PLANE AND SOLID

GEOMETRY

ON THE

SUGGESTIVE METHOD

WITH NUMEROUS EXERCISES

AND

A BRIEF COURSE ON LOCI OF EQUATIONS AND ON CONIC SECTIONS.

BY

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CONTENTS.

PREFACE.

This book is based on the "heuristic method". Boys and girls in the second year of the usual high school course can prove geometrical theorems for themselves with suitable aid and encouragement. Few of them can follow a proof written by another, and decide understandingly as to its validity. The active faculties are developed before the critical.

It is intended not merely to be "progressive" in itself, but to be helpful in making the entire course a progressive unit.

It aims to develop the ideas of magnitude already known to the pupil and to give them scientific precision. By his previous course in arithmetic and geography the pupil has become as familiar with the fact that a right angle is one fourth of 360° and one half of 180° as he is with the fact that an inch is one twelfth of a foot. The treatment of angular magnitude (see §§20—28) etc. is based upon these familiar ideas. The usual treatment, though perfectly adapted to those for whom Euclid wrote, confuses pupils as now trained, and gives them the false idea that geometry is a species of intellectual legerdemain. Using obscure ideas in proving facts thoroughly familiar is not pedagogical.

As algebra usually precedes geometry, a knowledge of it is assumed, and it may be thoroughly reviewed by the exercises applying it to geometry.

In selecting the subject matter from the great superabundance of material, those theorems and problems are selected that are the most useful for practical purposes, or for preparing pupils for subsequent studies. So much of the conic sections is introduced as is needed in preparing pupils for high school text-books on physics and astronomy. The treatment of mensuration is much fuller than in most text books, including everything that can profitably be taught without the aid of trigonometry or calculus. In the earlier part of the work considerable attention is paid to "field work". See pages 44—46, 117, 118. Teachers who use this material will probably be able to devise all the practical exercises in measuring solids for which their classes have time. The working plans for an accessible street cut or fill, which could usually be obtained from the city engineer of any large town, might be used to advantage in illustrating the use of the prismatoid. Most cisterns are a close approximation to some solid, and builders would usually permit their measurement at a suitable time.

Most young pupils find solid geometry more interesting than plane geometry. The book is accordingly so arranged that the teacher may either introduce solid geometry much earlier than usual, or give the course in the usual order. This is effected by making all the less important theorems either supplementary or else exercises, making it possible to postpone them to the review, without omitting anything needed in proving subsequent propositions. By adopting this arrangement, many boys who are compelled to begin learning a trade, before their course in geometry is completed, would have a knowledge of the parts of the subject of most practical value to them. This arrangement also makes it possible for the teacher who desires to do so to consider the future needs of pupils in assigning exercises. Those who are preparing for college may be given a larger proportion of theoretical exercises without disturbing the instruction of the others. I think, however, that a course in geometry, including a larger proportion of practical work than is usually given, would give a better preparation for college mathematics, than the course now generally given.

Euclid rightly laid stress on proving the possibility of operations, assumed to be performed. If we train our pupils to neglect this essential part of the proof in school, we may reasonably expect them to bring ridicule on themselves and on school education by neglecting it in life. They are likely to be unpractical, to be what is falsely called theoretical. The evil effects of such negligence are clearly seen in some of the best text books on geometry, which state, e. g.: "In every proportion the product of the extremes is equal to the product of the means." E. g.: 2 m: 4 m = 3 cu. m.: 6 cu. m.

Is $2 \text{ m} \times 6 \text{ cu. m.} = 4 \text{ m} \times 3 \text{ cu. m.}$? The Euclidean method of treating proportions, while strictly logical, is too difficult for many high school pupils. The treatment here given, see pages 68—73, has been successfully used in many classes. Teaching pupils to observe the limitations under which truths proved of numbers may be extended to magnitudes, as here used, has been found helpful in leading them to consider more carefully the limitations needed in extending geometrical truth to concrete magnitudes, in practical applications.

The methods used or suggested in this work are the results of long experience. The author has endeavored to teach each one of the forty-one classes he has instructed, either a part or the whole of the course in geometry, to reason independently. Twenty years experience in teaching college mathematics, and careful study of the results of geometrical teaching in practical life have modified his views. All of the methods and most of the proofs and exercises have been used in the class room.

The greatest obstacle in the way of success in proving theorems independently seems to be the difficulty in calling to mind the propositions to be used in the proof. The following expedients have been used to aid teachers and pupils in conquering this difficulty.

1. The more important corollaries and scholiums are made theorems and problems, the less important exercises.
2. Related theorems or problems are often grouped in one section.
3. The less important theorems and problems are made exercises or supplementary propositions.
4. The more important propositions are made familiar by making the exercises and supplementary work depend chiefly on them.
5. The free use of the principles of limits diminishes the strain on the memory by showing relations more clearly.
6. The copious index will often be of service. The theorems and problems should be frequently reviewed in such manner as will best promote the associations of ideas needed. The exact subject may be called for, e. g: "Give the theorem concerning vertical angles". All the theorems on a subject may be called for, e. g.: "In what way have we

found that two triangles may be proved equal?" An illustration may be given suggesting the theorem, in general using a figure, e. g.: "In the triangle A B C, A = 35° and B = 35°. What is true of a and b, and why?" A figure may be drawn and the pupil asked what theorem (or theorems) can be proved by its use. Many years ago I required the theorems and problems thoroughly memorized by number My reasons for discontinuing this practice are suggested by the note on page 9.

Pupils should be trained to do things in the easiest way in which they can be well done. Except in a few cases, for special reasons, the proofs suggested are the simplest and most direct I could devise, or find in non-copyrighted books. If written in full they would, on an average, be considerably shorter than those usually given The suggestions occupy much less space than full proofs. The method of arrangement also saves space. Though the book is small in size, the amount of matter is considerably larger than in most text-books. This makes it possible for the teacher not only to select a course adapted to the entrance requirements of any college, or technical school, or one best adapted to pupils who do not take a college course, but also to vary the exercises taught in different classes. This work is intended to help to train pupils to use all their powers in conquering difficulties, to emphasize the unity of mathematics, rather than to lay stress on the difference between algebraic and geometric reasoning.

The analytical method is used in treating the conic sections, for the following reasons;

1. Most pupils in geometry use their training in practical life as artisans, farmers, etc., rather than as public speakers. Analytical methods are as much superior in training men to discover the truths needed in their business as are geometrical methods in preparing them to present known truth in oratory.

2. The application of analytical methods to geometrical magnitude serves to show the pupil more clearly the relation between algebra and geometry, thus fixing both subjects more firmly in the mind.

3. Experience in a number of classes has proved that, even in as short a course as four weeks, I can teach more

about the conic sections by analytical methods than by geometrical, besides imparting the necessary instruction on the subject of loci.

To the best of my ability I have endeavored to avoid trespassing on the copyright rights of others, even rejecting several proofs in the work as first prepared, though I had used them in the class room long before the date of the copyright books containing them, but life is too short for exhaustive comparison.

I desire to acknowledge my indebtedness to many former pupils whose suggestions have had a modifying influence, but except in one case, §275, these suggestions have become so interwoven with my own ideas and with those of other pupils that it is impossible to make special acknowledgment.

I am particularly indebted to Professor A. O. Bersell, Ph. D., of Augustana College, for assistance in proof reading, and to Professor J. A. Bexell for drawing the figures and for valuable suggestions. Without their aid it would scarcely have been possible to complete the publication under existing circumstances. I am also under obligations to my publishers and especially to Mr. G. A. Gustafson, the patient and efficient compositor.

A. W. WILLIAMSON.

Augustana College, Rock Island, Ill., June 8, 1901.

SIGNS USED.*

$\equiv$	represents, is identical with.
$\triangleq$	is equivalent to, i. e., equal in area or volume.
$\cong$	coincides with.
$\sim$	similar to, is similar to.
$\doteq$	approaches as a limit.
$\parallel$	parallel to, is parallel to.
|, |s	a straight line, the straight line, straight lines; side (of a polygon), sides.
▱, ▱s	parallelogram, parallelograms.
▭, ▭s	rectangle, rectangles.
□ □s	square, squares.
⊙, ○, ○s	circle, circles.
Q. E. D.	quod erat demonstrandum, which was to be proved.
Q. E. F.	quod erat faciendum, which was to be done.
SIM.	similarly, in like manner it may be proved.

GREEK ALPHABET.

Letters.	Names.	Letters.	Names.
Α α	Alpha	Ν ν	Nu
Β β	Beta	Ξ ξ	Xi
Γ γ	Gamma	Ο ο	Omicron
Δ δ	Delta	Π π	Pi
Ε ε	Epsilon	Ρ ρ	Rho
Ζ ζ	Zeta	Σ σ ς	Sigma
Η η	Eta	Τ τ	Tau
Θ θ	Theta	Υ υ	Upsilon
Ι ι	Iota	Φ φ	Phi
Κ κ	Kappa	Χ χ	Chi
Λ λ	Lambda	Ψ ψ	Psi
Μ μ	Mu	Ω ω	Omega

* In addition to those in general use in elementary algebra and to some obvious abbreviations of words.

GEOMETRY.

PRELIMINARY DEFINITIONS.

Measure a chalk box: its length, breadth, and thickness. We call each of these ***dimensions.*** How many dimensions has a chalk box? Specify the dimensions of other objects; as, a book, a brick, etc. We can burn the box but we cannot think of the space it now occupies as destroyed. Like the box, the space has limits or boundaries. These limits separate it from the surrounding space. We can imagine the space without thinking of any object which has occupied it. We must think of this space as having length, breadth, and thickness.

1. DEFINITION. ***Geometry*** is the science of space.

2. DEF. ***Space*** has three dimensions, *length*, *breadth*, and *thickness.*

3. DEF. A ***solid*** is a limited portion of space.

How many flat sides has the chalk box? Measure the length and breadth of one flat surface. Has a surface any thickness? As we can think of the space the box occupied without thinking of the box, so we can think of a surface without thinking of the solid of which it is the limit or boundary.

4. DEF. A ***surface*** has two dimensions, length and breadth.

Limited portions of surface have various names, according to the kind of surface, which will be given as the surfaces are described.

Observe that each side of the chalk box is a surface limited, or bounded, by four lines, called *sides* of the surface and *edges* of the solid. How many edges has the chalk box? Measure an edge of the chalk box. Has it any breadth or thickness?

Think of a line without thinking of a surface. Think of lines joining various points in this room.

5. A ***line*** has one dimension, length.

Limited portions of a line are designated as follows:

By a letter at each end of the part taken; as, AB, BC, or AC; By one letter near the line; as, d, or e. We may also think of a line joining two points; as, F and G, without drawing it.

In how many points do the edges of the chalk box meet? Think of a point in the middle of one side. Place the point of your pencil one inch above the middle of your desk. Think of other points in the room.

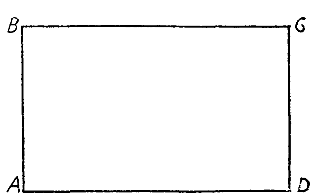
6. Def. A ***point*** has no dimension, but position only.

7. We represent points and lines by dots and lines on paper, to aid us in thinking of the real points and lines which exist in our minds only. Dots and lines on paper are physical solids, having length, breadth, and thickness.

8. We will more clearly understand the meaning of lines, surfaces, and solids in Geometry, by also considering them in reverse order.

1. A moving point generates a line.
2. A line moving sideways generates a surface.
3. A surface moving otherwise than along itself generates a solid.

We may think of the line AB as generated by a point moving from A to B; or, the line AD as generated by a point moving from A to D. We may think of the surface $ABCD$ as



generated by the line AB moving to the position DC; or by the line AD moving to the position BC. Raise a book from the desk. Think of the space it passes over as a solid.

EXERCISE 1. Construct a line on paper by moving the point of your pencil along the edge of your ruler.

Ex. 2. Move a crayon lying flat on the surface of the blackboard illustrating § 8 : 2.

9. DEF. A ***straight line*** is the shortest distance between any two of its points. In this work, the word line means a straight line, unless otherwise stated.

Ex. 3. Stretch a string and explain how this illustrates § 9.

Ex. 4. Mark two points and, if possible, draw two straight lines between them. How many straight lines can you draw between two points?

Ex. 5. Mark three points not in the same straight line. Draw as many lines as possible through them.

Ex. 6. Draw three lines intersecting in as many points as possible.

Ex. 7. Move the edge of your ruler so as not to generate a surface. Move it so as to generate a surface. In what way can a straight line be moved without generating a surface?

10. DEF. A ***plane*** is a surface, in which, if any two points be taken, the straight line joining them lies wholly in the surface.

11. DEF. A ***geometrical figure*** is any combination of points, lines, surfaces, and solids.

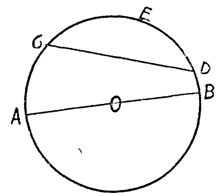
Ex. 8. Slide your book on the desk. What is the path of its under surface? What is the path of this surface when you raise the book?

Ex. 9. How do plasterers test the surface of a wall? What definition is illustrated? Have you seen other mechanics use this principle?

Ex. 10. Draw two unequal straight lines on paper. Subtract the less from the greater, using the dividers only.

Ex. 11. Construct on paper, 1) a geometrical figure consisting wholly of points; 2) one consisting wholly of lines; 3) one consisting of points and lines.

12. DEF. A ***circle*** is a figure bounded by a line called the ***circumference,*** every point of which is equally distant from a point within called the center. A line from the center to the circumference is called a ***radius,*** and a line through the center terminated by the circumference, a ***diameter.*** It follows that all radii are equal; that all diameters are equal; and that a diameter is twice the radius.



13. DEF. Any part of the circumference is called an ***arc***. The straight line joining the extremities of an arc is called a ***chord.*** In the above figure the chord CD *subtends* the arc CED and is *inscribed* in the circle O. The arc CED is *subtended by* the chord CD.

Ex. 12. Construct with your dividers a circle whose radius is 1 inch and inscribe in it a chord $1\frac{1}{2}$ inches in length.

14. DEF. ***Equal magnitudes*** are such as may be so placed that they will coincide throughout their whole extent.

15. DEF. ***Parallel lines*** are such as lie in the same plane, yet, can never meet however far produced.

Ex. 13. How many edges of the chalk box are parallel to a given edge? How many groups of parallel edges has it?

Ex. 14. Can a straight line be equal to the arc of a circle?

16. POSTULATES.

(Operations which we assume can be performed.)

1) A straight line may be drawn through any two points and produced to any length.

2) A circle may be described with any point as a center and with a radius of any length.

3) Any figure may be revolved in any direction and moved to any distance without altering its size or shape.

17. Axioms.

(Self-evident truths.)

1) Things equal to the same thing are equal to each other.

2) Performing the same operation on equals, the results are equal. *

3) Performing the same operation on unequals, the results are unequal in the same order.*

4) The whole is greater than its part.

5) The whole is the sum of its parts.

6) If, of three magnitudes, the first is less than the second, and the second is less then the third, then the first is less than the third.

7) Through two points, only one straight line can be drawn.

8) Through a given point, only one line can be drawn parallel to a given line.

Ex. 15. Illustrate Axiom 2, a) by adding equals to the equals 4 yards and 12 feet; b) by subtracting equals from the same; c) Multiplying both by equals; d) dividing both by equals; e) by raising the numbers 4 and 12 to equal powers; f) by extracting the square roots of the numbers 4 and 12. Of what denominations must the equals added to 4 and 12 be? The multipliers must be what kind of numbers? Can we extract the square root of feet?

Ex. 16. 1 yd $<$ 1 meter, 1 meter $<$ 1 fathom. What follows as to 1 yard and 1 fathom?

Ex. 17. 4 meters $>$ 12 feet. Illustrate Axiom 3 by performing the same 6 operations specified in Ex. 15.

Ex. 18. 1 meter = 39.37 in. Change 4 meters to feet, 5 meters, 8 meters.

Ex. 19. Change 2 decimeters to inches, 3 dm, 5 dm, 7 dm.

Ex. 20. Change 65 centimeters to inches, 75 cm, 85 cm.

Ex. 21. Change 6 kilometers to miles, 10 km, 50 km.

18. Def. ***Plane Geometry*** treats of figures all of whose points lie in the same plane.

* For the six algebraic axioms included in Axiom 2, see Ex. 15; for those in Axiom 3, see Ex. 17. We do not consider negative magnitudes in Elementary Geometry.

PLANE GEOMETRY.

BOOK I.

FIGURES BOUNDED BY STRAIGHT LINES.

19. Def. An ***angle*** is the difference of direction between two lines at a common point, called the ***vertex*** of the angle.

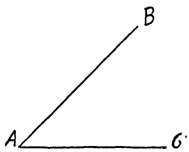


Fig. 1.

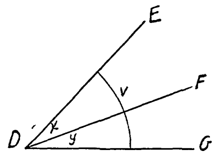


Fig. 2.

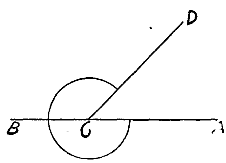


Fig. 3.

The lines AB and AC form an angle, which is often named BAC, always putting the letter at the vertex in the middle. If there is only one angle at a point, it may be named by the letter at the vertex. An angle may also be named by a letter within it. The angle x is the same as the angle EDF, $\angle y \cong \angle FDG$.* The arc on which v is placed indicates that the angle v extends the length of the arc, from ED to GD.

20. Def. Let C, one extremity of the line CD, remain fixed at C while the other extremity D revolves from the position AC to the position BC. DC is said to pass through a ***straight angle.*** All the angular magnitude about a point is called a ***perigon.*** If DC, after reaching the position BC,

* Read $\cong$, *coincides with*; $\angle$, *the angle.*

continues to revolve in the same direction till it reaches its first position AC, it is said to pass through a perigon.

21. Since, between two points, only one straight line can be drawn, if the straight angle ACB, above the line, be rotated on AB as an axis till it falls on the straight angle below, they coincide, that is, a straight angle is half a perigon, and a perigon is two straight angles.

22. DEF. A ***right angle*** is half of a straight angle.

23. DEF. A straight line is ***perpendicular*** to another when it meets it forming right angles.

Draw a straight line. Draw with the right angled ruler a line perpendicular to it above and near the middle. Extend this line across the first line. How many right angles have you formed in the perigon?

Observe, however, that these right angles are not geometrically constructed, as only the straight edge and dividers may be used in such constructions.

