sides are equal.* In other words, you can make an equation, and thereby find an unknown side.
2. If you know that the ratios of the corresponding sides of two triangles are equal, then you know that the angles of one triangle are equal respectively to the angles of the other triangle.
3. The corresponding angles of parallel lines are equal.
4. Unknown distances may be found by means of similar triangles.

### REVIEW EXERCISE 39

1. Translate into words: $4y + 3 = y + 21$.
2. If $A$, $B$, and $C$ represent the number of degrees in the respective angles of a triangle, we know that $A + B + C = 180°$. Why? What is $A$ if $B = 40°$ and $C = 65°$?
3. If five times a certain number is divided by 2.7, the result is 3. What is the number?
4. Given the formula $V = lwh$, find a formula for $l$; for $w$.
5. A boy receives $C$ cents an hour for regular work, and pay for time and a half when he works overtime. What will represent his earnings for 6 hr. overtime? Evaluate this when $C = 50$.

**6.** The number of years that a man at various ages may expect to live, as determined by insurance experts, is as follows:

| If a man is (age in years) | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|---|
| he may still live (years) | 49.6 | 45.2 | 41. | 37 | 33.1 | 29.2 | 25 6 | 22.2 | 18.9 |

Construct this graphically, representing ages on the horizontal axis.

**7.** From the graph, find how much longer a man 27 years old may expect to live; a man 32 years old.

# CHAPTER VII

## HOW TO FIND UNKNOWNS BY MEANS OF THE RATIOS OF THE SIDES OF THE RIGHT TRIANGLE

**Section 38. The advantage of the RIGHT TRIANGLE in finding unknowns.** It should be clear by this time that mathematics gives us methods of finding unknown quantities. The equation is the most important tool for doing this, for the reason that when we solve a problem we have to make an equation. This equation must contain the unknown quantity together with other known quantities which are related to it in some way.

In the last chapter we saw that an equation could be formed from the *ratios of corresponding sides of similar triangles* and that by that means we could find an unknown length. Two facts, however, make that method less satisfactory than the one we shall study in this chapter: (1) we must always be certain the triangles are *similar*, or we have no right to make an equation, and (2) the method is cumbersome because we must always use *two* triangles.

There is a particular kind of triangle whose properties can be used to find unknown distances *accurately* and at the same time more easily than by any other method. It is the RIGHT TRIANGLE. The most important fact about the right triangle is found in connection with the *ratios of its different sides*.

### I. THE TANGENT OF AN ANGLE

**Section 39. The ratio of the "side opposite" a given angle to the "side adjacent" the given angle, *i.e.* the TANGENT of the angle.** You will recall that in a right triangle one angle is 90° and the sum of the two *acute angles* equals 90°. (Why?) In finding unknown distances by means of right triangles we shall always deal especially with one of the *acute* angles. Therefore, in referring to the sides of a right triangle, when dealing with a given angle, we shall speak of them as they are described in Figs. 69 and 70. If angle $B$ is the acute angle with which we are concerned, then side $AC$ is the "side opposite" $\angle B$, and side $AB$ is the "side adjacent" $\angle B$. The side opposite the 90° angle is *always* called the HYPOTENUSE.

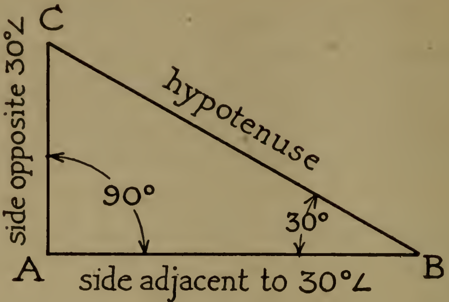


FIG. 69

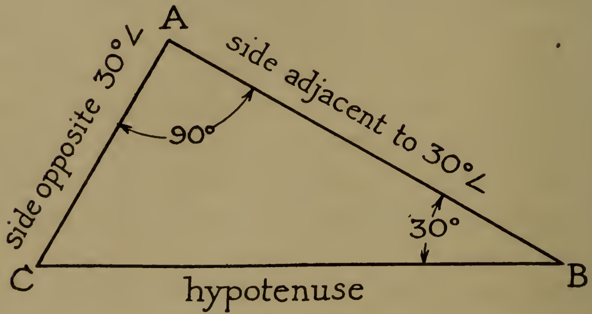


FIG. 70

Some exercises will show the importance of the *ratio*

$$\frac{\text{the ``side opposite''}}{\text{the ``side adjacent''}}$$

an acute angle of a *right triangle.*

## EXERCISE 40

### SOME EXPERIMENTS TO FIND THE NUMERICAL VALUE OF THE RATIO OF THE "SIDE OPPOSITE" TO THE "SIDE ADJACENT" A 30° ANGLE OF A RIGHT TRIANGLE

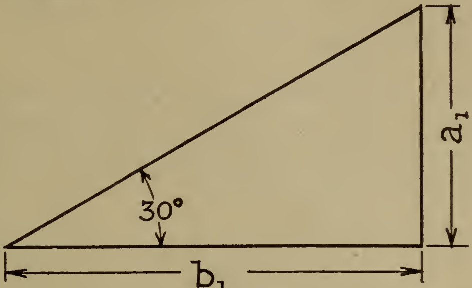


FIG. 71

**1.** In Fig. 71, $a_1$ is the "side opposite" the 30° angle and $b_1$ is the "side adjacent" the 30° angle. Measure $a_1$ and $b_1$. Now find the numerical value of the ratio of $a_1$ to $b_1$ by dividing the length of $a_1$ by the length of $b_1$. Record your results in Table 3.

**2.** In Fig. 72, $a_1$ and $b_1$ are respectively the "side opposite" and the "side adjacent" an acute angle of 30°. Measure each and compute the ratio $\frac{a_1}{b_1}$ to two decimal places. Record your results in Table 3.

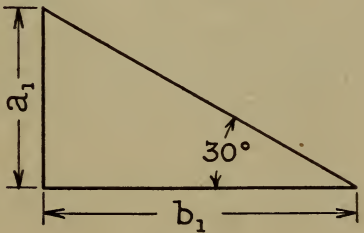


FIG. 72

3. Draw any other triangle similar to those above, but with much larger sides. Measure the "side opposite" and the "side adjacent" the 30° angle and compute their ratio as before. Record results, as before, in Table 3.

TABLE 3. Record here the results of measuring the sides of right triangles and of computing the ratio of the "side opposite" to the "side adjacent" an *acute angle of* 30°.

TABLE 3

| | Length of $a_1$ | Length of $b_1$ | Ratio of $a_1$ to $b_1$ (i.e., $\frac{a_1}{b_1}$) |
|---|---|---|---|
| Fig. | | | |
| Fig. | | | |
| Fig. | | | |
| Fig. | | | |

What do you notice in the table about the numerical values of the ratios

$$\frac{\text{the "side opposite" an acute angle of } 30^\circ}{\text{the "side adjacent" an acute angle of } 30^\circ} \text{ or of } \frac{a_1}{b_1}?$$

The members of the class should compare results, to see what result seems most likely to be the true one. If great care is taken in measuring, the ratio should be very close to .58 *in each triangle.* Why should it be *the same* in each triangle?

4. In Fig. 73, $CB$, $DE$, and $GF$ are perpendicular to $AB$. Is triangle $AFG$ similar to triangle $AED$? Why? Is either of the smaller triangles similar to the large triangle? Why? From this, why does the ratio of $GF$ to $AF$, or $\frac{GF}{AF}$, equal the ratio of $DE$ to $AE$, or $\frac{DE}{AE}$? If you measured these lines, and computed the ratios, what would you expect to be true of the results?

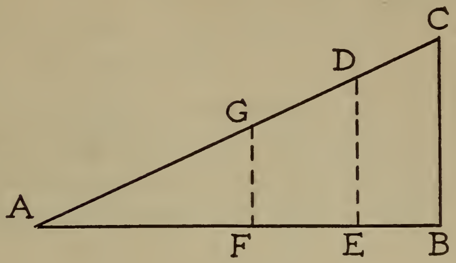


FIG. 73

**Section 40.** This last example is very important, because it shows, without measurement, that the ratio $\frac{GF}{AF}$ equals the ratio $\frac{DE}{AE}$. But this is the same as saying that **the ratio of the "side opposite" to the "side adjacent" a 30° angle in one right triangle is ALWAYS equal to the ratio of the "side opposite" to the "side adjacent" a 30° angle in any other right triangle.** The length of the sides may be far different, but the *ratios* should be the same. This shows that the ratios obtained in the table *should* have been the same, if it were possible to draw and measure without error.

We shall now make use of the fact that the numerical value of the ratio of the "side opposite" to the "side adjacent" an acute angle of 30° (in a right triangle) is approximately .58.

### EXERCISE 41

1. A man wishes to determine the height of a smokestack. He finds that the angle of elevation of the top of the smokestack, from a point 200 ft. from the base of the smokestack, is 30°.

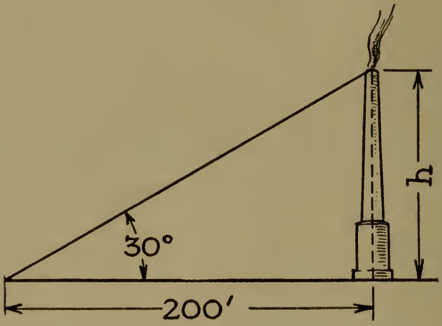


FIG. 74

**Solution:** $\frac{h}{200} = .58.$ **(Why?)**

$h = 200 \times .58.$ **(Why?)**

$h = 116$ **ft.**

**Note here that $\frac{h}{200}$ is the ratio of the "side opposite" to the "side adjacent" the 30° angle. From previous work we know that this ratio is .58. Thus, we can make the equation $\frac{h}{200} = .58.$**

2. In triangle $ABC$, angle $A$ is 30° and angle $C$ is 60°. Find $CB$ if $AB$ is 75 yards. What is $CB$ if $AB$ is 10 inches? Draw the figure.

3. In triangle $XYZ$, angle $X$ is 30° and angle $Z$ is 60°. Find $XY$ if $YZ$ is 116 ft.

4. In Fig. 75, *CD* bisects angle *C* and is perpendicular to *AB*. How many degrees in angle *BCD*? If *CD* is 100 cm., how long is *DB*?

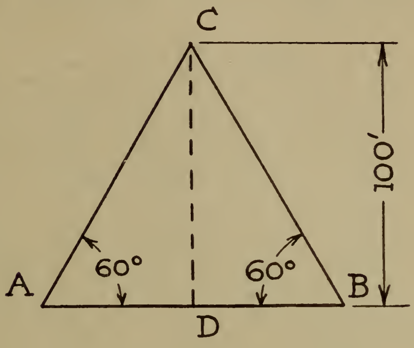


FIG. 75

5. In the right triangle *ABC*, angle *B* is 60°, angle *A* is 30°, and *BC* is 50 ft. Find *AC*.
6. In triangle *XYZ*, angle *X* is 30°. What do you know about the ratio $\frac{ZY}{XY}$? State definitely when this ratio is equal to .58.

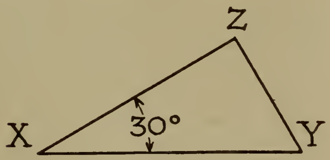


FIG. 76

**Section 41. It is convenient to name important ratios.** Since it is helpful to use the ratios of the various sides of a *right triangle*, very frequently in finding unknown distances, each is given a definite name. The ratio of the "side opposite" an acute angle to the "side adjacent" is called :

### THE TANGENT OF THE ANGLE

Its abbreviation is **tan**. Thus, in the above examples the **tangent of an angle of 30° is constant; it is approximately .58.**

### EXERCISE 42

1. Construct a right triangle such as Fig. 77, with *AB* equal to 4 cm. and angle *A* equal to 40°. Then measure *BC* and from that find the tangent of an angle of 40°. Compare results with those of other members of the class.
2. In a similar way find the tangent of an angle of 50°. (Use *AB* as 4 cm.) Also find the tangent of each of the following angles: 60°, 70°, and 20°.

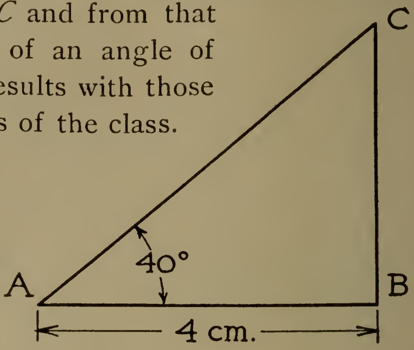


FIG. 77

**Section 42. Summary of steps in finding the tangent of an angle.** These examples show how to find the tangent of any angle. *Three* steps are necessary; namely: (**1**) measure the side opposite the particular angle; (**2**) measure the side adjacent the angle; (**3**) divide the first number obtained by the second. To do this, however, for angles of all sizes from very small to very large, would require a great deal of labor, and probably give, for a great many of you, inaccurate results. To save this trouble, and at the same time get very accurate results, these ratios or tangents have been computed very carefully and compiled in a table like Table 4. (See Table of Tangents on page 105.)

## TABLE 4

## TABLE OF SINES, COSINES, AND TANGENTS

### NUMERICAL VALUES OF THE TANGENTS, COSINES, AND SINES OF THE ANGLES FROM 0° TO 90° INCLUSIVE

| Deg. | tan | cos | sin | Deg. | tan | cos | sin |
|---|---|---|---|---|---|---|---|
| 0 | .000 | 1.000 | .000 | 46 | 1.04 | .695 | .719 |
| 1 | .017 | .999 | .017 | 47 | 1.07 | .682 | .731 |
| 2 | .035 | .999 | .035 | 48 | 1.11 | .669 | .743 |
| 3 | .052 | .999 | .052 | 49 | 1.15 | .656 | .755 |
| 4 | .070 | .998 | .070 | 50 | 1.19 | .643 | .766 |
| 5 | .087 | .996 | .087 | 51 | 1.23 | .629 | .777 |
| 6 | .105 | .995 | .105 | 52 | 1.28 | .616 | .788 |
| 7 | .123 | .993 | .122 | 53 | 1.33 | .602 | .799 |
| 8 | .141 | .990 | .139 | 54 | 1.38 | .588 | .809 |
| 9 | .158 | .988 | .156 | 55 | 1.43 | .574 | .819 |
| 10 | .176 | .985 | .174 | 56 | 1.48 | .559 | .829 |
| 11 | .194 | .982 | .191 | 57 | 1.54 | .545 | .839 |
| 12 | .213 | .978 | .208 | 58 | 1.60 | .530 | .848 |
| 13 | .231 | .974 | .225 | 59 | 1.66 | .515 | .857 |
| 14 | .249 | .970 | .242 | 60 | 1.73 | .500 | .866 |
| 15 | .268 | .966 | .259 | 61 | 1.80 | .485 | .875 |
| 16 | .287 | .961 | .276 | 62 | 1.88 | .469 | .883 |
| 17 | .306 | .956 | .292 | 63 | 1.96 | .454 | .891 |
| 18 | .325 | .951 | .309 | 64 | 2.05 | .438 | .899 |
| 19 | .344 | .946 | .326 | 65 | 2.14 | .423 | .906 |
| 20 | .364 | .940 | .342 | 66 | 2.25 | .407 | .914 |
| 21 | .384 | .934 | .358 | 67 | 2.36 | .391 | .921 |
| 22 | .404 | .927 | .375 | 68 | 2.48 | .375 | .927 |
| 23 | .424 | .921 | .391 | 69 | 2.61 | .358 | .934 |
| 24 | .445 | .914 | .407 | 70 | 2.75 | .342 | .940 |
| 25 | .466 | .906 | .423 | 71 | 2.90 | .326 | .946 |
| 26 | .488 | .899 | .438 | 72 | 3.08 | .309 | .951 |
| 27 | .510 | .891 | .454 | 73 | 3.27 | .292 | .956 |
| 28 | .532 | .883 | .469 | 74 | 3.49 | .276 | .961 |
| 29 | .554 | .875 | .485 | 75 | 3.73 | .259 | .966 |
| 30 | .577 | .866 | .500 | 76 | 4.01 | .242 | .970 |
| 31 | .601 | .857 | .515 | 77 | 4.33 | .225 | .974 |
| 32 | .625 | .848 | .530 | 78 | 4.70 | .208 | .978 |
| 33 | .649 | .839 | .545 | 79 | 5.14 | .191 | .982 |
| 34 | .675 | .829 | .559 | 80 | 5.67 | .174 | .985 |
| 35 | .700 | .819 | .574 | 81 | 6.31 | .156 | .988 |
| 36 | .727 | .809 | .588 | 82 | 7.12 | .139 | .990 |
| 37 | .754 | .799 | .602 | 83 | 8.14 | .122 | .993 |
| 38 | .781 | .788 | .616 | 84 | 9.51 | .105 | .995 |
| 39 | .810 | .777 | .629 | 85 | 11.4 | .087 | .996 |
| 40 | .839 | .766 | .643 | 86 | 14.3 | .070 | .998 |
| 41 | .869 | .755 | .656 | 87 | 19.1 | .052 | .999 |
| 42 | .900 | .743 | .669 | 88 | 28.6 | .035 | .999 |
| 43 | .933 | .731 | .682 | 89 | 57.3 | .017 | .999 |
| 44 | .966 | .719 | .695 | 90 | Inf. | .000 | 1.000 |
| 45 | 1.000 | .707 | .707 |  |  |  |  |

## EXERCISE 43

### FINDING ANGLES AND TANGENTS FROM THE TABLE OF TANGENTS

Find, from Table 4, each of the following:

1. tan 42°.
2. The angle whose tangent is .58.
3. tan 57°.
4. The angle whose tangent is .94.
5. tan 14°.
6. The angle whose tangent is $\frac{5}{4}$.
7. tan 25°.
8. The angle whose tangent is $\frac{2}{3}$.
9. tan 45°.

## EXERCISE 44

### EXAMPLES WHICH INVOLVE THE USE OF THE TANGENT OF AN ANGLE

1. **Illustrative example.** The brace wire *AC* of a telephone pole *BC*, Fig. 78, makes with the ground an angle of 62°. It enters the ground 15 ft. from the foot of the pole. Find the height of the pole *BC*.

Solution:

$$\frac{BC}{AB} = \text{tangent } 62°.$$

$$\frac{BC}{15} = 1.88 \text{ (from the Table).}$$

$$BC = 15 \times 1.88 = 28.2.$$

2. The angle of elevation of the top of a tree, from a point 75 ft. from its base (on level ground), is 48°. How high is the tree?

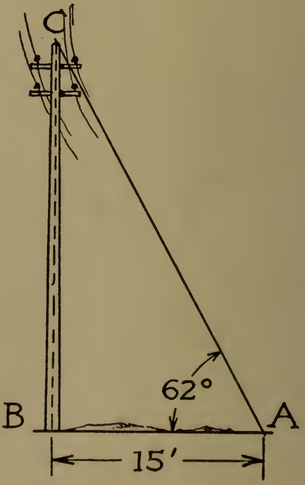


FIG. 78

**3.** From a vertical height of 1500 yd. a balloonist notes that the angle of depression of the enemy trench is 51°. Find the distance from the trench to the point on the level ground directly below the balloonist. Make a drawing.

**4.** The angle of elevation of an aëroplane at point *A* on level ground is 44°. The point *B* on the ground directly beneath the aëroplane is 450 yd. from *A*. How high is the aëroplane?

**5.** If a flagpole 42 ft. high casts a shadow 63 ft. long, what is the angle of elevation of the sun?

**6.** In Fig. 79, *CD* is perpendicular to *AB*. Find *AD* if angle $A = 60°$ and $CD = 20$.

**7.** From the point of observation on a merchant vessel, the angle of depression of the periscope of a submarine was 17°. How far was the submarine from the merchant vessel, if the observer was 40 ft. above the water?

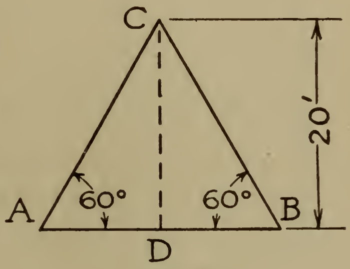


FIG. 79

**8.** Turn back to page 79 and solve problem 15 by this method. How do your results compare with those obtained by scale drawings?

## II. THE COSINE OF AN ANGLE

**Section 43. The ratio of the "side adjacent" the given angle to the hypotenuse of the triangle, *i.e.* the COSINE.** In the previous section we found that the *ratio* of the "side opposite" to the "side adjacent" an acute angle of a right triangle is always *constant* for any particular angle. This enabled us to find the length of the sides and the size of the acute angle. Now we come to another fact about right triangles. Let us examine a problem which *cannot* be solved by the use of the tangent.

**In Fig. 80, *BC* represents a telephone pole, *AC* an anchor wire, and *AB* the distance from the foot of the pole to the point at which the wire enters the ground, 20 ft. The wire makes an angle of 30° with the ground. How long is the wire?**

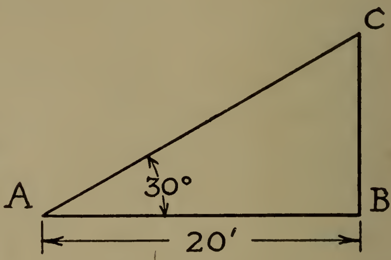


FIG. 80

Clearly, $AC$ cannot be found by means of the ratio which we called the tangent, because the tangent of 30° makes use only of $BC$ and $AB$, and we must get a ratio which contains $AC$. Therefore, to solve this problem we shall have to learn how to use the ratio of *the "side adjacent" the 30° angle, to the hypotenuse, or* $\frac{AB}{AC}$.

## EXERCISE 45

### EXPERIMENTS TO DETERMINE THE NUMERICAL VALUE OF THE RATIO $\frac{\text{THE "SIDE ADJACENT" A } 30^\circ \text{ ANGLE}}{\text{THE HYPOTENUSE OF THE TRIANGLE}}$, *I.E.* THE *COSINE*

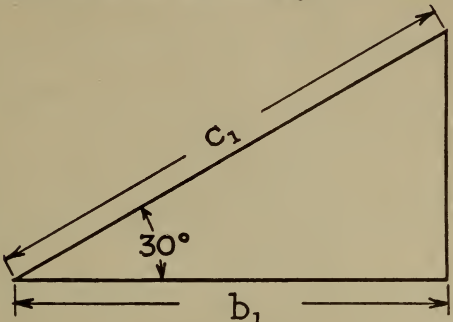


FIG. 81

1. Measure the length of $b_1$ and $c_1$ in Fig. 81. Then compute the ratio $\frac{b_1}{c_1}$ to two decimals.
2. Draw any other triangle similar to Fig. 81, but with much longer sides. Find, as in Example 1, the ratio of the side adjacent the 30° angle, to the hypotenuse. Compare your result with that of Example 1.

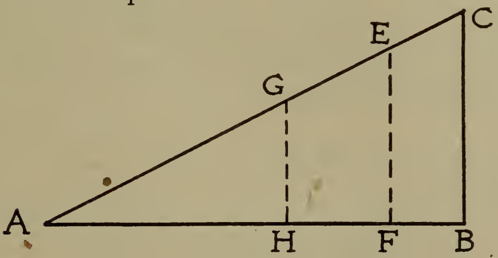


FIG. 82

3. In Fig. 82, $EF$ and $GH$ are perpendicular to $AB$. Why does $\frac{AH}{AG} = \frac{AF}{AE} = \frac{AB}{AC}$?

**Section 44. The COSINE of a particular angle is CONSTANT.** This shows that the ratio of the "side adjacent" a **30°** angle to the hypotenuse of one right triangle is equal to the same ratio in any other right triangle **which has an acute angle of 30°.** For this reason, you would get the same numerical value for $\frac{b_1}{c_1}$ in Examples 1 and 2, if it were not for errors in measurement.

Therefore, just as in the case of the tangent, so the **cosine,** *i.e.* $\frac{\text{"side adjacent" } 30^\circ \text{ angle}}{\text{hypotenuse}}$, is always constant, when the angle is 30°. It is approximately .86. *The right triangles may differ in size and position, but as long as they are similar* (that is, so long as the acute angles we are dealing with are the same size), *this ratio does not change.*

## EXERCISE 46

**PROBLEMS SOLVED BY APPLYING THE CONCLUSION ARRIVED AT ABOVE; NAMELY, THE RATIO OF THE "SIDE ADJACENT" A 30° ANGLE TO THE HYPOTENUSE IS .86.**

1. **Illustrative example.** The anchor wire $AC$, of a telephone pole, meets the ground 20 ft. from the foot of the pole, making an angle of 30° with the ground. Find the length of the wire $AC$.

Solution: $\frac{AB}{AC} = .86.$ (Why?)

$\frac{20}{h} = .86.$

$20 = .86\,h$, or $h = 23.2$ ft.

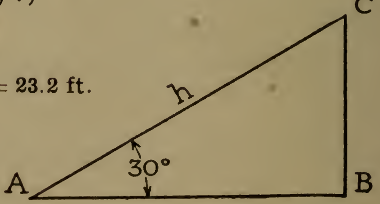


FIG. 83

2. The rope, *AC*, of the flagpole, *BC*, makes an angle of 30° with the ground, at a point 42 ft. from the foot of the pole. How long is the rope? Make a drawing.

3. The angle of elevation of the top of a tree from a point *A*, on level ground, 100 ft. from the base of the tree, is 30°. What is the distance from *A* to the top of the tree?

4. In the right triangle *ABC*, *AB* is 64 cm. and $\angle A = 30°$. Find *AC*.

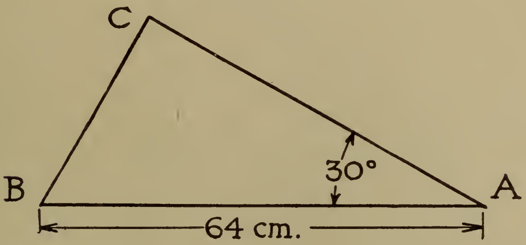


FIG. 84

5. Draw a right triangle such that angle $A = 60°$ and the hypotenuse $AC = 60$ cm. From this could you find *BC*?

6. The angle of depression of a boat, from the top of a cliff, is 30°. Find the distance from the observer to the boat, if the boat is 400 ft. from the foot of the cliff.

These examples have been solved by using the ratio of the "side adjacent" an acute angle of 30° to the hypotenuse, or, as we shall call it from now on, by using the COSINE OF THE ANGLE. The abbreviation for cosine is cos. Thus,

$$\cos \angle B = \frac{\textbf{ratio of ``side adjacent'' } \angle B}{\textbf{hypotenuse of the triangle}}.$$

### EXERCISE 47

1. Construct a right triangle similar to Fig. 85, with $A = 40°$ and $AB = 4$ cm. Then measure $c_1$ and compute the ratio $\frac{b_1}{c_1}$. By comparing your result with cos 40° as given in the table, see if you are within .05 of the correct result.

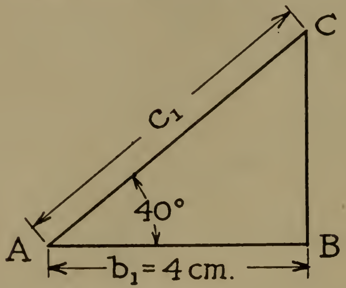


FIG. 85

2. How would you construct or draw the cos of a 60° angle? of an 80° angle?

3. Read from the table of cosines:
   (*a*) cos 67°.
   (*b*) The angle whose cos is .258.
   (*c*) cos 45°.
   (*d*) The angle whose cos is .573.
   (*e*) cos 2°.
   (*f*) The angle whose cos is .707.
   (*g*) cos 89°.
   (*h*) The angle whose cos is .629.

4. A surveyor desires to measure the distance $BC$ across a swamp. He surveys the line $BA$ perpendicular to $BC$. He extends this line $BA$ until he can measure from $A$ to $C$. If $AC$ is 400 ft.

and angle $C$ is 55°, show how he would compute the length of $BC$. Find $BC$.

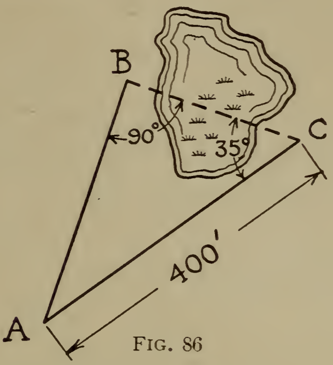


FIG. 86

**5.** A boy observes that his kite has taken all the string, 750 ft. Assuming that the string is straight and that it makes an angle of 34° with the ground, how far on level ground is it from the boy to the point directly below the kite?

**6.** The angle of elevation of the top of a tent pole, from a point 43.2 ft. from the foot of the pole, is 32°. Find the distance from the point of observation to the top of the pole.

**7.** Figure 87 is a right triangle. Find $AB$ if angle $A$ is 42° and $AC = 67$. HINT: What is the cosine of angle $A$?

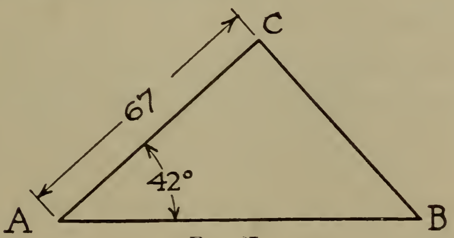


FIG. 87

8. How long a rope will be required to reach from the top of a flagpole to a point 19 ft. from the foot of the pole (on level ground) if the rope makes an angle of 63° with the ground?

9. The angle of depression of a boat from the top of a cliff is 37° when the boat is 1260 ft. from the foot of the cliff. Find the distance from the boat to the top of the cliff.

10. Find angle $A$ if $AB$ is 27 and $AC$ is 48. HINT: What is $\frac{27}{48}$ with respect to angle $A$?

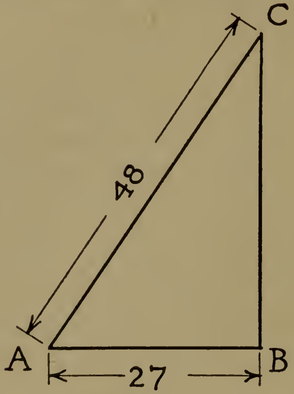


FIG. 88

11. A man starts at $O$ and travels in a direction which is 48° east of a north-south line. How far due north of $O$ will he be when he is 26 miles from $O$?

12. From the table find the cosine of 32°. Then find the cosine of an angle twice as large as 32°, and see if it is twice as large as the cosine of 32°. Does the cosine of an angle change or vary in the same way that the angle changes or varies?