24. DEF. A perigon is divided into 360 equal parts called ***degrees,*** a degree into 60 equal parts called ***minutes,*** and a minute into 60 equal parts called ***seconds.***

25. Being like parts of the same thing, all straight angles are equal, all right angles are equal, all degrees are equal, etc.

26. DEF. An ***acute angle*** is less than a right angle; an ***obtuse angle*** is greater than a right angle and less than a straight angle. All angles, except right angles, straight angles, and perigons are called ***oblique angles.***

Ex. 22. Make an acute angle, make an obtuse angle. Draw two lines cutting each other so as to form four oblique angles. How many of them are acute? How many are obtuse?

Ex. 23. How many degrees are there in a right angle? How many in a straight angle?

Ex. 24. How many minutes are there in a perigon? How many in a straight angle? How many in a right angle?

Ex. 25. How many seconds are there in a right angle?

Ex. 26. How many degrees are there in $\frac{2}{3}$ of a right angle? in $\frac{3}{4}$ of a straight angle? in $\frac{1}{8}$ of a perigon?

Ex. 27. Which of the following angles are acute, which obtuse, which right angles? 90°, 110°, 45°, 79°, 140°, 3600′, 7200′, 32400″, 18000″, 500000″, 324000″.

27. Def. When the sum of two angles is a right angle, each is said to be the ***complement*** of the other; when the sum is a straight angle, each is the ***supplement*** of the other; when the sum is a perigon, each is the ***conjugate*** of the other.

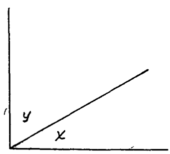


Fig. 1.

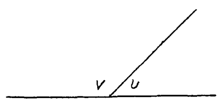


Fig. 2.

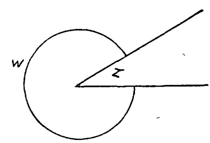


Fig. 3.

Name the complement of x, the supplement of u, the conjugate of z.

28. Since equals subtracted from equals leave equal remainders, the complements, supplements, and conjugates of the same angle, or of equal angles, are equal.

Ex. 28. Find the supplement and complement of 30°; 50°; 12′; 28″; 23° 27′ 25″.

Ex. 29. Find the angle, conjugate to 30°; 120°; 270°; 310°.

Ex. 30. Find the complement of the supplement of 120°.

Ex. 31. Find the angle which is 3 times its complement; 5 times; 8 times.

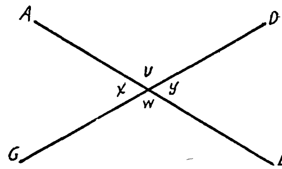
Ex. 32. Find the angle which is twice its supplement; 4 times; 9 times.

Ex. 33. Find the angle which is 3 times its conjugate; 8 times; 11 times.

Ex. 34. The sum of 4 times an angle, 3 times its complement, and twice its conjugate is 12 right angles. Find the angle.

THEOREM I.

29. *When two lines cut each other, the vertical angles formed are equal.*



Let AB cut CD, forming the vertical angles x and y; also the vertical angles w and v. Then the angle x is equal to the angle y and the angle w is equal to to the angle v.

For the angles x and y are each supplements of the angle v, and are therefore equal.

The supplements of equal angles are equal.

In like manner it may be proved that the angle w is equal to the angle v.

Proof, using abbreviations.

Let AB cut CD forming vert. $\angle$s x and y, also vert. $\angle$s, v and w. Then $\angle x = \angle y$ and $\angle v = \angle w$.

1. For x is the sup. of v and y is the sup. of v.
2. $\angle x = \angle y$.

   Sup. eq. $\angle$s are eq.
3. Sim. $\angle v = \angle w$. Q. E. D.

NOTE. Some demonstrations should be written in full, to aid the pupil in acquiring accuracy in the use of geometrical language, In general, it is better to write the abbreviated form. In reading the written exercise, be careful to use all the words for which the abbreviation stands. In step 3, read Sim., *In like manner it may be proved;* read in above for $\angle$, *the angle*, except after *vert.*

The theorems themselves, at least the thought, should be thoroughly memorized. It is better *at first* to memorize the exact language. In repeating them, or in thinking of them, think of the figure. Review them often, associating with them other theorems on similar subjects. Avoid referring to them, or thinking of them, by number, as far as possible. Thinking of theorems by number hinders the formation of other associations needed in recalling them for use.

To save the time of the pupil in cases in which he cannot think of the proposition needed, the number of the theorem or definition is often given. Endeavor to do your work with but little use of this help.

Do not use exercises in proving propositions. In a few cases, exercises are needed in working exercises, and they are then specially referred to.

Ex. 35. Prove Theorem I by using the angles w and v as vertical angles, instead of x and y.

Ex. 36. In the figure of §29, if the angle x is 35°, what are the other angles?

Ex. 37. If x were 90°, what would the other angles be? Illustrate by a figure.

Ex. 38. If, in fig. of §29, $x : v = 2 : 7$, find all the angles.

30. Def. A ***triangle*** is a plane figure bounded by three straight lines, called ***sides.*** Make a triangle.

31. Def. An ***equilateral triangle*** has all its sides equal. An ***isosceles triangle*** has two sides equal. A ***scalene triangle*** has no equal sides.

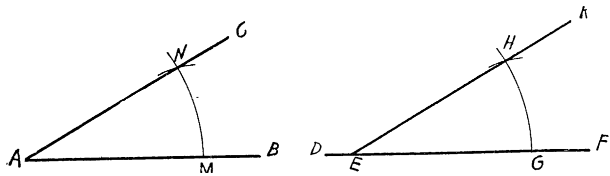
32. Def. An ***obtuse angled triangle*** has one obtuse angle. A ***right angled triangle*** has one right angle. An ***acute angled triangle*** has three acute angles.

33. Def. Any side of a triangle, but usually the lower side, is called the ***base.*** The angle opposite the base is called the ***vertical angle.*** The line from the vertical angle, perpendicular to the base, is called the ***perpendicular*** or ***altitude.*** In a right angled triangle, the side opposite the right angle is called the ***hypotenuse,*** the sides including the right angle are called the ***legs.***

Ex. 39. Draw as accurately as you can, with ruler and pencil only, one triangle of each of the six kinds defined in §§31 and 32. Specify orally the base and vertical angle of each triangle and the legs of the right triangle.

Problem I.

34. *To construct an angle, equal to a given angle.*



At E, in $|DF$, construct an $\angle = \angle A$.

1. With A as a center and a convenient radius AM const. $\smile MN$, limited by $|$s $\angle A$.*

2. With E as a center, and $EG=$ ~~MG~~ AM, as radius, const. $\smile GH$ not $< MN$.

3. Open the dividers, one pt. on M, one on N, and with this radius cut off $\smile GH = \smile MN$.

4. Draw EH, then $\angle HEG$ is the required $\angle$.

5. For apply AM to its equal EG, A on E, and M on G.

6. $\smile BC \cong \smile GH$, †
   every pt. in each being the same distance from E.

7. The pt. $N \cong H$,
   an equal part of each $|$ being cut off. Step 3

8. $\therefore AC$ falls both on E and H.

9. $\therefore AC \cong EH$.
   Between two pts. only one $|$ can be drawn.

10. $\therefore \angle A \cong \angle HEG$.

11. $\therefore \angle A = \angle HEG$.
    Magnitudes which $\cong$ are equal. Q. E. F.

Ex. 40. Extend the arcs MN and GH to complete the circles, and construct another angle equal to the angle A. Construct two angles in the opposite direction, each equal to the angle A.

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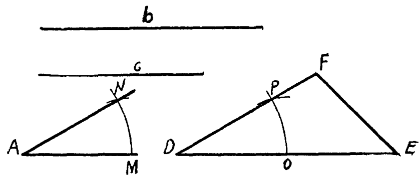
* Read const. $\smile$, *construct an arc;* $|$s, *the sides of.*

† Read $\smile$, *the arc;* $\cong$, *coincides with.*

PROBLEM II.

35. *To construct a triangle, two sides and the included angles being given.*

At D in the | DE construct △, having given . . . .
. . . . *



1. At D const. ∠ . . . .†
2. Const. $DB =$ . . . . .
3. Const. . . . .
4. Join . . .
5. △ DEF is the required △,‡
since by const. $\angle D = \angle A$, | $DE = b$, and | $DF = c$.¶

Q. E. F.

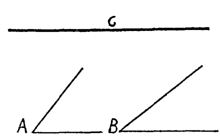
Ex. 41. Construct the triangle of §35, using A as one of its angles.

Ex. 42. Construct the triangle of §35, using b as one of its sides.

PROBLEM III.

36. *To construct a triangle, two angles and the included side being given.*

The pupil will construct and explain this by a method similar to that of Problem II.



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* The dots indicate that there are words or sentences omitted which the pupil is expected to fill in.

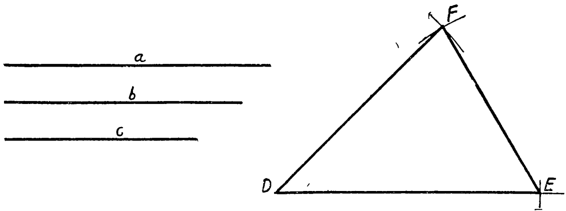
† Read ∠, *an angle.*

‡ Read △, DEF, *the triangle DEF;* required △, *required triangle.*

¶ Read ∠D, *the angle D;* |DE, *the side DE.*

Problem IV.

37. *To construct a triangle, the three sides being given.*



At D, in the line DE, construct a $\triangle$ whose |s shall be equal to |s, a, b, and c.

1. Lay off $DE = a$.
2. With b and c, as radii, and E and D, as centers, draw ⌣s intersecting in F.
3. Join . . . and . . .
4. DEF is the required $\triangle$.
5. For . . . .

Q. E. F.

Ex. 43. Complete the ⊙s whose centers are D and E. They will intersect in another point. Form a second $\triangle$, by joining this pt. to D and E.

Ex. 44. Make an angle greater than 90°, and construct an angle equal to it, by first producing one side, thus constructing its supplement, and then constructing an angle equal to its supplement. In what cases is this method preferable?

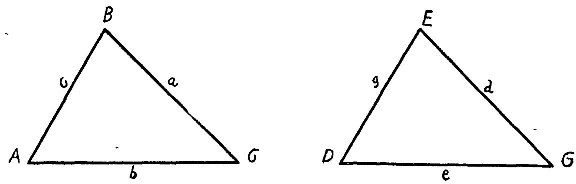
Ex. 45. From the conjugate of 48° subtract the supplement of 32°.

Ex. 46. Find the angle which is the sum of the complement of 73° 25′ and the supplement of 80° 25′.

Ex. 47. Draw two lines, a and b, on paper. Construct a third line $c = a + b$ using the dividers and straight edge only.

THEOREM II.

38. *If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles are equal.*



In the $\triangle$s ABC and DEG, let $b = e$, ... and . . . . Then $\triangle\ ABC = \triangle$ . . ., $\angle B$ . . ., $\angle C =$ . . . and a . . .

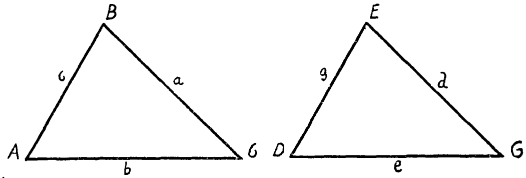
1. For apply $\triangle\ ABC$ to $\triangle\ DEF$, $\angle\ A$ on $\angle D$, b on e.
2. The pt. $C \cong$ . . Why?
3. $c \cong$ . .
   since $\angle\ A =$ . .
4. The pt. $B \cong$ . .
   since $c =$ . .
5. $a \cong$ . . Why? §17:7.
6. $\triangle\ ABC \cong \triangle\ DEF.$
7. $\therefore \triangle\ ABC$ . . . . . .
   Magnitudes which $\cong$ are equal.
8. Also $\angle\ B =$ . . ., $\angle\ C =$ . ., a . . .

Q. E. D.

39. Homologous parts of figures are those similarly situated. In the fig. to §38, A and D are homologous angles; name two other pair of homologous angles; a and d are homologous sides; name two other pair of homologous sides.

THEOREM III.

40. *If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles are equal.*



The pupil will state the theorem by the figure, making $b = e$.

1. For apply $\triangle\ ABC$ . . . . b on its equal e, A . . . . and C . . . .
2. c takes the direction . . . .
   since $\angle A =$ . . . .
3. a takes . . . . . . Why?
4. B falls both in g and d. It must fall at their intersection E.
5. $\triangle$ . . . . Why? § 14
6. Also $c =$ . . ., g . . . ., B . . . .

41. DEF. In the above theorem we have given the ***hypothesis*** or ***condition:*** *Two triangles have two angles and the included side of the one equal to two angles and the included side of the other.* We are to prove the ***thesis*** or ***conclusion:*** *The triangles are equal.* Before making any effort to prove a theorem we should always see clearly what is given, and what is to be proved. The author believes that it is better for the pupil to ascertain this for himself, and therefore usually omits to state it explicitly.

Ex. 48. State what is given in Theorem II; state what is to be proved.

Problem V.

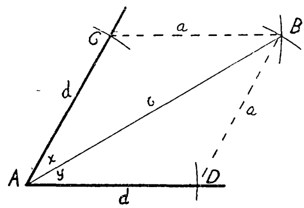
42. *To bisect an angle.*

Bisect $\angle A$.

1. On the |s of $\angle A$ lay off $AD = AC$, any convenient distance.

2. With the same radius a, construct ◡s intersecting in B.

3. Join AB; then $\angle x = \angle y$.



Proof omitted till Theorem VI. has been proved.

Ex. 49. State what is given in Theorem I; state what is to be proved.

Ex. 50. 1. Draw a straight line. 2. At any point in this line draw a line forming oblique angles on one side. 3. Bisect each of these supplementary angles. 4. Prove that the angle formed by the bisectors is a right angle. See §17 : 2 (addition) and §22.

Ex. 51. Prove Theorem III, making the hypothesis; (fig., §40.) $\angle B = \angle E$, $\angle C = \angle G$, $a = d$.

43. Def. If, in Theorem II., we change the word sides into angles and the word angles into sides, we have Theorem III. Such theorems are called ***reciprocal;*** thus Theorem II. is the reciprocal of Theorem III.

What theorem is the reciprocal of Theorem II?

Ex. 52. State the hypothesis (what is given), in Problem IV., also the conclusion (what is to be done.)

Ex. 53. Find 3 times the complement of 23° 27′.

44. The ***angle bisector*** of an angle of a triangle is the line which bisects the angle, limited by the opposite side. The ***median*** on a side of a triangle is the line to its mid-point from the vertex of the opposite angle.

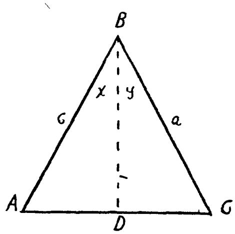
Draw a triangle whose sides are 1 in., 1½ in., and 2 in. Construct its three angle bisectors. Find the mid-points of its sides by measurement and draw its three medians.

THEOREM IV.

45. *1. The angles opposite the equal sides of an isosceles triangle are equal.*

2. The bisector of the vertical angle of an isosceles triangle bisects the base and is perpendicular to the base.

3. An equilateral triangle is equiangular.



I. and II. State the theorem by the figure.

1. For bisect $\angle B$ by BD, meeting AC in D.

Now in $\triangle$s ABD . . . .

2. $c =$ . . . . Why?
3. . . . . $\cong$ . . .
4. $\angle$ . . $=$ . . . Why?
5. . . . . $=$ . . . Why? §38
6. $AD =$ . . . .
   being hom. |s of eq. $\triangle$s.
7. $\angle ADB =$ . . . Why?
8. . . . . Why? §23

Q. E. D.

III. The pupil will construct a figure and state and prove this part. See §§37 and 45: 1.

Ex. 54. In a triangle $a = 6$, $c = 6$, $b = 8$. Find the length of the parts or segments into which b is divided by the bisector of angle B. See §45: 2.

Ex. 55. Prove that the bisectors of the angles of an equilateral triangle are equal.

Ex. 56. In a triangle ABC, $a = 3$, $b = 3$, $\angle A = 10°$. Find $\angle B$.

Ex. 57. Draw a triangle ABC making each side 1 inch; $\angle A = 60°$. Find $\angle B$ and $\angle C$.

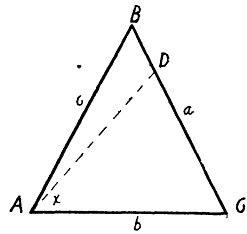
Ex. 58. The medians on the equal sides of an isosceles triangle are equal. See §§44 and 38.

Ex. 59. The medians on the sides of an equilateral triangle are equal.

THEOREM V.

46. *1. If two angles of a triangle are equal, the sides opposite these angles are equal.*

2. An equiangular triangle is equilateral.



I. State the theorem by the figure.

1. For, if not, one |, as a, must be the greater.
2. From a, cut off $CD = c$.
   Now in $\triangle$s ACD and ABC,
3. $CD = . . .$ Why?
4. $. . . \cong . .$
5. $\angle C = . . .$ Why?
6. $\triangle$ . . . . . Why? §38
7. A part = the whole, which is absurd. Why? §17: 1
8. $\therefore$ a cannot $>$ c. Sim. c cannot $>$ a.
9. $\therefore$ a and c cannot be unequal, and $a = c$. Q. E. D.

II. Construct a figure and prove this part.

47. The method of proof used in §46 is called ***reductio ad absurdum.*** It is much used by lawyers appealing to juries, and by public speakers generally, even among the rudest races. Nothing produces stronger conviction of the truth of a proposition than the fact that its denial leads to absurd consequences.

Ex. 60. Construct a triangle ABC, making each side 1 inch; $\angle A = 60°$. Find $\angle$s B and C.

Ex. 61. In $\triangle$ ABC, $\angle A = \angle B = \angle C = 60°$, $a = 5$; find b and c.

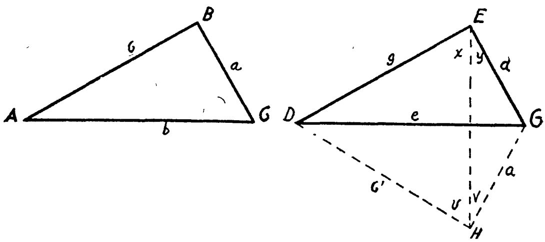
Ex. 62. In $\triangle$ ABC, sup. $A = 130°$, comp. $B = 40°$, $b = 6$; find a.

Ex. 63. Construct a triangle each side of which is 3 cm. What is true of its angles?

Ex. 64. Divide a perigon into the parts $2x$, $3x$, and $4x$.

Theorem VI.

48. *If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are equal.*



State the theorem by the figure.

1. For apply $\triangle ABC$ to $\triangle DEG$, b on its equal e, A on D, and C on G, but B at H opposite to E.

2. $d = a'$. Why?

3. $\therefore \angle y = \ldots\ldots$ Why? §45: 1

4. Sim. $\angle \ldots\ldots$

5. $\angle DEG \ldots\ldots$
   by adding (3) and (4).

6. $\angle B = \angle E$,
   each being $= \angle DHG$.

Now in $\triangle$s $ABC \ldots\ldots$

7. $c = \ldots\ldots$ Why?

8. $a \ldots\ldots$ Why?

9. $\angle B = \ldots\ldots$

10. $\triangle \ldots = \ldots$ Why? §38

11. $\angle A = \ldots, \angle C \ldots$ Why?

Q. E. D.

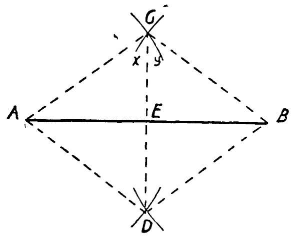
It is easier to remember and quote this proposition in the form: *Three sides determine a triangle.* Give the preceding propositions relating to triangles in similar short forms.

The pupil should now prove Problem V.

Problem VI.

49. *To bisect a given straight line.*

Bisect | *AB*.



1. With *A* and *B* as centers and $AC > \frac{1}{2} AB$ as radius describe . . . .

2. Join *CD*.

3. *E*, the intersection of . . . . . is the mid-pt. of *AB*.

For in $\triangle$s *ACE* . . . . .

4. $AC =$ . . Why?

5. . . $\cong$ . .

6. $\angle x = \angle y$,

since these $\angle$s were const. by the method . . . §42

7. $\therefore \triangle$ . . . . Why? §38

8. . . . .

being homologous |s of equal $\triangle$s. Q. E. F.

50. Def. If two statements are so related that the hypothesis of the first is the conclusion of the second, each is said to be the ***converse*** of the other. The converse of the proposition; *If this is a horse, it is an animal,* is: *If this is an animal, it is a horse;* the one statement true, the other false. The converse of a statement of equality or identity, including the converse of all correct definitions, is necessarily true. In other cases, the converse of a true statement, when true, requires proof.

Ex. 65. State the converse of each part of Theorem V. Do these statements require proof?

Ex. 66. State the converse of Theorem IV: 2.

Ex. 67. State the converse of Theorems II and III. Why is no proof of these required?

PROBLEM VII.

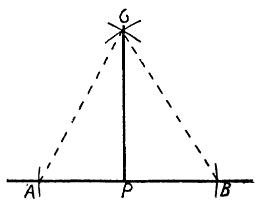
51. *1. To erect a perpendicular to a line from a point within the line.*

2. To let fall a perpendicular to a line from a point without the line.

I. At P in | AB, erect $PC \perp AB$.

First method: By the right angled ruler.

This merely copies the right angle constructed by the maker of the ruler.

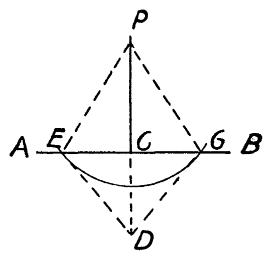


Second method: 1. From P lay off $PA = PB$.

2. With A and B as centers . . . .

3. Join CP. $CP \perp AB$.

4. Prove this by proving $\triangle$s equal. See §48. There should be five steps in the proof.



II. Construct and explain. In proof see §§48 and 45: 2.

Ex. 68. Construct a right angled triangle whose legs are 1 inch and ½ in. respectively.

Ex. 69. The median on the base of an isosceles triangle bisects the vertical angle and is perpendicular to the base. Apply §48.

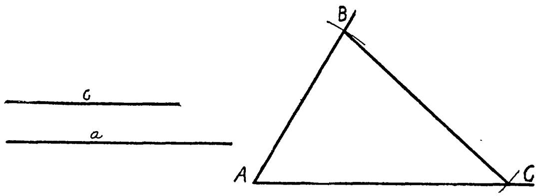
Ex. 70. An angle bisector of an equilateral triangle is both a median and an altitude on the opposite side.

Ex. 71. A right angle is divided into parts represented by $3x$, $4x$, and $5x$. Find the number of degrees and minutes in each part.

Ex. 72. A straight angle is divided in the ratio 7: 11. Compute the number of degrees in each part.

PROBLEM VIII.

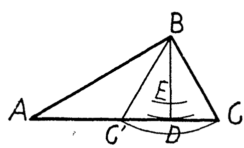
52. *To construct a triangle, two sides and the angle opposite the greater side being given.*



Given $\angle A$, $|\ c$, and $|\ a$; $a > c$. Construct $\triangle ABC$ using A as an $\angle$.

1. On one side of $\angle A$ lay off . . . .
2. With B as a center and a . . . . . .
3. Join BC. $\triangle ABC$ is by const. the required $\triangle$.

The circle of which B is the center and a the radius, if completed cuts $|\ AC$ in two points. Why can there not be two $\triangle$s? Complete the circle and explain.



Ex. 73. To construct a triangle, having given two sides and the angle opposite the less.

Show by construction, that there may be two triangles, one triangle, or no triangle possible, according to the length of the side opposite to the angle.

THEOREM VII.

53. *Any side of a triangle is less than the sum of the other two.*

Since by definition a straight line is the shortest distance between any two of its points.

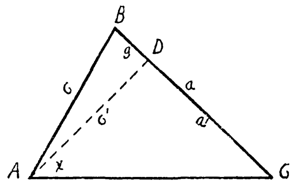
Illustrate the above by a figure.

Ex. 74. Why can you not construct a triangle whose sides are 2, 4, and 6?

THEOREM VIII.

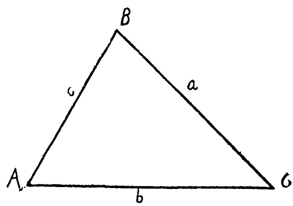
54. *1. The greater side of a triangle is opposite to the greater angle.*

2. Conversely: The greater angle is opposite to the greater side.



I. In $\triangle\ ABC$, let $\angle A > \angle C$, then $a > c$.

1. For from $\angle A$ cut off by c' $\angle x = \angle C$.
2. $c' = \ldots\ldots$ Why? §45: 1
3. $\ldots \cong \ldots$
4. $c' + g = \ldots$ Why?
5. but $c' + g > \ldots$ Why? §55
6. $a > c$. Why?



II. In $\triangle\ ABC \ldots\ldots\ldots$

1. For if $\angle A$ is not $> \angle C$, either $\angle A = \angle C$ or $\angle A < \angle C$.
2. But $\angle A$ is not $= \angle C$, for then a would $= c$.
3. $\angle A$ is not $< \angle C$, for then $a < c$;

both of which are contrary to the hypothesis.

4. Since $\angle A$ is not $= \angle C$, nor $< \angle C$; $\angle A > \angle C$.

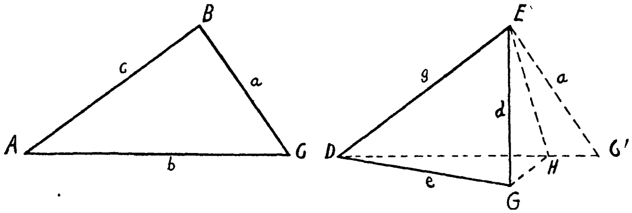
Q. E. D.

NOTE. We have here used an axiom obvious to any one of common sense, not included in the list of axioms previously given, viz: *a magnitude must be greater than another, be equal to it, or be less.* The formal statement and quotation of this class of axioms serves to confuse rather than help us in geometrical reasoning.

Ex. 75. Why can you not construct a triangle whose sides are 1, 2, and 5?

THEOREM IX.

55. *If two sides of one triangle be equal to two sides of another, and the included angles unequal, the third side of the one which has the greater included angle will be greater than the third side of the other.*



In the $\triangle$s ABC and DEG, let $c = g$ and $a = \ldots$, but the inc. $\angle$ . . . . then will $b > \ldots$

1. For apply $\triangle ABC$ . . c on its eq. g A on . . and . . on . .
2. Since $\angle B > \angle E$, a takes a position a' and C a position C' without $\triangle$ DEG.
3. Bisect $\angle GEC'$ by . . . . in H.
4, 5, 6, 7. Prove $\triangle$ $GEH = \triangle$ HEC'.
8. $GH = HC'$. Why?
9. . . . . $\cong$ . . . .
10. $DH + GH = DC'$. Why?
11. But . . . . . $> e$. Why? §53
12. $DC' > e$. Why?
13. $b > e$. Why?

Q. E. D.

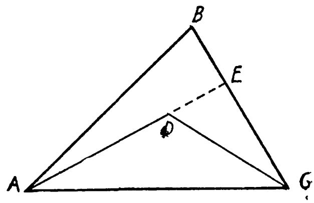
THEOREM X.

56. *If two sides of one triangle be equal to two sides of another each to each, but the bases unequal, the angle contained by the sides of the one which has the greater base will be greater than the corresponding angle of the other.*

Draw a fig. and prove the theor. by the method of §54: 2.

Theorem XI.

57. *If from a point within a triangle straight lines be drawn to the extremities of the base, the sum of these lines is less than the sum of the other two sides of the triangle.*



State the theorem by the figure.

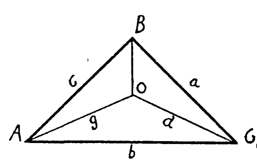
1. For produce AD . . . .
2. Now in $\triangle$ DEC, DC $<$ . . . Why? §53
3. $AD \cong$ . . . .
4. . . . . . . $< AE + EC$. Why? §17:3
5. Also in $\triangle$ ABE; $AE <$ . . . . . Why? §53
6. $EC \cong$ . . . . .
7. . . . $\angle$ $AB + BC$. Why? §17:3
8. Much more then . . . . Why? §17:6

Q. E. D.

58. Theorem XI may be stated: *If two lines envelop two others the sum of the enveloped lines is less than the sum of the enveloping lines.*

Specify in the above figure the enveloping lines and the enveloped lines.

Ex. 76. $d + g + OB > \frac{1}{2}(a + b + c)$



1. For $d + g >$ . . . . . Why?
2. . . . . . . $>$ . . . . .
3. . . . . . . $>$ . . . . .
4. $2d +$ . . $>$ . . . . . Why? §17:2
5. $d +$ . . . . .

State this exercise as a theorem.

Ex. 77. Prove by the above figure, that $a > b - c$ applying §53. State this exercise as a theorem beginning: *Any side of* $a \triangle >$ . . . .

Ex. 78. The medians on the equal sides of an isosceles triangle form another isosceles triangle. See §38 and 46: 1.

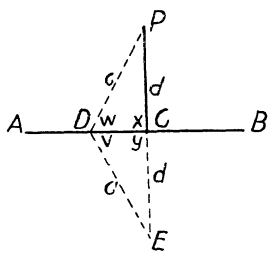
Ex. 79. The medians of an equilateral triangle bisect its angles.

Theorem XII.

59. *Through a given point only one line can be drawn perpendicular to a given line.*

I. At C in the $|$ AB only one $|$ CP can be drawn $\perp$ AB. For by definition the rt $\angle PCB = \frac{1}{2}$ the st $\angle ACB$.

II. *Through a pt P without AB only one $\perp$ as PC can be drawn to AB.*



1. For, if possible, let PD be another $|$, $\perp$ AB.
2. Produce PC to E making $CE = PC$.
3. Join DE.

In $\triangle$s PDC and EDC,

4. . . . . = ... Why?
5. . . . . $\cong$ . . . .
6. $\angle$ . . = ... Why? §25
7. $\triangle$ . . . . Why?
8. $\angle v$ = ... being hom. $\angle$s equal $\triangle$s.
9. $\angle PDE$ is a st $\angle$. Why?
10. Which is absurd. Why? §17:7

Q. E. D.

Theorem XIII.

60. *1. The perpendicular is the shortest distance from a point to a line.*

2. The shortest distance from a point to a line is the perpendicular.

I. State by the figure of §59.

1. Let $PD \equiv$ any other $|$ than PC.
2. Const. and prove $\triangle$s eq. as in §59, and that $c = c'$.

3. $c + c' > PE$. Why? §53

4. c which is $\frac{1}{2}$ of $(c + c') > d$ which is $\frac{1}{2}$ of $(d + d'.)$

Every other | $> d$; $\therefore d$ is the least dist. from B to AB.

II. Let d be the least distance from P to AB, then $d \perp AB$.

For let any other | as c be $\perp AB$, then $c < d$ which is contrary to the hypothesis.

61. The locus of a point is the place where all points satisfying a given condition is situated. To show that a given line or system of lines straight or otherwise is a locus we must prove:

1. That every point within the line or system of lines satisfies a given condition.

2. That no point without the line or system of lines satisfies it.

Ex. 80. Mark a point P on paper. Construct with the compass the locus of all points ½ inch from P, then the locus of a point ¾ inch from P, then the locus of a point 1 inch from P.

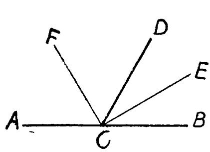


Fig. Ex. 81.

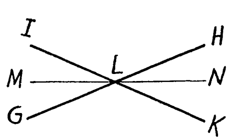


Fig. Ex. 82.

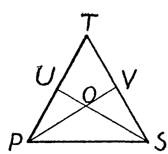


Fig. Ex. 83.

Ex. 81. Let EC bis. $\angle DCB$, and FC bis. the adj. $\angle ACD$, $FC \perp CE$. See §§22 and 23. State this as a theorem.

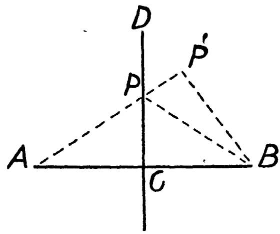
Ex. 82. Let ML bis. $\angle GLI$ and NL the vert. $\angle KLH$; MLN is a straight line. See §29. State this as a theorem.

Ex. 83. The bisectors of the base angles of an isosceles triangle, form another isosceles triangle. See §§45: 1 and 46: 1.

Ex. 84. Lines from the extremities of the base of an isosceles triangle to points on the opposite sides equally distant from the vertex are equal.

THEOREM XIV.

62. *The locus of a point equally distant from the extremities of a line, is the perpendicular to the line at its mid-point.*



Let $DC \perp AB$ at its mid-pt C.

I. If P is in DC; $PA = PB$.

Prove this. See §§23 and 38.

II. If P' is without DC, AP' cutting DC in some pt. as P; $AP' > BP'$.

1. For $AP = \ldots$ Why?
2. $PP' \cong \ldots$
3. $\therefore \ldots = \ldots$

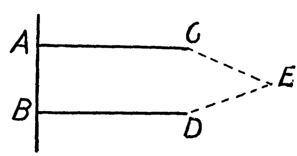
   adding noting that $AP + PP' = AP'$.
4. But $PP' + P'B > \ldots$ Why?
5. $\ldots > \ldots$ Why?

Q. E. D.

THEOREM XV.

63. *1. Lines perpendicular to the same line are parallel.*

2. A line perpendicular to one of two parallel lines is perpendicular to the other.



I. Let AC and $BD \perp AB$. Then . . . . .

1. For if not . . . . . . Why? §13
2. $\therefore$ From E we have 2 $\perp$s . . . . .

   which is absurd. Why?
3. A and B cannot meet and are $\|$.

II. Let $AB \perp AC$ . . . .

1. For $| \perp AB$ at B . . . . Why? part 1
2. $BD \cong | \perp AB$ at B. Why? §17: 8
3. . . . .

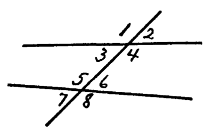
Q. E. D.

PROBLEM IX.

64. *Through a given point construct a line parallel to a given line.*

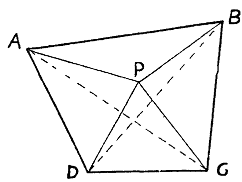
Construct by §51.
In proof, see §63.

65. DEF. A line cutting two or more lines is called a ***transversal*** or ***secant.*** ***Alternate interior*** angles are on opposite sides of the transversal between the lines cut by it. ***Alternate exterior*** angles are on opposite sides of the transversal without the lines cut by it. ***Corresponding angles*** are angles similarly situated as 7 and 3.



Name 3 other pair of corresponding angles. Name each of the 2 pair of alternate interior angles. Name each of the 2 pair of alternate exterior angles.

Ex. 85. Prove that $PA + PB + PC + PD > AC + BD$ and state this as a theorem.



Ex. 86. Prove that in the figure of Ex. 76; $g + d + OB < a + b + c$ and state this as a theorem.

Ex. 87. Construct an angle of 45° by bisecting a right angle.

Ex. 88. Draw a line AB. Through any pt P in AB, draw | DPC making $\angle CPA$, 45°. Through the pt Q, ½ in. from P, in CP draw $EQF \parallel AB$. Find by measuring the angles how many $\angle$s = $\angle CPA$. Find how many $\angle$s = $\angle CPB$. How many degrees in $\angle CPB$? Why? What is in this case true of the alternate exterior $\angle$s? What is true of the alternate interior $\angle$s? What is true of the corresponding $\angle$s? How many $\angle$s are there = 45°? How many = 135°? How many obtuse angles have you made in this figure? How many acute $\angle$s?

Theorem XVI.

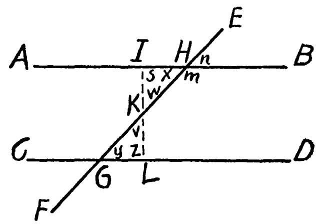
66. *Two lines cut by a transversal are parallel;*

1. If two alternate angles are equal;

2. If two corresponding angles are equal;

3. If the sum of the interior angles on the same side of the transversal is two right angles.

I. Let the transv. EF cut AB and CD making $\angle IHK = \angle$ . . . ., or $\angle$ . . . . $=$ $\angle$ . . . .; or $\angle EHI =$ . . . . or . . . . $=$ . . . . Then . . . .