## III. THE SINE OF AN ANGLE

**Section 45. The ratio of the "side opposite" an acute angle to the hypotenuse, or, the SINE of the angle.** We have now used two particular ratios of the sides of a right triangle, the *tangent* and the *cosine.* By using them we were able to determine unknown lines and angles. But these two ratios are not sufficient to find *any* side or *any* angle of a right triangle. For example, we have no ratio which involves *BC* and *AC* in Fig. 89. This brings us to the third (and last) important ratio:

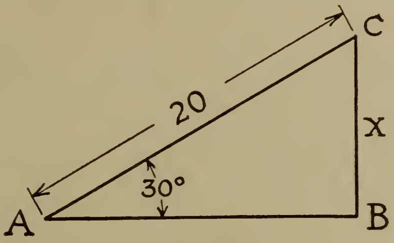


FIG. 89

$$\frac{\textbf{the "side opposite" an acute angle}}{\textbf{the hypotenuse}}.$$

In the same way as before, we can show that the numerical value of this ratio is *constant* for any given angle. Having discussed the tangent and cosine so completely, it is unnecessary to take the trouble to construct or to compute the value of this ratio. The numerical values of this ratio, for all acute angles, are given in the table of SINES.

## EXERCISE 48

### EXAMPLES SOLVED BY MEANS OF THE SINE OF AN ACUTE ANGLE

**1.** A man travels from *O* in a direction which is 50° east of a north-south line. How far is he from the north-south line when he has traveled 60 miles from the starting point, *O*?

**2.** How far was the man from an east-west line through the point *O*?

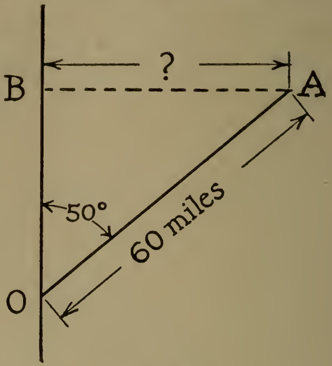


FIG. 90

**3.** To what height, on a vertical wall, will a 38-foot ladder reach, if it makes an angle of 58° with the ground?

**4.** An aviator, 4200 yd. directly above his own lines, takes the angle of depression of the enemy's battery. What must be the range of the enemy machine guns to endanger him, if the angle of depression is 29°?

**5.** What ratio gives the sine of ∠*A* in this figure? cosine *C*? If *A* is 60°, what is *C*? Compare sine 60° with cosine 30°, from the table. State in words your conclusion.

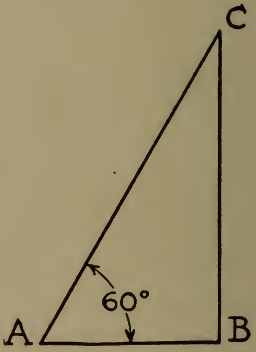


FIG. 91

### REVIEW EXERCISE 49

In this list of problems you will have to decide for yourself whether to use the tangent, the cosine, or the sine. Make a drawing for each problem; indicate the parts that you know, and the part you are to find.

1. A flagpole 50 ft. high casts a shadow 80 ft. long. What is the angle of elevation of the sun? What time of year is it?

2. A searchlight on the top of a building is 180 ft. above the street level. Through how many degrees from the horizontal must its beam of light be depressed so that it may fall directly on an object 400 ft. down the street from the base of the building?

3. From the top of a cliff 120 ft. above the surface of the water, the angle of depression of a boat is 20°. How far is it from the top of the cliff to the boat?

4. At a time when the sun was 55° above the horizon, the shadow of a certain building was found to be 98 ft. long. How high is the building?

5. A 40-foot ladder resting against a building makes an angle of 53° with the ground. Find the distance from the foot of the ladder to the building, and the distance from the top of the ladder to the base of the building.

6. A man starts at $O$ and travels in a direction which is 24° west of a north-south line through $O$, at the rate of 80 miles per day. At the end of 4 days how far north is he from an east-west

line through $O$? How far west is he from a north-south line through $O$?

7. What direction will a boy be from his starting point if he goes 40 miles due north and then 18 miles due east?

8. The gradient or slope of the railroad which runs up Pike's Peak is, in some places, 18%, *i.e.* in going 100 ft. horizontally it rises 18 ft. What angle does the road make with the horizontal?

# CHAPTER VIII

## HOW TO SHOW THE WAY IN WHICH ONE VARYING QUANTITY DEPENDS UPON ANOTHER

**Section 46. Quantities that change together.** We have already seen that there are many illustrations of quantities that change together. The amount of money paid out for rent at $30 per month *changes with*, or *depends upon*, the number of months; the time required to walk a certain distance, say 10 miles, *changes as*, or *depends upon*, the number of miles one walks per hour. In other words, there are **varying quantities** which are so RELATED that *a change in the value of one of them causes a change in the value of the other.*

This chapter will deal with quantities which change together. In addition to what you already know about these *varying* quantities, we shall now study just *how* these quantities vary. For example, does an *increase* in the value of one varying quantity cause a **corresponding increase** in the related quantity? Or does an *increase* in one varying quantity cause a **corresponding decrease** in the other? Can these be expressed (1) graphically, or (2) by tables, or (3) by formulas? These are the points which will be studied in the chapter.

**Section 47. Variables and constants.** In our study of time, rate, and distance problems we saw that the distance traveled by a train running at any given rate *changes* or *varies* as the time which it has been running *changes* or *varies*. If a train runs at the rate of 40 miles per hour, its movement is described by the equation

$$d = 40\,t.$$

In this equation, $d$ and $t$ change as the train progresses along its journey. The value of $d$ **depends upon** the value of $t$. This means that the **distance and time are variables,** while the **rate is constant.**

Table 5 shows the *tabular method* of representing the relation between these related variables. This shows that

TABLE 5

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| If the no. of hrs. is | 1 | 2 | 3 | 4 | 5 | 8 | 10 | 15 | 20 |
| then the distance is | 40 | 80 | 120 | 160 | 200 | 320 | 400 | 600 | 800 |

a change in the time causes a change in the distance, or that **a change in one variable causes a change in the related variable.**

### EXERCISE 50

1. In the above table, does an *increase* in the number of hours always cause an *increase* in the distance?
2. In the same table, find the ratio of each distance to its corresponding time. How do these ratios compare? Do the ratios change?
3. A man buys a railroad ticket at 3 cents per mile. Show by the tabular method the relation between the cost and the number of miles traveled. Show from the table that as the distance increases the cost increases, but that the *ratio* of the cost to the distance does not change. What equation will show the same thing the table shows?
4. Write the equation for the cost of any number of pounds of sugar at 9 cents per pound. What are the variables in your equation? Tabulate

the cost for 1, 2, 5, 8, and 10 pounds. Show from the table that the **ratio** of the cost to the number of pounds does not change; that is, it is *constant.*

5. A rectangle has a fixed base, 5 inches. Its altitude is subject to change. Tabulate its area if its altitude is 4, 6, 8, 10, and 12 inches. Compare the ratio of any two values of the area with the ratio of the two corresponding values of the altitude. If one altitude is three times another altitude, the one area is __?__ times the other area. Write the equation for its area.

6. A bicyclist rides 10 miles per hour. Show, by three methods, the relation between the number of miles he travels and the number of hours required. In 6 hours he travels __?__ times as far as he travels in 8 hours.

### I. DIRECT VARIATION, OR DIRECT PROPORTIONALITY

**Section 48.** The problems in the previous exercise illustrate **direct variation, or direct proportionality**. In each of the examples, one of the variables depended upon another variable for its value, and the *ratio of any two values of one variable was equal to the ratio of the two corresponding values of the other variable.* When two variables are related in this way, one is said to *vary as* or to be *directly proportional to* the other. Thus, to prove that two variables are *directly proportional*, or *vary directly*, we must show that

**The ratio of any two values of one variable is equal to the ratio of the two corresponding values of the other variable.**

## EXERCISE 51

1. **Illustrative example.** A man earns \$6 per day. Show that the amount he earns is *directly proportional* to the number of days he works.

Solution :

(1) $A = 6d$. (We write the equation first, from the conditions of the problem.)

(2) Tabulating :

TABLE 6

| If d is | 1 | 2 | 5 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| then A is | 6 | 12 | 30 | 48 | 60 | 72 |

(3) Now select any two values of $A$, say 12 and 60, and the two *corresponding* values of $d$, which are 2 and 10. If the ratio of these two values of $A$ is equal to the ratio of these two values of $d$, then in the equation $A = 6d$ we know that $A$ is directly proportional to $d$, or that $A$ varies directly as $d$. Does $\frac{12}{60} = \frac{2}{10}$? Yes.
Thus, *A is directly proportional* to $d$, or the amount a man earns at \$6 per day is directly proportional to the number of days he works. This is often written $\frac{A_1}{A_2} = \frac{d_1}{d_2}$. $A_1$ means some particular value of $A$, and $A_2$ means some other particular value of $A$; $d_1$ and $d_2$ mean those particular values of $d$ which *correspond* to the selected values of $A_1$ and $A_2$.

---

2. Write the equation for the area of a rectangle whose base is 10 inches. Then show by selecting particular values of $A$ and $h$ that the area is *directly proportional* to the altitude. In other words show that

$$\frac{A_1}{A_2} = \frac{h_1}{h_2}.$$

3. Write the equation for the circumference, $C$, of a circle whose diameter is $D$. Is $C$ directly proportional to $D$? Why?
4. Show that the area of a square is directly proportional to the square of its side.
5. Write the equation for the area of a circle. Show that the area varies directly as the square of the radius.
6. Show that the interest on $1000 at 6% is directly proportional to the time.
7. $x$ varies directly as $y$, and when $x = 10$, $y = 2$. Find the value of $x$ when $y = 7$.
8. $C$ varies directly as $d$, and when $d = 12$, $c = 48$. What is $d$ when $c = 72$?
9. Is your grade in mathematics directly proportional to the amount of time you spend in preparing your lessons?
10. Is the cost of a pair of shoes directly proportional to the size?

## II. INVERSE VARIATION

**Section 49. When quantities are inversely related to each other.** In the previous exercise the varying quantities were so related in any particular problem that an increase in one variable caused a corresponding increase in the other variable. Some variables, however, are so related that an **increase** in one is accompanied by a corresponding **decrease** in the other.

**An example:** An increase in the rate at which a train moves causes a decrease in the time required to travel a certain distance. If the train travels at the rate of 20 miles per hour, it will require 5 hours to cover 100 miles; but if it **increases** its rate to 30 miles per hour, it will **decrease** the time so that only $3\frac{1}{3}$ hours will be required to make the trip.

Let us illustrate this fact more in detail by tabulating the relation between the rate and the time of a train which makes a trip of 100 miles. Note from the table how a change in *one variable*, say the rate, is accompanied by a change in the *other variable*, the time.

TABLE 7

| If the rate is | 10 | $12\frac{1}{2}$ | 15 | 20 | 25 | 30 | $33\frac{1}{3}$ | 40 | 50 |
|---|---|---|---|---|---|---|---|---|---|
| then the time is | 10 | 8 | $6\frac{2}{3}$ | 5 | 4 | $3\frac{2}{3}$ | 3 | $2\frac{1}{2}$ | 2 |

This shows that an **increase** in the rate is accompanied by a **decrease** in the time. If we select *any* two values of the rate, say 20 and 50, and the *corresponding* values of the time, 5 and 2, we see that the ratio of the two values of the rate $\frac{20}{50}$ is *not* equal to the ratio of the corresponding values of the time $\frac{5}{2}$. Clearly, $\frac{20}{50}$ does **not** equal $\frac{5}{2}$, or, to use the more general form,

$$\frac{r_1}{r_2} \text{ does not equal } \frac{t_1}{t_2}.$$

These ratios *would* be equal, however, if we should invert one of them, *e.g.*

$$\frac{20}{50}=\frac{2}{5} \text{ or } \frac{r_1}{r_2}=\frac{t_2}{t_1}.$$

The fact that the ratio of any two values of one of the variables is equal to the *inverted* ratio of the corresponding values of the other variable leads us to say that one of them is *inversely proportional* to the other, or *varies inversely* as the other.

This gives the following principle:

**One variable is inversely proportional to another when the ratio of any two values of one of them is equal to the INVERTED RATIO of the two corresponding values of the other.**

### EXERCISE 52

1. The area of a rectangle is 200 sq. in. Show that the base varies inversely as the altitude, or that $\frac{b_1}{b_2}=\frac{a_2}{a_1}$.
2. The number of men doing a piece of work varies inversely as the time. If 10 men can do a piece of work in 32 days, in how many days can 4 men do the same work?
3. The variable $y$ varies inversely as $x$, and when $x=12$, $y=4$. Find $x$ when $y=16$.
4. Write an equation to show that the altitude and base of a rectangle, whose area is fixed, are inversely proportional.

**Section 50. Graphical method of representing inverse variation.** Figure 92, on the following page, shows graphically the relation between two numbers which are inversely proportional, or which vary inversely. It represents the base and altitude of a rectangle whose area is always constant, say 100 sq. ft.

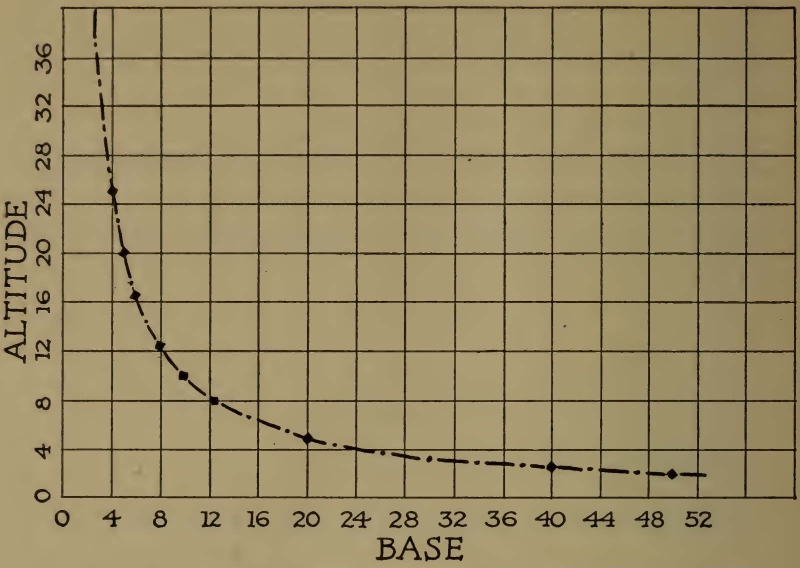


FIG. 92. **The line shows the relationship between two numbers which vary INVERSELY; in this case the relationship between the altitude and base of a rectangle whose area is constant, say 100 sq. ft. As the altitude INCREASES, the base DECREASES.**

To construct this graph, the following table was made:

TABLE 8

| If base is | 2 | 4 | 5 | 6 | 8 | 10 | 12.5 | 20 |
|---|---|---|---|---|---|---|---|---|
| then altitude is | 50 | 25 | 20 | 16.6 | 12.5 | 10 | 8 | 5 |

Note that as the base *increases*, the altitude *decreases*. How does the graph show this relation? In what way does this graph differ from those you have previously dealt with?

Show that the equation

$$\frac{a_1}{a_2}=\frac{b_2}{b_1}$$

describes the relation between the base and altitude of any rectangle whose area is constant, say 100 sq. ft.

## EXERCISE 53

### GRAPHICAL REPRESENTATION OF INVERSE VARIATION

1. The product of two variables, $x$ and $y$, is always 200. Tabulate 10 pairs of values of these variables, and from the table construct a graph showing the way in which the variables are related. Measure values of $x$ along the horizontal axis.

2. Some tourists decide to make a trip of 100 miles. Show graphically the relation between (1) the different rates at which they might travel, and (2) the time required at each rate.

## SUMMARY

1. **Two related variables or changing quantities are *directly* proportional, or *vary directly,* when a change in one is accompanied by a *corresponding* change in the other.**

   **To test for direct variation, it is necessary to see whether the *ratio* of any two particular values of one variable is equal to the ratio of the two corresponding values of the other variable.**

**2.** **Two related variables or changing quantities are *inversely* proportional, or *vary inversely*, when an increase in one is accompanied by a corresponding decrease in the other.**

**To test for inverse variation, it is necessary to see whether the ratio of any two values of one variable is equal to the *inverted* ratio of the two corresponding values of the other variable.**

**3.** **The graph of direct variation is a straight line, while the graph of inverse variation is a curve.**

### REVIEW EXERCISE 54

**1.** If 60 cu. in. of gold weighs 42 lb., how much will 35 cu. in. weigh?

**2.** If a section of a steel beam 10 yd. long weighs 840 lb., how long is a piece of the same material which weighs 1250 lb.?

**3.** At 40 lb. pressure per square inch, a given pipe discharges 160 gal. per minute. How many gallons per minute would be discharged at 65 lb. pressure?

**4.** A steam shovel can handle 900 cu. yd. of earth in 7 hr. At the same rate how many cubic yards can be handled in 5 hr.?

**5.** A train traveling at the rate of 50 miles per hour covers a trip in 5 hours. How long would it take to cover the same distance if it traveled at the rate of 35 miles per hour?

**6.** If 50 men can build a boat in 20 days, how long would it take 30 men to build it?

**7.** A wheel 28 in. in diameter makes 42 revolutions in going a given distance. How many revolutions would a 48-inch wheel make in going the same distance?

**8.** The volume, $v$, of a gas is inversely proportional to its pressure, $p$. Write an equation showing this fact.

**9.** If the volume of a gas is 600 cubic centimeters (cc.) when the pressure is 60 grams per square centimeter, find the pressure when the volume is 150 cubic centimeters.

**10.** When are two changing quantities or variables directly proportional? When do they vary inversely?

**11.** How can you test for direct variation? for inverse variation? Are $x$ and $y$ *directly* proportional in the equation $x = 2y + 5$?

**12.** If you know that

$$7b - 6 = 2b + 24,$$

then what is done to each side of the equation to get $7b = 2b + 30$?
What is the next step in solving this equation? Find the value of $b$. How do you check it?

**13.** How can you get rid of fractions in the equation

$$\frac{2}{3}x + 5 = \frac{1}{4}x + 29?$$

Why is 24 not the *most convenient multiplier?*

**14.** In order to save $d$ dollars in $n$ years, how much would your savings have to average per month?

# CHAPTER IX

## THE USE OF POSITIVE AND NEGATIVE NUMBERS

**Section 51. We need numbers to represent opposite qualities, or numbers of opposite nature.** The examples in the following exercises will illustrate what is meant by *opposite qualities*, or numbers of *opposite nature*. We shall take four different kinds of illustrations: (1) *opposite* numbers on a temperature scale, (2) *opposite* numbers on a distance scale, (3) *opposite* numbers to represent financial situations ("having" and "owing"), (4) *opposite* numbers on a *time* scale, to represent "time before" a beginning point and "time after."

### FIRST ILLUSTRATION: OPPOSITE NUMBERS ON A TEMPERATURE SCALE

### EXERCISE 55

1. The top of the mercury column of a thermometer stands at zero degrees (0°). During the next hour it *rises* 3°, and the next it *rises* 4°. What is the temperature at the end of the second hour?
2. The top of the mercury column stands at 0°. During the next hour it *falls* 3°, and in the next it *falls* 4°. What is the reading at the end of the second hour?
3. If it starts at 0°, *rises* 3°, and then *falls* 4°, what is the reading?
4. If it starts at 0°, *falls* 3°, and then *rises* 4°, what is the reading?

These examples show that we must distinguish two kinds of temperature readings, (1) those *above* zero and (2) those *below* zero. People have agreed to call readings above zero "*POSITIVE*," and readings below zero

"*NEGATIVE.*" Thus, if the mercury starts at zero and rises 4°, it will be at positive 4°, or, more briefly, + 4°. But if it starts at zero and falls 4°, it will be at negative 4°, or − 4°. In the remainder of these examples you should describe the mercury readings as *positive* or *negative*, rather than as *above* or *below* zero.

5. The temperature stands at zero. Its first change is described by the expression + 6°. Its next change is described by + 4°. What is the temperature at the end of the second change?
6. If the temperature reading is 0°, and it makes the change − 5°, then − 3°, what is the final reading?

### SECOND ILLUSTRATION: OPPOSITE NUMBERS ON A DISTANCE SCALE

### EXERCISE 56

1. An autoist starts from a certain point and goes east 10 miles, and then east 8 miles. How far and in what direction is he from the starting point?
2. If he had first gone west 10 miles, and then west 8 miles, how far and in what direction would he have been from his starting point?
3. If he had first gone east 10 miles and then west 8 miles, how far and in what direction would he have been from his starting point?
4. If he had first gone west 10 miles, and then east 8 miles, how far and in what direction would he have been from his starting point?

These examples show that we must distinguish between *opposite* distances, those *east* of some starting point, and those *west* of the starting point. People have agreed to call distances *east* of the starting point **positive** and distances *west* of the starting point **negative**. By this means a great deal of time can be saved, **because a positive or negative number tells both the *direction* and the *distance*** of a point on the distance scale, from some beginning point. Thus, on the distance scale, Fig. 93, point $A$ is *completely described* by the number $-5$.

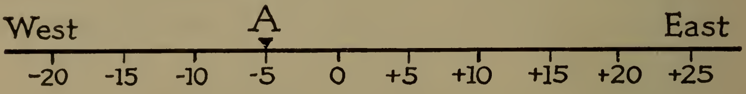


FIG. 93. **Points on a distance scale.**

This number, $-5$, tells that the point $A$ is 5 units *west* of, or to the left of, the starting point.

**5.** What would be the position on this distance scale of a man who starts at the zero point, goes east 60 units, and then west 15 units?

**6.** Where would you be if you started at zero, went $+8$ units, and then $-8$ units?

**7.** A man starts at 0; at the end of the first day he is at $+20$, and at the end of the second day he is at $-10$. What is the total distance he traveled? What number will completely describe his position at the end of the second day?

**8.** How far is it from $+9$ to $-6$? What direction is it?

**Section 52. Thus, positive and negative numbers are used to distinguish between opposite qualities.** The foregoing examples show that we need a brief, economical way to denote opposite qualities of numbers. This is done by *positive* and *negative* numbers, or, as we shall say from now on, by SIGNED NUMBERS. Thus, in referring to temperature readings, *e.g.* the "signed" number, + 10°, shows (1) how far and (2) in what direction the mercury stands from the zero point. In describing the location of a point on a distance scale, the "signed" number, – 6, tells how far and in what direction the point is from the starting or zero point; that is, 6 units to the left of, or to the west of, the zero point.

### THIRD ILLUSTRATION: OPPOSITE NUMBERS TO REPRESENT FINANCIAL SITUATIONS

**Section 53. Positive and negative numbers, or SIGNED NUMBERS, are used also to describe financial situations.** It has been agreed to consider money that you "have" as POSITIVE and money that you "owe" as NEGATIVE. Thus, if you *owe* 40 cents (*i.e.* – 40 cents) and *have* 55 cents (+ 55 cents), your *real* financial situation is + 15 cents. Why? Or, if you *owe* 90 cents (– 90 cents) and *have* 75 cents (+ 75 cents), your *real* financial situation is – 15 cents.

### FOURTH ILLUSTRATION: OPPOSITE NUMBERS ON A TIME SCALE

**Signed numbers are used also to distinguish "time before" from "time after" a given time.** For example, if time before Christ is *negative*, then time after Christ is *positive*. Thus, on the *time scale* below, since Christ's birth is regarded as zero, if a man was born 10 years be-

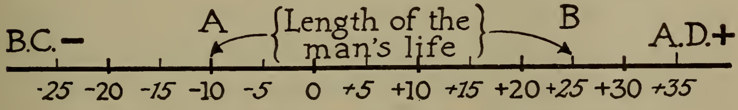


FIG. 94. **Points on a time scale.**

fore Christ and lived 35 years, *the distance* between the points *A* and *B* would represent the period of his life. Why?

### OTHER ILLUSTRATIONS OF THE USE OF SIGNED NUMBERS: FOR THE PUPIL TO DEVELOP

### EXERCISE 57

1. Show how signed numbers are helpful in dealing with latitude; with longitude. Illustrate each one.
2. Show that signed numbers are a convenience in keeping *scores* in games in which you either make or lose a certain number of points.
3. Can you think of any other illustrations of *opposite* numbers?

### EXERCISE 58

#### PRACTICE IN USING SIGNED NUMBERS

1. Your teacher's financial situation is – $250. What does this mean?
2. A man's property is worth $5200 and his debts amount to $3300. How can positive and negative numbers be used to represent these amounts? What number will describe his *net* financial situation?
3. The mercury at 8 A.M. was at – 6°. If it was rising 3° per hour, where was it at 9 A.M.? at 10 A.M.? at 11.30 A.M.?
4. Show on a time scale that Cæsar began to rule the Roman people 31 years B.C., and ruled for 45 years.

5. What is the total number of miles traveled by a man who starts at zero on the distance scale if he is at $+6$ at the end of the first day, $-2$ at the end of the second day, and at $-8$ at the end of the third day?

6. On the *distance scale*, where would you be if you started at $-4$ and went east 6 miles?

7. If your financial condition is $+60$ cents, $-15$ cents, and $-12$ cents, what single number will accurately describe your net financial situation?

8. What was your final score in a game in which you made the following single scores: $+15$, $-8$, $-10$, $+14$, and $+15$?

9. Represent on a *distance scale* (horizontal) the point where a man would be at the end of the third day if he started at zero and walked $+6$ miles on Monday, $-10$ on Tuesday, and $-3$ on Wednesday.

10. Find the net financial situation of a man who is worth the following: (*a*) $+\$5 + \$8 + \$10 - \$6$; (*b*) $+6d - 10d - 8d + 15d$.

**Section 54. Absolute value of positive and negative numbers.** The numerical value of a positive or negative number, without regard to its sign, is its *absolute value*. For example, the absolute value of $+6$ is 6; of $+17$ is 17; of $-9$ is 9, etc.

**Section 55. We need to be able to add, subtract, multiply, or divide signed numbers.** Now that we see clearly the practical ways in which positive and negative numbers are used we need to be able to solve problems which contain either kind. In all the examples which we have worked previously, *only positive numbers have been used.* Next, therefore, we must learn (1) how to *combine* signed numbers (*i.e.* add them); (2) how to *multiply* them; (3) how to find the *difference* between two signed numbers; and (4) how to *divide* signed numbers. We will take them up in that order.

## I. HOW TO COMBINE SIGNED NUMBERS — FINDING ALGEBRAIC SUMS

**Section 56. When the numbers are arranged vertically.** In the example: "Find the net financial situation of a man who is worth the following: $+\$5$, $+\$8$, $-\$10$, $-\$6$," we found *one* signed number which described the man's net financial situation; namely, $-\$3$. That is, we found one signed number which was the result of putting several signed numbers together. This process is called **combining signed** numbers, or **finding the algebraic sum.** Thus, to combine $+4$, $-2$, $-6$, and $+3$, we must find one signed number which is the result of putting all of these together. Evidently, this must be $-1$. Similarly, combining, or finding the algebraic sum of $+5d$ and $-11d$, we get $-6d$.

In each of the following examples, find the algebraic sum, *i.e.* find one number which will describe the result of putting all the separate numbers together.

### EXERCISE 59

|  |  |  |  |
|---|---|---|---|
| **1.** $+\$6$ | **2.** $-7d$ | **3.** $+4x$ | **4.** $5a$ |
| $-\$3$ | $+8d$ | $-3x$ | $4a$ |
| $+\$4$ | $-4d$ | $-x$ | $-6a$ |

**5.** $\begin{array}{r} 6y \\ -5y \\ -\ y \\ \hline \end{array}$

**6.** $\begin{array}{r} +8 \\ -5 \\ -6 \\ +7 \\ \hline \end{array}$

**7.** $\begin{array}{r} -10 \\ +13 \\ -\ \ 8 \\ +\ \ 5 \\ \hline \end{array}$

**8.** $\begin{array}{r} +3b \\ -5b \\ -6b \\ +2b \\ \hline \end{array}$

**9.** $\begin{array}{r} 4a \\ -6a \\ \hline \end{array}$

**10.** $\begin{array}{r} -3x \\ -5x \\ +8x \\ \hline \end{array}$

**11.** $\begin{array}{r} 15ab \\ 12ab \\ \hline \end{array}$

**12.** $\begin{array}{r} -\ \ 7xy \\ +11xy \\ -\ \ 2xy \\ \hline \end{array}$

**13.** $\begin{array}{r} +2\frac{1}{2}x \\ -7\ x \\ +2\ x \\ \hline \end{array}$

**14.** $\begin{array}{r} -\ \ 4 \\ +\ \ 3 \\ +11 \\ -\ \ 8 \\ +\ \ 1 \\ \hline \end{array}$

**15.** $\begin{array}{r} x \\ 2x \\ -6x \\ \hline \end{array}$

**16.** $\begin{array}{r} 3a \\ -4a \\ 7a \\ -9a \\ \hline \end{array}$

**Section 57. When the numbers are arranged horizontally.** The numbers to be combined are almost always written in a horizontal line, rather than in a vertical column. Combining these terms is done in the same way as if they were written in a vertical column. For example, $+6-5+4-9=-4$.

**17.** $6-8-5+11=?$

**18.** $-8-9+11+6$

**19.** $-6d+5d+9d-2d$

**20.** $5abc+6abc-7abc$

**21.** $4f-3f+6f-9f$

**22.** $+8t-9t-6t+12t$

**23.** How have you found the algebraic sum of these numbers?

**Section 58. Terms are either LIKE or UNLIKE: How to distinguish TERMS.** Any algebraic expression, such as $ax + b$ or $x^2 + 2xy + 5$, is made up of one or more numbers separated from each other by + signs or by − signs. These numbers thus separated from each other are called **terms.** Thus, in $ax + b$ there are *two* terms; $ax$ is one, $b$ is the other; while in $x^2 + 2xy - 5$ there are three terms, $x^2$, $2xy$, and 5. Note carefully that a "*term*" *includes everything* between + or − signs.

In many algebraic expressions these terms are all **like** terms, and, as we learned in the previous section, can be combined or put together into one number or term which we called their *algebraic sum.* Thus, $4d$, $+5d$, $-8d$, $+3d$ are **similar** or **like terms,** and their algebraic sum is $+4d$. It is important to understand that because each letter in the expression represents the same thing these are **like** terms. In many cases, however, the terms of an algebraic expression are not all *like terms.*

For example, consider: 4 boys, + 5 girls, + 8 boys, − 2 girls, or, using the initial letters of the words, $4b + 5g + 8b - 2g$. Evidently, these are **unlike terms** and *cannot* be combined into *one* number. However, the *like* terms in the expression *can* be combined; that is, the $+4b$ and $+8b$, giving $12b$, and the $+5g$ and $-2g$, giving $+3g$. Thus the expression $4b + 5g + 8b - 2g$ can be *simplified* or *expressed more briefly* by combining *like terms,* giving $12b + 3g$. From this illustration we see that the *like terms* of any algebraic expression can be combined, giving a simpler, briefer expression than the original one.