1. For through K the mid-pt of GH draw $IK \perp AB$.
2. In $\triangle$s IHK and HKL, $HK =$ . . . . . Why?
3. $\angle x =$ . . . . Why?
4. $\angle$ . . . $=$ . . . . . Why? §29
5. $\therefore \triangle$ . . . . $=$ . . . . Why? §40
6. $\therefore \angle KLG =$ . . . . . Why?
7. AB and CD are each $\perp$ AB. Why?
8. . . . . Why?
9. Sim. if $\angle KHB = \angle$ . . . . Why?

Give the proof in the case of the exterior alternate angles.

II. Let the $\angle EHI = \angle$ . . . or $\angle$ . . . . $=$ $\angle$ . . . . Then $AB \parallel CD$.

1. For $\angle EHI = \angle$ . . . Why? §29
   Complete the proof.

III. Let $\angle BHG +$ . . . . . $= 2$ rt $\angle$s. Then AB . . .

1. For $\angle IHK = \angle KGL$,
   each being the sup. of BHG.
2. $\therefore$ . . . . Part 1

Ex. 89. State the converse of Theorem XVI as a theorem.

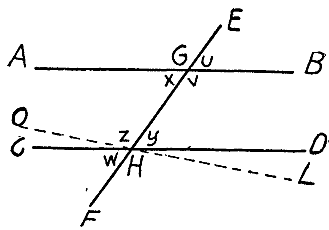
Theorem XVII.

67. *If a transversal cut two parallel lines;*

1. The alternate angles formed are equal.

2. The corresponding angles are equal.

3. The sum of the interior angles on the same side of the transversal is two right angles.



Let | EF cut . . . .

I. $\angle x =$ . . . and . . . . = the alt. ext. $\angle w$.

1. For through H draw OL making $\angle GHL = \angle x$.

2. $\therefore OL \parallel$ . . . . Why? 66: 1

3. $\therefore OL \cong$ . . . . Why? 17: 8

4. $\therefore \angle GHL \cong \angle y$.

5. $\therefore x =$ . . . . Why? 17: 1

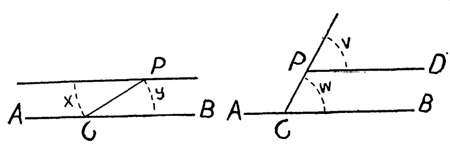
Prove that the $\angle u =$ the alt. ext. $\angle w$. See §§29 and 17: 1.

II. and III. See §29, note the method of proof of §§66: 2 and 66: 3.

Ex. 90. Let $y = 23° 27'$. Find the other 7 angles.

Ex. 91. Let $w = \frac{1}{2} v$. Find the other angles.

Ex. 92. Prove that the bisectors of x and y are parallel, and enunciate this statement generally as a theorem. See §67: 1.



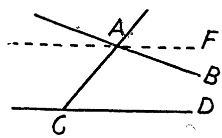
Ex. 93. Construct Problem IX by §66: 1.

Ex. 94. Construct Problem IX by §66: 2.

Ex. 95. If in fig. of §66; $EHB + FGD = 2$ rt $\angle$s; $AB \parallel CD$. Prove this statement and enunciate it as a theorem.

THEOREM XVIII.

68. *Two lines cut by a transversal making the sum of the angles on one side of it less than two right angles must meet on that side of the transversal.*



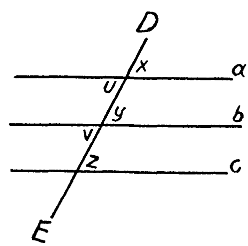
Let the . . . .

1. For const. . . . . || *CD*.
2. *AB* does not ≅ . .
 Why? §67: 3
3. *AB* is not || . . Why? §17: 8
4. *AB* cannot meet *CD* on the left . . . . Why? §17: 7
 since it would then cut *AF* in 2 pts., which is absurd.
5. But *AB* must meet *CD*. Why? §15
6. . . . .

Q. E. D.

THEOREM XIX.

69. *Lines parallel to the same line are parallel to each other.*



Let a || *c* and *b* || . . . Then . . . .

1. For draw *DE* making ∠s *x*, *y*, and *z*, with *a*, *b*, and *c* respectively.
2. $\angle x =$ . . . . Why? §67: 2
3. ∠ . . = ∠ . . . Why? §67: 2
4. ∴ ∠ . . . = ∠ . . .
 Why? §17: 1
5. . . . || . . . Why? §66: 2

Q. E. D.

Ex. 96. Prove §69 by applying §67: 1 instead of §67: 2.

Ex. 97. In the figure of §67 the bisectors of x and y are parallel.

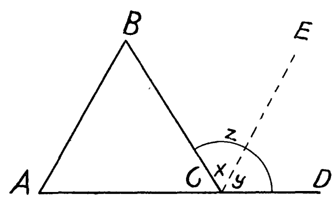
Ex. 98. If 2 || |s be cut by a transversal the bisectors of the alternate exterior ∠s are || and the bisectors of the corresponding ∠s are ||.

Ex. 99. If, in the figure of §69, $z = 34° 27'$, find all the other angles.

THEOREM XX.

70. 1. *If one side of a triangle be produced the exterior angle is equal to the sum of the interior and remote angles.*

2. *The sum of the angles of a triangle is two right angles.*



I. Let | AC of $\triangle$ . . . . .

1. For draw CE || . . .

$\angle x =$ . . Why? §67: 1

2. $\angle$ . . . Why? §67: 2

3. . . . . by adding (1) and (2).

II. Let ABC be $\triangle$; . . .

1. For $\angle A + \angle B =$ . . . . . Why? §70: 1

2. $\angle$ . . $\cong$ . . . . .

3. $\angle A + \angle B + \angle C = \angle C + \angle z$. Why?

4. But $\angle C + \angle z = 2$ rt $\angle$s. Why? §22

5. . . . . Q. E. D.

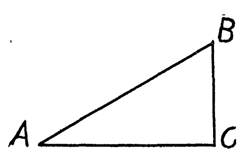
Ex. 100. How many degrees are there in the three $\angle$s of a $\triangle$?

Ex. 101. How many degrees in one $\angle$ of an equilateral $\triangle$?

Ex. 102. If in the figure of §70, $z = 110°$ and $B = 50°$; find A.

State this exercise as a theorem, beginning: If one side of a triangle be produced forming . . . . . .

THEOREM XXI.



71. *The acute angles of a rt-angled triangle are complements of each other.*

See §70: 2.

Ex. 103. $\angle A = 35° \ 20'$; find B; $A = 60°$; Find B; $A = 14° \ 40'$; find B.

Ex. 104. $\angle B = 3 \angle A$; find each $\angle$; $\angle A = 8 \angle B$; find each $\angle$.

Ex. 105. $\angle A$: $\angle B = 4$: 5. Find each angle.

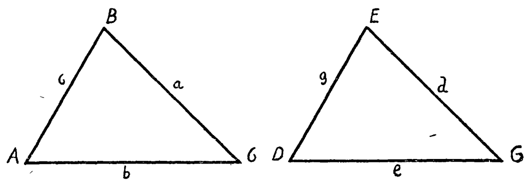
Ex. 106. $\angle A - \angle B = 20°$. Find A and B.

Theorem XXII.

72. *1. If two angles of a triangle are equal to two angles of another, each to each, the two triangles are mutually equiangular.*

2. If two angles and a side of one triangle are equal to two angles and the homologous side of the other, each to each, the two triangles are equal.

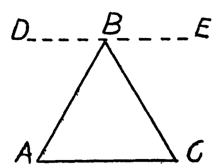
3. Homologous altitudes of equal triangles are equal.



I. See §70: 2.

II. Apply §§72: 1 and 40.

III. Draw a figure, and apply §70: 2.



Ex. 107. In a △ ABC, $A = 2B = 4C$. Find A, B, and C.

Ex. 108. If the angles of a △ are $3x$, $4x$, and $5x$; find each ∠.

Ex. 109. Prove §70: 2 by this figure. See §67: 1.

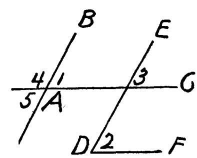
73. If the sides of two angles are so situated that, by revolving one angle about its vertex till one of its sides is parallel to the similarly situated side of the other, both the sides lie in the same direction from the vertex, the sides of the angles are said to lie in ***corresponding directions.***

Draw two angles whose sides are parallel and in the same direction from the vertex; Draw 2 ∠s whose |s are || and in opposite directions from the vertex. Draw 2 ∠s whose |s are ||, one in the same direction from the vertex, as the corresponding side of the other ∠, and the remaining |s in opposite directions from the vertex. Draw 2 ∠s whose |s are not ⊥ but lie in corresponding directions from the vertex. Point out in the figure of §74 the ∠ which conforms to each of the above conditions except one.

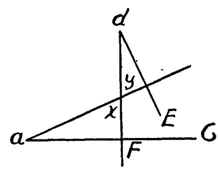
Theorem XXIII.

74. *1. Angles whose sides are parallel and lie in the same direction are equal.*

2. Angles whose sides are perpendicular and lie in corresponding directions are equal.



I. Let $\angle$s 1 and 2 have . . . . . .
1. For $\angle 1 =$ . . Why? §67: 2
2. $\angle 3 =$ . . . . Why? §67: 2
3. . . . . Why? §17: 1



II. Let $\angle$s a and d have | dF $\perp$ . . and . . $\perp$ . .; then . . . . .
1. For $\angle a$ is the comp. . . . Why? §71
2. $\angle$ . . . . Why? §71
3. $\angle x =$ . . . . . Why? §29
4. . . . . Why? §28

Q. E. D.

Ex. 110. In the figure to §74: 1, prove that $\angle 5 = \angle 2$ and state the fact as a theorem.

Ex. 111. Prove that $\angle 4$ is the sup. of $\angle 2$, and state the fact as a theorem.

Ex. 112. Prove §74: 2 by revolving $\angle d$ through 90°.

Ex. 113. If in the figure of §67, v and y are bisected, their bisectors are $\perp$.

Ex. 114. The vert. $\angle$ of an isosceles $\triangle = 40°$. Find the base $\angle$s.

Ex. 115. The vert. $\angle$ of an isosceles $\triangle$ = the sum of the base $\angle$s. Find each $\angle$.

Ex. 116. If the vert. $\angle$ of an isoseles $\triangle = A°$; find the base $\angle$s.

Ex. 117. If the base $\angle$ of an isosceles $\triangle = B°$; find the vertical $\angle$.

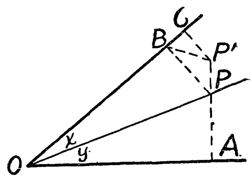
Ex. 118. In $\triangle ABC$, $A = 2C$ and $B = 3C$. Find each $\angle$.

Ex. 119. If in the fig. of §45: 1, $x = 32°$; find $\angle$s A and C.

Ex. 120. If in $\triangle$ of Ex. 83, $\angle PVS = 80°$, find all the other angles.

THEOREM XXIV.

75. *The bisector of an angle is the locus of a point equally distant from its sides.*



Let OP bisect the $\angle O$.

I. Let $P \equiv$ any pt. in OP, and PA and PB its distances from the $|$s of $\angle O$.

1. $PA \perp OA$ . . . . . Why?

2.—6. Prove that $\triangle OPA = \triangle OPB$. (See §72: 2).

7. . . . .

II. Let $P' \equiv$ any pt. without OP, and $P'A$ and $P'C$ . . .

1. For $P'A \perp$ . . . and . . . $\perp$ . . .
2. Join $P'B$.
3. $AP = BP$. Why?
4. $PP' \cong$ . . .
5. $AP' =$ . . . . . Why? §17: 2
6. But $PP' + P'B >$ . . Why?
7. $\therefore AP' >$ . .
8. Also $P'B > P'C$. Why?
9. Much more then . . . . Why? §17: 6

Q. E. D.

Ex. 121. Mark a point O on paper. Construct the locus of a point 1 inch from O. See §§12 and 61.

Ex. 122. Construct the locus of a point ½ inch distant from the circumference of the circle of Ex. 121. How many lines are there in the locus?

Ex. 123. Construct the locus of a point equidistant from two intersecting lines. Show that the locus consists of two lines intersecting at right angles.

Ex. 124. The exterior angle of an isosceles triangle is twice the adjacent interior angle. Prove that the triangle is equilateral.

Ex. 125. The vertical angle B of an isosceles triangle is equal to the sum of the base angles A and C; find each angle.

Theorem XXV.

76. *If, from any point on a perpendicular to a given line, oblique lines be drawn to that line:*

1. Lines cutting off equal distances from the foot of the perpendicular are equal.

2. Conversely: Equal lines cut off equal distances from the foot of the perpendicular.

3. Of two unequal lines the one cutting off the greater distance from the foot of the perpendicular is the greater.

4. Conversely: The greater line cuts off the greater distance from the foot of the perpendicular.

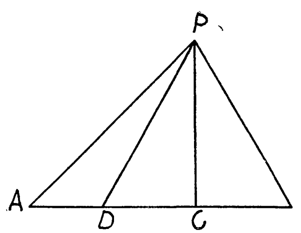
5. Only two equal oblique lines can be drawn.

From P let a $\perp$ PC. . . . .

I. Let $DC = CB$ then . . .
See §38. Make 5 steps.

II. May be proved by the method used in §54: 2.

III. Let $AC >$ . . ., then . . $>$ . . .



1. For $\angle ADP >$ the rt $\angle DCP$. Why? §70: 1
2. $\angle PAD <$ a rt $\angle$. Why? §70: 2
3. $\angle$ . . . $>$ . . . Why? §17: 6
4. $AP >$ . . . Why? §54: 1

IV. May be proved by the method of §54: 2.

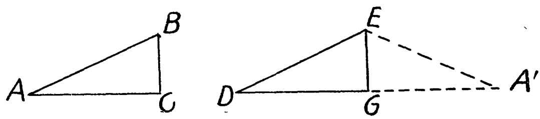
V. Only two eq. obl. |s, as PD . . . .
1. For if possible let PA . . . . .
2. Since PA does not $\cong$ PD, either $AC > DC$ or . . . .
3. $\therefore$ PA . . . $>$ or . . . $<$ . . . .
4. . . . .

Q. E. D.

Ex. 126. $\angle PDC - \angle PAC = 20°$; find $\angle APD$.

Theorem XXVI.

77. *Two right-angled triangles having the hypotenuse and a side equal in each are equal.*

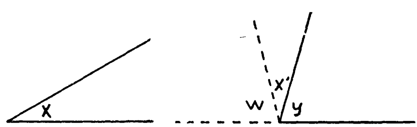


In the rt$\triangle$s . . . .

1. For apply $\triangle$ ABC to $\triangle$ DEG, BC on its equal EG, but A at A' opposite D.
2. DGA is a | . . . . Why? §22
3. $DG = GA'$. Why? §76: 2
4. . . . . Q. E. D.

Problem X.

78. *Given two angles of a triangle to construct the third angle.*



Given two $\angle$s of $\triangle$, x and y . . . .

79. Def. A ***quadrilateral*** is a rectilinear figure having four sides.

80. Def. A ***trapezium*** is a quadrilateral which has no parallel sides.

81. Def. A ***trapezoid*** is a quadrilateral which has two parallel sides. It is ***isosceles*** when the non-parallel sides are equal.

82. Def. A ***parallelogram*** is a quadrilateral whose opposite sides are parallel.

83. Def. A ***rhomboid*** is an oblique-angled parallelogram.

84. Def. A ***rhombus*** is an equilateral rhomboid.

85. Def. A ***rectangle*** is a right-angled parallelogram.

86. Def. A ***square*** is an equilateral rectangle.

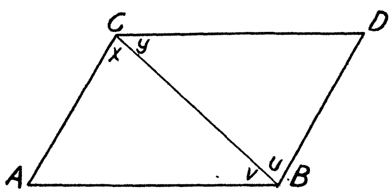
Ex. 127. Draw a trapezium, a trapezoid, a rhomboid, a rhombus, a rectangle, and a square. How many of the figures are quadrilaterals? how many parallelograms? how many rectangles?

Theorem XXVII.

87. *1. A diagonal divides a parallelogram into two equal triangles.*

2. The opposite sides and angles of a parallelogram are equal.

3. If two opposite sides of a quadrilateral are equal and parallel it is a parallelogram.



I. The diag . . . . .

1. For in $\triangle$s. . . .
   $\angle x$ . . . .
   Why? §67: 1

2. . . . .
   Why? §67: 1

3. . . . . $\cong$ . . . . .

4. $\triangle$ . . . . . . Why? §40

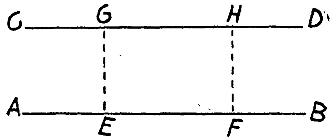
II. See part 1.

III. See §§67: 1, 38, and 66: 1.

Ex. 128. The sum of the angles of a quadrilateral is 4 rt $\angle$s. Apply §70: 2.

THEOREM XXVIII.

88. *Parallel lines are everywhere equally distant.*



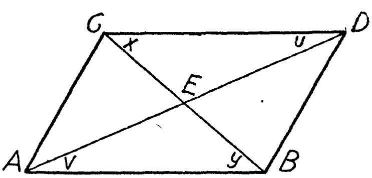
Let E and $F \equiv$ any pts. in AB, and EG and FH, their distances from CD, a $| \parallel AB$, then $EG = FH$.

1. For . . $\perp$ . . and . . $\perp$ . . Why? §60: 2
2. .·. . . $\parallel$ . . Why? §63: 1
3. .·. EH is a ▱. Why? §82
4. . . . . Why? §87: 2

Q. E. D.

THEOREM XXIX.

89. *The diagonals of a parallelogram bisect each other.*



Let AD and BC be the diag's of ▱ . . . .

1. For in $\triangle$s . . . $\angle x$ = . . and $\angle u$ . . . . Why? §67: 1
2. . . . . Why? §87: 2
3. $\triangle$ . . . . . . Why? §40
4. $CE =$ . . and . . $=$ . . Why?

Q. E. D.

Ex. 129. If the opposite sides of a quadrilateral are equal, it is a parallelogram. See §§48 and 66: 1.

Ex. 130. State what is given and what to be proved in theorems XV to XXV.

Ex. 131. The quadrilateral whose diagonals bisect each other is a parallelogram. Apply §§29, 38, 66: 1.