## EXERCISE 60

### FURTHER PRACTICE IN COMBINING SIGNED NUMBERS: NUMBERS HAVING LIKE OR UNLIKE TERMS

Write in the simplest or briefest form each of the following expressions:

1. $2a + 3a - 6a + 4a$
2. 5 ft. + 6 in. − 2 ft. − 4 in.
3. 7 yr. + 3 mo. − 2 yr. − 1 mo.
4. $4b + 5c - 8b - 2c + b$
5. $6a^2 + 3a^2 - 7a^2 + 4a^2$
6. $-2x^3 - 5x^3 - 8x^3 + 2x^3$
7. $ax + 5 + 4ax + 3$
8. $3xy + 5ab - 7xy - 11ab$
9. $2b^3 - 7b^3 + 5b^3 - b^3$
10. $2r + 8r + 3r - 10r$
11. $4a^2b + 5a^2b - 8a^2b - 3a^2b$
12. $-6 + 4 - 8 + 6 - 9 + 2$
13. $5x + 3 - 8x - 4 + 4x + 1$
14. $a^2b + 4a^2b - 6a^2b$
15. $xy + 3 - 8xy - 9 + 2xy + 7$
16. $p + 2q - 8p + 4q + 6p + 5q$

## SUMMARY OF IMPORTANT PRINCIPLES CONCERNING THE COMBINING OF SIGNED NUMBERS

**Section 59.** You have now worked many examples in finding algebraic sums. From your experience with such examples, complete these three sentences which tell how to combine signed numbers:

1. **To find the algebraic sum of two positive numbers, ___?___ the absolute values of the numbers, and give to the result a ___?___ sign.**
2. **To find the algebraic sum of two negative numbers, ___?___ the absolute values of the numbers, and give to the result a ___?___ sign.**
3. **To find the algebraic sum of two numbers having unlike signs, find the ___?___ of their absolute values, and give to the result the sign of the number having the ___?___ absolute value.**

### II. HOW TO MULTIPLY SIGNED NUMBERS

**Section 60. The four ways to multiply signed numbers.** In arithmetic it was found that multiplication shortened the work of addition. For example, in adding $3 + 3 + 3 + 3 + 3 + 3 + 3$, the result is found most easily by multiplying 3 by 7, because 3 is taken 7 times. So, in algebra, it is equally desirable to multiply one signed number by another.

There are *four* different ways in which we may have to multiply signed numbers. These are:

(1) **plus** times **plus**, as in the example $+4$ times $+2 = ?$

(2) **plus** times **minus**, as in the example $+4$ times $-2 = ?$

(3) **minus** times **plus**, as in the example $-4$ times $+2 = ?$

(4) **minus** times **minus**, as in the example $-4$ times $-2 = ?$

By considering the following problems we can tell what meaning must be given to the multiplication of signed numbers.

## A. ILLUSTRATIVE QUESTIONS BASED UPON THE SAVING AND WASTING OF MONEY

### EXERCISE 61

1. If you save \$5 a month (+ \$5), how much better off will you be 6 months from now (+ 6)?
   Evidently you will be \$30 better off (+ \$30).
   Thus, + 5 times + 6 = + 30.

2. If you have been saving \$5 a month (+ \$5), how much better off were you 6 months ago (– 6)?
   Evidently you were \$30 worse off (– \$30) than you are now. Thus, + 5 times – 6 = – 30.

3. If you are wasting \$5 a month (– \$5), how much better off will you be in 6 months from now (+ 6)?
   Evidently you will be \$30 worse off (– \$30).
   Thus, – 5 times + 6 = – 30.

4. If you have been wasting \$5 a month (– \$5), how much better off were you 6 months ago (– 6)?
   Evidently you were \$30 better off (+ \$30).
   Thus, – 5 times – 6 = + 30.

Summarizing: These problems based upon saving and wasting money have led to the following illustrative statements:

1. + 5 times + 6 = + 30.
2. + 5 times – 6 = – 30.
3. – 5 times + 6 = – 30.
4. – 5 times – 6 = + 30.

### B. ILLUSTRATIVE QUESTIONS BASED UPON THERMOMETER READINGS

### EXERCISE 62

1. If the mercury is now at zero and *is rising* 2° per hour (+ 2), where *will it be* 4 hours from now (+ 4)?

   Evidently it will be 8° above zero (+ 8). Thus, + 2 times + 4 = + 8.

2. If the mercury *has been rising* 2° per hour (+ 2°) and is now at zero, where *was it* 4 hours ago (– 4)?

   Evidently it was 8° below zero (– 8°). Thus, + 2 times – 4 = – 8.

3. If the mercury is now at zero and *is falling* 2° per hour (– 2°), where *will it be* 4 hours from now (+ 4)?

   Evidently it will be 8° below zero (– 8°). Thus, – 2 times + 4 = – 8.

4. If the mercury is now at zero and *has been falling* 2° per hour (– 2°), where *was it* 4 hours ago (– 4)?

   Evidently it was 8° above zero (+ 8°). Thus, – 2 times – 4 = + 8.

Summarizing: these problems based upon the thermometer have led to the following illustrative statements:

1. + 2 times + 4 = + 8.
2. + 2 times – 4 = – 8.
3. – 2 times + 4 = – 8.
4. – 2 times – 4 = + 8.

A careful study of these illustrations will enable you to complete the following statements concerning multiplication of signed numbers :

1. A positive number multiplied by a positive number gives as a product a ____?____ number.
2. A positive number multiplied by a negative number gives as a product a ____?____ number.
3. A negative number multiplied by a positive number gives as a product a ____?____ number.
4. A negative number multiplied by a negative number gives as a product a ____?____ number.
5. The product of two numbers having like signs is ____?____.
6. The product of two numbers having unlike signs is ____?____.

### 11*a*. PARENTHESES ARE USED TO INDICATE MULTIPLICATION

**Section 61.** Multiplication of two or more numbers is often indicated by placing the numbers within parentheses. Thus, "$+4$ times $-35$" is often written "$(+4)(-35)$." It is important to note that *no* sign or symbol is placed between the parentheses when multiplication is indicated.

## EXERCISE 63

### PRACTICE IN MULTIPLYING SIGNED NUMBERS

1. $(+3)(+5)$
2. $(+6)(-2)$
3. $(+10)(-2\frac{1}{2})$
4. $(+6)(-9)$
5. $(-2)(-5)$
6. $(+8)(-\frac{1}{4})$
7. $(+12)(+6)$
8. $(+5)(+4)(-2)$
9. $(-3)(-6)(+2)$
10. $(-4)(-10)(-3)$
11. $(+\frac{2}{3})(-\frac{6}{8})$
12. $(-\frac{1}{2})(+\frac{5}{8})$
13. $(-7)(-6)(+2)$
14. $(-2)(-2)(-2)$

**15.** $(-3)(-3)(-3)$

**16.** $(+1)(+1)(+1)+(+1)$

**17.** $(-2)(-2)$

**18.** $(-2)(-2)(-2)(-2)$

**19.** $(3)(4)(5)(2)$

**20.** $(+2)(+\$5)$

**21.** $(+3)(+7D)$

**22.** $(+6)(+5\text{ ft.})$

**23.** $(-8)(+6y)$

**24.** $(+\frac{2}{3})(18a)$

**25.** $(-12)(+\frac{1}{2})$

**26.** $(\frac{2}{3})(-18)$

**27.** $(27)(-\frac{2}{9})$

**28.** $(+\frac{6}{5})(-25)$

**29.** $(-32)(-\frac{5}{8})$

**30.** $(-\frac{3}{4})(+24d)$

**31.** $(-\frac{2}{3})(+\frac{9}{8})$

**32.** $(-\frac{5}{7})(-\frac{14}{15})$

**33.** $(-2)(-2)(+2)$

**34.** $(+\frac{4}{5})(-\frac{10}{12})$

**35.** $(\frac{1}{5})(-10)$

**36.** $-5 \cdot 8$

**37.** $-7 \cdot 21$

**38.** $(-6)(-\frac{1}{6})$

**39.** $2 \cdot 2 \cdot (-3)$

**40.** $(-1)(-1)(-1)(-1)$

**41.** $(2)(-3)(+4)(-5)$

**42.** $(6)(\frac{1}{2})(\frac{1}{3})$

**43.** $(10)(5)(-\frac{1}{20})$

**44.** $(\frac{1}{2})(-\frac{4}{3})$

### II *b*. HOW TO USE EXPONENTS IN MULTIPLICATION

**Section 62.** Suppose we had to find the product of $3x^2$ and $5x^4$. It is important to keep in mind the meaning of exponents. $3x^2$ means $3 \cdot x \cdot x$ and $5x^4$ means $5 \cdot x \cdot x \cdot x \cdot x$. Hence, $3x^2$ times $5x^4$ or $(3x^2)(5x^4)$ means $3 \cdot x \cdot x \cdot 5 \cdot x \cdot x \cdot x \cdot x$, or $15x^6$. By the same reasoning, the product of $+6x^5$ and $-7x^4$ is $-42x^9$.

## EXERCISE 64

### PRACTICE IN USING EXPONENTS IN MULTIPLYING SIGNED NUMBERS
### (ORAL)

**1.** $(5a^2)(6a^4)$

**2.** $(+7b)(-9b^5)$

**3.** $(+8)(2y^3)$

**4.** $(6ab)(2a^2)$

**5.** $(+3abc)(5ab)$

**6.** $4x^3 \cdot 5x^2 \cdot 2x^5$

**7.** $y^4 \cdot 5y^3$

**8.** $\frac{1}{2}x^3 \cdot 10x^7$

**9.** $2ab^2 \cdot 3ab$

**10.** $a^2b \cdot ab^2 \cdot a^3b^4$

**11.** $x \cdot 2x^3$

**12.** $-6a \cdot 3a$

**13.** $-2y \cdot 3y^2$

**14.** $(-5b)(-2b^2)$

**15.** $(-6x^2)(-7x^2y)$

**16.** $(-8x^2)(\frac{1}{4}x)$

**17.** $(+10y^2x)(-2yx^2)$

**18.** $16a^2b \cdot (\frac{1}{4}ab^3)$

**19.** $a \cdot b \cdot b \cdot a \cdot a$

**20.** $x^4 \cdot 2x$

**21.** $y \cdot 5y^3$

**22.** $b^2c \cdot bc$

**23.** $5 \cdot x^2$

**24.** $10y^2 \cdot \frac{1}{10}y \cdot y^3$

## III. HOW TO FIND THE DIFFERENCE BETWEEN SIGNED NUMBERS: SUBTRACTION

**Section 63. How "differences" are found in practical work.** Clerks in stores have a method of making change or of finding the difference between two numbers that is very helpful in finding the difference between two *signed* numbers. For example, if a customer gives the clerk 50 cents in payment for a 27-cent purchase, the clerk begins at 27 and counts out enough money to make 50 cents. If we use the same terms as were used in arithmetic, — namely, the *subtrahend*, *minuend*, and *difference*, — then we say, "The clerk begins at the subtrahend, 27 cents, and counts to the minuend, 50 cents."

**First illustrative example.** To illustrate this method of finding the difference between two signed numbers, let us consider this problem:

On a certain day the mercury stands at – 4° in Chicago and at + 13° in St. Louis. How much warmer is it in St. Louis, or what is the difference between + 13° and – 4°? Naturally, we do the same thing the clerk does, *begin at the subtrahend and count to the minuend*, *i.e.* we count from – 4° to + 13°, giving us + 17°. The difference is called *positive* because we counted *upward.* If we counted *downward*, the difference would be called *negative.* This example is written as follows:

$$\begin{array}{rl} +13^\circ & \text{minuend} \\ -\ \ 4^\circ & \text{subtrahend} \\ \hline +17^\circ & \text{difference} \end{array}$$

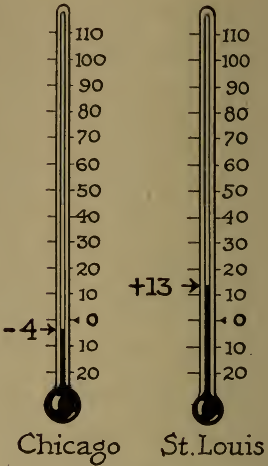


FIG. 95 *a*

**Second illustrative example.** Subtract + 10 from – 5 by referring to the number scale. This means to find the distance from the subtrahend to the minuend or from + 10 to – 5. The distance from 10 above to 5 below is clearly 15; and since the direction is downward, the *difference* is – 15. This example is written:

$$\begin{array}{rl} -5 & \text{minuend} \\ +10 & \text{subtrahend} \\ \hline -15 & \text{difference} \end{array}$$

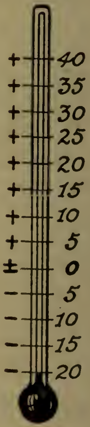


FIG. 95 *b*

These illustrations are given merely to show that **the difference between two signed numbers can always be found by counting on a number scale from the subtrahend to the minuend.** The difference will be positive or negative, depending upon whether the direction of counting is *upward* or *downward.*

## EXERCISE 65

### PRACTICE IN FINDING THE DIFFERENCE BETWEEN TWO SIGNED NUMBERS: SUBTRACTION

**1.**
$$\begin{array}{rrrrrrrr} +6 & -4 & +8 & +\ 3 & +5 & -7 & +13 & +\ 9 \\ -2 & +5 & +2 & +10 & -8 & +4 & +\ 4 & +14 \\ \hline \end{array}$$

**2.**
$$\begin{array}{rrrrrr} +\$4 & -6^\circ & +7\,d & +10 \text{ ft.} & -4 \text{ in.} & +\ 3\,x \\ -\$8 & +9^\circ & -2\,d & -\ \ 6 \text{ ft.} & -7 \text{ in.} & -10\,x \\ \hline \end{array}$$

**3.**
$$\begin{array}{rrrrrr} -\ \ 3\,a & +4\,b & -5\,c & 10\,x^2 & -3\,x^3y & +10\,abc \\ -11\,a & -2\,b & +6\,c & -2\,x^2 & +5\,x^3y & -\ \ 4\,abc \\ \hline \end{array}$$

**4.**
$$\begin{array}{rrrrr} -2\,t^3 & +\ \ 5\,xy^2 & 5\,a & 3\,x & -2\,bc \\ +3\,t^3 & +11\,xy^2 & -2\,a & x & bc \\ \hline \end{array}$$

**5.** From $-7\,a$ take $+5\,a$.

**6.** Take $-13\,b^2$ from $+2\,b^2$.

**7.** From $x$ take $7\,x$.

**8.** Take $-bc$ from $2\,bc$.

## IV. HOW TO DIVIDE SIGNED NUMBERS

**Section 64. Division is the opposite of multiplication.** You will have little or no difficulty in the *division* of signed numbers if you understand that division is just the opposite of multiplication. For example, if $4 \times 2 = 8$, then $\frac{8}{2} = 4$. In this case 8 is the dividend, 2 is the divisor, and 4 is the quotient. In signed numbers, as well as in arithmetic, the *dividend equals* the *quotient times* the *divisor.*

$+8 \div -2 = -4$; or $\frac{+8}{-2} = -4$, because $(-2)(-4) = +8$.

$-8 \div -2 = +4$; or $\frac{-8}{-2} = +4$, because $(-2)(+4) = -8$.

$+8 \div +2 = +4$; or $\frac{+8}{+2} = +4$, because $(+2)(+4) = +8$.

$-8 \div +2 = -4$; or $\frac{-8}{+2} = -4$, because $(+2)(-4) = -8$.

## EXERCISE 66

### PRACTICE IN FINDING THE QUOTIENTS OF SIGNED NUMBERS

Find the quotient in each of the following:

**1.** $\frac{+10}{+2}$; $\frac{+18}{-2}$; $\frac{-16}{+4}$; $\frac{-30}{-10}$; $\frac{-14}{+7}$; $\frac{+6}{+2}$; $\frac{-8}{-1}$; $\frac{+16}{-2}$.

**2.** $\frac{+15\,d}{-3}$; $\frac{-18 \text{ ft.}}{+3}$; $\frac{+16 \text{ mo.}}{-8}$; $\frac{+25\,x}{-5}$; $\frac{-10\,a}{+2}$.

**3.** $(21) \div (-3)$; $(-36) \div (-9)$; $(-54) \div (+27)$; $(-96) \div (+12)$; $(-21) \div (-7)$; $(-60) \div (+12)$.

**4.** $\frac{10\,x^3}{2\,x}$; $\frac{21\,y^4}{3\,y^2}$; $\frac{18\,b^5}{6\,b^4}$; $\frac{12\,c^4}{6\,c^4}$; $\frac{16\,p^3}{16\,p}$. How can you prove each of these?

**5.** $\frac{+20\,y^2}{-4\,y}$; $\frac{-27\,x^6}{-9\,x^4}$; $\frac{-34\,b^2c^2}{-17\,b^2}$; $\frac{-50\,x^3y}{+25\,y}$; $\frac{-72\,y^6w^5}{-6\,yw^5}$.

## EXERCISE 67

### COMPLETING STATEMENTS ABOUT DIVISION

1. A positive number divided by a positive number gives as a quotient a ___?___ number.
2. A negative number divided by a positive number gives as a quotient a ___?___ number.
3. The quotient of two numbers having like signs is ___?___.
4. The quotient of two numbers having unlike signs is ___?___.

## EXERCISE 68

### A REVIEW OF ADDITION, MULTIPLICATION, SUBTRACTION, AND DIVISION OF SIGNED NUMBERS

This is a very difficult but a very important exercise.

1. From the sum of $2a$ and $-5a$ take the difference between $-3a$ and $+8a$.
2. Add the product of $-3$ and $+5$ to the quotient of $-18$ and $-2$.
3. Take the sum of $-7b$ and $+4b$ from the difference between $-6b$ and $+11b$.
4. To the quotient of $-21$ and $+3$ add the product of $-6$ and $+7$.
5. From the sum of $7t$ and $-10t$ take the difference between $-4t$ and $+11t$.
6. Add the product of $-6$ and $+9$ to the quotient of $-28$ and $-4$.
7. Take the sum of $-9x$ and $+3x$ from the difference between $-5x$ and $+13x$.

**8.** To the product of $-7$ and $+11$ add the quotient of $-33$ and $+3$.

**9.** From the sum of $8c$ and $-14c$ take the difference between $-5c$ and $+16c$.

**10.** Add the product of $-12$ and $+5$ to the quotient of $-32$ and $-4$.

**11.** Take the sum of $-8y$ and $3y$ from the difference between $-7y$ and $+12y$.

**12.** To the product of $-8$ and $+9$ add the quotient of $-36$ and $+6$.

**13.** From the sum of $12b$ and $-16b$ take the difference between $-7b$ and $+8b$.

**14.** Add the product of $-8$ and $+9$ to the quotient of $-40$ and $-5$.

**15.** Take the sum of $-11t$ and $7t$ from the difference between $-9t$ and $+10t$.

**16.** To the product of $-6$ and $+13$ add the quotient of $-42$ and $+7$.

## SUMMARY

**This chapter shows the need of signed numbers. It teaches how to:**

1. ***Combine* or *add* signed numbers.**
2. **Find the *difference* between two signed numbers; that is, to subtract one signed number from another.**
3. **Find the *product* of signed numbers.**
4. ***Divide* one signed number by another.**

### REVIEW EXERCISE 69

1. The formula $h = 25 + \frac{3}{2}(G - 4)$ is used to determine the proper height of the chalk trough in a schoolroom. If $h$ stands for the height in inches, and $g$ stands for the number of the grade, find the height for Grade VIII; that is, when $g = 8$. What is the proper height for a third-grade room?

2. Evaluate the expression $ab^2 + a^2b$ if $a = 2$ and $b = -3$.

3. Show that the sum of any two numbers having unlike signs, but the same absolute value, is zero. Give some illustrations.

4. In a class of 25 pupils, 2 were conditioned and 6 failed. Express the ratio of the number of pupils that succeeded to the total number in the class. What percentage is this?

5. The ratio of $y + 1$ to 9 is equal to the ratio of $y + 5$ to 15. Find $y$.

6. The number of posts required for a fence is 84 when they are placed 18 feet apart. How many would be needed if they were placed 12 feet apart?

7. If I am now $x$ years old, what does the following expression tell about my age: $2x + 5 = 55$?

# CHAPTER X

## THE COMPLETE SOLUTION OF THE SIMPLE EQUATION

**Section 65. What we have already learned about the equation.** Since the equation is the most important operation in mathematics, we must be able to solve quickly and accurately equations of any kind. Thus far we have learned two very important facts about equations:

1. That if we do anything to one side of an equation, we must **do the same thing to the other side.**

2. That an equation is **SOLVED** when a value of the unknown is found which **satisfies** the equation; that is, one which makes the numerical value of one side equal to the numerical value of the other side.

Furthermore, we have learned: (1) how to solve simple equations of the type

$$6b + 3 = 45,$$

or,

$$c + 5c = 20 + c, \text{ etc.};$$

(2) how to get rid of fractions in an equation, *e.g.* of the type $\frac{2}{3}x + \frac{3}{4}x - 1 = 3$;

(3) how to solve word problems, first by translating them into equations and second by solving the equations. These methods, which you have now mastered, are important first steps in the more important problem of learning how to solve equations of *any kind.* That is your task in studying this chapter.

### I. SOLVING EQUATIONS WHICH CONTAIN NEGATIVE NUMBERS

**Section 66.** There are just two more steps that we must learn in using equations. *First,* we must be able to solve equations which contain negative numbers; *second,* we must be able to solve equations which contain parentheses.

Negative numbers occur very commonly in equations. The following examples illustrate this fact.

## EXERCISE 70

Write as equations, and solve each of the following examples:

1. What number multiplied by 7 equals $-28$?
2. What number multiplied by $-5$ equals 20?
3. If a certain number be added to 13, the result is 8. Find the number.
4. A certain number increased by 10 equals $-5$. Find the number.
5. If 7 be subtracted from a certain number, the result is $-3$. What is the number?
6. If negative four times a certain number gives 22, what is the number?

These examples show how negative numbers occur in equations. Throughout the remainder of the work, equations which are satisfied by negative numbers will occur very commonly. The next exercise contains many examples of this kind.

## EXERCISE 71

### SOLUTION OF EASY EQUATIONS WHICH CONTAIN NEGATIVE NUMBERS

Solve each of the following equations. You should be able to tell exactly what must be done to each side of the equation.

1. $x+5=3$
2. $2y=-16$
3. $b+7=2$
4. $-3a=15$
5. $2x+16=2$
6. $-4y=12$
7. $10y+2=-18$
8. $2b-1=9$

9. Three times a certain number, increased by 10, gives 6. What is the number?

10. If twice a certain number is added to 16, the result equals the number increased by 6. Find the number.

11. $\frac{2x}{3} + 5 = \frac{1}{4}x + \frac{1}{6}x + 2.$

12. $12 - 2x = 8.$

13. The sum of two thirds of a certain number and three fourths of the same number is $-17$. Find the number.

**A new kind of equation.**

**14.** $-2x - 12 = 5x - 40.$

The equations which you have just solved are of the kind in which you can easily see what to do to each side. With examples like 14, however, in which both *knowns* and *unknowns* occur on *each* side and which *include negative numbers* on one or both sides, we need special and systematic practice.

**Section 67. We need to get "knowns" on one side and "unknowns" on the other.** Just as clerks in stores always place the known weights on one scale pan and the unknown weights on the other scale pan, so we, in solving equations, always get the **known numbers**, or terms, **on one side of the equation**, and the **unknown terms on the other side.**

Usually we get all the **unknown** terms on the **left** side, and all the **known** terms on the **right** side. Thus, in the equation above,

$$-2x - 12 = 5x - 40,$$

we do not want $-12$ on the left side. Therefore we get rid of the *known* on the left side by adding $+12$ to *each* side, giving the equation $-2x=5x-28$. We also do not want the $5x$ on the right side. Therefore we subtract $5x$ from *each* side, giving the equation $-7x=-28$. Dividing each side of this equation by $-7$, we find that $x=4$.

**Section 68. Equations should be solved in a systematic order.** In learning to solve equations which require several steps, **pupils make many mistakes** because their work is not arranged in a set order. For some time to come, therefore, you will find it very important to use a form like the following in solving equations:

**Illustrative example.** Solve the equation

$$-2x-12=5x-40.$$

(1) Adding $+12$ to each side, gives

$$-2x=5x-28.$$

(2) Subtracting $5x$ from each side, gives

$$-7x=-28.$$

(3) Dividing each side by $-7$, gives

$$x=4.$$

(4) Check: Substituting 4 for $x$ to check the result, gives

$$-8-12=20-40.$$
$$-20=-20.$$

---

EXERCISE 72

Solve each of the following examples, *writing out each step* exactly as in the solution of the illustrative example:

1. $-3x-8=8x-30$
2. $5y-6=9y+42$
3. $-6b+11=2b+43$
4. $x-20=50-6x$
5. $-2c+10=4$
6. $10-3x=-20$
7. $6-4y=2$
8. $14=2y+20$
9. $-7a+4=+8a-41$
10. $-2x-7=-8x-19$
11. $+5y-3=8y-16$
12. $5+2y=0$
13. $10x+22=12$
14. $0=4x+20$

## II. HOW TO SOLVE EQUATIONS WHICH CONTAIN PARENTHESES

**Section 69.** We saw in the last chapter that *parentheses* were used to indicate multiplication. Thus, to show that $-4$ is to be multiplied by $-6$, we use the *parentheses*, as follows: $(-4)(-6)$. Multiplication is usually indicated in this way. Take this example to illustrate the way in which parentheses will be used in a practical way:

*A.* **Illustrative example.** Oranges cost 10 cents more per dozen than lemons; the cost of four dozen lemons and two dozen oranges is $2. What is the price per dozen of each?

*B.* Solution:

Let $x$ = no. cents per dozen paid for lemons;
then $x + 10$ = no. cents per dozen paid for oranges;
then $4x$ = no. cents paid for 4 doz. lemons;
and $2(x + 10)$ = no. cents paid for 2 doz. lemons.
Therefore $4x + 2(x + 10)$ = no. cents paid for both;
or $4x + 2(x + 10) = 200$.
Note the use that is made of *parentheses*; that is, to show that the expression $x + 10$ must be multiplied by $+2$. Performing this multiplication,

(1) or removing parentheses, gives

$$4x + 2x + 20 = 200.$$

(2) Combining terms gives

$$6x + 20 = 200.$$

(3) Subtracting 20 from each side gives

$$6x = 180.$$

(4) Dividing each side by 6 gives

$$x = 30 \text{ cents per dozen for lemons,}$$

and therefore

$$x + 10 = 40 \text{ cents per dozen for oranges.}$$

(5) Substituting in the original word-statement 30 cents and 40 cents respectively for the cost of lemons and oranges enables us to check the result:

$$4 \cdot \$.30 + 2 \cdot \$.40 = \$2$$

## EXERCISE 73

### PRACTICE IN SOLVING EQUATIONS WHICH CONTAIN PARENTHESES

Solve and check each of the following equations:

1. $2(x+10)=42$
2. $5(y-2)=15$
3. $3(2b-4)=18$
4. $4x+5(x+2)=46$
5. $2(x-3)+3(x-2)=8$
6. $5b+2(4-b)=32$
7. $-3x+6(x-4)=9$
8. $-7b+4(2b-3)=16$

Note that in all the foregoing examples the number before the parenthesis has been *positive*. **If negative numbers occur, however, we proceed just the same,** remembering how to multiply a negative number.

**Illustrative example.** Solution of an equation involving REMOVAL OF PARENTHESES.

9. $8x-2(2x-7)=x+8.$

The expression $2x-7$ is to be multiplied by $-2$.

(1) Performing this multiplication, or *removing parentheses,* gives

$$8x-4x+14=x+8. \quad \text{(Why is it } +14\text{?)}$$

(2) Combining terms gives

$$4x+14=x+8.$$

(3) Subtracting $+x$ from each side gives

$$3x+14=8.$$

(4) Subtracting $+14$ from each side gives

$$3x=-6.$$

(5) Dividing each side by 3 gives

$$x=-2.$$

(6) Check: Substituting $-2$ for $x$ throughout the equation gives

$$-16-2(-4-7)=-2+8.$$
$$-16+8+14=-2+8.$$
$$6=6.$$

EXERCISE 73 (*continued*)

10. $5b - 3(4 - 2b) = 2b + 42$
11. $6(x - 3) - 4(x + 2) = 4 - x$
12. $7(b - 2) - 2(3 + b) = 0$
13. $4(2y - 5) + 15 = 3(y + 10)$
14. $9y - 3(2y - 4) = 6$

**Section 70. A difficult form of multiplication.** A form of multiplication (as shown by parentheses) that gives pupils difficulty is the kind represented by $-(4 - 5x)$ in the equation:

15. $5 - 2(x - 6) = -(4 - 5x)$

**When no multiplier appears immediately before the parentheses, the multiplier 1 is understood.** Therefore in this case the multiplier is **$-1$.** It is just as though the right side of the equation read

$$-1(4 - 5x).$$

**Illustrative example.**

Therefore the complete set of steps required to solve this equation includes:

(1) Removing parentheses gives

$$5 - 2x + 12 = -4 + 5x. \text{ (Why is it } +5x\text{?)}$$

(2) Combining terms gives

$$-2x + 17 = -4 + 5x.$$

(3) Subtracting $+17$ from each side gives

$$-2x = -21 + 5x.$$

(4) Subtracting $-5x$ from each side gives

$$-7x = -21.$$

(5) Dividing each side by $-7$ gives

$$x = 3.$$

(6) Substituting 3 for $x$, throughout the *original* equation to *check* the result, gives

$$5 - 2(3 - 6) = -(4 - 5 \cdot 3).$$

or,

$$5 - 6 + 12 = -4 + 15.$$

$$11 = 11.$$

There are **two important and difficult points** in this last example. *First*, you should note that in the expression $5-2(x-6)$ the $-2$ is NOT to be subtracted from the 5. The expression in parentheses must be *multiplied* by $-2$. *Second*, if no multiplier is written before the parentheses, as in the expression $-(4-5x)$, it is understood that the multiplier is 1. In this case it is $-1$. If there had been no sign before the parentheses, as $(4-5x)$, the multiplier would be understood to be $+1$.

### EXERCISE 73 (*continued*)

**16.** $7x-(x-4)=25$

**17.** $-5y-(2-y)=18$

**18.** $6x-(x+7)=-2x+35$

**19.** $5-2(x-4)=23$

**20.** $7-12(3-b)=31$

**21.** $2b-7(3-b)=b+8$

**22.** $1(2x+3)=-17$

**23.** $-1(6-2x)=43$

**24.** $-(6-2x)=24$

**25.** $16=(2x+4)$

### A. SUMMARY OF THE STEPS REQUIRED TO SOLVE EQUATIONS WHICH CONTAIN PARENTHESES

**Section 71.** Look back to the illustrative examples, 9 and 15, and compare the steps in the solution that is worked out for each one with the steps in the solution in each of those which you have just worked. You will note that to solve such an equation the following steps are always included:

I. Removing the parentheses (*i.e.* multiplying).

II. Combining like terms on each side.

III. Getting rid of all known terms on one side and all unknown terms on the other side.

IV. Dividing each side by the coefficient of the unknown, to give the numerical value of the unknown.

V. Substituting the obtained value in the original equation to *check* the result.

## III. HOW TO SOLVE EQUATIONS WHICH CONTAIN FRACTIONS

**Section 72.** If, however, we should be given the equation

$$\frac{2(x-3)}{5}-\frac{x-6}{4}=\frac{x}{2}-\frac{16}{5},$$

we need to add another step; namely,

*Getting rid of fractions by multiplying each term by the most convenient multiplier.*

**Illustrative example.** 1. Solution of an equation which involves getting rid of fractions.

$$\frac{2(x-3)}{5}-\frac{x-6}{4}=\frac{x}{2}-\frac{16}{5}.$$

(1) Removing parentheses gives

$$\frac{2x-6}{5}-\frac{x-6}{4}=\frac{x}{2}-\frac{16}{5}.$$

(2) Multiplying each term by the most convenient multiplier, 20, gives

$$20\frac{2x-6}{5}-20\frac{x-6}{4}=20\frac{x}{2}-20\frac{16}{5}.$$

(3) Reducing fractions gives

$$8x-24-5x+30=10x-64. \quad \text{(Why }+30\text{?)}$$

(4) Collecting terms gives

$$3x+6=10x-64.$$

(5) Subtracting 6 from each side gives

$$3x=10x-70.$$

(6) Subtracting $10x$ from each side gives

$$-7x=-70.$$

(7) Dividing each side by $-7$ gives

$$x=10.$$

(8) Substituting the obtained value of the unknown to *check* the result.

### B. SUMMARY OF THE STEPS REQUIRED TO SOLVE EQUATIONS WHICH CONTAIN BOTH PARENTHESES AND FRACTIONS

The solution of this example illustrates *all* the steps that are ever included in solving simple equations. The complete list now includes:

I. Removing parentheses.
II. Getting rid of fractions by multiplying each term by the most convenient multiplier.
III. Combining like terms.
IV. Getting rid of all the known terms on one side and all the unknown terms on the other side.
V. Dividing each side by the coefficient of the unknown.
VI. Substituting the obtained value in the original equation to *check* the result. All these steps are not required unless the equation includes parentheses and fractions.

Now that we have learned all of the steps that are necessary in solving simple equations we need to practice so as to be very proficient in this work. Nothing is more important in high school mathematics. The next exercise is included to provide that practice.

### EXERCISE 74

#### PRACTICE IN THE COMPLETE SOLUTION OF EQUATIONS

**1.** $\frac{3(x-4)}{7}=\frac{x+8}{3}$ **2.** $\frac{x}{2}-\frac{x-4}{5}=\frac{11}{10}$

What step in the above list is omitted in working this example?

3. $\frac{5(x-2)}{3}-\frac{2(x+1)}{5}=\frac{7}{3}$

4. $\frac{2(3x+1)}{7}-\frac{(x+2)}{14}=\frac{13}{14}$

5. $\frac{3b}{4}=\frac{2(b-10)}{5}+\frac{5b-19}{3}$

6. $\frac{1}{2}(4x+8)-\frac{2}{3}(6x-9)=6$

7. $\frac{2}{3}(6x-9)-\frac{3}{4}(8x+4)=-13$

8. $\frac{5(y-2)}{3}-\frac{7(2y-3)}{4}=-2y+\frac{12}{4}$

9. $\frac{2(x+8)}{7}-\frac{3(x-8)}{2}=\frac{-3(2x+8)}{1}$

10. $3(4-y)=5(6-3y)$

11. $\frac{x}{2}-\frac{x}{3}-x-\frac{x-2}{6}=\frac{x}{36}$

12. $\frac{2b}{5}-\frac{3b}{4}-\frac{b-5}{10}=\frac{b-37}{20}$

## THE ALGEBRAIC SOLUTION OF WORD PROBLEMS

**Section 73. Review of important steps in translating word problems into algebraic statements.** We have taken a great deal of time to learn how to solve any kind of simple equation because we need to be able to use equations skillfully in solving actual problems later. The problems as a rule will not be stated for us, in algebraic or equational form, all ready for solution. They will be stated merely in words. First, therefore, we shall always have to *translate* the word problem into an equation. Beyond this first step the work is the mere solution of the equation.

Our second principal task in this chapter, therefore, is to become skillful in translating word problems into algebraic form. We learned in our work with Chapter III the important steps in translating word problems. Since we are to learn in the next few lessons how to *translate* a great many different kinds of word statements, let us review these steps here:

**First step**: See clearly which things in the problem are **known** and which are **unknown.**

**Second step**: Represent one of the unknowns, most conveniently the smallest one, by some letter.

**Third step**: Represent all of the others by using the same letter.

**Fourth step**: By careful study of the RELATIONS between the parts of the problem, express the word statement in algebraic form.

Sometimes this will mean an *equation* and sometimes not.

For the next few lessons, therefore, you will work many word problems. The exercises are included to give practice in translating many different kinds, **so that you will be able to use the method in solving any kind that you may happen to meet later.** For convenience they will be arranged by types, examples of the same type being studied together.

## I. PROBLEMS IN WHICH A NUMBER IS DIVIDED INTO TWO OR MORE PARTS

**Section 74.** The solution of a great many problems depends upon our being able to separate a number into two or more parts. For example, if a man has a certain sum of money to invest, he may invest part of it in one thing, and part in another. The solution of such an example requires that we be able to divide a number into two or more parts algebraically.

## EXERCISE 75

### PRACTICE IN DIVIDING A NUMBER INTO TWO OR MORE PARTS

**1.** The sum of two numbers is 20.

(*a*) Express in algebraic form the second one if the first one is 12.

(*b*) Express in algebraic form the second one if the first one is $n$.

(*c*) Express in algebraic form the fact that the second one exceeds the first one by 4.

**2.** There are 36 pupils in a mathematics class.

(*a*) Express algebraically the number of boys if there are 19 girls.

(*b*) Express algebraically the number of girls if there are $n$ boys.

(*c*) State algebraically that there were 6 more girls than boys.

**3.** A farmer has two kinds of seed, clover seed and blue grass seed. If he has 100 lb. of both, express:

(*a*) the number of pounds of clover seed if there were 24 lb. of blue grass seed;

(*b*) the number of pounds of clover seed if there were $n$ lb. of blue grass seed;

(*c*) the value of the clover seed ($n$ lb.) at 20 cents per pound; and the value of the blue grass seed at 15 cents per pound.

(*d*) State by an equation that the value of both kinds together was $19.

4. Divide 20 into two parts such that the larger part exceeds the smaller part by 4.

5. A boy paid 48 cents for 20 stamps; some cost two cents each and the remainder cost three cents. How many of each kind did he buy?

6. During one afternoon a clerk at a soda fountain sold 200 drinks, for which he received $16. Some were 5 cents each; the others were 10 cents each. Find the number of each kind.

7. A grocer has two kinds of coffee, some selling at 30 cents per pound and some selling at 50 cents per pound. How many pounds of each kind must he use in a mixture of 100 pounds which he can sell for 34 cents per pound?

**Section 75. Need for tabulating the data of word problems.** Many problems involve so many different statements that it is practically **necessary to arrange the steps in the translation in very systematic tabular form.** Take an example like this:

> **John's age exceeds James's by 20 years. In 15 years he will be twice as old as James. Find the age of each now.**

Before we can write this statement in the form of an equation we must express in algebraic form *four* different things: (1) John's age *now;* (2) James's age *now;* (3) John's age in 15 years; and (4) James's age in 15 years. These four facts can best be stated in a table like this:

**(First step)** Let $n$ represent James's age NOW.

**(Second step)** Tabulate the data:

TABLE 9

| | Age now | Age in 15 years |
|---|---|---|
| John's age | $n + 20$ | $n + 20 + 15$ |
| James's age | $n$ | $n + 15$ |

With all the facts expressed in letters we can now state the equation which *tells* the same thing as the original word statement; namely:

**(Third step)** $n + 20 + 15 = 2(n + 15)$.

We are now ready for the

**(Fourth step)** the solution of the equation; the steps are as follows:

(1) $n + 35 = 2n + 30$.

(2) $-n = -5$.

(3) $n = 5$.

Therefore James's age **now** is 5, and John's age **now** is $n + 20$, or 25.

(4) Check the accuracy of this result thus:
**In 15 years** John will be 40 and James will be 20; or John will be twice as old as James, as the problem states.

To be proficient in solving such problems, therefore, we first need practice in tabulating such facts as "age *now*," "age some other time," as in this example. Other types which involve the same need for tabulation will be taken up later.

## PRACTICE IN REPRESENTING RELATIONS BETWEEN NUMBERS

### II. PROBLEMS RELATING TO AGE

### EXERCISE 76

**1.** A man is now 25 years of age. What expression will represent his age:

(*a*) 10 years ago? (*c*) $x$ years ago?

(*b*) 8 years from now? (*d*) $m$ years from now?

**2.** C is now $n$ years of age. What expression will represent his age:

(*a*) 12 years from now? (*c*) $y$ years ago?

(*b*) 7 years ago? (*d*) $m$ years from now?

**3.** A is now $x$ years old. B's present age exceeds A's age by 8 years. What expression will represent:

(*a*) B's present age?

(*b*) the sum of their ages?

(*c*) the age of each 10 years ago?

(*d*) the age of each 5 years from now?

(*e*) the sum of their ages in 5 years?

**4.** A is now $n$ years of age; B is three times as old. Express algebraically:

(*a*) B's present age;

(*b*) the age of each 4 years ago;

(*c*) the age of each 9 years from now.

(*d*) State algebraically that B's age 4 years ago was 5 times A's age then.

5. A's present age exceeds B's present age by 25 years. In 15 years he will be twice as old as B. Find their present ages.

6. C is six times as old as D. In 20 years C's age will be only twice D's age 20 years from now. What are their present ages?

7. A man is now 45 years old and his son is 15. In how many years will he be twice as old as his son?

8. A father is 9 times as old as his son. In 9 years he will be only 3 times as old. What is the age of each now?

9. A's present age is twice B's present age; 10 years ago A's age was three times B's age then. Find the age of each now.

## III. PROBLEMS BASED ON COINS

**Section 76.** Another illustrative type of word problem which gives practice in tabulating data and thus in solving difficult word problems is the "coin problem." Take this example:

> **Illustrative example.** A man has 3 times as many dimes as quarters.
>
> How *many* of each has he if the value of both together is $11?
>
> Here there are four distinct numbers to be expressed, as in the case of the age problem: (1) the *number* of quarters; (2) the *number* of dimes; (3) the *value* of the quarters in terms of a common base (for example, cents); (4) the *value* of the dimes in the same base (cents). The steps in the solution are clear, therefore, from the following illustrative solution:

(1) Let $n =$ the *number* of quarters.

(2) Then

TABLE 10

| | Number | Value |
|---|---|---|
| quarters | n | 25 n |
| dimes | 3n | 30 n |

(3) $25\,n + 30\,n = 1100$ cents.

(4) $\therefore\ n = 20$, number of quarters.

(5) $3\,n = 60$, number of dimes.

(*a*) *Value* of the quarters $= \$5$.

(*b*) *Value* of the dimes $= \$6$.

Total value $= \$11$, as stated in the example.

---

## EXERCISE 77

### PRACTICE IN EXPRESSING THE VALUE OF VARIOUS NUMBERS OF COINS

**1.** Express the value *in cents* of:

(*a*) $d$ dimes;
(*b*) $3\,d$ quarters;
(*c*) $2\,d$ nickels;
(*d*) $4\,d$ half dollars;
(*e*) $d$ dollars;
(*f*) of all the coins.

**2.** Express the *value* in cents of:

(*a*) $n$ nickels;
(*b*) $(3 - n)$ dimes;
(*c*) $(n + 5)$ quarters;
(*d*) $(12 - n)$ half dollars;
(*e*) $(30 - n)$ nickels.

**3.** A purse was found which contained nickels and dimes, 20 in all. Find the number of each if the value of both was $1.60.

**4.** I received at a candy counter twice as many dimes as quarters, and 6 more nickels than dimes and quarters together. How many of each coin did I receive if the value of all was $7.50?

**5.** A debt of \$72 was paid with 5-dollar bills and 2-dollar bills, there being twice as many of the latter as of the former. Find the number of each kind of bill.

**6.** 18 coins, dimes and quarters, amount to \$2.25. Find the number of each kind of coin.

**7.** A cab driver received twice as many quarters as half dollars, and three times as many dimes as half dollars; in all he had \$13. How many of each coin did he receive?

## IV. PROBLEMS BASED ON TIME, RATE, AND DISTANCE

**Section 77.** In Chapter IV we saw that the motion of a train could be represented graphically. Now we shall learn how to solve this kind of problem by means of the equation.

### EXERCISE 78

### PRACTICE IN SOLVING PROBLEMS BASED ON RELATIONS BETWEEN TIME, RATE, AND DISTANCE

**1.** Express the distance covered by an automobile in 10 hours if its rate is:

(*a*) 18 miles per hour;
(*b*) 5 miles per hour;
(*c*) $(r+3)$ miles per hour;
(*d*) $(2x-5)$ miles per hour.

**2.** A train runs for $t$ hours. Express the distance it will cover at the rate of:

(*a*) 35 miles per hour;
(*b*) $m$ miles per hour;
(*c*) $(r+6)$ miles per hour;
(*d*) $t$ miles per hour.

3. An automobile tourist sets out on a 400-mile trip. Express the time required if he goes at the rate of:

   (*a*) 40 miles per hour;
   (*b*) 5 miles per hour;
   (*c*) $(r + 10)$ miles per day;
   (*d*) $(2r - 3)$ miles per day.

4. How long will it require to make a trip of $D$ miles at the rate of 15 miles per hour? 5 miles per hour?

5. At what rate must one travel to go $D$ miles in 10 hours? In $t$ hours? In $t + 3$ hours?

6. A slow train travels at the rate of 5 miles per hour; a fast train travels 15 miles more per hour. Express:

   (*a*) the rate of the fast train;
   (*b*) the distance passed over by each in 5 hours.
   (*c*) State algebraically that the two trains together traveled 275 miles in 5 hours.

7. Two trains leave Chicago at the same time, one eastbound, the other westbound. The eastbound train travels 10 miles less per hour than the westbound train. Express:

   (*a*) the rate of each;
   (*b*) the distance traveled by each in 4 hours.
   (*c*) Form an equation stating that they were 440 miles apart at the end of 4 hours.

8. Two trains, 350 miles apart, travel toward each other at the rate of 40 and 35 miles per hour, respectively.

(*a*) Express the distance traveled by each in $t$ hours.

(*b*) Form an equation stating the fact that the trains met in $t$ hours.

9. Make formulas for $d$, for $t$, and for $r$, that can be used in any problem based upon uniform motion.

10. **Illustrative example.** **Two bicyclists, 200 miles apart, travel toward each other at rates of 12 and 8 miles per hour respectively. In how many hours will they meet?**

**(1) Let $t$ represent the number of hours until they meet.**

**(2)** TABLE 11

| | Time in hours | Rate per hr. in miles | Distance in miles |
|---|---|---|---|
| For slow one | t | 8 | 8t |
| For fast one | t | 12 | 12t |

**(3) Then** $8t + 12t = 200.$

**(4)** $\therefore t = 10.$

11. Two men start from the same place, one going south and the other going north. One goes twice as fast as the other. In 5 hours they are 120 miles apart. Find the rate of each.

12. An eastbound train going 30 miles per hour left Chicago 3 hours before a westbound train going 36 miles per hour. In how many hours, after the westbound train left, will they be 519 miles apart?

**13.** A bicyclist traveling 15 miles per hour was overtaken 8 hours after he started by an automobile which left the same starting point $4\frac{1}{2}$ hours later. Find the rate of the automobile.

**14.** A starts from a certain place, traveling at the rate of 4 miles per hour. Five hours later B starts from the same place and travels in the same direction at the rate of 6 miles per hour. In how many hours will B overtake A?

## V. PROBLEMS INVOLVING PER CENTS

**Section 78.** Many problems involving per cents may be solved by algebraic methods.

### EXERCISE 79

### PRACTICE IN SOLVING PERCENTAGE PROBLEMS

**1.** What does 10 % mean? 5 %? $r$ %?

**2.** Indicate 4 % of $600; 5 % of $275.

**3.** Express decimally 5 % of $p$; 8 % of $c$; $6\frac{1}{2}$ % of $b$.

**4.** A man paid $c$ dollars for an article. He sold it at a gain of 25 %. Express:

(*a*) the gain in dollars;

(*b*) the selling price.

(*c*) State algebraically that he sold the article for $2.50.

**5.** A merchant sold a suit for $25, thereby gaining 25 %. If the cost is represented by $c$ dollars, what will represent:

(*a*) the gain in dollars?

(*b*) the selling price in terms of $c$?

(*c*) State algebraically that the selling price was $25.

6. Solve each of the following equations:

(*a*) $.20x = 180$
(*b*) $x + .06x = 3.18$
(*c*) $c + .10c = 495$
(*d*) $m - .15m = 21.25$
(*e*) $p + .04p = 520$
(*f*) $x - .50x = 18.75$
(*g*) $2 - 3x - .5x = 7$
(*h*) $1.75x - \frac{1}{2}x = 1000$

7. Find the cost of an article sold for $156 if the gain was 10 %. (Use *c* for the cost.)

8. What number increased by $66\frac{2}{3}$ % of itself equals 150?

9. After deducting 15 % from the marked price of a table, a dealer sold it for $21.25. What was the marked price?

10. A dealer made a profit of $3690 this year. This is 18 % less than his profit last year. Find his profit last year.

11. A number increased by 12.5 % of itself equals 243. What is the number?

12. A shoe dealer wishes to make 25 % on shoes. At what price must he buy them in order to sell them at $4.50 per pair?

13. A furniture dealer was forced to sell some damaged goods at 14 % less than cost, and sold them for $129. How much did they cost?

14. A man sold a suit of clothes for $30.25. What per cent did he gain if the clothes cost him $25?

## VI. INTEREST PROBLEMS

**Section 79.** Many interest problems can be more easily solved by algebraic equations than by the methods of arithmetic.

### EXERCISE 80

1. Express the interest on $150 at 5% for 1 year; for 3 years; for $t$ years.
2. Express the interest on $P$ dollars at 6% for 1 year; for 3 years; for $t$ years.
3. Express the simple interest on $500 for 1 year at $r$%; for 4 years.
4. A man borrowed a certain sum of money at 6%. Express:
   (*a*) the interest for 2 years.
   (*b*) State algebraically that the interest for three years was $48.
5. What principal must be invested at 6% to yield an annual income of $57?
6. For how many years must $2800 be invested at 7% simple interest to yield $833 interest?
7. What is the interest on $P$ dollars at $r$% for $t$ years?
8. A man invests part of $1000 at 4%, and the remainder at 6%. If $x$ represents the number of dollars invested at 4%, express:
   (*a*) the annual income on the 4% investment;
   (*b*) the amount of the 6% investment;
   (*c*) the annual income on the 6% investment.
   (*d*) State algebraically that the annual income on the 4% investment exceeds the annual income on the 6% investment by $20.

9. Part of \$1200 is invested at 5 % and the remainder at 7 %. The total annual income from the two investments is \$67. What was the amount of each investment?

10. Ten thousand dollars' worth of Liberty Bonds yield an annual interest of \$370. Some pay $3\frac{1}{2}$ %, and the remainder pay 4 %. Find the amount of each kind of bond.

11. A 5 % investment yields annually \$5 less than a 4 % investment. Find the amount of each investment if the sum of both is \$800.

## VII. PROBLEMS CONCERNING PERIMETERS AND AREAS

**Section 80.** The following examples are based on squares and rectangles.

### EXERCISE 81

1. The length of a rectangle exceeds twice its width by 12 in. Represent its width by $w$.
   - (*a*) Make a drawing to represent it.
   - (*b*) Express its length.
   - (*c*) Express its area.
   - (*d*) Express its perimeter.
   - (*e*) State that its perimeter is 84 in.

2. The length of a rectangle is 9 in. more, and the width is 6 in. less, than the side of a square.
   - (*a*) Make a drawing for each.
   - (*b*) Express the dimensions of the square.
   - (*c*) Express the dimensions of the rectangle.
   - (*d*) Express the perimeter of the rectangle.
   - (*e*) Express the area of the rectangle.
   - (*f*) State algebraically that the sum of the perimeters is 168 in.

3. The base of a triangle exceeds its height by 10 inches.
   (*a*) Make a drawing for the figure.
   (*b*) Express its base and height.
   (*c*) Express its area.
   (*d*) State that its area is equal to the area of a rectangle whose dimensions are 8 in. and 5 in.

4. The length of a rectangle is 4 feet more, and its width is 2 feet less, than a square whose perimeter is $P$ inches. Express:
   (*a*) the side of the square;
   (*b*) the dimensions of the rectangle;
   (*c*) the perimeter of the rectangle.
   (*d*) Find the value of $P$ if the perimeter of the rectangle is 44 inches.

## VIII. PROBLEMS BASED ON LEVERS

**Section 81.** A teeter board is one form of *lever.* The point on which the board rests or turns is the fulcrum; the parts of the board to the right of and to the left of the fulcrum are the *lever* arms.

FIG. 96

If a boy at $A$, who just balances a boy at $B$, moves to the left while $B$ remains stationary, it is clear that the left side goes down. But if the boy at $B$ moves closer to the fulcrum while *A* remains stationary, then *A* goes down. It is also clear that boys of unequal weight cannot teeter unless the heavier boy sits closer to the fulcrum. There is a mathematical relation between the weight on the lever arm and its distance from the fulcrum. Two boys will balance each other when the *weight* of one *times* his *distance* from the fulcrum is equal to the *weight* of the other *times* his *distance*, or in general, when

> ***weight times distance*** **on one side equals** ***weight times distance*** **on the other side.**

This law or relation may be tested by placing equal coins at different positions on a stiff ruler balanced on the edge of a desk. Try this experiment. See whether 2 pennies placed 4 inches from the fulcrum (at the center of the lever) will balance 1 penny placed 8 inches from the fulcrum. See whether 6 pennies placed 2 inches from the fulcrum will balance 3 pennies placed 4 inches from the fulcrum.

Thus, to make a lever balance, the *product* of *weight* and *distance* from the fulcrum on one side *must equal* the *product* of *weight* and *distance* from the fulcrum on the other side.

### EXERCISE 82

Problems based on levers. Make a drawing for each.

1. John weighs 80 lb. and sits 4 ft. from the fulcrum. Where must Robert sit if he weighs 90 lb.?

2. A, weighing 120 lb., sits $4\frac{1}{2}$ ft. from the fulcrum, and balances B, who sits 5 ft. from the fulcrum. What is B's weight?

3. A hunter wishes to carry home two pieces of meat, one weighing 40 lb. and the other 60 lb. He puts them on the ends of a stick 4 ft. long and places the stick across his shoulder. Where must the fulcrum (his shoulder) be placed to make the weights balance?
4. Two children play teeter, one on each end of a board 9 ft. long. Where must the fulcrum be if the children weigh 60 and 80 lb. respectively?
5. Could three children teeter on the same board? How?
6. A and B sit on the side of the fulcrum. A weighs 100 lb. and sits 5 ft. from the fulcrum; B weighs 80 lb. and sits 3 ft. from the fulcrum. Where must C sit to balance the other two, if he weighs 150 lb.?
7. Show that $\frac{W_1}{W_2}=\frac{d_2}{d_1}$ is the formula or law for a balance on a teeter board. Are the weight and distance *directly* proportional?

## SUMMARY

**This chapter has taught all the steps involved in solving a simple equation:**

1. **Removal of parentheses.**
2. **Getting rid of fractions.**
3. **Collecting terms on each side.**
4. **Getting rid of known terms on one side and unknown terms on the other side.**
5. **Dividing each side by the coefficient of the unknown.**

6. Checking by substituting the obtained value of the unknown in the original equation.

Many kinds of word problems have been solved. Tabulating the information or data of such problems is a great help in solving them. A *systematic method* always pays big dividends in any kind of work.

## EXERCISE 83

### MISCELLANEOUS PROBLEMS

1. A grocer has two kinds of tea, — some worth 60 ¢ per pound and some worth 75 ¢ per pound. He has 20 lb. more of the former than of the latter kind. How many pounds of each kind has he, if the value of both kinds is $45.75?

2. I bought 45 stamps for $1.05. If part of them were 2-cent stamps and part 3-cent stamps, how many of each did I buy?

3. The sum of the third, the fourth, and the eighth parts of a number is 17. What is the number?

4. John has $\frac{1}{3}$ as many marbles as Harry. If John buys 120 and Harry loses 23, John will then have 7 more than Harry. How many has each boy?

5. A clerk spends $\frac{1}{4}$ of his yearly salary for board and room, $\frac{1}{8}$ for clothes, $\frac{1}{6}$ for other expenses, and saves $880. What are his annual expenses?

6. A father left one third of his property to his wife, one fifth to each of his three children, and the remainder, which was $1200, to other relatives. Find the value of his estate.

**7.** Ten years ago A was one third as old as he is at present. Find his age now.

**8.** Evaluate the formula $C = \frac{5(F-32)}{9}$ if $F = 20$.

**9.** A merchant bought goods for \$500, and sold them at a gain of 5 %. What was the selling price?

**10.** If in problem 9 the merchant had sold the goods at a gain of $x$ per cent, what would have been the selling price?

**11.** A 6-foot pole casts a shadow $4\frac{1}{2}$ ft. in length. At the same time how long is the shadow of an 8-foot pole?

**12.** The ratio of two numbers is $\frac{3}{4}$. Find each number if their sum is 56.

**13.** Two numbers differ by 70; the ratio of the larger to the smaller is $\frac{7}{2}$. Find each number.

**14.** In Fig. 97, $\angle C = 90°$, $\angle A = 37°$, and $AC = 24$. Find $AB$, $BC$, and $\angle B$.

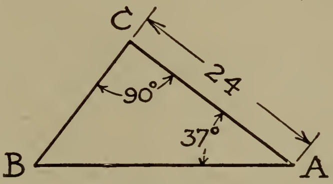


FIG. 97

**15.** In general, which is the larger, the *sine* or the *tangent* of an angle? Show by a drawing.

**16.** The highest office building in the world (the Woolworth Building, New York City) casts a shadow 1240 ft. long at the same time that a boy 5 ft. tall casts a shadow 8 ft. long. What is the height of the building?

**17.** The table below gives the annual cost of premium per $1000 life insurance, at various ages.

| Age | 21 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|
| Premium | 18.25 | 20.04 | 22.60 | 26.40 | 30.50 | 36.10 | 45.20 |

Show this graphically. Measure *age* along the horizontal axis.

# CHAPTER XI

## HOW TO SOLVE EQUATIONS WHICH CONTAIN TWO UNKNOWNS

### I. GRAPHICAL SOLUTION

**Section 82. Importance of skill in drawing the picture or graph of an equation.** In Chapter IV we learned how to represent and to determine the **relationship** between quantities that change together. Three methods of doing this were studied: (1) the tabular method; (2) the graphic method; (3) the equational or formula method. One of the most important facts for us to recall is that the graph and the equation *tell exactly the same thing*. For example, on page 51, the line $BC$ and the equation $C = .12n$ tell exactly the same thing. Any information that you get from the equation you can also get from the graph. *Furthermore, relationships can be seen more easily from graphs than from tables or equations.* For these reasons, and since much of our later work in mathematics makes use of graphic methods, we need to be skillful in drawing the line which stands for an equation.

**Section 83. We need to know how to locate or to "plot" points.** But a line may be regarded as a series of points. Thus, to represent or locate a line we have to locate a series of its points. It happens that much of our elementary work, furthermore, deals with *straight* lines. This kind of line, clearly, can be fully determined by locating any *two* of its points.

### HOW TO LOCATE OR PLOT POINTS

Thus, we see that the important thing in "graphing" is how to locate, or to represent, points. In every graph that you have already constructed you have had to *locate* *points* through which to draw the line. For example, in constructing a cost graph it is necessary to locate several points representing the cost of different numbers of units of the article. Let us study more carefully how points are located.

**Section 84. How points are located on maps.** (1) Points are located on maps by means of latitude and longitude. Any point on the earth's surface is *definitely* located by stating its distance *east* or *west* of the prime meridian, and its distance *north* or *south* of the equator.

Thus, to the nearest degree, the location of New York is 74° W. and 41° N. because it is 74° west of the prime meridian and 41° north of the equator. Similarly, the position of Chicago is 88° W. and 42° N.; that of Paris, 2° E. and 49° N.

(2) This same method is used by many cities in numbering their houses. Two streets, which make right angles with each other, are selected as reference streets. Any house or building is completely located, then, by stating the number of blocks it is east or west, and north or south, of these reference streets.

**Section 85. How points are located on drawings.** By a method similar to that above, we locate points on paper. Instead of using the equator and the prime meridian as our reference lines, we **take two lines**, — for convenience, one *horizontal* and the other *vertical*, — which make a right angle with each other. Any point may be located, then, by stating its distance to the *right* of, or to the *left* of, the vertical reference line; and its distance *above* or *below* the horizontal reference line.

Thus, in Fig. 98, point $A$ is 2 units to the *right* of, and 1 unit *above*, the reference lines; point $B$ is 2 units to the *left* of, and 3 units *below*, the reference lines; point $C$ is 3 units to the *left* of, and 2 units *below*, the reference

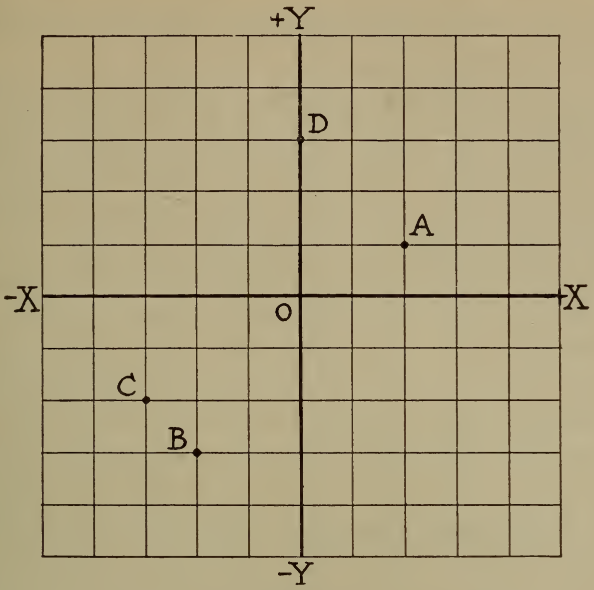


FIG. 98

lines; and point *D* is 0 units to the right or left of, and 3 units *above*, the reference lines.