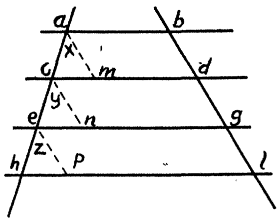
Ex. 132. The diagonals of a rectangle are equal.

Ex. 133. The diagonals of a rhombus bisect each other at right angles.

Ex. 134. State and prove the converse of theorem XXVIII.

Theorem XXX.

90. *A system of parallel lines which divides one transversal equally divides all transversals equally.*



Let bl cut $\|$ $|$s ab, . . . making $bd =$ . . ., $=$ . . ., and also ah cut . . . . . .; then $ac =$ . . $=$ . .

1. For draw $am \|$ . . . .
2. $am =$ . ., cm . . ., and . . . Why? §87: 2
3. $am \|$ . . $\|$ . . Why? §69
4. Now in $\triangle$s amc and cne, $am =$ . . Why? §17: 1
5. $\angle x =$ . . . and $\angle acm$ . . . Why? §67: 2
6. $\triangle$ . . . . . . Why? §38
7. $ac =$ . . Why?
8. Sim. . . . .

Q. E. D.

Ex. 135. State the converse of theorem XXVII: 1.

Ex. 136. State the converse of exercise 132 beginning: The parallelogram whose . . . . . . Prove this to be true.

Ex. 137. State and prove the converse of Ex. 133.

91. Def. A ***polygon*** is a figure bounded by straight lines, called ***sides,*** which taken together constitute its ***perimeter.***

92. Classes of polygons by the number of sides.

NO. OF SIDES.	NAME.	NO. OF SIDES.	NAME.
3.	Triangle.	8.	Octagon.
4.	Quadrilateral.	9.	Nonagon.
5.	Pentagon.	10.	Decagon.
6.	Hexagon.	11.	Undecagon.
7.	Heptagon.	12.	Dodecagon.

93. DEF. A ***diagonal*** of a polygon joins two angles not adjacent.

Draw a polygon of 5 sides and all possible diagonals.

94. A ***regular polygon*** is one that is equilateral and equiangular.

95. A ***convex polygon*** is one in which each interior angle is less than two right angles.

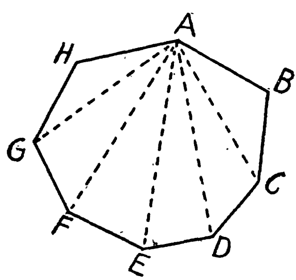
96. A ***concave polygon*** is one which has one or more interior angles called ***reentrant angles*** greater than two right angles.

Ex. 138. Draw as many diagonals as possible in a hexagon.

97. DEF. Two polygons whose sides taken in the same order are equal, each to each, are ***mutually equilateral.*** If their angles are similarly equal they are ***mutually equiangular.***

THEOREM XXXI.

98. *The sum of the interior angles of a polygon of n sides is 2(n—2) right angles.*



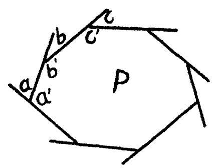
Let $AB \ldots \equiv$ a polygon of n |s; then $\angle A + \ldots\ldots =$ 2(n—2) rt $\angle$s.

1. For from A draw . . . . .
2. Each | of . . . except AB . . . may be regarded as the base of $\triangle$ whose vertex is A.
3. Since the polygon has n |s . . . . .
4. All the $\angle$s of all the $\triangle$s . . . . §70: 2
5. But the sum of . . = the sum of . . . Why? §17: 5
6. The sum . . . . . . Why? §17: 1

Q. E. D.

Theorem XXXII.

99. *The sum of the exterior angles of any polygon is four right angles.*



Let $P \equiv$ a polygon of n |s, a, b, c, etc., its ext. $\angle$ s; then $a + b + c +$ etc. $= 4$ rt $\angle$ s.

1. For $a + a' = 2$ rt $\angle$ s. Why?
2. Sum of $\angle$ s at all the vertices $=$ 2 n rt $\angle$ s. Why?
3. But sum of $\angle$ s of $P = \ldots$ Why? §98
4. $a + b + c$ etc. $= 4$ rt $\angle$ s. Why?

Q. E. D.

Ex. 139. Draw a pentagon having one reentrant angle.

Ex. 140. Draw a pentagon having two reentrant angles.

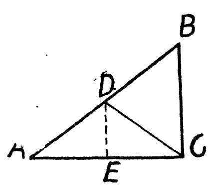
Ex. 141. Draw a hexagon having three reentrant angles.

Ex. 142. Find the sum of the angles of a quadrilateral; a hexagon; an octagon; a decagon; a dodecagon.

Ex. 143. Find one angle of each of the following regular polygons: a hexagon; a pentagon; a heptagon; a nonagon; an undecagon.

Ex. 144. Find in degrees, minutes, and seconds one exterior angle of a regular heptagon; a regular undecagon.

Ex. 145. If a line parallel to the base of a triangle bisect one of the other sides, it bisects both. Suggestion. Draw a third parallel through the vertex and apply §90.



Ex. 146. The mid-point of the hypotenuse of a right triangle is equidistant from the vertices of the three angles. Apply §63: 1, Ex. 145, §76: 1.

Ex. 147. If the median to a side of a triangle is equal to half that side, the angle opposite it is a right angle. Apply §45: 1 twice, then §70: 2.

Ex. 148. On a given line construct a regular dodecagon. Suggestion. Construct an exterior angle by bisecting an angle of an equilateral triangle.

PROBLEM XI.

100. *On a given line as a side:*

1. Construct an equilateral triangle.

2. Construct a square.

3. Construct a hexagon.

I. See §37.

II. See §51: 1.

III. Construct ∠s by applying §100: 1 twice.

FIELD WORK.

101. The name Geometry originally meant measuring the earth. With due care to secure accuracy, the application of the science to actual measurements in the fields is not only intensely interesting to young pupils, and useful in manual training, but of great value in developing intellectual faculties whose culture is too much neglected.

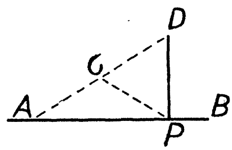
Apparatus need not be expensive but should be provided in such abundance as to give employment to every member of the class. Surveyors chains, tapes, pins and ranging poles are desirable, yet most of the work of providing inexpensive substitutes may be performed by the class, and serve as useful manual training. Straight sticks one to two feet long, for pins, and others five to ten feet long, for ranging poles, preferably of hard wood, may sometimes be secured without cost, as carpenters' waste material, and in any case cost little. A ball of strong twine may be cut into lengths slightly exceeding 50, 100, and 66 feet. Every 10 feet (or 10 links) may be marked by tieing on a knot made of colored office tape. Feet (or links) may be marked by knots of colored druggists' twine.

The measuring and marking should be done by the pupils, being in itself a valuable exercise. If no suitable fields are available, streets little travelled, may be used.

102. By the exercise of ingenuity all problems may be adapted to field work, but it is best to spend most of the time on such problems as a surveyer might be called on to perform. The ability to run a straight line and to measure it accurately is the foundation of all surveyors work and should receive due attention. We add a few exercises. In most cases, where lines are given or required, it is better to work them using only two or more ranging poles as instruments.

Ex. 149. To produce a straight line. Place two ranging poles at points in the line and sighting by these place a third pole, then using those already placed extend the line as far as desired.

Ex. 150. At a given point in a given line erect a perpendicular to the line.

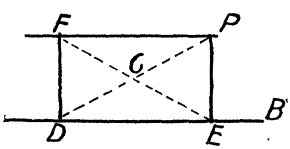


Hold one end of the chain at P, the given point, the other at A on the given line, and the middle at C, without the line. Still holding fast the end at A and the middle at C revolve the part CP till it is in the same straight line with AC. $DA \perp AB$. Prove this by Ex. 147.

Ex. 151. At a given point without a line construct a perpendicular to that line.

Let D, fig. of Ex. 150, be the given pt, AB the given line. Stretch the chain from D to A, a point in the line. Complete the construction.

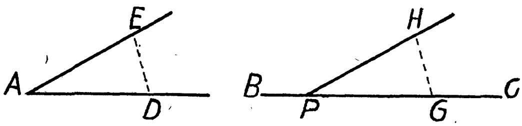
Ex. 152. Construct through a given point a line parallel to a given line.



Through P construct a line parallel to DB. Stretch the chain from P to D in the given line. Revolve the chain about its center, held fast at C till the end at P is at E in | DB and the part CD in the position CF, making | $FP \parallel CB$. Prove this.

103. Evidently a part of the chain, tape or string may be used instead of the whole. Lines should be determined not by stretching a string but by points, e. g., in the figure of Ex. 152, ascertain whether D, E and B are exactly in one line by standing at D and observing whether E is exactly between D and B.

Ex. 153. To construct an angle equal to a given angle. At B in the line BC construct an angle equal to $\angle A$.



Hold three points of the chain at A, E, and D pts., at the vertex and on the sides of the given $\angle$ and an end long enough to reach from E to D. Stretch a part of this end from E to D and note the length of each side of the $\triangle$ ADE. Complete the construction and prove it.

Ex. 154. To find the distances between two objects visible from each other, but separated by a pond or other obstacle to measurement.

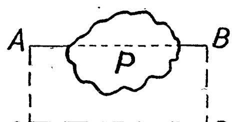


Fig. Ex. 154.

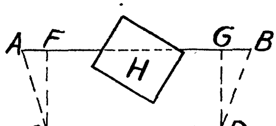


Fig. Ex. 155.

Ex. 155. To find the distance between two objects, neither of which can be seen from the other.

In this case it is impossible to make $AC \perp AB$, exactly. We therefore lay off $AC = BD$, and as nearly $\perp AB$ as possible, D being visible from C. The measured distance CD must then be corrected by adding or subtracting AF and GB the distances from A and B to the feet of $\perp$s to CD from C and D. This method is usually sufficiently exact, but if greater accuracy is required especially if AF and GB should be more than one or two rods, let fall $\perp$s from A and B to the $\|$ CD and correct the positions C and D by taking equal distances from A and B on those $\perp$s.

SUPPLEMENTARY THEOREMS AND EXERCISES.

THEOREM XXXIV.

104. *A triangle is determined.*

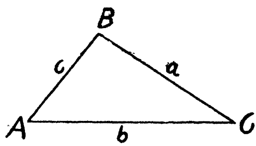
1. *By three sides.* §48

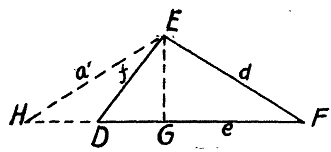
2. *By two angles and a side.* §72: 2

3. *By two sides and the included angle.* §38

4. *By two equal sides and an angle.*

5. *Two sides and the angle opposite the greater.*





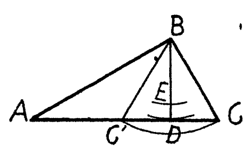
IV. See §§45: 1 and 70: 2.

V. In △s . . . let c . ., a = . ., and ∠A, opposite the gr. | = . . .; then . . . .

1. For draw EG ⊥ . .
2. Now apply △ . . ., ∠A on its eq. . . . . ., $c \cong$ . . .
3. Either $a \cong d$. Why?
4. Or a meets DF . . . . Why?
5. But a cannot take the position a'. Why?
6. $a \cong d$; △$ABC \cong$ . . ., and . . . Why? §14

Q. E. D.

Ex. 156. When it is known whether a triangle is acute angled or obtuse angled, two sides and the angle opposite the less determine a triangle.

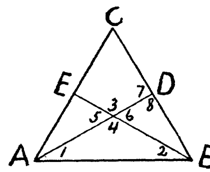


Ex. 157. Const. 2 ‖ |s and the locus of a point equally distant from them.

Ex. 158. Const. the complete locus of a point 1 cm distant from a given limited line 3 cm long.

Ex. 159. Const. the complete locus of a point distant from the perimeter of an eq. △, ¼ the length of one side.

Ex. 160. Const. an eq. △ and divide it into 9 eq. △s, thus trisect. each |. See §§42, 64, 67: 1, and 72:2.



Ex. 161. In △ ABC, $AC = BC$, AD and BE are angle-bisectors: 1) If $B = 70°$, find ∠s 1, 3, and 8; 2) If $B = \frac{2}{3}C$; find ∠s 2, 5, and 7. 3) If ∠4 = 110°; find C. 4) If $B = B°$; find all the other ∠s.

Ex. 162. (Figure of §76). If from a pt. without a |, a ⊥ and also oblique |s, be drawn: 1) Oblique |s making equal ∠s with the ⊥ are equal, and conversely. 2) Obl. |s making uneq. ∠s with ⊥ are uneq., and conversely.

Ex. 163. If from a pt. without a |, obl. |s be drawn: 1) Eq. obl. |s make eq. ∠s with the base, and conversely. 2) Of two uneq. obl. |s, the gr. makes the less ∠ with the base, and conversely. 3) Only two |s can be drawn making eq ∠s with the base.

Ex. 164. An ext. base ∠ of an isos. △ = 1 rt ∠ + the vert. ∠.

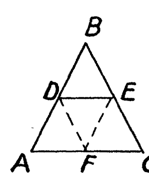
105. Def. When three or more lines pass through the same point they are said to be ***concurrent***.

106. Def. A ***unique point*** is the only one possessing a given property; as the intersection of two lines.

Ex. 165. Draw five concurrent lines.

Ex. 166. What kind of a point is the intersection of six concurrent lines.

Theorem XXXV.

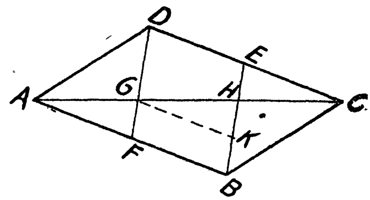


107. *The line joining the mid-points of two sides of a triangle is parallel to the third side and half of its length.*

Ex. 167. The |s joining the mid-points of the adj. |s of a quadrilateral form a ▱ whose |s are ∥ the diagonals of the quadrilateral and half their length.

THEOREM XXXVI.

108. *Lines drawn from two opposite angles of a parallelogram to the mid-points of two opposite sides trisect the diagonal they cut.*



Prove $\triangle AGF = \triangle GHK$ and $\triangle GHK = \triangle EHC$. See §§87:2, 67:2, 67:1, and 72:2.

Ex. 168. The |s joining the mid-pts of adj. |s of a rhombus form a ▭.

Ex. 169. The bisectors of the base ∠s of an isosc. trapezoid form an isosc. △

Ex. 170. Homologous medians of eq. △s are equal.

109. DEF. A ***median*** of a polygon of 2n sides joins the mid-points of opposite sides. A median of a polygon of 2n+1 sides joins the mid-point of a side to the vertex of the opposite angle.

Illustrate this by drawing polygons of 3, 4, 5, 6, and 7 sides, all possible medians to two of them, and one median to each of the others.

Ex. 171. If $n = 5$, find the number of medians in a polygon of 2n sides; in a polygon of $2n + 1$ sides.

Ex. 172. The medians of a ▱ ∥ its |s and = its |s.

Ex. 173. The medians of a ▱ bisect each other.

Ex. 174. The medians of a ▱ bisect its diagonals.

Ex. 175. The intersection of the medians and diagonals of a parallelogram is a unique point. See §106.

Ex. 176. Homologous angle bisectors of equal triangles are equal, also the segments of sides made by them.

Ex. 177. Homologous segments of sides of equal triangles made by homologous perpendiculars are equal.

Ex. 178. Construct an angle of 60°. (Const. an eqtl. △.)

Ex. 179. Construct angles of 30°, 15°, and 7½°. See §42.

THEOREM XXXVII.

110. *In any triangle, the following lines are concurrent:*

1. The three angle bisectors.

2. The three perpendiculars to the mid-points of the sides.

3. The three perpendiculars from the angles to the opposite sides.

4. The three medians; the greater segment of each being twice the smaller.

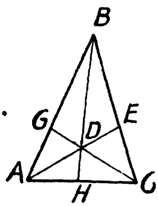


Fig. 1.

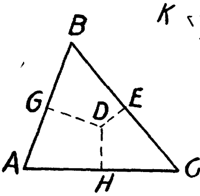


Fig. 2.

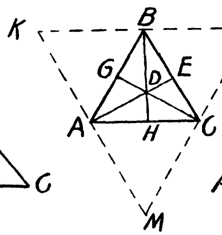


Fig. 3.

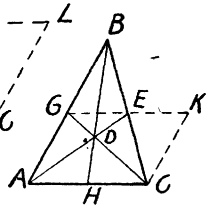


Fig. 4.

I. Apply §75.

II. Apply §62.

III. Apply §§51, 63 1:, 87: 2, 17: 1 and 110: 2.

IV. Let AE . . . .

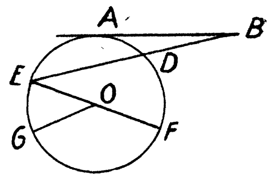
1. For join GE and produce it making $GK = AB$.
2. $GK \parallel AC$. Why? §107
3. AK is a ▱. Why? §87: 3
4. E is the mid-pt of GK. Why?
5. $GD = \frac{1}{3}\ GC$. Why? §108
6. Sim. by joining EH etc., it may be shown that BH cuts off $\frac{1}{3}$ of GC and passes through D.
7. ∴ . . . . Q. E. D.

111. By methods similar to those used in proving §72: 3 and Ex. 176 and 177, it may be shown that all homologous lines and parts of equal triangles are equal.

BOOK II.

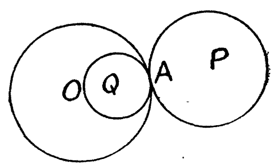
THE CIRCLE.

REVIEW §§11 and 12.



112. DEF. A ***tangent*** to a circle (or curve) is a line which meets it in a point but being produced does not cut it.

Similarly the circles O and Q are *tangent* to ○ P *externally* and the circle Q to ○ O *internally.*



113. DEF. A ***secant*** is a line that cuts a circle.

Name the secant in the fig. above.

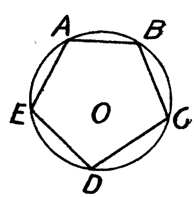
114. DEF. A ***segment*** is the part of a circle included between an arc and its chord.

Name the smaller and greater segm. formed by the chord ED.

115. DEF. A ***sector*** is a part of a circle included between two radii and the circumference.

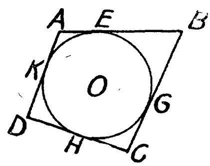
The radii EO and OG form two *sectors* one greater than a semicircle and one less. In this book the word *sector* will be used for the smaller *sector*, only, unless otherwise specified.

116. DEF. The angle formed by two radii is called a ***central angle.***



117. DEF. An ***inscribed*** polygon has the vertices of all its angles in the circumference. The circle is then ***circumscribed*** about the polygon.

118. Def. A polygon all of whose sides touch a circle is said to be ***circumscribed*** about it. The circle is then said to be ***inscribed*** in the polygon.



THEOREM I.

119. *1. The diameter of a circle is greater than any other chord.*

2. A diameter bisects the circle and its circumference.

I. Apply §§12 and 53.

II. Let AB be . . . .

1. For revolve the segment ACB about AB as an axis till it falls on ADB.

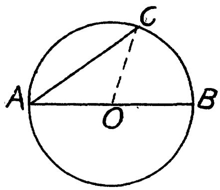


Fig. 1.

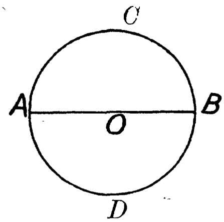


Fig. 2.

2. $\smile ACB \cong$ . . . and the segm. . . . . . .

For if any pt. in $\smile ACB$ does not $\cong \smile ADB$ it must either be more remote from the center, or nearer to the center, either of which is contrary to the definition of a circle.

3. $\smile$ . . = . . and the segm. . . . = . . .

4. . . . . Q. E. D.

THEOREM II.

120. *A straight line can meet the circumference of a circle in only two points.*

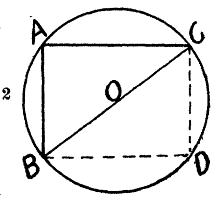
Since only two equal lines can be drawn from the center to the line.

Illustrate this by a figure, and find the theorem referred to.

THEOREM III.

121. *An angle inscribed in a semicircle is a right angle.*

Let $\angle A$ . . . .



1. For from C lay off $\smile CD = \smile AB$, join . . .
2. $\smile BAC = \smile$ . . Why? §119: 2
3. $\therefore \smile BD = \smile AC$. Why?
4. Now apply $\smile BAC$ to $\smile DCA$, B on D; since $\smile BA = \dots$ and $\smile \dots = \dots$, A falls on C and C on A.
5. $| \dots \cong \dots$ and $\dots \cong \dots$ Why? §17: 7
6. $\therefore \angle \dots \cong \dots$
7. $\angle A = \dots$ Why? §14
8. Sim. $\angle A = \angle D = \angle B$.

Complete the proof. See §98.

Ex. 180. Why can you not draw a chord 3 inches long in a circle whose radius is 1 inch?

Ex. 181. How many chords 2 inches long can be drawn parallel to a given line in a circle whose radius is 1 inch?

Ex. 182. Construct the locus of the vertex of a right triangle having a given hypotenuse one inch long.

Ex. 183. State as a theorem the general principle of which Ex. 182 is a special case beginning: The locus of the vertex of the rt $\angle$ of a rt$\triangle$ of which a given | is the . . . .

Ex. 184. Find the acute $\angle$s of an isosc. rt$\triangle$.

Ex. 185. If the adjacent extremities of two $\perp$ diameters of a ○ be joined a □ is constructed.

Ex. 186. Construct a □ in a ○ whose radius is 1 inch and find by measurement the length of one of its |s.

Ex. 187. Construct a square whose diagonal is 1 inch.

Ex. 188. Construct an equilateral triangle whose altitude is 1 inch.

Ex. 189. $\angle BAC = 70°$, $\angle BAD = 110°$, the vertex A is common. What kind of a line is BAD.

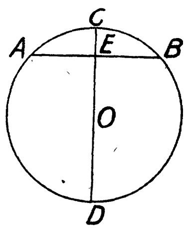
Ex. 190. In two $\triangle$s ABC and $A'B'C'$: $a = a'$, $b = b'$, $C = C'$, $A = 70°$, $B = 60°$, $c = 70$. Find A', B', c'.

THEOREM IV.

122. *1. The diameter which is perpendicular to a chord bisects the chord and also the arc which it subtends.*

2. The diameter which bisects an arc is perpendicular to the chord, which subtends it, at its mid-point.

3. The perpendicular to a chord at its mid-point is a diameter.



I. Let the diam. *CD* . . . . .

1. For revolve the semi ⊙ *CAD*.
2. ◡ *CA* falls on . . . . Why?
3. | *EA* falls on . . . . Why? §59
4. ∴ The pt *A* falls both in . . .
5. It falls at . . . . Why? §120
6. ∴ | *AC* ≅ . . . and ◡ *EA* ≅ . . .
7. ∴ . . . = . . . and . . . = . . .

II. Let the diam. . . . . .

May be proved in a manner similar to §122: 1.

III. Let *CD* ⊥ . . . .

1. For the diam. ⊥ *AC* cuts *AB* in *E*.
2. ∴ *CD* ≅ the diam. ⊥ *AB* in *E*. Why? §59
3. ∴ *CD* is a diameter of ⊙ *O*. Q. E. D.

Ex. 191. If the bisector of an arc also bisects the chord subtending it, it is the diameter ⊥ the chord. Of what theorem is this exercise the converse?

Ex. 192. The diameter which bisects a chord is perpendicular to the chord and bisects its arc.

Ex. 193. In two △s *ABC* and *A′B′C′*; $A = A'$, $B = B'$, $c = c'$, $a = 4$, $b = 5$, $C = 65°$. Find a', b', C'.

Ex. 194. In a △ *ABC*; $a = b$ and $C = 50°$. Find *A* and *B*.

Ex. 195. In a △ *ABC* $a = b$, $c = 22$. Find the segments of *c* made by the bisector of the ∠*C*; also the angles which this angle bisector makes with *c*.

Theorem V.

123. *In the same circle or in equal circles:*

1. Equal arcs are subtended by equal angles at the center.

2. Conversely: Equal angles at the center subtend equal arcs.

3. Equal arcs are subtended by equal chords.

4. Conversely: Equal chords subtend equal arcs.

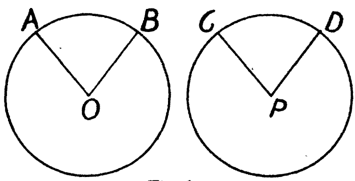


Fig. 1.

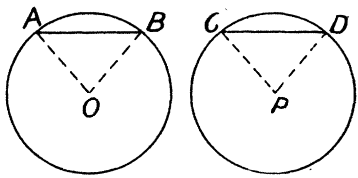


Fig. 2.

I. Let $\smile AB$ . . . ., then $\angle O$ . . . .

1. For apply $\odot\ O$ to . . . . ., O on . ., A on . . .; since $\smile AB$ . . ., B falls . . .

2. $OA \cong$ . . and . . $\cong$ . . Why? §17: 7

3. $\therefore \angle O \cong$ . . and $\angle$ . . = . . . Why? §14

Q. E. D.

II. May be proved similarly by superposition.

III. May be proved by superposition; may also be proved without superposition by §§123: 1 and 38.

IV. Apply §§48 and 123: 2.

Ex. 196. Equal angles at the center subtend equal chords.

Ex. 197. State and prove the converse of Ex. 196.

Ex. 198. State the reciprocal of Ex. 196.

Ex. 199. The greater angle at the center subtends the greater chord, and conversely, when both angles are less than straight angles.

Ex. 200. If an angle bisector of a triangle is perpendicular to a side it bisects that side and the triangle is isosceles.

Ex. 201. In $\triangle ABC$, $a = 4$, $b = 5$, and $c = 6$. Name the $\angle$s in order beginning with the largest, and give the reason.

Theorem VI.

124. *In the same circle or in equal circles:*

1. The greater arc is subtended by the greater chord, each being less than a semi-circle.

2. Conversely: The greater chord subtends the greater arc.

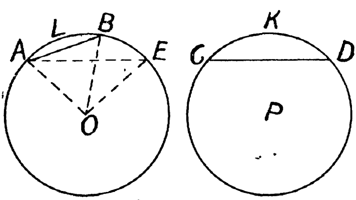


Fig. 1.

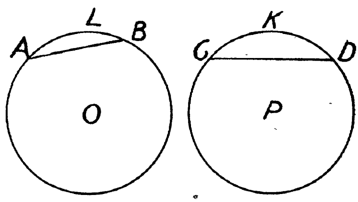


Fig. 2.

In the equal . . . .

I. Let $\smile CD >$ . . . . ., then the chord . . . .

1. For apply . . . . ., center P on . . . ., C on . . . and $\smile CD$ . . . .

2. Since $\smile CD >$ . . . D falls at some pt. on circumf. of O beyond B.

3. Join OA . . ., and . .

Now in $\triangle$s . . . . .

4. $OB =$ . ., and . . . . Why?

5. $\angle AOE >$ . . . . Why? §17: 4

6. . . . . Why? §55

7. . . . .

Q. E. D.

II. State by Fig. 2 and prove by the method of §54: 2 applying §§123: 4 and 124: 1.

Ex. 202. Prove §124: 2 by a method similar to that used in §124: 1, applying §56.

Ex. 203. State the theorem corresponding to §124, both arcs being greater than semicircles, and prove it.

Ex. 204. In eq. $\bigcirc$s, of any number of $\smile$s some gr. than $\frac{1}{2}\bigcirc$s and some less, the one nearest a $\frac{1}{2}\bigcirc$ is subt. by the greatest chord.

THEOREM VII.

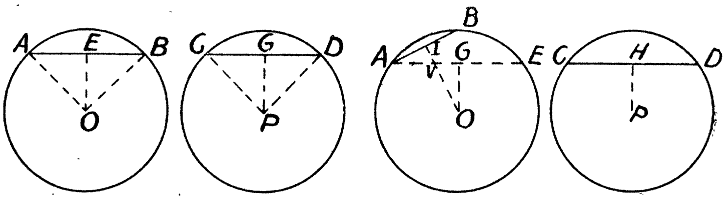
125. *In the same circle or in equal circles:*

1. Equal chords are equally distant from the center.

2. Conversely: Chords equally distant from the center are equal.

3. The greater of two unequal chords is nearest to the center.

4. Conversely: Of two chords unequally distant from the center, the one nearest to it is the greater.



In the . . . .

I. Let the chord . . . . . ., then OE the dist. . . . .

1. For join . . . ., ..., ..., and . . . .
2. $OE \perp$ . . .

Complete the proof applying §§48 and 72: 3.

II. (Fig. 1). Let OE be the dist. of . . . ., then . . . .

1. For join OB and PD.
2. $OE \perp$ . . . . and . . . . Why? §60: 2

Complete the proof applying §77.

III. In the $\odot$s. . . . . .

1. For apply $\odot$ . . . . . P on . . . and C on A.
2. D falls at some pt. as E more . . . . . Why? §124: 2
3. Draw $OG \perp$ . . . and OI . . ., OI cutting AE in V.
4. $OG >$ . . . . Why? §60: 1

5. and 6. Apply §17: 4 and 6.

IV. (Fig. 2.) The method of §54: 2 may be used.

PROBLEM I.

126. *1. To construct a line tangent to a given circle at a given point.*

2. To let fall a tangent on a given circle from a given point without it.

3. To draw a common tangent to two circles.

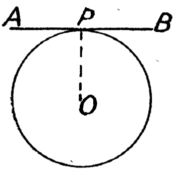


Fig. 1.

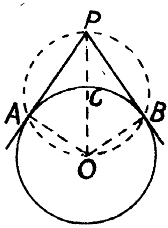


Fig. 2.

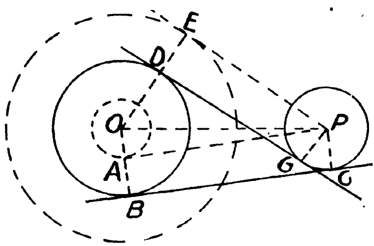


Fig. 3.

I. (Fig. 1). Apply §51: 1 in construction; §§12, 60: 1, and 112, in proof.

Study §127 before proving the other parts.

II. From P . . . .

1. Bisect OP in C. With C . . . . . describe $\frac{1}{2}\odot$ . . . . cutting . . . Join . . .

2. For $\angle OBP$ is . . . Why? §121

3. . . . . Why? §127: 1

Q. E. F.

III. See Fig. 3.

Show when there are four common tangents, when three, when two, when one, and when there can be no common tangent.

Ex. 205. State as a theorem the reciprocal of §123: 3.

Ex. 206. State as a theorem the reciprocal of §126: 1.

Ex. 207. State as a theorem the converse of §119: 2.

Ex. 208. If in eq. $\bigcirc$s, one $\smile > \frac{1}{2}\bigcirc$, and another $\smile < \frac{1}{2}\bigcirc$, the chord subtending the one which differs least from $\frac{1}{2}\bigcirc$ is $>$ the chord subtending the other.

THEOREM VIII.

127. *1. The line perpendicular to a radius at its extremity is tangent to the circle.*

2. Conversely: The perpendicular to the tangent at the point of contact passes through the center of the circle.

3. The diameter perpendicular to a tangent meets it at the point of contact.

I. Prove as in §126: 1.

II. (Fig. 1 of §126). Let OP be $\perp$ . . .

1. For the | $\perp$ . . . Why? §127: 1
2. No other tang. than . . . .
   For any | not $\perp$ PO must have a point nearer to C than P since . . . . . . §60: 1
3. . . . . Q. E. D.

III. Apply §60: 2.

THEOREM IX.

128. *The two tangents to a circle from the same point without it are equal.*

Use fig. 2 of §126 and apply §77.

Prove that only two tangents can be drawn from one point.

Ex. 209. Through what points can only one tangent be drawn to a circle? Through what points can no tangent be drawn?

Ex. 210. Tangents to a circle from points equally distant from its center are equal.

Ex. 211. Construct the locus of a point from which a tangent one inch long can be drawn to a given circle one inch in diameter.

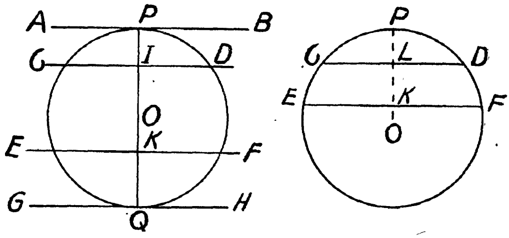
Ex. 212. Generalize Ex. 211 as a theorem beginning: The locus of points from which equal tangents . . . . . .

Ex. 213. Construct the locus of the mid-point of a chord $1\frac{1}{2}$ inches long in a circle whose radius is 1 inch. Apply §125: 1.

Ex. 214. State and prove the theorem of which Ex. 213 is a special case.

Theorem X.

129. *Parallels cutting or touching a circle intercept equal arcs, and conversely.*



Let the || |s AB . . . meet $\odot$ O in . . ; then $\smile$ LP=. ., $\smile$ . . . .

1. For through P draw . . . . $\perp$ . . . .
2. PQ $\perp$ . . . . Why? § 63: 2
3. PQ passes through . . . Why? §127: 2
4. $\smile$ CP = . ., $\smile$ . . = . ., and . . . . Why? §122: 1

II. State the converse first in words then by the figure. It may be proved either by applying §122: 2 or by superposition.

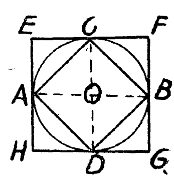
Problem II.

130. *1. Inscribe a square in a circle.*

2. Circumscribe a square about a circle.

I. Apply §§51, 38, and 121.

II. Apply §§51, 126: 1, 63: 1, and 87: 2.



Problem III.

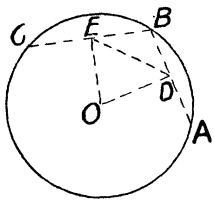
131. *1. To inscribe a regular octagon in a circle.*

2. To circumscribe a regular octagon about a circle.

Apply §§130, 51: 2, 122: 1, 123: 3, and 48.

THEOREM XI.

132. *Through any three points not in a straight line one circumference may be made to pass, and but one.*



Let A, B, and C be . . .; then . . . .

1. For join AB and BC, draw . . . ⊥ . . . .
2. $\angle OED < 1$ rt$\angle$ and $\angle$ . . . . Why? §17: 4
3. ∴ . . . . . $<$ 2 rt$\angle$s Why?
4. DO and . . . at O. Why? §68
5. O the common pt. of . . . is the locus . . Why? §62
6. ∴ O is equidist. from . . . . and the only pt. . . . .
7. ∴ Through A, B . . . . Q. E. D.

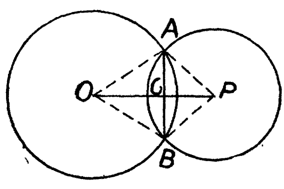
PROBLEM IV.

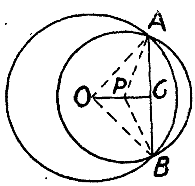
133. *To find the center of a given circle or arc.*

Apply the method used in §132.

THEOREM XII.

134. *If two circles cut each other the line joining their centers bisects their common chord at right angles.*





Let ○s . . . . . .

1. For join . . . . .
2. OA = . . and . . = . . Why? §12
3. | joining . . . . Why? §62

Q. E. D.

PROBLEM V.

135. *Construct with a given radius a circle tangent to a given circle at a given point.*

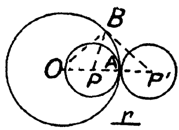


Fig. 1.

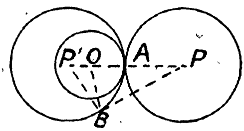


Fig. 2.

Construct a ⊙ tang. to ⊙ O at A with the radius r.

1. Join OA and produce it.
2. Lay off AP and AP' each = r.
3. With P and P' as centers . . . .
4. ⊙ P and P' are tang. to ⊙ O.
5. For, A is common to ○s O and P. Why?
6. No other pt. can be com . . . for if possible let B . .
7. Then, $OB + BP' = OA + AP'$ and $OB—BP = OA—AP'$. Why? §12
8. Which is absurd. Why?
9. ⊙s O and P meet in A, and A only, and. . Why? §112

Q. E. D.

The pupil will prove the proposition for the case in which r > OA (Fig. 2.)

THEOREM XIII.