**Section 86. The point from which distances are measured: The origin.** The point in which the two axes meet, or their intersection point, is called the *origin.* It is the point from which we measure distances, either way. The origin is usually lettered with a capital *O*, as in Fig. 98.

**Section 87. How distances are distinguished from each other.** It would be laborious to state that a particular point is "to the right of" or "to the left of" some reference line, each time we refer to it. To avoid this, it has been agreed to call *distances to the right of the Y-axis* ***positive***, and *distances to the left of the Y-axis* ***negative.*** Similarly, *distances above the X-axis are* ***positive***, and *distances below the X-axis are* ***negative.*** It is very important to remember these facts because we use them so frequently in graphic work.

Thus, in Fig. 98, the position of point $A$ is described by the numbers $+2$ and $+1$, or by $(2, 1)$. This means that point $A$ is 2 units to the right of the $Y$-axis, and 1 unit above the $X$-axis. Similarly, the position or location of point $B$ is described by the numbers $-2$ and $-3$, or by $(-2, -3)$; this means that point $B$ is 2 units to the left of the $Y$-axis and 2 units above the $X$-axis. In the same way, the position of point $C$ is described by the numbers $-3$ and $-2$, or $(-3, -2)$; this means that point $C$ is 3 units to the left of the $Y$-axis and 2 units below the $X$-axis.

At this time the student should note that in stating the location of a point, its distance to the right of, or to the left of, the $Y$-axis is *always* given *before* its distance above or *below* the $X$-axis. This is done to avoid confusion. *That is, the x-distance is always first*, the $y$-distance second.

**Section 88. Plotting a point.** By "plotting a point" we mean the locating, on cross-section paper, of a point whose $x$-distance and $y$-distance are known.

Thus, to plot $A$, whose $x$-distance is $+3$ and whose $y$-distance is $+4$, usually written $(3, 4)$, means to locate on the cross-section paper a point 3 units to the *right* of, and 4 units *above*, the origin, as in Fig. 99. In the same way, the point $(-2, 1)$ is point $B$ on the graph.

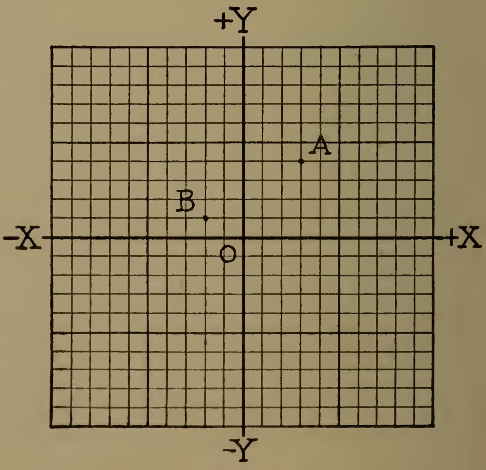


FIG. 99

## EXERCISE 84

### PRACTICE IN PLOTTING POINTS

1. The $x$-distance of a point is $+3$, *i.e.* it is 3 units to the right of the vertical axis. Is it *definitely* located? Why?
2. The $y$-distance of a point is $-4$, *i.e.* it is 4 units below the horizontal axis. Is it *definitely* located? Why?
3. A certain point is on both axes. What are its $x$- and $y$-distances?
4. Plot the points whose position is determined by the following: $(4, 2)$, $(5, 6)$, $(-3, 2)$, $(-4, -1)$, and $(-6, 2)$.
5. Plot the following: $(2, 8)$, $(3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4)$, $(8, 2)$, $(10, 0)$.
6. Plot the following: $(12, -2)$, $(15, -5)$, $(18, -8)$, $(10, 0)$, $(5, 5)$.
7. Plot the following: $(2\frac{1}{2}, 3)$, $(1\frac{3}{4}, 5)$, $(-2\frac{1}{3}, 3)$.

## HOW TO DRAW THE GRAPH OF AN EQUATION WHICH CONTAINS TWO UNKNOWNS

**Section 89. The picture of an equation.** Now that we have learned how to locate, or plot, points, we come to the main purpose of the chapter: *to show how equations can be solved graphically.*

**First illustrative example.** Let us take an equation which contains two unknowns, such as,

$$y = 2x + 3.$$

In this equation the value of $y$ changes as the value of $x$ changes. Clearly, the value of $y$ *depends upon* the value of $x$. For example, if $x = 1$, then $y = 5$; if $x = 2$, then $y = 7$, etc. A table will help to show this *relation* between the unknowns, $x$ and $y$.

TABLE 12

| If x equals | 1 | 2 | 3 | 4 | 5 | 0 | -1 | -2 | -3 | -4 | -5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| then y equals | 5 | 7 | 9 | 11 | 13 | 3 | 1 | -1 | -3 | -5 | -7 |

If we select any particular value of $x$, and the corresponding value of $y$ which accompanies it, such as 1 and 5, or 2 and 7, we may think of them as completely describing the position of points on a graph. Thus, (1, 5), (2, 7), (3, 9), etc., *definitely locate the position of the points.* Plotting these points with respect to an $X$- and $Y$-axis, we get a series of points, such as Fig. 100. By joining these points we obtain a straight line, *which is the picture or the graphical representation of the equation* $y = 2x + 3$.

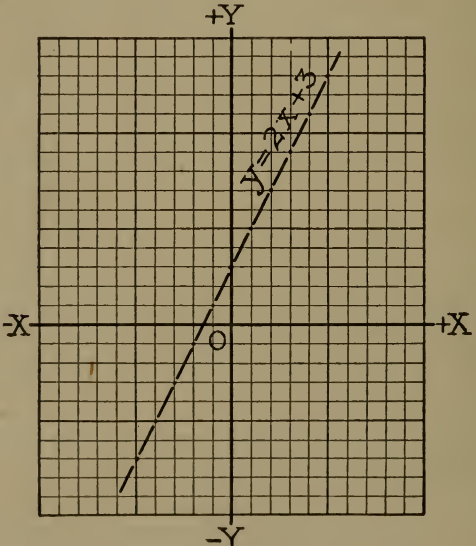


FIG. 100

**Second illustrative example.** For a second example, let us consider an equation in two unknowns which shows that the sum of two numbers is always 10, such as $x + y = 10$. It is clear that one number, $x$, *might* be 2, and if so, that $y$ *must* be 8 ; or that $x$ *might* be 4, and if so, the other number, $y$, *must* be 6. Thus the two unknowns, *$x$ and $y$, may have many different values.* A table helps to show this.

TABLE 13

| If x equals | 1 | 2 | 3 | 4 | 0 | -1 | -2 | -3 |
|---|---|---|---|---|---|---|---|---|
| then y equals | 9 | 8 | 7 | 6 | 10 | 11 | 12 | 13 |

Now we may think of any pair of related numbers, such as 1 and 9, or 2 and 8, as describing the position of points on this line. Thus, (1, 9), (2, 8), (3, 7), (4, 6), etc., show the location of points on the graph. Plotting these points, we have Fig. 101. By joining these points we obtain in this illustrative example the "graph" or picture of the equation $x + y = 10$. Expressed in another way, we have represented graphically the relation between two numbers whose sum is always 10.

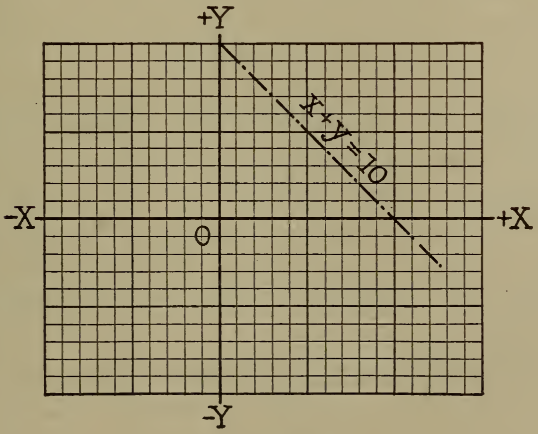


FIG. 101

## EXERCISE 85

### PRACTICE IN REPRESENTING GRAPHICALLY EQUATIONS WHICH CONTAIN TWO UNKNOWNS

1. In the equation $y = 2x + 5$, find the value of $y$ when $x = 1$; when $x = 2$; when $x = 4$; when $x = 0$; when $x = -3$. Make a table similar to the one above, showing these related pairs of numbers.
2. In the equation $x = y - 4$, find the value of $x$ when $y = 0$; when $y = 1$; when $y = 4$; when $y = 6$; when $y = -2$; when $y = -6$. Tabulate.
3. Plot the equation given in Example 2.
4. Plot the equation $x + y = 6$. HINT: First tabulate related values of $x$ and $y$. Use only four points.
5. Plot, or graph, $y = 5 + x$.
6. Plot $2x + y = 6$, or $y = 6 - 2x$.
7. Graph $x - 2y = 5$, or $x = 2y + 5$.
8. Plot $3x - y = 8$, or $y = 3x - 8$.

**Section 90. An easier method of plotting a line.** A straight line is definitely determined or located if any *two* of its points are known. If these points are not too close together, they fix the plotted position of the line just as accurately as eight or ten points. Therefore, in plotting a straight line, it is sufficient to *plot only two points*, unless they are quite close together.

The **easiest points to plot are those on the axes**; that is, the points where the line cuts the $x$-axis and the $y$-axis. By referring to Fig. 101, or to the graph of any line, you will see that the *$x$-distance of the point in which the line cuts the vertical, or $y$-axis, is always* 0, and *that the $y$-distance of the point in which the line cuts the horizontal, or $x$-axis, is always* 0. Thus, if we let $x$ be 0 in any equation, such as $2x-y=6$, we find the point in which the line cuts the $y$-axis. If $x$ is 0 in $2x-y=6$, we see that $y$ equals $-6$, *which shows that the line cuts the $y$-axis at a point* $(0, -6)$; *that is,* 6 *units below the origin.* In the same way, if we let $y$ be 0 in any equation, we find the point in which the line cuts the $x$-axis. In this particular equation, $2x-y=6$, if $y$ is 0, then $x$ is 3, which shows the point in which the line $2x-y=6$ cuts the $x$-axis.

This shorter method requires only the following brief table:

TABLE 14

| | | |
|---|---|---|
| x equals | 0 | ? |
| y equals | ? | 0 |

### EXERCISE 86

1. If $x=0$, what is $y$ in the equation $2x+y=8$? What is $x$ if $y=0$? From these two sets of values for $x$ and $y$, plot the equation.
2. Given $4x-2y=8$. Plot by finding where the line cuts the axes.
3. Where does the graph of $5x+2y=10$ cut the $x$-axis? the $y$-axis?
4. Where does the graph of $2x-3y=-6$ cut the $x$-axis? the $y$-axis? Plot.
5. Graph $2\frac{1}{2}x+y=5$.

## HOW TO SOLVE GRAPHICALLY EQUATIONS WITH TWO UNKNOWNS

**Section 91. When is an equation with two unknowns solved?** In equations with only one unknown, such as $3x+4=19$, we found that there was only *one* value for $x$ which would satisfy the equation; namely, $x=5$. If we substitute 5 for $x$ in this equation, giving $15+4=19$, we find that 5 *satisfies the equation.* Any other number substituted for $x$ would not "satisfy the equation."

But now consider an equation which has *two* unknowns, such as

$$x+y=8.$$

Here we see that $x$ *might* be 3 and $y$ *would* be 5; or $x$ *might* be 6 and $y$ *would* be 2; or $x$ *might* be 10 and $y$ *would* be $-2$. Thus, there are a great **many sets of values of $x$ and $y$ which could satisfy the equation** $x+y=8$. This will be made clear as you work the following examples.

### EXERCISE 87

1. Give four sets of values of $x$ and $y$ that will satisfy the equation $x-y=6$.
2. Will $x=4\frac{1}{2}$ and $y=3$ satisfy the equation $4x-y=15$? Does $x=5$ and $y=4$ satisfy it?
3. If the equation $x+y=8$ is plotted, would the points $(5, 3)$ lie on the line representing the equation? $(3, 4)$? $(10, -2)$? $(1, 7)$?
4. Does the graph of the equation $2x-y=7$ pass through the point $(5, 3)$? $(4, 2)$?

We have shown that an equation with two unknowns is solved when a set of values for the unknowns is found which SATISFIES the equation.

**Section 92. Linear equations.** The fact that the graph of an equation which contains two unknowns, *each of the first degree* (*i.e.* no squares or cubes), is **always** a straight line, has led to the name *linear equations.* Thus, $2x+y=5$, $x+5=10$, etc., are *linear equations.*

### TWO LINEAR EQUATIONS MAY BE EASILY SOLVED BY PLOTTING THEM ON THE SAME AXES

**Section 93.** It is a very common problem in mathematics to have to find **one set of values** which will satisfy each of two equations having two unknowns. For example, what single set of values will satisfy each of these equations?

$$\begin{cases} x+y=8 \\ 2x-y=7 \end{cases}$$

It is clear that $x=6$ and $y=2$ or (6, 2) will satisfy the first equation, but not the second one; in the same way $x=4$ and $y=4$ or (4, 4) satisfies the first equation, but not the second one; $x=6$ and $y=5$ satisfies the second equation, but not the first one.

### OUR PROBLEM IS TO FIND ONE SET OF VALUES THAT WILL SATISFY BOTH EQUATIONS

This can be done, *graphically*, by plotting both equations on the same axes, because in that way we can find a point common to the two lines; that is, the **point in which two lines intersect.** *The coördinates of this point will satisfy both equations.* Figure 102, on the following page, shows both equations plotted on the same axes. Note that the

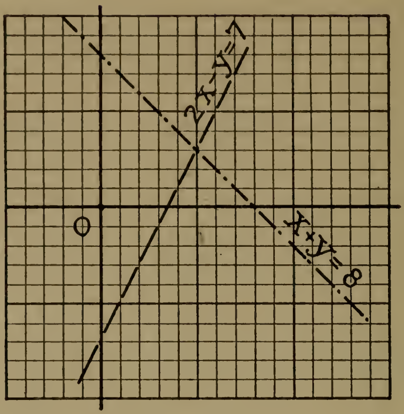


FIG. 102

two lines intersect at the point (5, 3). **This point of intersection of the two lines gives a single set of values, $x = 5$ and $y = 3$, which satisfies both equations.** (Show that $x = 5$ and $y = 3$ checks for each of the equations.)

EXERCISE 88

Find a set of values that will satisfy each of the following pairs of equations, by finding the intersection point of their graphs:

**1.** $\begin{cases} x + y = 6 \\ 2x - y = 3 \end{cases}$

**2.** $\begin{cases} y = 2x + 3 \\ y = x + 7 \end{cases}$

**3.** $\begin{cases} x - y = 5 \\ 2x + y = 7 \end{cases}$

**4.** $\begin{cases} 2x + 3y = 5 \\ x - 3y = -2 \end{cases}$

**5.** $\begin{cases} y = x + 6 \\ y = -x + 2 \end{cases}$

**6.** $\begin{cases} a + b = 7 \\ a - 2b = -5 \end{cases}$

**7.** $\begin{cases} x + y = 10 \\ 3x + 3y = 6 \end{cases}$

**8.** $\begin{cases} x = y + 6 \\ 2x - 2y = 12 \end{cases}$

**9.** $\begin{cases} x + y = 1 \\ x - y = 5 \end{cases}$

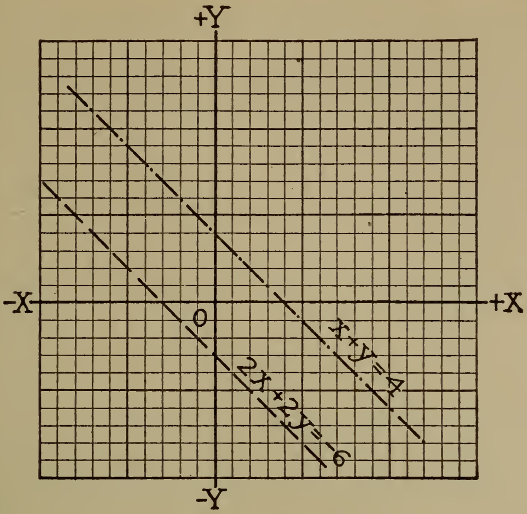


FIG. 103

**Section 94. Equations whose graphs are parallel lines, *i.e.* inconsistent equations.** Figure 103 shows the graphs of the equations represented below.

$$\begin{cases} x+y=4 & (1) \\ 2x+2y=-6 & (2) \end{cases}$$

Note that the lines do not intersect, but are parallel. What single set of values of $x$ and $y$ will satisfy each of these equations? Evidently there is none, *for they have no point in common.* Such equations are generally called *inconsistent*, to distinguish them from the kind that are satisfied by some set of values. The latter kind, those whose graphs intersect, are often called **simultaneous equations.**

### EXERCISE 89

Graph each of the following pairs of equations to determine which pairs are inconsistent and which are simultaneous:

1. $\begin{cases} y - x = 4 \\ x - 6 = y \end{cases}$

2. $\begin{cases} 2x - 3 = y \\ 2y + 10 = 4x \end{cases}$

3. $\begin{cases} x + y = 4 \\ x - y = 6 \end{cases}$

4. $\begin{cases} y + 4 = x \\ x + y = 12 \end{cases}$

### SUMMARY

This chapter should make it clear that:

1. **An equation may be plotted or graphed by locating a series of points, the $x$ and $y$ values of which will satisfy the equation.**
2. **Two equations are solved graphically by finding the $x$ and $y$ values of the point of intersection of their lines.**

### REVIEW EXERCISE 90

1. From the sum of $-6$ and $+10$ take $-8$.
2. The product of two numbers is $-40y$; one of them is $+10$. What is the other?
3. State the four principles, or axioms, used in solving equations. Illustrate in solving the equation $5y - 8 = +2y - 50$.
4. If $A = 4x + 3y$ and $B = 4x - 3y$, what does $A + B$ equal? What does $A - B$ equal?
5. Does $\frac{5}{7} = \frac{5 \cdot 2}{7 \cdot 2}$? Does $\frac{a}{b} = \frac{ac}{bc}$? Does $\frac{1}{2} = \frac{5}{10}$? State the principle involved in these examples.

| Age | Boys who leave school at the age of 14 earn weekly wages as indicated | Boys who leave school at the age of 18 earn weekly wages as indicated |
|---|---|---|
| 14 | $ 4.00 | 0 |
| 16 | 5.00 | 0 |
| 18 | 7.00 | $10.00 |
| 20 | 9.50 | 15.00 |
| 22 | 11.00 | 20.00 |
| 24 | 12.00 | 24.00 |
| 25 | 13.00 | 30.00 |

6. Studies have been made to determine the money value of a high school education. The table above shows the average weekly earnings for boys who leave school at the age of 14, and for those who remain in school until they are 18 years old.
Graph the earnings for each class of boys on the same axes. Measure *age* along the horizontal axis.

   Interpret the graph. If a boy knew that he would live to be only 25 years old, would it pay him, in dollars, to go to high school? How much?

7. The areas of two circles are directly proportional to the squares of their radii. Compare their areas if the radius of one circle is 3 times the radius of the other.

8. In solving a particular problem, how do you tell whether to use the sine, cosine, or tangent? Illustrate by specific examples.

# CHAPTER XII

## HOW TO SOLVE EQUATIONS WITH TWO UNKNOWNS

(*Continued*)

## II. SOLUTION BY ELIMINATING ONE UNKNOWN

**Section 95. The need for a shorter method of solving equations with two unknowns.** In the previous chapter we saw that equations with two unknowns can be solved by graphic methods. The exclusive use of that method, however, would require a great deal of time, and would necessitate that we have cross-section paper at all times. Fortunately, there is *a shorter method* which can be used. This is a method by which we ELIMINATE ONE OF THE UNKNOWNS. By eliminating or getting rid of *one* of the unknowns, we obtain an equation with only *one* unknown. The following illustrative examples will explain the different ways by which one of the unknowns is eliminated. This chapter will show three methods of elimination.

## I. ELIMINATION BY COMPARISON; THAT IS, BY EQUATING VALUES OF ONE OF THE UNKNOWNS

**Section 96. Equating values of one of the unknowns.** This method is illustrated by the following example:

**First illustrative example.** Find the value of $x$ and of $y$ in the following equations:

$$\begin{cases} x + y = 10, & (1) \\ x - 3y = -6. & (2) \end{cases}$$

Solution:

Solving for $x$ in equation (1),

$$x = 10 - y \qquad (3)$$

and in equation (2),

$$x = 3y - 6. \qquad (4)$$

Comparing or equating the values of $x$,

$$3y - 6 = 10 - y, \qquad (5)$$

which gives

$$4y = 16, \qquad (6)$$

or

$$y = 4. \qquad (7)$$

Substituting 4 for $y$ in (1) or (2) gives

$$x = 6.$$

Checking in equations (1) and (2),

$$\begin{aligned} 6 + 4 &= 10. \\ 6 - 12 &= -6. \end{aligned}$$

---

*Elimination by comparison* is based upon a fact which was illustrated in the graphical solution of equations with two unknowns; namely, that *at the point of intersection of the two lines*, the value of $x$ in one equation is the same as the value of $x$ in the other equation, and the value of $y$ in one equation is the same as the value of $y$ in the other. For this reason, we may form an equation by equating, or placing equal to each other, the two values of $x$, which were $10 - y$ and $3y - 6$. This process gets rid of, or ELIMINATES, the unknown $x$ and gives us an equation with only one unknown, $y$.

**Second illustration of the method of eliminating one unknown.** The same results could have been obtained by finding the value of $y$ in each of the two given equations:

$$\begin{cases} x + y = 10. & (1) \\ x - 3y = -6. & (2) \end{cases}$$

From (1), $$y = 10 - x. \tag{3}$$

From (2), $$-3y = -6 - x, \tag{4}$$

or $$y = \frac{6 + x}{3}. \tag{5}$$

Comparing the values of $y$ $$10 - x = \frac{6 + x}{3}. \tag{6}$$

Multiplying by 3, $$30 - 3x = 6 + x, \tag{7}$$

or $$-4x = -24, \tag{8}$$

or $$x = 6. \tag{9}$$

Substituting in (1) or in (2), $$y = 4. \tag{10}$$

## EXERCISE 91

### PRACTICE IN ELIMINATING ONE OF THE UNKNOWNS BY COMPARING, OR EQUATING, ITS VALUES AS OBTAINED FROM THE TWO EQUATIONS

Solve and **check** each of the following:

**1.** $\begin{cases} x = 2y - 3 \\ x = 5y - 21 \end{cases}$

**2.** $\begin{cases} x - y = 10 \\ x = 16 - 2y \end{cases}$

**3.** $\begin{cases} y + 2x = 12 \\ 5x + y = 42 \end{cases}$

**4.** $\begin{cases} s - 3t = -3 \\ 4s + t = 14 \end{cases}$

**5.** $\begin{cases} 2b + 3c = 6 \\ b = 5c + 16 \end{cases}$

**6.** $\begin{cases} x + 5y = 1 \\ 2x + 6y = -2 \end{cases}$

**7.** The sum of two numbers is 14; the larger exceeds the smaller by 2. Find each number.

**8.** Twenty coins, dimes and nickels, have a value of $1.70. Find the number of each.

**9.** A boy earns $2 per day more than his sister; the boy worked 8 days and the girl worked 6 days. Both together earned $44. What did each earn per day?

**10.** $\begin{cases} y = \frac{3}{2}x - 5 \\ y + x = 10 \end{cases}$

**11.** $\begin{cases} 2x + 3y = 5 \\ 3x - y = 2 \end{cases}$

**12.** $\begin{cases} \frac{1}{2}x = 10 - \frac{1}{3}y \\ \frac{2}{3}x = 14 - \frac{1}{2}y \end{cases}$

**13.** $\begin{cases} y = 2x - 10 \\ x = 2y - 14 \end{cases}$

### II. ELIMINATION BY SUBSTITUTION; THAT IS, BY SUBSTITUTING THE VALUE OF $X$ FROM ONE EQUATION IN THE OTHER EQUATION

**Section 97.** This method of elimination will be illustrated by working the **same problem** which we used in the previous section.

**Illustrative example.** Find the value of $x$ and of $y$ in the following equations:

$$\begin{cases} x + y = 10, \\ x - 3y = -6. \end{cases} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Solving equation (1) for $x$, we get

$$x = 10 - y. \qquad (3)$$

Substituting $10 - y$ for $x$ in (2) gives

$$10 - y - 3y = -6, \qquad (4)$$

or $$-4y = -16, \qquad (5)$$

or $$y = 4. \qquad (6)$$

Substituting 4 for $y$ in (1) or (2) gives

$$x = 6. \qquad (7)$$

Here, as when we eliminate one unknown by "comparison," our real aim is to get an equation which contains only *one* unknown. We found from equation (1) that $x = 10 - y$. This value of $x$ must be true for both equations. (Recall that $x$ is the same for both equations, or for both lines, *at their point of intersection.*) For this reason we may substitute $10 - y$ in place of $x$ in the second equation. This gives an equation in one unknown; namely, $y$.

The same results could have been obtained by finding the value of $y$, instead of the value of $x$, from one of the equations and substituting it in the other equation. For example:

**Second illustration of the method of eliminating by substitution.**

$$\begin{cases} x + y = 10, \\ x - 3y = -6. \end{cases} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

From equation (1), $$y = 10 - x. \qquad (3)$$

Substituting $10 - x$ for $y$ in (2)

$$x - 3(10 - x) = -6, \qquad (4)$$

or $$x - 30 + 3x = -6, \qquad (5)$$

or $$4x = 24, \qquad (6)$$

or $$x = 6, \qquad (7)$$

and $$y = 4, \text{ as before.}$$

### EXERCISE 92

Solve by the method of substitution and check each result:

**1.** $\begin{cases} x - 2y = 10 \\ 3x + 2y = 6 \end{cases}$

**2.** $\begin{cases} x - 3y = -1 \\ 4x - y = -15 \end{cases}$

**3.** $\begin{cases} 5r - 4s = 18 \\ r = 2s + 7 \end{cases}$

**4.** $\begin{cases} y = x \\ 3x + 4y = 7 \end{cases}$

**5.** $\begin{cases} x + y = 8 \\ 2x - y = 10 \end{cases}$

**6.** $\begin{cases} a - 2b = -13 \\ b - a = 9 \end{cases}$

**7.** The sum of two numbers is 102; the greater exceeds the smaller by 6. Find the numbers.

**8.** The difference between two numbers is 14, and their sum is 66. Find the numbers.

**9.** 12 coins, nickels and dimes, amount to $1.05. Find the number of each kind of coin.

**10.** The perimeter of a rectangle is 158 inches; the length is 4 feet more than twice the width. Find the dimensions of the rectangle.

**11.** Bacon costs 10 cents per pound more than steak. Find the cost per pound of each if 4 pounds of bacon and 7 pounds of steak together cost $3.48.

**12.** A part of $4000 is invested at 4% and the remainder at 5%. The annual income on both investments is $185. Find the amount of each investment.

**13.** The quotient of two numbers is 2, and the larger exceeds the smaller by 7. Find the numbers.

**14.** Oranges cost 20 cents per dozen more than apples. A customer bought 10 dozen oranges and 4 dozen apples and received 20 cents in change from a 5-dollar bill. Find the price per dozen of each.

**15.** $\begin{cases} 2x + 3y = 5 \\ 4x - y = 3 \end{cases}$

**16.** $\begin{cases} \frac{1}{2}p - \frac{2}{3}q = -1 \\ \frac{3}{4}p = q - \frac{3}{2} \end{cases}$

**17.** $\begin{cases} \dfrac{x}{2} - \dfrac{y}{3} = 9 \\ \dfrac{2x}{3} - \dfrac{3y}{4} = \dfrac{7}{3} \end{cases}$

**18.** $\begin{cases} \frac{1}{6}x - \frac{1}{9}y = 1 \\ \frac{1}{2}y - 2x = 3 \end{cases}$

## III. ELIMINATION OF ONE UNKNOWN BY ADDING OR BY SUBTRACTING THE EQUATIONS

**Section 98.** A great many equations in two unknowns can be most easily solved by this method. The following example illustrates it.

**Illustrative example.** Find the value of $x$ and $y$ in these equations:

$$\begin{cases} 2x - y = 5, & (1) \\ x + y = 13. & (2) \end{cases}$$

Adding equation (1) and equation (2) gives

$$3x = 18, \tag{3}$$

or

$$x = 6. \tag{4}$$

Substituting 6 for $x$ in (1) and (2) gives

$$y = 7. \tag{5}$$

Check: Substituting 6 for $x$ and 7 for $y$ in (1) and (2) gives

$$12 - 7 = 5. \tag{6}$$

$$6 + 7 = 13. \tag{7}$$

---

It happens in this example that one of the unknowns, $y$, is eliminated by adding the corresponding members of the given equations. In many examples it is possible to eliminate one of the unknowns by subtracting the members of one equation from the corresponding members of the other. In many other examples, however, it is impossible to eliminate one of the unknowns directly, either by adding or by subtracting the members of the two equations. For illustration, take this set of equations:

**Illustrative example.**

$$\begin{cases} x - 2y = 8, & (1) \\ 2x + y = 6. & (2) \end{cases}$$

If we add the corresponding members of the two equations, we get the equation $3x - y = 14$. But this does *not* eliminate either of the unknowns. In the same way, if we subtract (2) from (1), we get the equation $-x - 3y = 2$. Again, this does *not* eliminate either one of the unknowns. This shows that addition or subtraction of the members to the equations will *not* eliminate one of the unknowns *unless* one of them, the $x$ or the $y$, has the same *coefficient* in both equations.

Now let us *make* $y$ in the second equation have the same coefficient as $y$ in the first equation. To do so, the second equation must be multiplied through by 2. This gives

$$\begin{cases} 4x + 2y = 12, & (3) \\ x - 2y = 8. & (1) \end{cases}$$

Now, by adding (3) and (1), we get rid of $y$, obtaining:

$$5x = 20, \quad (4)$$

or $$x = 4. \quad (5)$$

Substituting in (1) or (2), $$y = -2. \quad (6)$$

Check: $$16 - 4 = 12. \quad (7)$$

$$4 + 4 = 8. \quad (8)$$

An important question naturally arises here: When do we eliminate by *addition* and when by *subtraction?* This can be answered by referring to an example.

$$\begin{cases} x + y = 11, & (1) \\ 2x + y = 4. & (2) \end{cases}$$

Would either $x$ or $y$ be eliminated by adding the corresponding members of these equations? Certainly not, for that would give $3x + 2y = 15$. Now, would either $x$ or $y$ be eliminated by subtracting the members of one equation from those of the other? Yes, for we should have $-x = 7$. However, if the second equation (2) above had been $2x - y = 4$, then we should eliminate $y$ by *adding* (1) and (2).