136. *If two circles are tangent, the line joining their centers passes through the point of contact.*

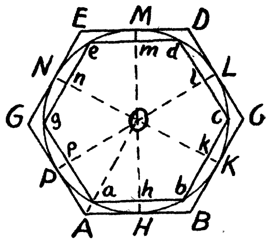
Use the figure of §135 and prove by a similar method.

Ex. 215. If two circles are tangent, the perpendicular to the line joining their centers erected at the point of contact is a common tangent.

Ex. 216. If two circles are tangent the distance between their centers is equal to the sum, or to the difference, of their radii.

Ex. 217. Find the locus of the centers of all circles tangent to a given circle at a given point.

PROBLEM VI.



137. *1. Inscribe a regular hexagon in a circle.*

2. Circumscribe a regular hexagon about a circle.

I. Suggestions: Const. *ab*, *bc*, etc., each = *Oa*.
Apply §§45: 3 and 70: 2.

II. Apply §§51: 2, 126: 1, 67: 2.

PROBLEM VII.

138. *1. Inscribe an equilateral triangle in a circle.*

2. Circumscribe an equilateral triangle about a circle.

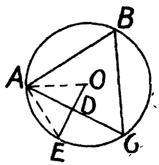
See §137.

Ex. 218. Explain how regular polygons of 12, 24, and 48 sides may be inscribed in a circle or circumscribed about it.

THEOREM XIV.

139. *The side of a regular hexagon is equal to the radius of the circumscribed circle.*

See §137.



Ex. 219. A side of an equilateral triangle inscribed in a circle bisects the radius perpendicular to it.

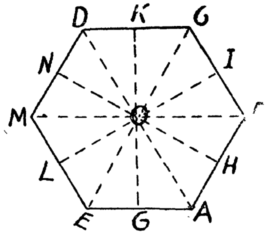
Ex. 220. What part of the diameter is *DE*?

Ex. 221. The radius of the circle inscribed in an equilateral triangle is one half the radius of the circle circumscribed about it.

Ex. 222. The angles of a quadrilateral are 2m, 3m 4m, and 6m. Find each angle.

Ex. 223. The angles of a hexagon are 4a, 5a, 6a, 7a, 8a, and 10a. Find each angle.

Problem VIII.



140. *1. Circumscribe a circle about a regular polygon.*

2. Inscribe a circle in a regular polygon.

I. Apply §§42, and 40.

II. Apply §§42 and 51: 2 in construction, §72: 3 in proof.

141. Def. The radius of the circle inscribed in a regular polygon is called the ***apothem.***

Name six equal lines, any of which may be called the apothem, in the above figure.

Ex. 224. Explain how regular inscribed and circumscribed polygons of 16, 32, and 64 sides may be constructed in and about a given circle.

Ex. 225. Find the locus of the centers of all circles having a common chord.

Ex. 226. Find the locus of the centers of all circles touching a given line in a given point.

Ex. 227. Find the locus of the centers of all circles passing through two given points.

Ex. 228. Find the locus of the centers of all circles touching each of two intersecting lines.

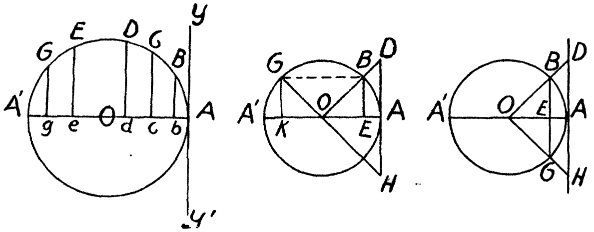
Ex. 229. Find the locus of the centers of all circles touching each of two parallel lines.

Ex. 230. The less acute $\angle$ of a rt $\triangle = \frac{1}{4}$ the greater $\angle$. Find each angle.

Ex. 231. In $\triangle$ ABC, $A = 23° 27' 5''$, $B = 80°$; find C.

142. Def. A chord parallel to the tangent at the extremity of a diameter is called a ***double ordinate,*** and the part of it on one side, an ***ordinate*** to that diameter.

Name the ordinates to AA'. Show that an ordinate to a diameter of a circle is perpendicular to the diameter and half the corresponding chord. Name the double ordinates in fig. §129.



143. DEF. If OB, the radius at one extremity of the arc AB, be produced to meet AD, the tangent at A, in some point, as D; AD is called the ***tangent*** and OD the ***secant*** of the ***arc*** BA. The ordinate BE is called the ***sine*** of the arc BA. Define the terms sine, tangent, and secant of an arc, without reference to a figure.

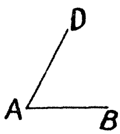
MEASUREMENT AND CONSTRUCTION OF ARCS AND ANGLES.

144. Euclidean geometry recognizes only the straight edge and dividers as admissible instruments, yet no pupil should complete a course in elementary geometry, without learning the use of the protractor, scale of chords, and scale of equal parts.

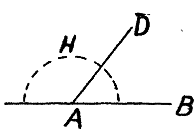
Ex. 232. At A, in the line AB, construct an arc of 63° with the protractor. Make the st. edge ≅ AB the center on A. If the figure 0 on the circle of figures ≅ AB, mark the pt. 63° on the outside. If 180° is on | AB, mark 117°. Why? Complete the construction.

Ex. 233. Construct an arc of 56° with the scale of chords. At A in the line . . . .

This scale is marked ch. First put the steel pt. on 0° and the pencil pt. on 60°, and with



Ex. 232.



Ex. 233.

A as a center and this chord of 60° . . . . Sim. find on scale a chord of 56° and lay it off from B as BD. BD is the required arc

145. As it will be proved hereafter, that central angles are measured by their intercepted arcs, angles may be constructed by the method of Ex. 232, or 233.

Ex. 234. Given: $A = 62°$, $b = 2.4$, $c = 4.3$. Constr. the $\triangle$. Find by measurement B, C, and a.*

Ex. 235. Given: $C = 118°$, $a = 2.4$, $c = 4.3$. Find A, B, and b.

Ex. 236. Given: $A = 27°$, $C = 38°$, $b = 5.4$. Find a, c, and B.

Ex. 237. Given: $C = 90°$, $c = 8$, $b = 4$. Find B, A, and a.

Ex. 238. Given: $C = 52° \ 40'$, $a = 6.2$, $c = 5.4$. Constr. two $\triangle$s and find by measurement two sets of answers.

SUPPLEMENTARY THEOREMS AND EXERCISES.

Theorem XIV.

146. *Two unequal circles may have five positions with reference to each other. Let O and $P \equiv$ two $\odot$s, r and r', their radii, and d, the dist. between their centers.*

1. Let $d > r + r'$; P is wholly without O.
2. Let $d = r + r'$; P touches O externally.
3. Let $d > r - r'$ and $d < r + r'$; P cuts O.
4. Let $d = r - r'$; P touches O internally.
5. Let $d < r - r'$; P is within O.

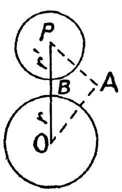


Fig. 1.

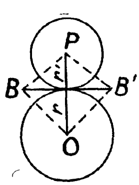


Fig. 2.

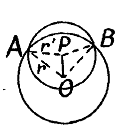


Fig. 3.

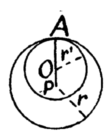


Fig. 4.

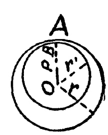


Fig. 5.

Ex. 239. If two ○s intersect each other the distance between their centers < the sum and > the difference of their radii.

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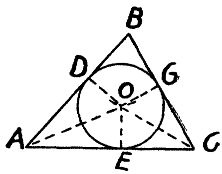
* a, b, c, etc. $\equiv$ the |s opp. the $\angle$s A, B, C, etc.

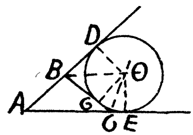
PROBLEM IX.



147. *Inscribe a circle in a given triangle.*

Apply §42 twice, also §51: 2. In proof apply §72: 2.

148. DEF. An ***escribed circle*** is tangent to one side of a triangle and to the other two produced.



Ex. 240. Construct an escribed circle, on | *c* of △ *ABC.*

Ex. 241. How many circles can be escribed about a given triangle?

Ex. 242. Construct a line tangent to a given circle and parallel to a given line. Through the center of the ○ draw a | ⊥ the given line, etc.

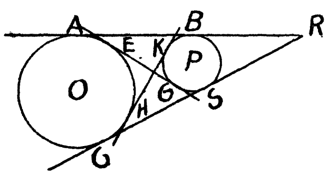
Ex. 243. Inscribe a circle in a given square.

Ex. 244. Construct a square on a line 1 inch long, as a side, and circumscribe a circle about it.

Ex. 245. Construct a □ whose diag. is 1 inch, and circumscribe a ○ about it.

Ex. 246. Construct a reg. hexagon on a line ½ inch long, as a side, inscribe a ○ within it, and circumscribe one about it.

Ex. 247. Construct the locus of a point ½ inch distant from a given line 1 inch long.



Ex. 248. In the ○s *O* and *P* the exterior com. tang. *AB* = the ext. com. tang. *CS*, and the interior com. tang. *EG* = the int. com. tang. *HK*. See §128.

Ex. 249. Define *exterior common tangent* and *interior common tangent*, as the terms are used in Ex. 248.

Ex. 250. State Ex. 248 without reference to a figure.

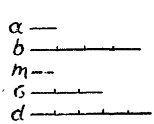
Ex. 251. If two circles touch each other the interior common tangent bisects each of the exterior common tangents.

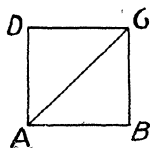
Ex. 252. If two circles touch each other the exterior tangents drawn from any point in the interior common tangent are equal.

BOOK III.

RATIO, PROPORTION, AND LIMITS.

148. In comparing two magnitudes, three cases may arise:





1. One magnitude may be exactly contained in another; a is contained in b 4 times; that is, $b = 4a$; if $b = 1$, $a = 4$.

2. Two magnitudes may have a common measure exactly contained in each; m is contained 3 times in c, and 5 times in d.

3. Two magnitudes, as AB, the side of a square, and AC, its diagonal, may be so related that no measure, however small, is exactly contained in each of them.

DEF. ***Commensurable quantities*** are such as have a common measure. ***Incommensurable quantities*** are such as have no common measure.

The above magnitudes are lines, but the principles of ratio and proportion are equally applicable to angular measure, areas, volumes, and all other geometrical magnitudes.

Ex. 253. Give numerically, commensurable angular magnitudes, commensurable lengths, and commensurable volumes. Find two common measures of 13 inches and 17 inches.

149. DEF. ***Ratio*** is the relation between two numbers, or two like magnitudes, expressed by their quotient. A ***proportion*** is the statement that two ratios are equal. A proportion may be written in three ways:

1) $a : b = c : d$; 2) $a : b :: c : d$; 3) $\frac{a}{b} = \frac{c}{d}$. The second is merely an old way of writing the first and is

now little used by mathematicians. Any of the three forms may be read, either: a is to b as c is to d, or a divided by b is equal to c divided by d.

150. DEF. The first term of each ratio is called its ***antecedent;*** the second term, its ***consequent;*** the first and last terms of a proportion are called ***extremes;*** the second and third, ***means.***

151. DEF. When the third term of a proportion is the same quantity as the second, it is called a ***mean proportional,*** and the last term a ***third proportional.***

152. Since the quotient of a magnitude by a magnitude is a number, two ratios composed of unlike magnitudes may be equal. We cannot correctly say 3 in. : 9° but; 3 in. : 6 in. = 9° : 18°.

Ex. 254. Make a true proportion, in which the terms of the first ratio, or couplet, are degrees, and the terms of the second ratio are miles.

153. AXIOM 9. Multiplying or dividing both terms of a ratio by the same number does not change its value.

This axiom is stated in Arithmetic: Multiplying or dividing both terms of a fraction by the same number does not change its value. It is sometimes also stated: Multiplying and dividing a quantity by the same number does not change its value.

Ex. 255. Illustrate axiom 9 by multiplying and dividing both terms of the ratio, 18° : 6°, and finding the numerical value of each result.

THEOREM I.

154. *Let a, b, c, d, m, and n be numbers, and $a : b = c : d$:*

1. $na : nb = c : d$.
2. $a : b = nc : nd$.
3. $ma : mb = nc : nd$.
4. $\frac{a}{m} : \frac{b}{m} = \frac{c}{n} : \frac{d}{n}$.

Apply §153.

THEOREM II.

155. *Let a, b, c, d, m, and n be numbers, and $a : b = c : d$.*

1.	$na : b = nc : d.$	Mult. each side by n.	§17: 2
2.	$a : nb = c : nd.$	Dividing each side by n.	§17: 2
3.	$\frac{a}{n} : b = \frac{c}{n} : d.$	Dividing each side by n.	
4.	$a : \frac{b}{n} = c : \frac{d}{n}.$		Why?
5.	$a^n : b^n = c^n : d^n.$		Why?
6.	$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$		Why?
7.	$b : a = d : c.$	Inverting each fraction.	§17: 2
8.	$a : c = b : d.$	Mult. each side by $b \div c$.	§17: 2
9.	$ad = bc.$	Mult. each side by bd.	§17: 2
10.	$a = b\,\frac{c}{d}.$	Mult. each side by b.	§17: 2
11.	$\frac{a+b}{b} = \frac{c+d}{d}$	Add. 1 to each side	§17: 2
12.	$\frac{a-b}{b} = \frac{c-d}{d}.$	Subt. 1 fr. each side	§17: 2
13.	$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$	Div. (11) by (12).	

Let $a : b = b : c$; a, b, and c being numbers:

14.	$ac = b^2.$	Why?	§17: 2
15.	$b = \sqrt{ac}.$	Why?	
16.	$a = b.\ \frac{b}{c}.$	Why?	

156. It is easier to remember and use these equations, or formulas, than to remember and use them enunciated in words. The pupil should, however, practice expressing these formulas in words. We thus state five of them, all parts of Theorem II, to enable the pupil to define terms contained in their statement.

If four numbers are in proportion:

7. They are in proportion by ***inversion.***
8. They are in proportion by ***alternation.***
11. They are in proportion by ***composition.***
12. They are in proportion by ***division.***
13. They are in proportion by ***composition*** and ***division.***

The terms composition and division, as above used, are unsatisfactory, but there is as yet no general agreement on other terms among those who discard them.

157. The proofs of §§154 and 155 being based on axioms universally valid, are as applicable to geometrical magnitudes as to numbers, provided we prove that their application involves only possible operations.* From our knowledge of geometrical magnitudes gained in books I and II, it is evident that:

1. Like magnitudes only can be added or subtracted.
2. Magnitudes can be multiplied by numbers only.†
3. Magnitudes can be divided only by numbers, or by like magnitudes.
4. Roots and powers of magnitudes are impossible.†

158. 1. In §154: If a, b, c, and d are magnitudes and m and n numbers, the operations are possible.

Illustrate each case, using feet as magnitudes.

2. §155: 1 to 4 are valid, when a, b, c, and d are magnitudes and n a number. Illustrate each case.

3. §155: 5, 6, 14, and 15 are not valid as to geometrical magnitudes. Why not?

---

* The pupil should take the utmost care to ascertain certainly that an operation is possible before assuming its performance. Carelessness in this in study often leads to a like carelessness in life, which makes education worse than worthless.

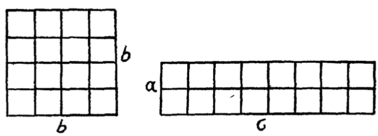
† The definitions of multiplication and division, will in a future section be extended to make multiplication of lines and surfaces, by lines, possible.

4. §155: 7 is valid as to magnitudes; §161: 8 as to magnitudes of the same kind. Illustrate and explain.

5. §155: 9 is valid, when only one couplet consists of geometrical magnitudes, the other of numbers. Why?

6. §155: 10–13 and 16 are valid as to proportions consisting of geometrical magnitudes. Give proof in each case. Observe that multiplying a number by a magnitude is the same as multiplying a magnitude by a number.

Ex. 256. Illustr. §158 by the proportion: 6 ft : 9 ft = 8° : 12°.



159. In the adjacent figure the same unit of measure is contained in a, twice; in b, 4 times; and in c, 8 times. $a : b = b : c$ is evidently true, while $b^2 = ac$ is absurd as multiplication is so far defined, but if a, b, and $c \equiv$ the numbers 2, 4, and 8 then $b^2 = ac$; that is: The number of units of surface in the square on b is a mean proportional between the number of units of length in the lines a and c. By analogous methods, those principles not valid as to geometrical magnitudes are yet made useful in geometrical reasoning.

THEOREM III.

160. *When the product of two numbers is equal to the product of two others, either pair may be taken as extremes and the other pair as means in a proportion.*

Apply §17: 2 (division); make four proportions.

Ex. 257. Make a correct proportion from the equation $2 \times 6\text{m} = 3 \times 4\text{m}$, and state the principle involved as a theorem.

Ex. 258. Square each term of the proportion $3 : 6 = 5 : 10$; cube each term.

Ex. 259. Find x in the proportion: $25 : 36 = 100 : x^2$.

Ex. 260. Illustrate §§154: 1–4 and 158 by examples in which a and b are degrees, c and d, centimetres, and n, a number. Illustrate each of the other parts of §158 by numerical examples.

THEOREM IV.

161. *In a series of equal ratios consisting of numbers, or of magnitudes of the same kind, the sum of the antecedents is to the sum of the consequents as any antecedent to its consequent.*

Let $a : b = c : d = e : g = r$; then $a + c + e : b + d + g = a : b$.

1. For $a = br$. Why?
2. $c = \ldots$ Why?
3. $\ldots = \ldots$ Why?
4. $a + c + e = (b + d + e)\, r$. Why?

Complete the proof.

Ex. 261. Show that the restriction in §161 is necessary by letting the two first ratios $\equiv$ metres, the third ratio, degrees, and attempting to perform the operations required in the proof.

Ex. 262. 1) Multiply the antecedents of, $3° : 6° = 9$ ft: 18 ft by 5; 2) the consequents by 3; 3) each term of the first ratio by 3; 4) each term of the second ratio by 2; 5) every term of the proportion by 10; 6) divide each antecedent by 3; 7) each consequent by 3; 8) every term by 3; 9) each term of the second ratio by 10; 10) each consequent by 2. 11) Show what part of §154 or §155 is illustrated by each operation.

Ex. 263. Show which formulas are valid as to the proportion: 15 cm : 12 cm $= 10° : x°$.