From these examples we come to the following conclusions about eliminating one of the variables:

I. If the variable we wish to eliminate has the same sign in both equations, then it is eliminated by subtracting the members of one equation from the members of the other equation.

II. If the variable we wish to eliminate has different signs in the two equations, it is eliminated by adding the corresponding members of the equations.

III. No variable can be eliminated either by addition or by subtraction unless it has the same *coefficient* in both equations. If it does not have the same coefficient in both equations, then we must multiply one, or both, of the equations by such a number, or numbers, as will make that variable have the same coefficient. Thus, to eliminate $x$ in the following equations:

$$\begin{cases} 2x - y = 8, & (1) \\ 3x + 4y = 23. & (2) \end{cases}$$

It is necessary to multiply (1) by 3 and to multiply (2) by 2. This gives the following equations:

$$\begin{cases} 6x - 3y = 24, & (3) \\ 6x + 8y = 46. & (4) \end{cases}$$

Now the variable $x$ can be eliminated by subtracting (4) from (3), which gives

$$-11y = -22,$$

or

$$y = 2.$$

## EXERCISE 93

### ELIMINATION BY ADDITION OR SUBTRACTION

Solve and check each of the following:

**1.** $\begin{cases} x+y=5 \\ x-y=2 \end{cases}$

**2.** $\begin{cases} 2x+3y=14 \\ 3x-3y=1 \end{cases}$

**3.** $\begin{cases} 2r+s=9 \\ r-s=12 \end{cases}$

**4.** $\begin{cases} x-y=-8 \\ x+y=-4 \end{cases}$

**5.** $\begin{cases} 3a-b=-2 \\ 4a+b=-12 \end{cases}$

**6.** $\begin{cases} 2x-y=11 \\ x-3y=13 \end{cases}$

**7.** $\begin{cases} 3b+2c=5 \\ 2b+c=3 \end{cases}$

**8.** $\begin{cases} 4x-3y=8 \\ x-7y=2 \end{cases}$

**9.** Find two numbers whose sum is 100 and whose difference is 18.

**10.** In an election of 642 votes an amendment was carried by a majority of 60 votes. How many voted *yes* and how many *no?*

**11.** The admission to a school play was 25 cents for adults and 15 cents for children. The proceeds from 267 tickets were $50.05. How many tickets of each kind were sold?

**12.** A purse containing 18 coins, dimes and half dollars, amounts to $6.20. Find the number of each denomination.

**13.** $\begin{cases} p-q=8 \\ 3p+4q=10 \end{cases}$

**14.** $\begin{cases} \frac{1}{2}x+y=3 \\ 2x+3y=11 \end{cases}$

**15.** $\begin{cases} 2x+5y=12 \\ y-3x=-1 \end{cases}$

**16.** $\begin{cases} x+2y=11 \\ 5x-3=3y \end{cases}$

**17.** $\begin{cases} 3x-y=-2 \\ x-3y=10 \end{cases}$

**18.** $\begin{cases} 2a-8=-b \\ 3a+4b=7 \end{cases}$

19. $\begin{cases} \frac{1}{2}x + 3 = -4y \\ x - \frac{1}{2}y = 11 \end{cases}$ 20. $\begin{cases} \frac{3}{4}x + \frac{2}{3}y = 12 \\ \frac{1}{2}x - \frac{4}{3}y = -8 \end{cases}$

In the remaining examples of this exercise, use any method of elimination.

21. $\begin{cases} y = x + 4 \\ 2x + y = 16 \end{cases}$ 25. $\begin{cases} x = -2y - 1 \\ y = x + 14.5 \end{cases}$

22. $\begin{cases} 2x = 3y + 24 \\ x - y = 10 \end{cases}$ 26. $\begin{cases} b = \frac{1}{2}c \\ 2c - 3b = 7 \end{cases}$

23. $\begin{cases} \frac{1}{2}x - \frac{1}{5}y = 9 \\ x = y + 2 \end{cases}$ 27. $\begin{cases} \frac{2}{3}x - \frac{3}{4}y + 1 = 0 \\ \frac{1}{4}y + x = -15 \end{cases}$

24. $\begin{cases} 4x = 3y + 3 \\ 5y = 6x \end{cases}$ 28. $\begin{cases} 2x + 2y = 0 \\ x = y + 12 \end{cases}$

## SUMMARY

**This chapter has taught three methods of solving a pair of equations which contain two unknowns:**

1. **Elimination of one of the unknowns by *substitution*;**
2. **Elimination of one of the unknowns by *comparison*;**
3. **Elimination of one of the unknowns by *addition* or *subtraction*.**

## REVIEW EXERCISE 94

1. If one tablet costs $b$ dollars, what will $x$ tablets cost?
2. If $a$ books cost $b$ dollars, what will one book cost? $c$ books?
3. What is the perimeter of a rectangle whose width is $a$ and whose length is $b$? What is its area?
4. What is the width of a rectangle whose perimeter is $p$ and whose length is $x$?

5. The area of a triangle is $k$. Its base is $b$. What is its altitude?

6. The sum of two numbers is $s$. If one is $d$, what is the other?

Solve each of the following pairs of equations by any method of elimination:

7. $\begin{cases} x - 2y = 8 \\ 3x + 2y = 8 \end{cases}$

8. $\begin{cases} a - 2b = -1 \\ 4a - b = 10 \end{cases}$

9. $\begin{cases} \dfrac{5x}{6} + \dfrac{y}{4} = 7 \\ \dfrac{2x}{3} - \dfrac{y}{8} = 3 \end{cases}$

10. A collection box contained 63 coins, nickels and quarters. How many of each kind were there if the total amount was $8.35?

11. Express algebraically that the weight, $W$, of the water in a tank varies as the volume, $V$, of the water. If 6 cu. ft. weigh 374.4 lb., how much will 11 cu. ft. weigh?

12. In weighing with a spring balance scale, the principle is applied that the amount of stretch, $s$, of the spring varies as the weight, $W$, which is suspended to the scales. Express this more briefly. If a 20-pound weight produces a stretch of $\frac{3}{4}$ inch, what weight will produce a stretch of 2 inches?

13. What is the best method of eliminating one of the unknowns in an equation?

14. The volume, $V$, of a sphere varies as the cube of its radius, $r$. Express this more briefly. If an orange with a radius of 2 inches is worth 10 cents, what could you afford to pay for an orange with a 4-inch radius?

**15.** The table below shows how much money (to the nearest dollar) you would have at the end of a certain number of years if you saved 10 cents a day and deposited it in a bank which pays 3 % interest.

| At the end of (years) | 1 | 2 | 3 | 5 | 8 | 10 | 14 | 17 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| total amt. saved is | 37 | 75 | 115 | 197 | 330 | 425 | 635 | 809 | 999 |

Represent this graphically, measuring the *time* on the horizontal axis.

Estimate the total savings at the end of 4 yr.; 6 yr.; 25 yr.

**16.** Is it ever possible to get from a graph information which could not be obtained from the table? Illustrate.

**17.** The distance from the base to the top of a hill, up a uniform incline of 40°, is 800 ft. What is the altitude of the top above the base?

# CHAPTER XIII

## HOW TO FIND PRODUCTS AND FACTORS

**Section 99. Why you should be able to find products.** Suppose you wanted to find the area of a rectangle whose dimensions are $3x+5$ and $2x$. To do so, it would be necessary to multiply $3x+5$ by $2x$, or to find the *product* of these two algebraic expressions. One way to do this is to divide the rectangle into smaller rectangles, as indicated in Fig. 104. This gives two rectangles, the dimensions of one being $2x$ by $3x$, and of the other $2x$ by 5. From what you already have learned about multiplication you can see that the areas of these are $6x^2$ and $10x$, be-

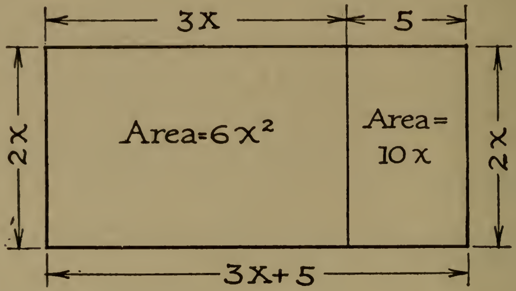


FIG. 104

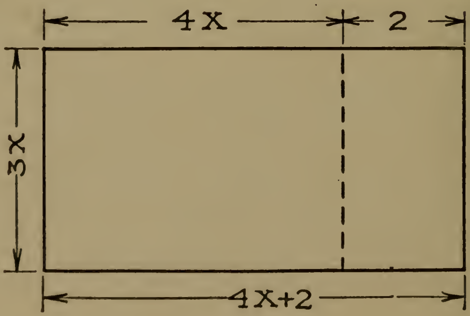


FIG. 105

cause $3x$ times $2x$ is $6x^2$ and 5 times $2x$ is $10x$. Thus the area of the original rectangle is $6x^2 + 10x$. Similarly, the area of the rectangle in Fig. 105 is what? What would its area be if the dimensions were $6a + 4$ and $5a$?

These illustrations are given to make clear the *need of learning how to find products.* Other illustrations might have been taken. For example, what is the cost of $15b + 3$ articles at $4b$ cents each? How much could you earn in $6y + 4$ days at $3y$ dollars per day?

## A NEW WAY OF INDICATING MULTIPLICATION

**Section 100.** As you progress through your mathematics, you will find that it is a language which tells more in fewer words or symbols than any other language. For example, instead of writing "*find the product of* $3x + 5$ *and* $2x$," it has been agreed to express this by means of the parentheses, ( ). Thus, $2x(3x + 5)$ means "*to find the product of* $3x + 5$ *and* $2x$," or, "*to multiply* $3x + 5$ by $2x$. It is *important* to note that there is no sign between the $2x$ and the expression in the parentheses. Similarly, to state algebraically the problem in the second illustration, Fig. 105, you would write $4x(7x + 3)$, putting no sign between the $4x$ and the parentheses. Thus, $5b(3b + 7)$ means to multiply *each* of the numbers in the parentheses by $5b$.

## ORAL EXERCISE 95

### PRACTICE IN FINDING PRODUCTS

In the following examples, multiply each term within the parentheses by the number which immediately precedes the parenthesis, or, remove parentheses.

**Illustrative example.** $3x(5x^2 + 7x + 8) = 15x^3 + 21x^2 + 24x.$

---

1. $4y(3y + 9)$
2. $6b(2b + 1)$
3. $7c(3 + 5c)$
4. $5x(x - 4)$
5. $9(2x^2 + 7x - 4)$
6. $4a(a^2 + 3a + 7)$
7. $6y(3y^2 - 5y + 2)$
8. $1(2b^2 + 3)$
9. $5t(6 - t)$
10. $8y(y - 8)$
11. $3r(r + 3)$
12. $7b(1 - b)$
13. $8(b^2 - 8b + 12)$
14. $6(2a - 3b + c)$
15. $ab(a + b + 1)$
16. $xy^2(x + y + xy)$
17. $-1(4 - 5y)$
18. $-3x(6x - 4)$
19. What algebraic expression will represent the area of a rectangle whose length is 10 inches more than its width?
20. What algebraic expression will represent the total daily earning of 4 men and 7 boys, if each man earns $2 per day more than each boy?
21. $-6(2x - 7)$
22. $-(10 - x)$
23. $-4y^2(2y - 3)$
24. $(16y^2 - 7)$
25. $(a + b)$
26. $-(b - c)$

In this exercise you have learned how *parentheses* are used to indicate that *each* of the terms in an expression must be multiplied by another number.

**Section 101. More difficult multiplication.** Most products which you will need to find are more difficult than those of the preceding exercise. For example: How many square feet of floor area in a dining room whose dimensions are $4x + 3$ ft. and $5x + 4$ ft.? To find the

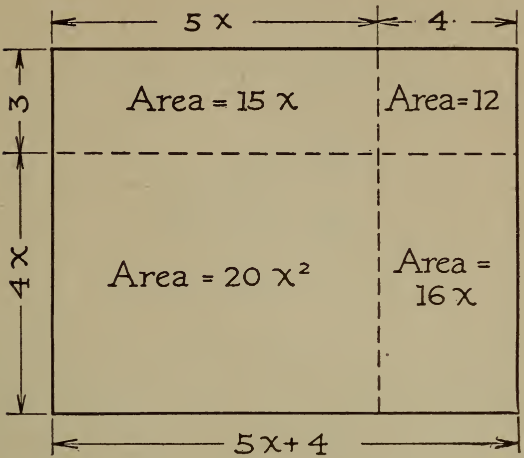


FIG. 106

*product* of these factors requires something you have not yet learned. You know how to multiply $5x + 4$ by $4x$ or $4x + 3$ by $5x$, but you have not learned how to multiply such expressions as $4x + 3$ by $5x + 4$. The drawing shows one way to do this; namely, the *geometrical method* of dividing the entire area into smaller rectangles, the area of each of which you can find. This method gives four rectangles whose areas we can find. Thus, we get four rectangles whose areas are $20x^2$, $16x$, $15x$, and 12, or, *collecting terms*, an entire area of $20x^2 + 31x + 12$.

Another method of finding the product of $4x + 3$ and $5x + 4$ (which is generally written as $(4x + 3)(5x + 4)$) makes no reference to rectangles. It is very much like the method of multiplication used in arithmetic. To illustrate, this same example could be solved as follows:

**First illustrative example.**

$$(4x+3)(5x+4).$$

$$\begin{array}{lll}
4x & +3 & \\
5x & +4 & \\
\hline
20x^2 & +15x & \\
 & +16x & +12 \\
\hline
20x^2 & +31x & +12
\end{array}$$

The $20x^2+15x$ is the result of multiplying $4x+3$ by $5x$, and the $16x+12$ is the result of multiplying $4x+3$ by 4. This latter method is much more generally used than the geometrical method. Let us take another illustration.

**Second illustrative example.**

$$(7a+3)(4a-5).$$

$$\begin{array}{lll}
7a & +3 & \\
4a & -5 & \\
\hline
28a^2 & +12a & \\
 & -35a & -15 \\
\hline
28a^3 & -23a & -15
\end{array}$$

Note that $7a+3$ was first multiplied by $4a$, giving $28a^2+12a$. Then $7a+3$ was multiplied by $-5$, giving $-35a-15$. How was the final product obtained?

Which of these two methods, do you think, should be learned?

## EXERCISE 96

### PRACTICE IN FINDING PRODUCTS

**Illustrative example.**

$$(2x^2+3x+4)(3x-7).$$

$$\begin{array}{llll}
2x^2 & +3x & +4 & \\
3x & -7 & & \\
\hline
6x^3 & +9x^2 & +12x & \\
 & -14x^2 & -21x & -28 \\
\hline
6x^3 & -5x^2 & -9x & -28
\end{array}$$

---

**1.** $(3a+2)(4a+5)$

**2.** $(7b+5)(b+6)$

**3.** $(y+4)(5y+3)$

**4.** $(t-5)(2t+8)$

**5.** $(8y+5)(7y+3)$

**6.** $(6c+4)(4c+6)$

**7.** $(4a-3)(4a-8)$

**8.** $(x+3)(x-9)$

**9.** $(3c-4)(3c-9)$

**10.** $(6t-7)(3t-9)$

**11.** $(y+6)(y+6)$

**12.** $(3b^2+1)(2b^2+5)$

**13.** $(4x^3+3)(3x^3+10)$

**14.** $(5a+3)(5a-3)$

**15.** $(7c+5)(7c-5)$

**16.** $(a+7)(a-7)$

**17.** $(3a-2)(3a-2)$

**18.** $(x^2+2x+1)(x-3)$

**19.** $(y^2+6y+9)(y+3)$

**20.** $(x^3+5)(x^3-5)$

**21.** If you should multiply $x+7$ by $x+10$, what would be the *first* term of your product? What would be the *last* term?

**22.** Can you tell, at a glance, the *first* and *last* terms of the product which you would obtain by multiplying $2x+7$ by $5x+4$?

### A SHORTER METHOD OF MULTIPLYING

**Section 102.** There is a much shorter method of finding products like those in Exercise 96. Mastery of this short cut will not only save a great deal of time, but it will help you in the later work of this chapter. For an illustration, take the example

$$(3b+5)(2b+7).$$

You have no difficulty in seeing that the first term of the product is $6b^2$ (*i.e.* $3b \times 2b$) and that the last term is $+35$ (*i.e.* $5 \times 7$). So if there were some method by which you could tell the middle term, you could write the product at once, without using the longer method of placing one factor under the other and multiplying in the regular way. To make the new method clear, it is necessary to refer again to the regular way. Let us illustrate with the example:

**Illustrative example.**

$$(3b+5)(2b+7).$$

**By the old method :**

$$\begin{array}{l} 3b+5 \\ \quad\times \\ \underline{2b+7\quad} \\ 6b^2+10b \\ \underline{\qquad+21b+35} \\ 6b^2+31b+35 \end{array}$$

The arrows show the *cross-multiplications* or cross-products that make up the middle term. One "cross-product" is $2b$ times $+5$, or $+10b$, and the other cross-product is $+7$ times $+3b$, or $+21b$. Combining the cross-products, we get the middle term, $+31b$.

By the shorter method we get

$$(\overbrace{3b+\underbrace{5)(2b}_{+10b}+7}^{+21b})=6b^2+31b+35.$$

The curved lines indicate the cross-multiplication, or **cross-products**, which must be combined to give the middle term, $+21b$ and $+10b$, giving $+31b$. Thus, in this new method, it is assumed that you can tell, at a glance, the first and last terms of the product. Then you can **get the middle term by finding the sum of the cross-products.** The curved lines are drawn to help you see the cross-products.

**Second illustrative example.**

$$(4b+3)(7b-8).$$

THE LONG METHOD

$$\begin{array}{rrr} & 4b & +3 \\ & \underline{7b} & \underline{-8} \\ 28b^2 & +21b & \\ & \underline{-32b} & \underline{-24} \\ 28b^2 & -11b & -24 \end{array}$$

THE SHORT METHOD

$$\overbrace{(4b+\underbrace{3)(7b}_{+21b}-8)}^{-32b}=28b^2-11b-24$$

It is important to recognize that *the curved lines indicate cross-products in just the same way that the arrows in the long method refer to cross-products.* The new method, which from now on we shall call the CROSS-PRODUCT METHOD, enables you to do **mentally** in much less time what was **written down** by the old method. To give you practice in this important method of finding the product of two factors the following exercise has been included.

## EXERCISE 97

Find the products of the following factors by the *cross-product* method:

1. $(x+2)(x+5)$
2. $(2y+4)(3y+5)$
3. $(b+6)(2b+7)$
4. $(x+4)(3x+5)$
5. $(t-8)(t-3)$
6. $(5b+3)(b+1)$
7. $(9x+2)(2x+1)$
8. $(4c+3)(7c+10)$
9. $(5s+3)(5s-3)$
10. $(a+9)(a+9)$

**11.** $(s + 2)(s - 5)$
**12.** $(ab + 6)(ab + 3)$
**13.** $(x^2 + 3)(x^2 + 6)$
**14.** $(abc + 8)(abc - 10)$
**15.** $(2x + 3y)(2x + 3y)$
**16.** $(a + b)(a + b)$
**17.** $(c + d)(c + d)$
**18.** $(f + s)(f + s)$
**19.** $(x + 5)(x - 5)$
**20.** $(x + 10)(x - 10)$
**21.** $(y + 4)(y - 4)$
**22.** $(3t + 7)(3t - 7)$
**23.** $(4a + 3)(4a + 3)$
**24.** $(7b + 9y)(7b + 9y)$
**25.** $(y + 3)(y + 5)$
**26.** $(x - 8)(x + 8)$
**27.** $(c + 9)(c + 9)$
**28.** $(t - 4)(t - 4)$
**29.** $(3a + 2)(2a + 4)$
**30.** $(4y - 2)(3y + 5)$
**31.** $(2x + 4)(5x - 9)$
**32.** $(d + 4)(d + 7)$
**33.** $(c - 9)(c + 9)$
**34.** $(y + 7)(y + 7)$
**35.** $(a - 2)(a - 2)$
**36.** $(5a + 1)(a - 2)$
**37.** $(2x + 1)(x - 2)$
**38.** $(4y + 3)(3y - 9)$
**39.** $(t^2 + 3)(t^2 + 5)$
**40.** $(p - 9)(p + 2)$
**41.** $(r^2 - 3)(r^2 - 5)$
**42.** $(y^2 + 3)(y^2 + 3)$
**43.** $(b^3 - 11)(b^3 - 11)$
**44.** $(e^2 + 3)(e^2 + 3)$
**45.** $(4y + 10)(5y - 8)$
**46.** $(ab + 2c)(ab + 3c)$
**47.** $(6abc - d)(2abc + d)$
**48.** $(7x^2t + y)(2x^2t - 5y)$
**49.** $(9x^2 + 3y)(9x^2 + 3y)$
**50.** $(8a - 4y)(8a - 7y)$
**51.** $(f + s)(f + s)$
**52.** $(f - s)(f - s)$

**53. Illustrative example.** What expression will represent the area of a square if each side is $x + 6$ inches?

Solution: Evidently the area is $(x + 6)(x + 6)$, or $x^2 + 12x + 36$. This is usually written, however, not as $(x + 6)(x + 6)$ but as $(x+6)^2$. The exponent, 2, shows that $x + 6$ is used twice as a factor. Thus, $(3x + 5)(3x + 5)$ is usually written as $(3x + 5)^2$. In the same way, $(3x + 5)(3x + 5)(3x + 5)$ would be written as $(3x + 5)^3$.

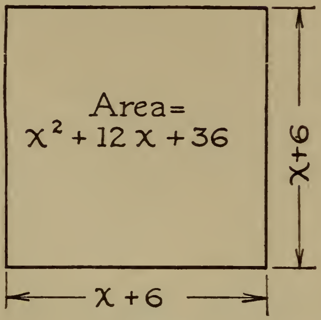


FIG. 107

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54. $(x + 5)^2$
55. $(3b + 5)^2$
56. $(2y + 7)^2$
57. $(4a - 3)^2$
58. $(x + y)^2$
59. $(x - y)^2$
60. $(a + b)^2$
61. $(a - b)^2$
62. $(f + s)^2$
63. $(f - s)^2$
64. $(2x + 3y)$
65. $(4b - 2)^2$
66. $(a + b)(c + d)$

**Section 103. Further practice in translating from algebraic symbols into word statements.** You have had some practice in translating from algebraic statements into word statements. For example, you translated the algebraic expression "$a + b$" into the word statement "the sum of two numbers," and the expression "$xy$" into "the product of two numbers." It is important to be able to translate into word statements some of the examples which you did in the last exercise.

### EXERCISE 98

Write out the word statement which means the same thing as each of the following algebraic expressions:

1. $a - b$
2. $a^2$
3. $x^3$
4. $(2y)^3$
5. $c + d$
6. $a^2 + b^9$
7. $(a + b)^2$
8. $(a - b)^2$
9. $(a + b)(a - b)$
10. $r^3 + s^3$
11. $(f + s)^2 = f^2 + 2fs + s^2$
12. $(f - s)^2 = f^2 - 2fs + s^2$

**Section 104. How to solve equations which involve products.** The solution of a great many equations depends upon your being able to find products like those you have just been finding. To illustrate, consider the problem:

**Illustrative example.** The length of a rectangle is 6 inches more than, and the width is 2 inches less than, the sides of a square; the area of the rectangle exceeds the area of the square by 20 square inches. What are the dimensions of each?

Solution: Translating into algebra, we have the equation:

$$(s + 6)(s - 2) = s^2 + 20.$$

Multiplying, or removing parentheses, gives

$$s^2 + 4s - 12 = s^2 + 20.$$

Subtracting $s^2$ from each side gives

$$4s - 12 = 20.$$

$$\therefore s = 8.$$

Thus, the sides of the rectangle are 14 and 8.

Check this result.

---

### EXERCISE 99

#### PRACTICE IN SOLVING EQUATIONS WHICH INVOLVE PRODUCTS

Solve and *check* each of the following equations:

1. $(x + 6)^2 = x^2 + 96$
2. $(y + 3)(y + 6) - y^2 = 63$
3. $(2b + 3)(b + 4) = (b + 1)(2b + 10)$

4. One number is 5 larger than another; the square of the larger exceeds the square of the smaller by 55. Find each number.

5. The length of a rectangle is 8 inches more than, and its width is 3 inches less than, the sides of a square; the area of the rectangle exceeds the area of the square by 26 square inches. Find the dimensions of the rectangle.

6. $(y+5)(y-5)=(y-6)(y+2)$

7. $2(x-8)-3(x-4)=-5x$

8. $(b+4)^2-(b-2)^2=10$

9. $(x+3)^2-(x-1)^2=40$

10. $(y+5)^2-(y+4)^2=-1$

11. $(x-8)^2=(x-12)^2$

12. $2(f+3)^2=2(f-8)^2$

13. $3(x+6)(x+4)=(3x+1)(x+9)$

## II. HOW TO FIND THE FACTORS OF AN ALGEBRAIC EXPRESSION

### A. FINDING THE *COMMON* FACTOR

**Section 105. Meaning of the word FACTOR.** If you know that the area of a rectangle is 24 square inches, what might be its dimensions? You readily see here that the dimensions *might* be 4 inches and 6 inches; or 3 inches and 8 inches; or 12 inches and 2 inches, because the *product* of 4 and 6, or of 3 and 8, or of 12 and 2, is in each case 24. *This process of finding the numbers which, when multiplied together, give another is called FACTORING.* The numbers you find are called the *FACTORS.* Thus, 4 and 6 are factors of 24; also 3 and 8, or 12 and 2.

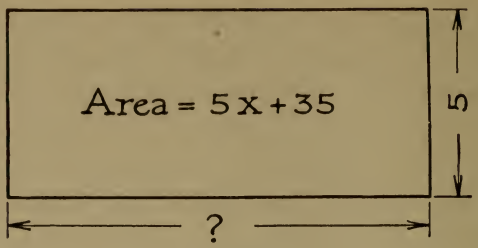

FIG. 108

The same reasoning is used in algebra as in arithmetic. For example, suppose the area of a rectangle is $5x + 35$ square units. If the width is 5 units, what must the length be? Similarly, if the area is $4x^2 + 28x$ and the width is $4x$, what is the length?

**Section 106. What is a common factor?** Now let us take a more difficult illustration. In the previous two cases, you knew both the *area* and one dimension, or the product and one of its factors. But in most *factoring problems* you do not know *any* of the factors. For example, what are the dimensions of the rectangle whose area is $7x + 21$, or, in other words, *what are the factors of* $7x + 21$? A study of the $7x$ and 21 shows that 7 is a factor of each, or is a *COMMON FACTOR* of both terms. What, then, must 7 be multiplied by to give $7x + 21$, or, what must be the length of this rectangle if its width is 7? Evidently, it must be $x + 3$. This problem should be written

$$7x + 21 = 7(x + 3).$$

A second illustration should make clear what is meant by factoring in algebra.

Find the factors of $ax + ay + aw$, or find the dimensions of a rectangle if its area is $ax + ay + aw$.

By observing each term, we see that $a$ is a factor COMMON to each.

Dividing each term of the expression by $a$ gives the other factor, $x+y+w$. Hence, $ax+ay+aw=a(x+y+w)$, and $a$ is one factor and $x+y+w$ is the other one.

These illustrations are intended to make clear how to factor an expression which contains a common factor.

### EXERCISE 100

Factor each of the following expressions:

**1.** $3x+12$
**2.** $5a-20$
**3.** $ax+bx$
**4.** $7a-21$
**5.** $17x+34$
**6.** $6a+9$
**7.** $5+15a$
**8.** $9+6x^2$
**9.** $4y^2+12$
**10.** $8a+12b$
**11.** $ab+ac+ah$
**12.** $5+10a+15a^2$
**13.** $2x^2-4xy+2y^2$
**14.** $2-8d^2$
**15.** $5x^2-5y^2$
**16.** $4a^2+8ab+4b^2$
**17.** $x^2+x$
**18.** $a^2+20a^3$
**19.** $x^2+5x^3$
**20.** $a+ab+a^2$
**21.** $x^2-5x^4$
**22.** $4a^3-12a^5$
**23.** $5x^2y-5xy^2$
**24.** $7b^3-21b^2$
**25.** $12ab+6a$
**26.** $x^2+2xy$
**27.** $2-20x$
**28.** $64x^2-21x^5$
**29.** $6ax^2-12ax^3$

Can you check the examples in this exercise? Check this one:

$$5x^2-15x^4=5x^2(1-3x).$$

## B. THE CROSS-PRODUCT METHOD OF FACTORING

**Section 107.** In the previous section all the expressions which you factored had a *common factor.* But most expressions which you will need to factor are much more difficult than those; and they do not always contain a common factor.

**Illustrative example. Factor $2x^2 + 5x + 3$, or find the dimensions of a rectangle having this area.**

From what you learned about products, you can see that this expression was very likely made by multiplying two factors together. Also, you can see that the first terms of the two factors must be $2x$ and $x$. To help you to get the correct result, always **write the blank form of the two parentheses** *first,* thus: (  )(  ). Then as you determine each term of each factor, you can write it in the appropriate place. Later you will doubtless be able to do all the work in your head and not have to write out the steps. *Second,* therefore, write the **first terms** in the blank form, thus:

$$2x^2 + 5x + 3 = (2x \quad)(x \quad).$$

*Third,* you have to find the **second terms** of each factor. From your previous work in finding products, you know that the last term of the expression, $+3$, was obtained by multiplying together the second terms of the two factors. Then the second terms must be such that their product is $+3$. Obviously, they are 1 and 3 or 3 and 1. To tell whether the 3 or the 1 belongs in the first factor we have to *try* it, and test or check to see if the middle term will be correct ($+5x$). Trying this out, we have:

$$2x^2 + 5x + 3 = (\overbrace{2x + \underbrace{1)(x}_{x} + 3}^{6x}).$$

Checking, — that is, multiplying the two factors together, — we see that this does *not* give the correct middle term, for $+6x$ and $+x$ are not $5x$. But, we might interchange the 1 and 3. Trying this, we get

$$2x^2+5x+3=(\overbrace{2x+\underbrace{3)(x}_{3x}+1}^{2x}).$$

Multiplying these together shows that our factors are correct, for their product gives the original expression, $2x^2+5x+3$.

Let us try another example:

**Second illustrative example.** **Factor $5x^2-36x+7$.**

First write the blank form thus: (         )(         ).

Next we can tell at once that the first terms of our factors are $5x$ and $x$, giving

$$5x^2-36x+7=(5x\qquad)(x\qquad).$$

Examining the last term of the expression, $+7$, we see that the second terms of the required factors must be 1 and 7 or 7 and 1. Trying out the 1 and 7 gives

$$5x^2-36x+7=(5x+1)(x+7).$$

But the check shows that the sum of the cross-products is $+36x$, whereas it should be $-36x$. This can be corrected by changing the sign of the second terms to $-1$ and $-7$, giving

$$(5x-1)(x-7).$$

---

The result of checking shows these to be the correct factors.

**Section 108. The sum of the cross-products must equal the middle term.** These two explanations have been given to show the importance of getting, as factors, expressions such that the sum of the cross-products will give the middle term of the expression to be factored. It is assumed that you can tell, at a glance, what the first terms *might* be, and what the second terms *might* be by looking at the first and last terms of the expressions which you want to factor.

For example, in factoring

$$6x^2 + 13x + 6,$$

the first terms *might* be $3x$ and $2x$, or $6x$ and $x$; the second terms *might* be 3 and 2, 2 and 3, or 6 and 1. But since the product of the factors must give the original expression, we can tell by trying these various possible combinations that the factors are

$$(\underbrace{3x + \overbrace{2)(2x}^{4x} + 3}_{9x}).$$

No other arrangement of numbers will give the correct middle term, **$+ 13x$**.

From this explanation you should be able to factor the expressions in Exercise 101. Don't be discouraged if you have to try more than once before you succeed. Difficult tasks often require many trials.

## EXERCISE 101

### FACTORING BY THE CROSS-PRODUCT METHOD

Check each example carefully.

(The parentheses are written here to suggest to you how to begin.)

**1.** $x^2 + 5x + 6 = (\qquad)(\qquad)$
**2.** $y^2 + 10y - 21 = (\qquad)(\qquad)$
**3.** $2x^2 + 7x + 5 = (2x + ?)(x + ?)$
**4.** $c^2 + 6c + 9 = (\qquad)(\qquad)$
**5.** $x^2 - 8x + 12 = (\qquad)(\qquad)$
**6.** $5y^2 + 16y + 2 = (5y + ?)(y + ?)$
**7.** $a^2 + 12a + 36 =$
**8.** $5x^2 + 8x + 3 =$
**9.** $x^2 - 2x - 24 =$
**10.** $m^2 - m - 20 =$
**11.** $2b^2 + 13b + 15 =$
**12.** $3x^2 - 13x + 4 =$
**13.** $t^2 - 5t - 40 =$
**14.** $10y^2 + 13y - 3 =$
**15.** $4x^2 + 20x + 25 =$
**16.** $a^2 + a - 72 =$
**17.** $y^2 - 16 =$
**18.** $b^2 - 25 =$
**19.** $15c^2 - 31c + 14 =$
**20.** $21b^2 - b - 2 =$
**21.** $5x^2 - 6x + 1$
**22.** $3x^2 + 4x + 1$
**23.** $2y^2 - y - 28$
**24.** $2a^2 + 7a + 3$
**25.** $x^2 - 11x + 24$
**26.** $y^2 - 10y + 20$
**27.** $a^2 + 6a + 9$
**28.** $y^2 - 8y + 16$
**29.** $p^2 + 10$
**30.** $c^2 + c - 30$
**31.** $3x^2 + 8x + 5$
**32.** $2x^2 - 5x + 3$

**Section 109. Importance of finding the prime factors.** Any algebraic expression that *cannot* be *factored* is *prime.* For example, $3x + 5$ is prime, because there are no integral expressions which can be multiplied together to produce it.

But $9x + 6$ is *not prime* because it can be obtained by multiplying 3 and $x + 2$.

It is important that you should always find *prime factors.* To illustrate:

**First illustrative example.** Factor $3b^2 - 21b + 36$.

By inspection, we see that 3 is a common factor.

Removing it, we have

$$3(b^2 - 7b + 12).$$

Now, unless we remember that *prime* factors should be found, we are likely to leave the example in this incomplete form. The $b^2 - 7b + 12$ can be factored further, however, giving

$$(b - 3)(b - 4).$$

Thus, the original example should be factored as follows:

$$\begin{aligned} 3b^2 - 21b + 36 &= 3(b^2 - 7b + 12) \\ &= 3(b - 4)(b - 3). \end{aligned}$$

---

**Second illustrative example.** Another illustration will make clear the importance of finding *prime factors.* Suppose we wish to factor $2x^2 - 50$. As in the previous example, we *always first look for a common factor.* This gives

$$2(x^2 - 25).$$

Now, again, we are apt to leave the example in this incomplete form, not remembering to see if we can further factor $x^2 - 25$. We see, however, that we can. The complete solution is:

$$\begin{aligned} 2x^2 - 50 &= 2(x^2 - 25) \\ &= 2(x + 5)(x - 5). \end{aligned}$$

---

These explanations are given to help you keep in mind that in all factoring work **there are two absolutely essential steps**; namely,

1. LOOK FOR A COMMON FACTOR.
2. FIND PRIME FACTORS; *I.E.* FACTOR COMPLETELY.

## EXERCISE 102

### PRACTICE IN FACTORING COMPLETELY

**1.** $2a^2 + 14a + 24$
**2.** $5y^2 - 45$
**3.** $st^2 - st - 20s$
**4.** $7a^2 - 14a - 105$
**5.** $3x^2 + 12x + 45$
**6.** $x^2 - 6x + 9$
**7.** $6t^2 - 15t^3$
**8.** $2 - 128t^2$
**9.** $ab^2 - ab - 72a$
**10.** $6x^2 + 13x + 6$
**11.** $3a^2 + a - 2$
**12.** $2a^2 - 5a + 3$
**13.** $7b^2 - 17b - 12$
**14.** $q^2 - 12q - 28$
**15.** $2x^2 - 14x + 24$
**16.** $y^4 - 6y^2 - 16$
**17.** $49c^2 + 70c + 25$
**18.** $5a^2 - 80$
**19.** $x^3 - x^2$
**20.** $6x^2 - 18x^3$
**21.** $3y^2 - 3y - 36$
**22.** $12x^2 + 37x - 10$
**23.** $10x + 25 + x^2$
**24.** $y^2 + 10$
**25.** $x^2 + 36$
**26.** $2a^2 - 8$
**27.** $3b^3 + 27$
**28.** $3x^2 - 12x - 180$
**29.** $2x^2 + 10x - 168$
**30.** $x^2 - x - 110$
**31.** $2y^2 - y - 1$
**32.** $6a^2 - 4a - 2$
**33.** $3x^2 + 4x + 1$
**34.** $20x^2 + 70x + 60$
**35.** $2b^2 - b - 3$
**36.** $2a^2 + 18$
**37.** $5y - 15y^2$
**38.** $a^2 - b^2$
**39.** $3y^2 - y - 10$
**40.** $2x^2 + 3x - 9$
**41.** $6y^2 + y - 15$
**42.** $8x^2 - 2$
**43.** $18t^2 - 50$
**44.** $9x^2 + 17x - 2$
**45.** $6a^2 - a - 12$
**46.** $7y^2 - 9y - 10$
**47.** $2x^2 - 36x + 64$
**48.** $y^2 - 3y - 4$
**49.** $p^2 - 16$
**50.** $p^2 + 16$

## SUMMARY

**This chapter has taught the following methods:**

1. **A short method of multiplying which we call the "cross-product" method.**
2. **How to find the *factors* of an algebraic expression.**

## REVIEW EXERCISE 103

1. What is the area of a square formed by adding 4 ft. to the sides of a square $x$ ft. long?
2. What does $(x-4)(x+6)$ represent, if $x$ represents the side of a square?
3. A rectangular field $5y$ rods long has a perimeter of $24y$ rods. What expression will represent the area of the field in square rods? in acres?
4. If the quotient is represented by $q$, the divisor by $d$, and the remainder by $r$, what will represent the dividend?
5. If a park is $w$ rods wide and 1 rod long, how many miles would you walk in going around it $n$ times?
6. How do you divide a product of several factors by a number? For example, in dividing $12 \cdot 3 \cdot 6$ by 2, would you divide each factor by 2?
7. How do you multiply a product of several factors by a number? Give an illustration.
8. The product of four factors is 60. Three of them are 2, 3, and 5. Find the fourth factor.
9. How much do you increase the area of a square whose side is $x$, if you increase its side 4 units?

**10.** Translate into words: $(f+s)^2=f^2+2fs+s^2$.

**11.** Solve for $x$, explaining each step:

$$-4x+6=2x-18.$$

**12.** In what way is *factoring* like *division?* How is it like multiplication?

**13.** If an automobile uses $8\frac{1}{2}$ gallons of gas in going 120 miles, how many gallons will it use in going 250 miles?

**14.** Solve: $\begin{cases} 6y-x=7+4y, \\ 5x+8y=1. \end{cases}$

**15.** Make up five examples for the class to factor, and then give them to the class to work.

**16.** How many terms do you get when you square the *sum* of *two numbers*, *e.g.* $(2x+3y)^2$? when you square the *difference* of *two numbers*, *e.g.* $(4a-3b)^2$?

**17.** Evaluate $(a+b)^3$ if $a=-3$ and $b=+1$.

**18.** Does $(a+b)^2=a^2+b^2$? Show by using 4 for $a$ and 5 for $b$.

**19.** If $\frac{V_1}{V_2}=\frac{p_2}{p_1}$, what is the value of $V_1$ when $V_2=40$, $p_1=8$, and $p_2=12$?

**20.** A tree stands on a bluff on the opposite side of a river from the observer. Its foot is at an elevation of 45° and its top at 60°. Which has the greater height, the bluff or the tree? What measurement would you have to make to find the height of the tree? the width of the river?

**21.** Translate: $a^2-b^2=(a+b)(a-b)$.

# CHAPTER XIV

## HOW TO SOLVE EQUATIONS OF THE SECOND DEGREE

**Section 110. What are quadratic equations?** In all the previous chapters you have solved equations of the *first degree;* that is, equations in which the *unknown* (or *unknowns*) did not have exponents greater than 1. This chapter will show how to solve equations of the *second degree,* equations in which the unknown occurs to the *second power.* To illustrate, you will learn how to solve equations such as

$$x^2 + 8x = 20.$$

The fact that the unknown, $x$, in this equation occurs in the second power or second degree (as, $x^2$) leads us to speak of the equation as a SECOND-DEGREE, or QUADRATIC, EQUATION.

*Three* ways of solving second-degree or quadratic equations will be explained. These, in the order in which we shall discuss them, are:

**I. Solution by graphical representation.**
**II. Solution by factoring.**
**III. Solution by completing the square.**

Before we take up the first method, it will be necessary to study how to represent graphically algebraic **expressions** of the second degree **which are not equations.**

### A. HOW TO REPRESENT GRAPHICALLY AN **EXPRESSION** OF THE SECOND DEGREE IN **ONE** UNKNOWN

**Section 111. The value of an expression depends upon what value is assigned to $x$.** Let us consider the second degree or quadratic *expression*

$$x^2 - 8x + 12.$$

Has this expression a definite numerical value? What is it? 10? 18? If not, what? Evidently, we cannot tell what the *value of the expression* is unless we know what the *value of $x$ is.* That is, the numerical value of the expression *depends upon what value is assigned to $x$.* In other words, as $x$ changes, the value of the expression $x^2-8x+12$ changes. Thus, we are dealing here with *two variable quantities*; **$x$ itself is one variable, and the value of the expression $x^2-8x+12$ is the other one.**

**Section 112. (1) Tabulating values of the two variables.** Thus, to represent an algebraic *expression* of the second degree graphically we first have to determine what numerical value the expression has for various *assigned* values of $x$. That is, we will let $x$ be 0, say, and we find by *evaluating* $x^2-8x+12$ when $x=0$, that the expression is 12. In the same way, when $x=1$, the expression is 5; when $x=2$, $x^2-8x+12$ is 0, etc.

We have already learned that to represent the way in which two related quantities change together, it is best to **tabulate first** the values of the two variables. For the present example we get the following table:

TABLE 15

| If $x$ is | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | -1 | -2 |
|---|---|---|---|---|---|---|---|---|---|---|
| then $x^2-8x+12$ is | 12 | 5 | 0 | -3 | -4 | -3 | 0 | 5 | 21 | 32 |

etc.

**Section 113. (2) Plotting the graph of the two variables.** With the table of values of the two variables [(1) the unknown, (2) the expression which contains it] once made, we can plot the points representing these pairs of values. Let us agree for convenience to plot the *unknown, say $x$,* on the horizontal axis and the *expression* (in this case, $x^2 - 8x + 12$), on the vertical axis. Reading the pairs of values 0, 12; 1, 5; 2, 0; etc., from the table, we obtain as the graphic representation of $x^2 - 8x + 12$ the curve of Fig. 109.

Note that from the graph we can find the value of $x^2 - 8x + 12$ for any value of $x$ represented on the $x$-axis. For example, if $x$ is $2\frac{1}{2}$, then $x^2 - 8x + 12$ is $-3\frac{3}{4}$; if $x$ is 0, then $x^2 - 8x + 12$ is 12, etc.

Note also from the curve that $x^2 - 8x + 12$ is *zero* for two particular values of $x$; namely, $x = +2$ and $x = +6$, *i.e. the curve cuts the x-axis at +2 and +6.*

The graph shown in Fig. 109 is called a graph of an algebraic expression of the second degree, or a *quadratic* expression.

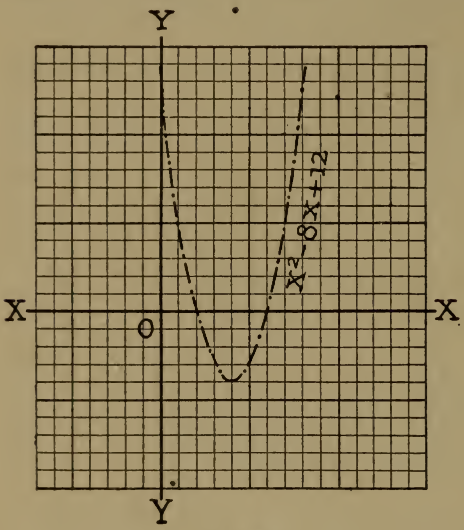

FIG. 109

**Section 114. Graphs of second-degree equations are always curved lines.** Turn back to Figs. 14, 16, 17, etc. What difference do you notice between each of the graphs in these figures and the one in Fig. 109? If you will turn to Fig. 110, you will check your conclusion. One new fact of importance is, therefore, that the graphs of expressions of the first degree are *always straight* lines, whereas graphs of expressions of the second degree are *curved lines.*

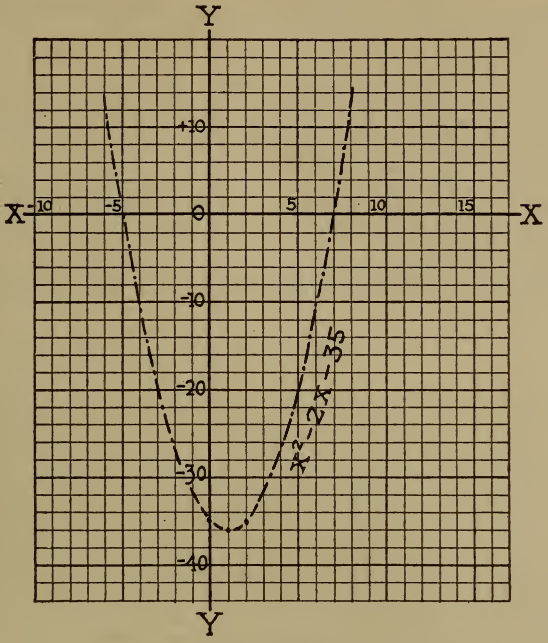

FIG. 110

*Graphs of such quadratic expressions are U-shaped and are commonly called parabolas.*

Figure 110 is a graphical representation of the quadratic expression $x^2 - 2x - 35$. By referring to it, you will be able to answer the following questions:

## EXERCISE 104

1. How many different values may the expression have?
2. What is the value of $x$ when the value of the expression is zero?
3. For what value or values of $x$ is the expression equal to $-10$?

4. What is the lowest or least value of the expression?
5. How were the points which fix the position of this curve located? Are two points enough to determine the graph?
6. What is measured along the horizontal axis? the vertical axis?
7. What variables are represented in this graph?

## EXERCISE 105

### PRACTICE IN CONSTRUCTING GRAPHS OF QUADRATIC EXPRESSIONS

In each case, determine from the graph what values of $x$, if any, will make the expression equal to zero:

1. $x^2 - 6x + 8$
2. $x^2 - 4x + 3$
3. $x^2 + 7x + 10$
4. $x^2 + x - 12$
5. $x^2 - 2x - 15$
6. $x^2 - 9$
7. $x^2 - 8x + 16$
8. $x^2 + 4x + 2$

Now that we have seen how to graph a quadratic *EXPRESSION*, we are ready to consider the *first method of solving quadratic equations:*

## B. HOW TO SOLVE QUADRATIC EQUATIONS

### I. GRAPHICAL SOLUTION

**Section 115. To solve a quadratic equation is to find values of $x$ which will make the expression equal to zero.** In the previous section we graphed quadratic **EXPRESSIONS.** It is important to note that they were *not* quadratic **EQUATIONS.** Thus, $x^2 - 8x + 12$ is a quadratic *expression,* — that is, an algebraic expression of the second degree in one unknown, — but it is *not* an *equa-*

*tion.* If, however, we should write $x^2 - 8x + 12 = 0$, then we have a *quadratic equation,* — an equation of the second degree. To solve this equation is to find the *value* or *values* of $x$ which will make $x^2 - 8x + 12$ equal to zero, **because such values will make one member of the equation equal to the other; *i.e.* 0 = 0.** Graphically, this can be done by plotting the values of the expression $x^2 - 8x + 12$, as we did in the previous section. Thus, to graph the quadratic EQUATION $x^2 - 8x + 12 = 0$, we graph the EXPRESSION $x^2 - 8x + 12$, and *note from the graph what values of $x$ will make the expression equal to zero.*

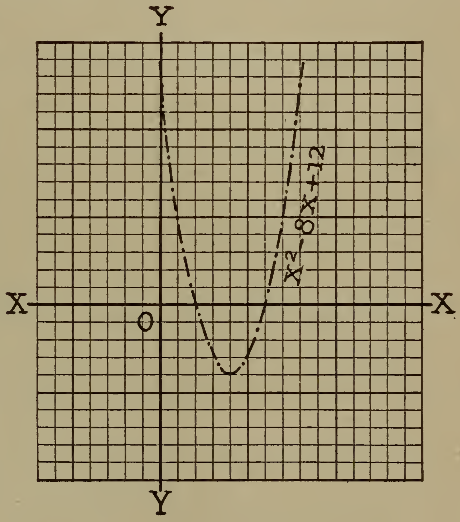

FIG. 111

Thus, from the graph we see that $x^2 - 8x + 12$ is zero when $x = 2$ or when $x = 6$. Thus, we see that $x$ can have *two* values in the *quadratic equation*

$$x^2 - 8x + 12 = 0.$$

Checking, we find that **2 and 6 each satisfies the equation.**

This suggests the following method for solving a quadratic equation graphically:

1. Graph the quadratic *expression* which forms one member of the equation. (The other member should be zero.)

2. From the graph find for what values of $x$ the expression is zero. In other words, *find at what value of $x$ the graph cuts the $x$-axis.* These values are the values of $x$ which will *satisfy* the equation.
3. Check your result by substituting these values of the unknown in the original equation to see if they do *satisfy* it.

## EXERCISE 106

### PRACTICE IN SOLVING QUADRATIC EQUATIONS GRAPHICALLY

Solve the following quadratic equations by drawing graphs of the expressions:

1. $x^2 - 5x - 14 = 0$
2. $x^2 + 3x = 40$
3. $y^2 - y = 20$
4. $2x = 48 - x^2$
5. $x^2 - 6x + 9 = 0$
6. $2y^2 + 5y + 3 = 0$
7. $x^2 + 8x + 16 = 0$
8. $y^2 + 3y + 1 = 0$
9. $(x - 2)^2 + 6x = 12$
10. $x^2 + 4 = 0$

---

In these examples, did you find any graph that did *not* cut the $x$-axis in two places? What conclusion would you draw if the graph *just touched* the $x$-axis? if it did *not* even touch it?

What seem to be the disadvantages of the graphical method of solving quadratic equations? the advantages?

## II. HOW TO SOLVE QUADRATIC EQUATIONS BY FACTORING THE EXPRESSION

**Section 116.** By making use of a principle that you already know, — namely, *that* **the product of any number of factors is zero if one of them is zero,** — we have a very easy method of solving quadratic equations. Recall that $4 \cdot 5 \cdot 0 = 0$, or that $6 \cdot 0 \cdot 26 = 0$ or that $a \cdot b \cdot c = 0$ if either $a$, or $b$, or $c$ is 0. Why? Because any number multiplied by 0 is 0. Under what conditions is the product, $xyz$, zero? Evidently, if *either* $x$, $y$, or $z$ is zero; that is, if *one of the factors is* 0. In the same way, the expression $(x-4)(x+6)$, which is the *product* of *two factors*, could be 0 if $x-4$ were 0, *or* if $x+6$ were 0. Now let us apply this to the solution of a quadratic equation.

**Illustrative example.** The square of a certain number, increased by twice the number, gives as a result 48. What is the number?

Translating into algebraic language, we have the quadratic equation

$$x^2 + 2x = 48. \qquad (1)$$

In order to solve the equation we want to get a *product equal to zero.* Hence the equation $x^2 + 2x = 48$ should read,

$$x^2 + 2x - 48 = 0. \qquad (2)$$

Factoring the left member to form a product,

$$(x+8)(x-6) = 0. \qquad (3)$$

Note here that we have a *product equal to* 0. *But, in order that a product can be* 0, *one of the factors must be* 0.

Therefore, if $x + 8 = 0$, $x$ must equal $-8$, or if $x - 6$ is to be 0, $x$ must be 6.

*Summarizing the solution,* we have the following steps:

$$x^2 + 2x = 48. \qquad (1)$$

Making one side 0, $x^2 + 2x - 48 = 0$. (2)

Forming a product, $(x+8)(x-6) = 0$. (3)

Making each factor 0, $x + 8 = 0$, or $x = -8$. (4)

$x - 6 = 0$, or $x = 6$. (5)

Checking, by substituting 6 and $-8$ in equation (1),

$$64 - 16 = 48,$$

or

$$36 + 12 = 48.$$

### EXERCISE 107

Solve the following quadratic equations by factoring. *Check* each one.

1. $x^2 - 5x + 6 = 0$
2. $y^2 - y = 20$
3. $a^2 + 9a = 22$
4. $b^2 - 36 = 0$
5. $2x^2 - 5x + 3 = 0$
6. $c^2 - 8c = -16$
7. $m^2 = m + 2$
8. $t^2 + t = 56$
9. $\frac{1}{2}x^2 + x = 12$
10. $y^2 + \frac{5y}{2} = 6$
11. $a^2 + a = 20$
12. $\frac{b^2}{2} + \frac{b}{4} = 9$
13. $c^2 + 2c = 0$
14. $x^3 - 2x^2 = x$
15. $12 = x^2 + x$
16. The altitude of a triangle exceeds its base by 4 inches; the area is 96 square inches. Find the base and altitude.
17. Find two consecutive even numbers whose product is 120.
18. The sum of two numbers is 10; the sum of their squares is 52. Find each number.

---

Which of the two methods of solving a quadratic equation that we have considered thus far do you think is better? State the reasons for your choice. Could you solve ANY quadratic equation by the *Factoring* Method? Solve $x^2 + 3x = 12$ by this method.

---

**Before we can consider the third method of solving quadratic equations, — namely, solution by " completing the square," — it will be necessary to learn how to find the *square root* of algebraic expressions.**

## SQUARE ROOT

**Section 117. The need for square root.** In arithmetic you learned how to find a side of a square when its area was known. For example, if you knew that the area of a square was 81 square inches, you learned to find one side of the square by finding *one* of the *two equal factors* of 81, or, in other words, by finding the *square root* of 81. Just now we cannot learn the best method of solving quadratic equations until we learn how to find the *square root* of algebraic expressions and of arithmetic numbers.

**Section 118. What is square root?** The square root of a number is one of the *two equal factors* of that number. Let us illustrate with the number 100. We might *factor* it in several ways, as follows:

$$4 \times 25 = 100$$
$$5 \times 20 = 100$$
$$10 \times 10 = 100$$
$$8 \times 12.5 = 100, \text{ etc.}$$

Remembering that the *square root* of a number is *one* of its *two* EQUAL *factors*, we see that neither 4 nor 25 is the square root of 100, because they are not *equal* factors. Evidently, 10 must be the square root. In the same way, $-10$ is also a square root of 100 because it is *one* of *two equal factors* of 100.

In the same way the cube root of a number is *one* of the *three equal factors* of the number. Thus, $-2$ is the cube root of $-8$ because $(-2)(-2)(-2) = -8$.

**Section 119. How to indicate the root of a number.** It has been agreed to indicate the root of a number by the symbol $\sqrt{\quad}$, called a *radical sign*. To designate *what* root is meant, a small number called an INDEX, is placed in the radical sign. Thus, $\sqrt[4]{16}$ means the fourth root of 16, *i.e.* 2, *because* $2 \cdot 2 \cdot 2 \cdot 2 = 16$; $\sqrt[3]{27}$ means the cube root of 27, *i.e.* 3, because $3 \cdot 3 \cdot 3 = 27$; and $\sqrt[2]{25}$ means the square root of 25. It is customary, however, to omit the index when *square* root is meant. Thus, $\sqrt{25}$, without an index, is always understood to mean the *square* root of 25.

### EXERCISE 108

**FIND, BY TRIAL, THE ROOTS, WHICH ARE INDICATED, OF THE FOLLOWING EXPRESSIONS**

1. $\sqrt{9}$
2. $\sqrt{x^6}$
3. $\sqrt{25\,a^2}$
4. $\sqrt[3]{8}$
5. $\sqrt[3]{27\,x^3}$
6. $\sqrt[3]{b^{12}}$
7. $\sqrt{64\,x^8}$
8. $\sqrt{100\,y^6}$
9. $\sqrt{\frac{1}{4}}$
10. $\sqrt{\frac{9}{64}}$
11. $\sqrt{\frac{36\,x^2}{49\,y^2}}$
12. $\sqrt[3]{\frac{8}{27}}$
13. $\sqrt{144\,a^{10}}$
14. $\sqrt{400\,x^2y^4}$
15. $\sqrt[3]{125\,b^6c^9}$
16. $\sqrt[4]{16\,x^8}$
17. $\sqrt[5]{243\,y^{10}}$
18. $\sqrt[4]{256\,x^{12}}$

**Section 120. How to find the square root of algebraic expressions.** We have seen that $(a + b)^2$ or $(a + b)(a + b)$ $= a^2 + 2\,ab + b^2$. From this *it is evident* that the square root of $a^2 + 2\,ab + b^2$ must be $a + b$, or $\sqrt{a^2 + 2\,ab + b^2}$ $= (a + b)$. In the same way $\sqrt{x^2 + 10\,x + 25}$ is $x + 5$, because $(x + 5)(x + 5)$ gives $x^2 + 10\,x + 25$. From these illustrations we see that it is possible to extract the square root of an algebraic expression if we can show that it can be obtained by *squaring* some other expression; that is, if we can show that it is the product of *two equal factors*.

### EXERCISE 109

Find the square root of the following expressions, where it is possible to do so. Check each.

1. $x^2 + 2xy + y^2$
2. $a^2 + 6a + 9$
3. $b^2 - 4b + 4$
4. $t^2 - 10t + 25$
5. $4a^2 + 12a + 9$
6. $16 + x^2 + 8x$
7. $1 + 21x + 100x^2$
8. $y^2 + 6y + 20$
9. $25a^2 + 40a + 16$
10. $x^2 + 16$
11. $y^2 - 49$
12. $12x + 36 + x^2$
13. $t^2 + u^2 + 2tu$
14. $r^2s^4 - 6rs^2 + 9$

If any of the expressions above are not perfect squares, make the necessary changes to transform them into perfect squares.

## HOW TO FIND THE SQUARE ROOT OF ARITHMETICAL NUMBERS

**Section 121.** We have learned that the square root of a number is *one* of its *two equal factors*. Hence, to find the square root, we need to find one of its two equal factors.

If a number is divided by its square root, the quotient will be the same as the divisor. Thus, if 36 is divided by its square root, 6, the quotient will be the same as the divisor, 6. But if 36 is divided by a number which is *smaller* than its square root, the quotient will be *larger* than its square root. Thus, if 36 is divided by 4, the quotient is 9. The square root is somewhere between the divisor, 4, and the quotient, 9. If the divisor and quotient are the same, either one is the square root of the number, but if they are not the same, then the square root is some number between them. The next exercise will illustrate *this* TRIAL method of finding square root.

Find the square root of each of the following numbers by the **trial method**:

**First illustrative example.** Find the square root of 55. Since this number is between the two perfect squares, 49 and 64, its square root will be between 7 and 8; let us try 7.5 and use it as a divisor (one factor) to find the quotient (the other factor).

$$
\begin{array}{r}
7.333 \\
7.50\overline{)55.0} \\
52\ 5 \\
2\ 50 \\
\underline{2\ 25} \\
250 \\
\underline{225} \\
250
\end{array}
$$

This shows that the square root of 55 is between 7.500 and 7.333. Let us *try* a number halfway between them, say 7.416.

$$
\begin{array}{r}
7.4164 \\
7.4160\overline{)55.000} \\
\underline{51\ 912} \\
3\ 0880 \\
\underline{2\ 9664} \\
12160 \\
\underline{7416} \\
47440 \\
\underline{44496} \\
29540 \\
\underline{29664}
\end{array}
$$

The square root, then, of 55 is 7.4162 This can be checked by multiplication or by division.

**Second illustrative example.** Find the square root of $12\frac{1}{2}$. Since the number is between the two squares, 9 and 16, its square root will be between 3 and 4; let us try 3.5 and use it as a divisor.

$$\begin{array}{r}
3.57\\
3.5\overline{)12.5}\\
\underline{10\ 5}\\
2\ 00\\
\underline{1\ 75}\\
250\\
\underline{245}\\
5
\end{array}$$

This shows that the square root of 12.5 is between 3.50 and 3.57. A value closer than either of these is the number halfway between them, say 3.535.

$$\begin{array}{r}
3.536\\
3.535\overline{)12.5}\\
\underline{10\ 605}\\
1\ 8950\\
\underline{1\ 7675}\\
12750\\
\underline{10605}\\
21450\\
\underline{21210}
\end{array}$$

This shows that the square root is between the two factors 3.535 and 3.536. Using the number halfway between them, 3.5355, we have the square root correct to 4 decimals.

## EXERCISE 110

By using this method, find the square root of each of the following numbers:

1. 18
2. 52
3. 200
4. 500
5. 16.80
6. 150
7. 965
8. 3820
9. .2640
10. 14.75
11. 2025
12. 8
13. 3
14. 2
15. 10

**Section 122. The old method of finding square root.**[1] Many pupils have learned in arithmetic another method of finding the square root of numbers. It is illustrated in the following example:

**Illustrative example.** Find the square root of 200.

Note the following steps:
(1) The number is separated into periods of two figures each, counting from the decimal point.
(2) You find the greatest square in the left-hand period, and write its square root for the first figure of the root.
(3) Subtract this square from the left-hand period, and with the remainder place the next period for a new dividend. (This is 100 in the example.)
(4) Double the part of the root already found ($2 \times 1 = 2$) for your trial divisor. Divide the dividend, exclusive of the right-hand figure (10) by the trial divisor, 2. Write the quotient obtained, 4, as the next figure of the root and the divisor. Multiply the complete divisor, 24, by the last term of the root, 4. Subtract the product, 96, from the dividend, 100. To the remainder, 4, annex the next period, 00 for a new dividend. Repeat this process until all periods are used, or until any required degree of accuracy is obtained.

$$\begin{array}{r}
200.0000)\underline{14.14} \\
\underline{1}\phantom{00000000000} \\
24)\overline{100}\phantom{000000} \\
\underline{96}\phantom{000000} \\
281)\overline{400}\phantom{0000} \\
\underline{281}\phantom{0000} \\
11900\phantom{00} \\
2824)\underline{11296}\phantom{00} \\
604\phantom{00}
\end{array}$$

## EXERCISE 111

Solve the examples of Exercise 110 by this method.

---

[1] It is believed that this traditional method is much more difficult to *rationalize* for the pupil than the so-called *trial* or *estimate* method. For those teachers who insist upon its use, the *trial* method may be omitted. In the interest of experimentation, however, the authors hope the proposed method will be fairly tested out.

## III. QUADRATIC EQUATIONS SOLVED BY THE MOST GENERAL METHOD: COMPLETING THE SQUARE

**Section 123. The third method of solving quadratic equations.** Only a few easy quadratics can be solved by the second method which we studied, *i.e. only those which can be factored.* Furthermore, the first method shows that graphical solutions are too slow and often give only approximate results. Consequently we need a more general method — *one that is applicable to all quadratics, and one that gives accurate results.* Let us illustrate such a general method of solving quadratic equations.

**First illustrative example.** The area of a rectangle, which is 6 inches longer than it is wide, is 55 square inches. Find its dimensions.

Translating into algebraic form, we have

$$x^2 + 6x = 55. \qquad (1)$$

Adding 9 to each side to make the left side a perfect square,

$$x^2 + 6x + 9 = 64. \qquad (2)$$

Extracting the square root of each side,

$$x + 3 = +8 \text{ or } -8. \qquad (3)$$

Using the $+8$, $x = 5.$ (4)

Using the $-8$, $x = -11.$ (5)

Checking the result: etc.

**Section 124. We must make the left side a perfect square.** It is *important* to see why 9 was added to each side of equation (1). Why not add 10 or 20 or any other number, to each side? *Because the left side would not be a perfect square* if any other number were added. Success with the method of *completing the square* depends upon knowing what number to add to each side to make the left side a perfect square. Note that the left side *must* be a perfect square, *because you cannot extract the square root of an algebraic expression which is not a perfect square.*

**Second illustrative example.** Find a number such that its square decreased by 3 times itself shall be 10.

Translating into algebraic form gives:

$$x^2 - 3x = 10. \qquad (1)$$

Adding $\frac{9}{4}$ *to each side* in order to make the left side a perfect square,

$$x^2 - 3x + \tfrac{9}{4} = 10 + \tfrac{9}{4} = \tfrac{49}{4}. \qquad (2)$$

Extracting the square root of each side,

$$x - \tfrac{3}{2} = \tfrac{7}{2} \text{ or } -\tfrac{7}{2}. \qquad (3)$$

Using $+\frac{7}{2}$, $\quad x = \frac{10}{2}$ or 5. $\qquad (4)$

Using $-\frac{7}{2}$, $\quad x = -\frac{4}{2}$ or $-2$. $\qquad (5)$

Check: etc.

∴ The unknown number is either $+5$ or $-2$.

---

Here again, as in the first illustrative example, the *most important and most difficult step is completing the square, i.e.* to know what to add to $x^2 - 3x$ to *make it a perfect square.* We need to know why $\frac{9}{4}$, rather than some other number, was added to each side. Remember that the algebraic expression, the left side of the equation, *must* be a perfect square, for otherwise we could not extract the square root of it. But we can extract the square root of the right side (because it is an arithmetical number) even if it is not a perfect square.

**Section 125. How to complete the square of any quadratic expression: add the square of one half the coefficient of $x$.** Because we use the method so frequently, it will be worth while to learn the general method of completing the square.

If we square $x + \frac{p}{2}$, we get the expression

$$x^2 + px + \frac{p^2}{4},$$

that is,

$$\left(x + \frac{p}{2}\right)^2 = x^2 + px + \frac{p^2}{4}.$$

Note here that the last term of the expression is $\frac{p^2}{4}$. It is the square of $\frac{p}{2}$. But $p$ represents the coefficient of $x$ in the expression which we wish to change into a perfect square. Thus, if we had the expression

$$x^2+px$$

and desired to change it into a perfect square, *i.e.* to complete the square, we should have to add $\frac{p^2}{4}$; **that is, add the square of one half the coefficient of $x$.**

To complete the square in the expression $x^2+6x$ (see first illustrative example), we should add the square of one half the coefficient of $x$; that is, the square of $\frac{1}{2}$ of 6, or 9. In the same way, to complete the square in the expression $x^2-3x$, we must *add the square of one half the coefficient of* $x$; that is, the square of $\frac{1}{2}$ of 3 or $(\frac{3}{2})^2$, which is $\frac{9}{4}$.

## EXERCISE 112

Solve each of the following quadratic equations by the method of completing the square:

1. $x^2+6x=40$
2. $x^2-8x=84$
3. $y^2+2y=15$
4. $p^2-10p=-16$
5. $t^2+3t=10.$
6. $y^2+7y=8$
7. $x^2+x=56$
8. $c^2+6c=11$
9. $p^2-4p=36$
10. $10x+x^2=-9$
11. A rectangular field is 2 rods longer than it is wide, and it contains 6 acres. Find the length of the sides.
12. Find two consecutive even numbers whose product is 80.

**13.** Find the value of $x$ in the equation

**Illustrative example.**

$$3x^2 - 13x + 4 = 0. \qquad (1)$$

**Solution : The equation must be in the form**

$$x^2 + px = n.$$

**In other words, the coefficient of $x^2$ must be 1.**

**Dividing each side of (1) by 3 gives**

$$x^2 - \tfrac{13}{3}x = -\tfrac{4}{3}. \qquad (2)$$

**Adding $(\frac{13}{6})^2$ to each side gives**

$$x^2 - \tfrac{13}{3}x + \tfrac{169}{36} = \tfrac{169}{36} - \tfrac{4}{3} = \tfrac{121}{36}. \qquad (3)$$

**Extracting the square root of each side gives**

$$x - \tfrac{13}{6} = \tfrac{11}{6} \text{ or } -\tfrac{11}{6}. \qquad (4)$$

**Using $+\frac{11}{6}$,** $\qquad x = \frac{13}{6} + \frac{11}{6} = \frac{24}{6} = 4.$

**Using $-\frac{11}{6}$,** $\qquad x = \frac{13}{6} - \frac{11}{6} = \frac{2}{6} = \frac{1}{3}.$

**Substituting $\frac{1}{3}$ for $x$ in equation (1), to check gives**

$$3 \cdot \tfrac{1}{9} - 13 \cdot \tfrac{1}{3} + 4 = 0.$$

$$\tfrac{1}{3} - \tfrac{13}{3} + \tfrac{13}{3} = 0.$$

$$\tfrac{13}{3} - \tfrac{13}{3} = 0.$$

---

**The pupil should check for $x = 4$.**

**14.** $2x^2 + 10x = 72$

**15.** $3a^2 + 6a = 45$

**16.** $\dfrac{x^2}{2} + \dfrac{x}{4} = 9$

HINT: Get rid of fractions.

**17.** $y^2 - 40 = 8y$

**18.** $\dfrac{2x^2}{3} - x = 3$

**19.** $\dfrac{x}{2} - \dfrac{3}{x} = \dfrac{47}{10}$

**20.** $\dfrac{x^2}{2} + \dfrac{3x}{5} = \dfrac{31}{2}$

**21.** $3y^2 + 5y = 22$

**22.** $5b^2 + 16b + 3 = 0$

**23.** The difference of two numbers is 4, and the sum of their squares is 210. Find the number.

**24.** A farmer has a square wheatfield containing 10 acres. In harvesting the wheat, he cuts a strip of uniform width around the field. How wide a strip must be cut in order to have the wheat half cut?

**25.** Divide 20 into two parts whose product is 96.

**26.** The sum of two numbers is 20, and the sum of their squares is 208. Find the numbers.

**27.** I went to the grocery for oranges. The clerk said they had advanced 10 cents per dozen. I got $\frac{1}{2}$ dozen fewer oranges for a dollar. What was the original price per dozen?

**28.** A piece of tin in the form of a square is taken to make an open-top box. The box is made by cutting out a 3-inch square from each corner of the piece of tin and folding up the sides. Find the length of the side of the original piece of tin if the box contains 243 cubic inches.

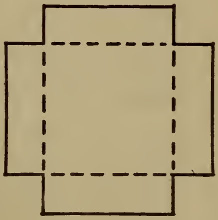

FIG. 112

**29.** A rectangular park 56 rods long and 16 rods wide is surrounded by a boulevard of uniform width. Find the width of this street if it contains 4 acres.

**30.** The members of a high-school class agreed to pay $8 for a sleigh ride. As 4 were obliged to be absent, the cost for each of the rest was 10 cents greater than it otherwise would have been. How many intended to go on the sleigh ride?

**31.** Solve by all three methods the following quadratic equation:

$$2x^2 + 5x = 18.$$

### REVIEW EXERCISE 113

1. The length of a 10-acre field is 4 times its width. What are its dimensions?

2. Does $x = -\frac{3}{4}$ satisfy the equation $4x^2 + 11x = -6$?

3. The space passed over by a body falling $t$ seconds is expressed by the formula $S = 16t^2$, where $S$ is the number of feet the body falls. Construct a graph for this formula, using for $t$ the values 0, $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, and 3. Plot the values of $t$ along the horizontal axis.

4. Does $\sqrt{9} + \sqrt{16} = \sqrt{25}$? Does the *sum* of the square roots of two numbers equal the square root of the sum of the numbers? Does $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$?

5. Is the square of a number always larger than the number? Illustrate.

6. Evaluate $\sqrt{s(s-a)(s-b)(s-c)}$ if $s = 20$, $a = 8$, $b = 14$, and $c = 18$.

7. What is the square root of $-25$? of $-x^2$?

8. Does $(x + y)^2 = x^2 + y^2$? Test by using 4 for $x$ and 3 for $y$.

9. If you wanted to divide a product of several factors, such as $6 \cdot 8 \cdot 10 \cdot 12$, by some number such as 2, would you divide each of the factors by 2?

10. Complete the statement: To divide a product by a number, divide ____ of the factors by that number.

**11.** If a square piece of tin 10 inches on each side sells for 60 cents, what should a 15-inch square piece of the same thickness sell for?

**12.** A girl went to the bakery for pies. If she could buy pies 5 inches in diameter for 15 cents each, or 10 inches in diameter (the same thickness) for 35 cents, which would be cheaper for her to buy?

## SUMMARY

**This chapter has taught:**

1. **The meaning of a quadratic equation.**
2. **How to solve quadratic equations by three methods:**
   - (*a*) **By graphical representation.**
   - (*b*) **By factoring.**
   - (*c*) **By completing the square.**
3. **Square root.**

# CHAPTER XV

## FURTHER USE OF THE RIGHT TRIANGLE: HOW TO SOLVE QUADRATIC EQUATIONS WHICH CONTAIN TWO UNKNOWNS

**Section 126. Previous use of the right triangle.** We have already seen how right triangles can be used to find unknown distances. If we knew *one* side, and *one* acute angle, we were able to find any other side. Now we come to another method of dealing with right triangles; namely, *when two sides are known, but when no acute angle is known.* This method will be illustrated by the following problem:

**What is the longest straight line you can draw upon a rectangular blackboard 28 in. wide and 36 in. long ?**

Evidently the longest straight line is the *diagonal* of the blackboard, or the *hypotenuse of the right triangle*, Fig. 113. Thus, we need to know how to find the hypotenuse of a right triangle when the other two sides are known. This leads to the following:

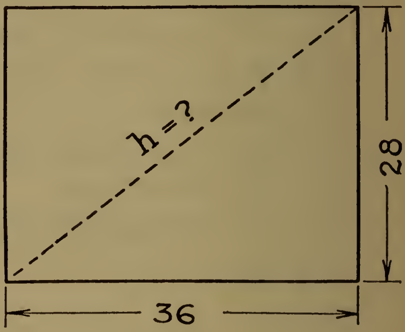

FIG. 113

### THE PYTHAGOREAN THEOREM

**Section 127.** This important relation between the sides of a right triangle was discovered by the celebrated Greek mathematician Pythagoras, after whom it has been named. This theorem or law states that **the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides**, or, in the above problem, that

$$h^2 = 36^2 + 28^2.$$

This relation between the sides of a right triangle can be seen from Fig. 114. The base, $AB$, and altitude, $AC$, of a right triangle are drawn so that they contain a common unit an integral number of times. $AB$ contains the common unit 4 times and $AC$ contains it 3 times. Then by actual measurement $BC$ will contain the same unit 5 times. By constructing squares on the sides of the triangle, you can see by counting that the sum of the squares on $AB$ and $AC$ is equal to the square on $BC$.

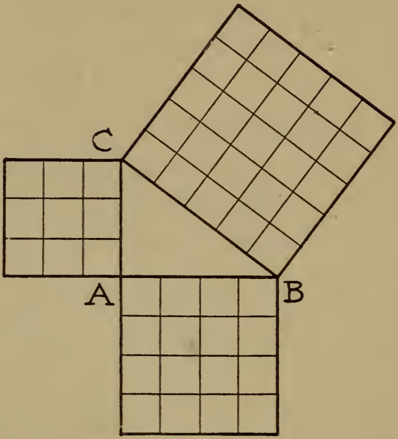

FIG. 114

To test this further, the pupil should construct a right triangle with the base 12 units and the altitude 5 units. Then *actually measure* the hypotenuse, and note whether the square on the hypotenuse is equal to the *sum* of the *squares* of the other two sides. Now we are ready to go back to the problem of finding the longest line that can be drawn upon the blackboard. By making use of the truth which was just studied we have:

$$h^2 = 28^2 + 36^2$$

$$\text{or } h^2 = 784 + 1296$$

$$\text{or } h^2 = 2080$$

$$\text{or } h = \sqrt{2080} = 45.66 \text{ in.}$$

This relation between the sides of a right triangle is more widely used by engineers, carpenters, mechanics, and builders than any other mathematical law. Historical records show that the knowledge of this important relation is as old as civilization itself.

### EXERCISE 114

Problems based on the Pythagorean Theorem.

1. A rectangular schoolroom floor is 32 feet long and 28 feet wide. What is the longest straight line that could be drawn upon the floor?
2. How much walking is saved by cutting diagonally across a rectangular plot of ground which is 25 rods wide and 42 rods long?
3. A tree 100 feet high was broken off by a storm. The top struck the ground 40 feet from the foot of the tree, the broken end remaining on the stump. Find the height of the part standing, assuming the ground to be level. Make a drawing.
4. What is the diagonal of a square whose sides are each 10 in.?
5. Find the side of a square whose diagonal is 20 inches.
6. Two vessels start from the same place, one sailing due northwest at the rate of 12 miles per hour, and the other sailing due southwest at the rate of 16 miles per hour. How far apart are they at the end of 3 hours?
7. The foot of a 36-foot ladder is 13 ft. 6 in. from the wall of a building against which the top is leaning. How high on the wall does the top reach?
8. A rope stretched from the top of a 62-foot pole just reaches the ground 16 feet from the foot of the pole. Find the length of the rope.

**9.** The side of a square room is 21.5 feet. Find its diagonal correct to two decimals.

**10.** What is the perimeter of a square whose diagonal is 12 inches?

**11.** The side of a square is $a$. What represents its area? its perimeter? its diagonal?

**12.** A rectangle is four times as long as it is wide. Find its diagonal if its area is 576 square inches.

**13.** Figure 115 is an equilateral triangle, and $CD$ is perpendicular to $AB$. Find $CD$ if each side of the triangle is 20 inches. Then find the area of triangle $ABC$.

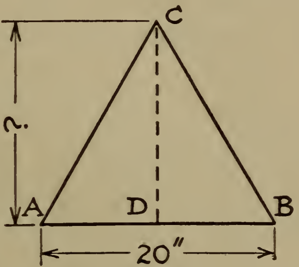

FIG. 115

**14.** The area of a right triangle is 24 square inches. Its base is 6 inches. Find its altitude and hypotenuse.

**15.** The diagonal of a square is $d$. Show that $s$ (side) is $\frac{d}{\sqrt{2}}$.

**16.** $CD$, the altitude of equilateral triangle $ABC$, is 16 inches. Find the sides of the triangle and its area.

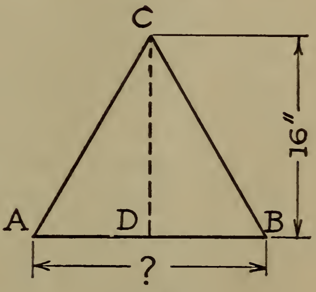

FIG. 116

**17.** How long an umbrella will lie flat down on the bottom of a trunk whose inside dimensions are 27 inches by 39 inches?

**18.** Can a circular wheel 8 feet in diameter be taken into a shop if the shop door is $4\frac{1}{2}$ feet wide and $6\frac{1}{2}$ feet high?

**19.** If $A$ represents the area of a square, what will represent its perimeter? its diagonal?

**20.** The sides of a triangle are 12, 16, and 24 inches. Is it a right triangle? Why?

**21.** If you know two sides of a triangle, can you *always* find its area? Explain.

**22.** The hypotenuse of a right triangle is 10 feet, and one of its sides is 2 feet longer than the other. Find the length of the sides.

**23.** Find the area of a square whose diagonal is 12 inches longer than one of its sides.

**24.** Will an umbrella 30 inches long lie flat down in a suit case whose inside dimensions are 18 by 25 inches?

**25.** A rectangle is 12 by 18 inches. How much must be added to its length to increase its diagonal 4 inches?

**26.** In a right triangle one side is one unit less than twice the other side. The hypotenuse is 17 units. What is the area of the triangle?

**27.** One side of a right triangle is 3 times as long as the other. The hypotenuse contains 30 inches. Find the area of the triangle.

**28.** The dimensions of a certain rectangular blackboard, and the longest line which can be drawn upon it, are represented in feet by three consecutive even numbers. Find the dimensions of the blackboard.

## SOLVING QUADRATIC EQUATIONS WITH TWO UNKNOWNS

**Section 128.** In the last chapter we solved quadratic equations in one unknown. Many quadratic equations, however, contain two, or more, unknowns. For example, consider the equation

$$y = x^2,$$

which states that one number is equal to the square of another number. This equation contains two unknowns, $x$ and $y$, and is at the same time a second-degree or quadratic equation. A graph will help to *show the relation between the variables or unknowns* in this equation.

Tabulating:

TABLE 16

| If x is | 0 | 1 | 2 | 3 | 4 | 5 | -1 | -2 | -3 | -4 |
|---|---|---|---|---|---|---|---|---|---|---|
| then y is | 0 | 1 | 4 | 9 | 16 | 25 | 1 | 4 | 9 | 16 |

These values of $x$ and $y$ are plotted in Fig. 117. It is evident both from the graph and from the table that *there*

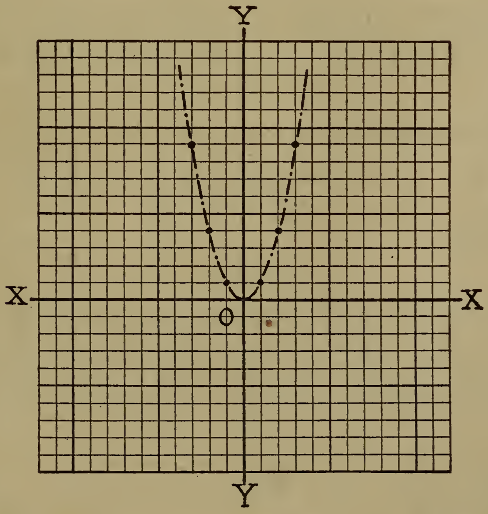

FIG. 117. The line shows relationship between two variables, when one equals the square of the other.

*is an indefinite number of sets of values of $x$ and $y$ which will satisfy the equation.* Note that values of $x$ are plotted along the horizontal axis, and values of $y$ along the vertical axis.

The graph of this equation may be thought of as answering the question which is suggested by the following:

**What numbers are so related that one of them is equal to the square of the other?**

## EXERCISE 115

Graph each of the following quadratic equations:

1. $y = x^2 + 4x$
2. $y = x^2 - 4x$
3. $y = (x + 2)^2$
4. $x = y^2$
5. $x = y^2 + 6y$
6. $x = (y - 3)^2$
7. $xy = 60$
8. $y = x^2 - x - 6$

9. **Illustrative example.** **Graph the equation which states that the sum of the squares of two numbers is 16.**

$$x^2 + y^2 = 16.$$

**Solving the equation for $x$ gives**

$$x^2 = 16 - y^2,$$

**or**

$$x = \pm\sqrt{16 - y^2}.$$

**Tabulating:**

TABLE 17

| If $y$ is | 0 | 1 | 2 | 3 | 4 | 5 | -1 | -2 | -3 | -4 | -5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| then $x$ is | ±4. | ±3.8 | ±3.4 | ±2.6 | 0 | * | ±3.8 | ±3.4 | ±2.6 | 0 | * |

* If $y$ is larger than 4, then $y^2$ is greater than 16 and the expression under the radical becomes a negative number. But we cannot extract the square root of a negative number. We call it *imaginary*.

The graph of this equation is a circle whose radius is 4 units, as in Fig. 118.

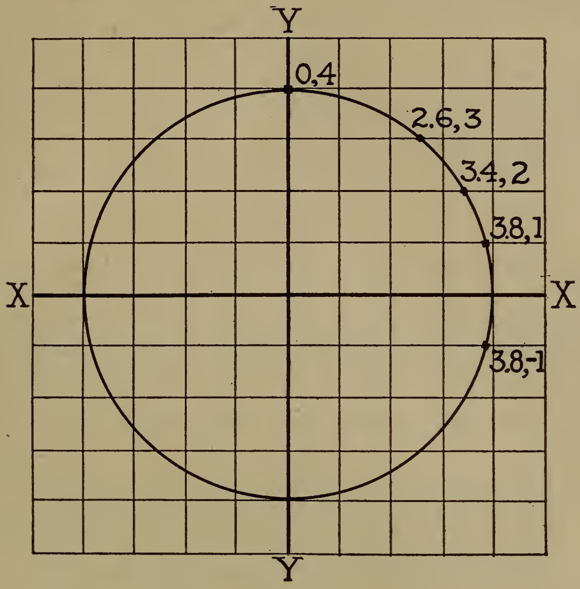

FIG. 118

**10.** $x^2 + y^2 = 25$ **11.** $y^2 + x^2 = 40$

GRAPHICAL SOLUTION OF A PAIR OF EQUATIONS

**Section 129. In solving equations there must always be as many equations as there are unknown quantities.** The examples which you have just graphed were quadratic equations in two unknowns, or two variables. There are *many* sets of values of the unknowns which will satisfy any *one* of these equations, just as there were many sets of values which would satisfy a first-degree equation in two variables, such as $x + y = 10$. To obtain a *single set* of values, *or a limited number of sets of* values which satisfy a quadratic equation in two unknowns, we must have *two* equations. **(There must always be as many different equations as there are unknowns.)**

We shall now consider *two* equations:

**Illustrative example.**

$$\begin{cases} y^2 = 4x + 4. & (1) \\ x + y = 2. & (2) \end{cases}$$

What set of values of $x$ and $y$ will satisfy *both* of these equations? Let us first solve them graphically.

Tabulating equation (1)

$$y^2 = 4x + 4.$$

TABLE 18

| If x is | 0 | 2 | 3 | 8 | -1 | -2 |
|---|---|---|---|---|---|---|
| then y is | ±2. | ±3.4 | ±4. | ±6 | 0 | impossible |

Plotting these points gives the curve shown in Fig. 119.

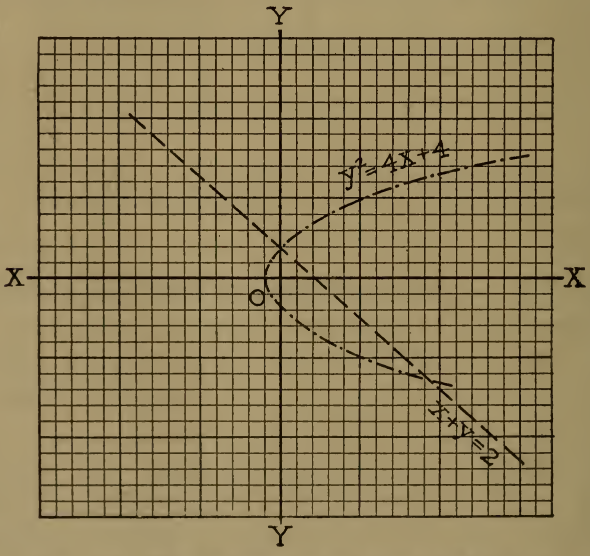

FIG. 119. The $x$-distance and the $y$-distance of the points of intersection of the two lines give the value of $x$ and of $y$ which satisfy the two equations.

Plotting $x + y = 2$ as in Chapter 11 gives the straight line. From Fig. 119 we see that the graphs intersect in two points, (0, 2) and (+ 8, − 6). Hence, there are two sets of values, and only two, which satisfy both equations :

$$\begin{cases} x = 0 \\ y = 2 \end{cases} \text{ and } \begin{cases} x = +8 \\ y = -6 \end{cases}$$

Show by checking that these values do satisfy both equations.

---

The *important step* in solving equations graphically is to **determine the intersection points of their graphs.** If their graphs intersect in only *one* point, then there is only *one* set of values of the unknowns which will satisfy both equations. There will be as many solutions as there are intersection points of their graphs.

### EXERCISE 116

Solve graphically the following pairs of equations :

1. $\begin{cases} x^2 = 4y \\ 2x + y = 12 \end{cases}$

2. $\begin{cases} y^2 + 2 = x \\ x - 3y = 0 \end{cases}$

3. $\begin{cases} x^2 + y^2 = 16 \\ x + y = 11.31 \end{cases}$

4. $\begin{cases} y = x^2 - 4x \\ 2x - y = 5 \end{cases}$

### ALGEBRAIC SOLUTION

**Section 130. The algebraic solution is much easier than the graphic solution.** To illustrate, take the first example in the previous exercise :

**Illustrative example.**

$$\begin{cases} x^2 = 4y, & (1) \\ 2x + y = 12. & (2) \end{cases}$$

From (2), $y = 12 - 2x$. (3)

Substituting $12 - 2x$ for $y$ in (1) gives

$$x^2 = 4(12 + 2x), \quad (4)$$

or $x^2 = 48 - 8x$. (5)

Solving by factoring,

$$x^2 + 8x - 48 = 0. \quad (6)$$

$$(x + 12)(x - 4) = 0.$$

$$x = -12 \text{ or } +4.$$

If $x = -12$, then $y = 36$.

If $x = 4$, then $y = 4$.

The pupil should check each pair of values.

---

## EXERCISE 117

Solve by the algebraic method each of the following pairs of equations. Check each.

1. $\begin{cases} y^2 = 3x \\ x + y = 6 \end{cases}$

2. $\begin{cases} x^2 + 2x = y \\ 2x + y = 12 \end{cases}$

3. $\begin{cases} x^2 + y^2 = 17 \\ y = x - 3 \end{cases}$

4. $\begin{cases} a + b = 6 \\ ab = 5 \end{cases}$

5. $\begin{cases} x^2 + xy = 14 \\ y = 2x + 1 \end{cases}$

6. $\begin{cases} 2ab = -8 \\ a - b = 5 \end{cases}$

7. $\begin{cases} x + y = 2 \\ xy = -15 \end{cases}$

8. $\begin{cases} \frac{x}{2} + \frac{y}{3} = 6 \\ 2x^2 - y^2 = -9 \end{cases}$

9. Find two numbers whose difference is 9 and the sum of whose squares is 221.

10. The area of a rectangular field is 216 square rods, and its perimeter is 60 rods. What are its dimensions?

**11.** The sum of two numbers is $\frac{19}{6}$ and their product is $\frac{5}{3}$. Find the numbers.

**12.** The hypotenuse of a right triangle is 25 feet. Find the other two sides if you know that their sum is 35 feet.

**13.** A piece of wire 30 inches long is bent into the form of a right triangle whose hypotenuse is 13 inches. Find the other sides of the triangle.

**14.** The area and the perimeter of a rectangle are each 25. What are its dimensions?

**15.** A photograph, 8 inches by 10 inches, is enlarged until it covers twice the original area, keeping the ratio of the length to the width unchanged. Find the sides of the enlarged photograph.

**16.** In placing telephone poles between two places, it was found that if the poles were set 10 feet farther apart than originally planned, 4 poles fewer per mile were needed. How far apart were the poles placed at first?

### REVIEW EXERCISE 118

**1.** By substituting any value for $x$ in $x^2 - 1$, $2x$, and $x^2 + 1$, show that the three numbers which result are sides of a right triangle.

**2.** If the sides of a right triangle are 6 inches and 8 inches, then the hypotenuse must be ______?______ inches.

**3.** How can you tell when a triangle is a right triangle without measuring its angle? Is the triangle whose sides are 5, 12, and 13 a right triangle? Why?

4. What is the area of an equilateral triangle each of whose sides is 30 inches?

5. How would you find the side of a square which had the same area as a circle with a radius of 12 inches?

6. Write a formula for $b$ if $a$, $b$, and $c$ are the altitude, base, and hypotenuse of a right triangle; similarly, a formula for $a$.

7. Express the hypotenuse of a right triangle whose altitude exceeds its base by 6 inches.

8. Solve the pair of equations: $\begin{cases} x + y = 12, \\ x^2 + 2y = 39. \end{cases}$

9. When is it impossible to find the square root of a number?

A full and complete index will be supplied for the regular edition of this book. It has been decided to forego an index in the present Experimental Edition.