Ex. 264. Find x in: 4 dm : x dm $= x$ ft : 9 ft.

Ex. 265. Find x in the proportion: $125 : 1000 = 64 : x^3$.

Ex. 266. Find $x : y$ by composition in: $x - 3 : 3 = y - 7 : 7$.

Ex. 267. Find $x : y$ by division in: $x + 5 : 5 = y + 8 : 8$.

Ex. 268. Find $p : q$ in: $p + 3 : p - 3 = q + 2 : q - 2$. Apply §155: 13. In what cases is the result valid when p, q, 2, and 3 are replaced by geometrical magnitudes?

Ex. 269. If $a : b = c : d$; show that: 1) $a^3 + b^3 : a^3 - b^3 = c^3 + d^3 : c^3 - d^3$; 2) $a^2 - ab + b^2 : a^2 + ab + b^2 = c^2 - cd + d^2 : c^2 + cd + d^2$. Is this applicable to geometrical magnitudes?

Ex. 270. If $c^2 + d^2 : c^2 - d^2 = x^3 + y^3 : x^3 - y^3$; show that: $c^4 + c^2 d^2 + d^4 : c^4 - c^2 d^2 + d^4 = x^6 + x^3 y^3 + y^6 : x^6 - x^3 y^3 + y^6$.

LIMITS.

162. If a boy walk 2 yards each second, how far will he walk in 2 seconds? in 3? in 4? in 5? Of the time rate and distance walked, which vary, and which is constant?

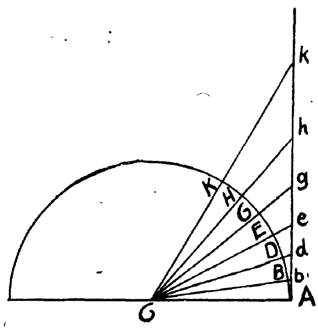
DEF. A ***variable*** is a quantity whose value changes.

A ***constant*** is a quantity whose value does not change in the same problem or discussion.

163. DEF. Let the point P move half the distance from A to B the first second, half the rest the next, etc.; can it ever reach B? How near to B does it approach? A ***limit*** of a variable is a constant it approaches indefinitely near, though it may never reach it.

Ex. 271. If a body move 4 ft. the first second, 2 ft. the next, 1 the next, etc., find: 1): the distance it moves in 15 sec.; 2) its limit. Ans. 1) $7\frac{4095}{4096}$ ft.; 2) 8 ft.

164. Geometrical magnitudes may be so related that the rate of increase of one depends on the rate of increase of another.



Let the radius CH move from the position CA at the rate of 10° in a second towards the position CK, what arc does it describe in 1 second? in 2? in 3? in 4? in 5? What line ≡ the secant each second? Does the tangent or arc increase most rapidly at A? at B? at D? at K? How does the tangent increase as the arc approaches 90°? Explain the rate of the secant at various points, and as the arc approaches 90°.

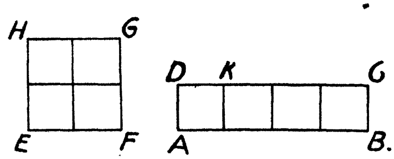
165. A variable may approach its limit by successive steps, thus: $.333$, etc. $\doteq \frac{1}{3}$; $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, etc. $\doteq 1$.

What part of the preceding step is any step in each example? The steps by which a variable approaches its limit are not necessarily like parts of the preceding step. By obtaining successive figures extracting the square root of 2, we find: 1, 1.4, 1.41, 1.414, 1.4142, etc. $\doteq \sqrt{2}$.

166. DEF. An ***incommensurable number*** is one which cannot be exactly measured in ordinary units. Ratios consisting of incommensurable magnitudes (see §148) are incommensurable numbers.

Ratios between incommensurable numbers may be commensurable: $\sqrt{8} : \sqrt{2} = \sqrt{4} : 1 = 2 : 1$. Show that: $\sqrt{32} : \sqrt{2}$; $\sqrt{50} : \sqrt{18}$; $\sqrt[3]{81} : \sqrt[3]{3}$, and $\sqrt[3]{24} : \sqrt[3]{375}$, are commensurable.

167. ***Equivalent magnitudes*** are such as contain the same *unit* of measure an equal number of times.



In ▭ AC and $\square$ EG, let $AB = 4AD$ and $EF = 2\,AD$. How many times is $\square$ AK, cont. in ▭ AC? in $\square$ EG? The unit of length is the line AD. The unit of surface is $\square$ AK. Show that the square and rectangle in the figure of §158 are equivalent.

Ex. 272. As two adjacent angles approach equality, they approach right angles as limits.

168. In applying the principles of limits, we must be careful to prove that the supposed limit is the true limit. A series may continually approach a constant which is not its true limit. The limit of the circulating decimal $.\dot{9}99\,99\dot{8}$ is not 1, but $999\,998 \div 999\,999$.

* Read $\doteq$, *approaches as a limit.*

THEOREM V.

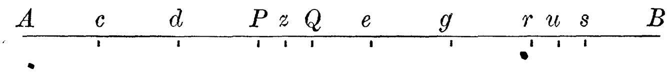
169. *1. Two variable magnitudes, which are always equivalent when commensurable, increase or decrease together.*

2. Two variable magnitudes, which are always equivalent when commensurable, are equivalent when incommensurable.

3. If two variable magnitudes are always equivalent, their limits are equivalent.

Let m°, n°, x ft., and y ft. be vbl. magnitudes so related that, when cmbl., $m^\circ : n^\circ = x$ ft.: y ft.; then, when incmbl., $m^\circ : n^\circ = x$ ft. : y ft.

$m^\circ : n^\circ = m : n$ and x ft. : y ft. $= x : y$; m, n, x and y being the numerical measures of m°, etc. (See §158.) $m : n$ and $x : y$ being numbers may $\equiv$ by lines (§158).



I. Let AP be any value of $m : n$; then if at P, $m : n$ incr., $x : y$ incr., and if $m : n$ decr., $x : y$ decr.

1. For let $m : n$ incr. from AP to AQ.
2. Continually bisect Ae, a cmbl. value of $m : n$, and each part, a pt. z must at last be found between P and Q where $Az = m : n$ is cmbl.
3. $\therefore m : n = x : y$ and $x : y$ has incr.
4. Sim. if $m : n$ decr., $x : y$ must decr.

II. Let Ar be incmbl.; then at r, $m : n = x : y$.

1. For if $x : y$ is not $= Ar$, it must $=$ some other | as As.

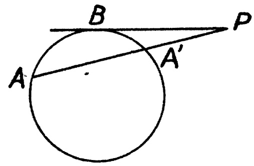
The proof may be completed as in part 1.

III. Let $m : n \doteq AB$, then $x : y \doteq AB$.

Mark the pts. required near B and prove as in part 2.

THEOREM VI.

170. *The limit of a secant, as its points of intersection approach each other, is a tangent.*



Apply §112.

Ex. 273. $.6 : .3 = 2$; $.66 : .33 = 2$; $.666 : .333 = 2$ etc.; find by §169: 2, $.\dot{6} : .\dot{3}$.

Ex. 274. Show by §169: 2 that lt. .5151 etc.$=3$ (lt. 1717, etc.)

Ex. 275. Given lt. .1818, etc. $= \frac{2}{11}$. Find lt. .7272, etc.

Ex. 276. If we define a straight line as a line which does not change its direction at any point, what is the limit of a curve as the change of direction at every point is indefinitely diminished?

Ex. 277. As ABA' is diminished $\smile ABA' \doteq$ chord AA'. (See fig. of §170.)

Ex. 278. Lines whose point of intersection becomes continually more distant without limit, approach to parallel lines as a limit.

171. DEF. A ***curve*** is a line which changes its direction at every point. A ***tangent*** to a curve at any point is the limit of a secant through that point as another intersection approaches indefinitely near that point.

Fig. 1.

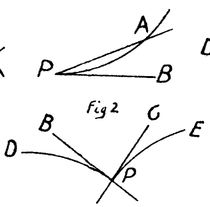


Fig. 3.

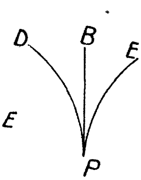


Fig. 4.

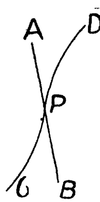


Fig. 5.

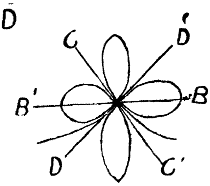


Fig. 6.

Fig. 1 illustrates the definition of a tangent generally.

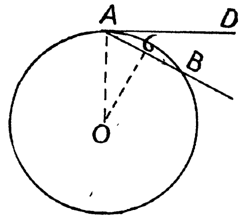
In fig. 2, the curve terminates at the point of tangency.

In fig. 3 and 4, two branches terminate at the pt. of tangency.

In fig. 5, the tangent cuts the curve at the point of tangency.

Explain Fig. 6.

172. As the limit of a secant, as its points of intersection approach, is a tangent, a tangent to a circle by §171 is a tangent by §112. The converse is also true and the definitions are equivalent.

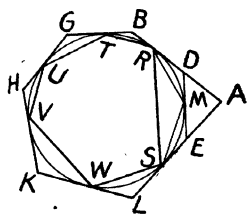


1. For join OA which is $\perp$ . . . Why?
2. Draw $OC \perp$ . .
3. $\angle AOC = \angle$ . . Why?
4. As $B \doteq A$, $\angle AOC \doteq$ . .
5. . . . .

Q. E. D.

Theorem VII.

173. *The limit of a polygon inscribed in a curve, or circumscribed about it, as the number of sides is increased and the length of each indefinitely diminished, is the curve.*



Let $C \equiv$ a curve; P and $p \equiv$ the perimeters of the inscr. and circumscr. polyg's; the |s of p joining the points of contact of P, then $p \doteq C$ and $P \doteq C$.

1. For let RS be a | of p and BRA a | of P.

2. Increasing the number of |s and dimin. their length $\angle ARS \doteq 0$.

The lt. of a secant as the pts., etc. §170

3. Sim. every $\angle$ between |s of p and $P \doteq 0$; and $p \doteq P$.

4. Since C is at all times between . . . . . Q. E. D.

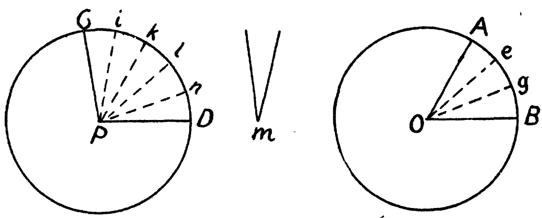
174. Remark. It is not enough to show that the distance between p and $C \doteq 0$. Unless they coincide in direction also, they might be unequal in length.

THEOREM VIII.

175. *In equal circles or in the same circle:*

1. Central angles have the same ratio as their intercepted arcs.

2. Sectors have the same ratio as their angles or arcs.



I. Let . . . .

1. For let m be a measure . . . .

2. Draw the radii . . . . .

3. $\smile Ae = \smile . . = \smile . .$, etc. Why? §123: 2

4. $\smile AB : \smile . . = 3 : 5$.

5. But $\angle . . : \angle . . = 3 : 5$.

6. . . . .

7. Sim. $\angle AOB : \angle CBD = \smile AB : \smile CD$, whatever number of times each contains the common measure.

8. Since $\angle AOB : \angle CPD = \smile AB : \smile CD$ when commensurable, the same is true when incommensurable. Why? §168: 2

Q. E. D.

176. In similar demonstrations hereafter, steps 7 and 8, which are the same in all, will not be repeated.

177. §175: 1 gives a convenient measure of an angle and may be briefly expressed:

Central angles are measured by their intercepted arcs.

Ex. 279. **As the number of sides of a regular polygon increases indefinitely, its angles approach straight angles as limits.**

THEOREM IX.

178. *1. An inscribed angle is measured by half the intercepted arc.*

2. An angle between a chord and a tangent is measured by half the intercepted arc.

3. Angles inscribed in the same segment are equal.

4. An angle inscribed in a segment less than a semicircle is obtuse.

5. An angle inscribed in a segment greater than a semicircle is acute.

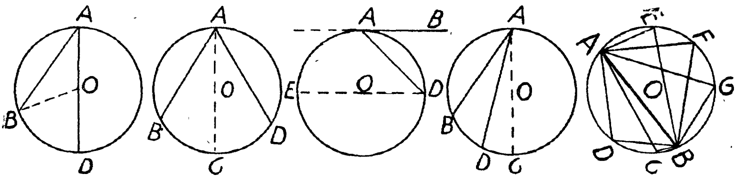


Fig. 1. Fig. 2. Fig. 4. Fig. 3. Fig. 5.

I. Figures 1, 2, and 3. Let $\angle BAD$ . . . .

a) Let O, the center, be on | AD. Join . . Apply §§70: 1, 45: 1, and 177.

b) (Fig. 2.) The center O within the $\angle$.

c) (Fig. 3.) The center O without the $\angle$.

II. (Fig. 4.) Let $\angle DAB$ . . . .

1. For draw $DE \parallel AB$.

Apply §§67: 1, 129, and 178: 1.

III, IV, and V. (Fig. 5.) Apply §178: 1.

Ex. 280. Find the angles intercepting the following arcs, 90°; 60°; 180°; 40° 20′; 61° 41′ 25″.

Ex. 281. Find the arc intercepted by an angle of 2°; 3°; 17°; 23° 27′; 38° 40′; 57° 15′ 56″.

Ex. 282. An equilateral polygon inscribed in a circle is regular.

Ex. 283. An equiangular polygon inscr. in a circle is regular.

Ex. 284. The opposite angles of a quadrilateral inscribed in a circle are supplementary.

THEOREM X.

179. *1. The angle formed by two chords is measured by half the sum of the intercepted arcs.*

2. The angle formed by two secants, by a secant and a tangent, or by two tangents, meeting without a circle, is measured by half the difference of the intercepted arcs.

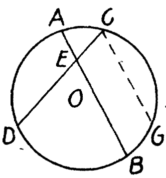


Fig. 1.

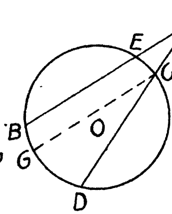


Fig. 2.

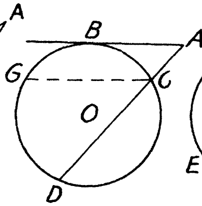


Fig. 3.

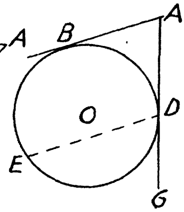


Fig. 4.

I. Apply §§67: 2, 178: 1, and 129.

II. Make three cases: a) Two secants, b) a secant and a tangent, c) two tangents.

Ex. 285. The arcs intercepted by two tangents are as 3 : 5. Find the angle between the tangents. Ans. 45°.

Ex. 286. The angle between 2 secants is 40° and the ratio of the intercepted arcs, 3 : 5. Find the intercepted arcs.

Ex. 287. The angle inscribed in the smaller of two segments, into which a circle is divided, is twice the angle inscribed in the greater segment. Find the angles and arcs.

Ex. 288. The arc intercepted between a chord and a tangent is 80°. Find the angle.

Ex. 289. A chord divides the circumference of a ○ in the ratio 3 : 7. Find the angle it makes with a tangent at one extremity.

Ex. 290. (Fig. 1.) If $\smile AC = 40°$ and $\smile DB = 80°$; find $\angle AEC$.

Ex. 291. (Fig. 2.) If $\smile EC = 40°$ and $\smile BD = 120°$; find $\angle A$.

Ex. 292. (Fig. 3.) If $\smile BC = 50°$ and $\smile BD = 160°$; find $\angle A$.

Ex. 293. (Fig. 4.) If $\smile BD = 110°$ and $\smile BED = 250°$; find $\angle A$.

Ex. 294. State exercises 290 to 293 in words without reference to a figure.

Ex. 295. The angle between two tangents is 25° and one of the intercepted arcs is 40°. Find the other intercepted arc.

THEOREM XI.

180. *1. Any three parallel lines divide all transversals proportionally.*

2. A line parallel to the base of a triangle divides the sides proportionally.

3. Any number of parallel lines divide two or more transversals proportionally.

I. Let the .... cut the transversals ...... in .....; then $AC : CE = BD : DF$; $AE : AC = BF : BD$; etc.

1. For let $EC : AC = 3 : 4$.

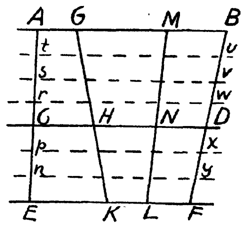


Fig. 1.

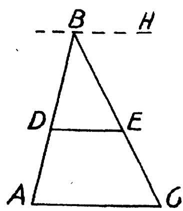


Fig. 2.

2. Divide EC into 3 parts, and AC into 4 parts each = the common measure En.
3. Through n, p ....
4. Then $Fy = yx = ...$ etc. Why? §90
5. FD contains ...... and DB ......
6. $\therefore FD : DB = 3 : 4$. Why?
7. .... Why? § 17: 1
8. AE : .... Why? §155:11

Q. E. D.

II. Apply §180: 1. Use fig. 2.

III. Apply §180; 1. Construct a figure.

Ex. 296. A system of parallel lines cuts three transversals. The segments of one are 1, 2, 4, and 5. The entire length of the other intercepted lines are respectively 18 and 30. Find each segment of the two latter.

Ex. 297. The segments of the altitude of a triangle made by lines parallel to its bass are 2, 3, and 4. The segments of the sides corresponding to 2 are 5 and 6. Find the length of each side, also its segments corresponding to 3 and 4.

Problem I.

181. *1. Divide a line into segments proportional to those of a given line.*

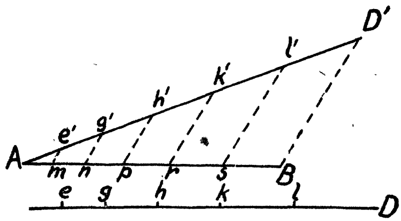
2. Divide a line into any number of equal parts.

I. To divide | AB

. . . .

1. Draw $AD' = AD$ making any convenient $\angle$ . . . . .

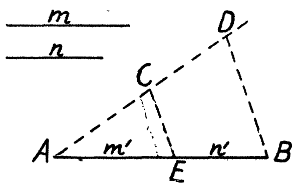
Complete construction and proof. Apply §§64 and 180.



II. Draw a line, give statement, construction, and proof.

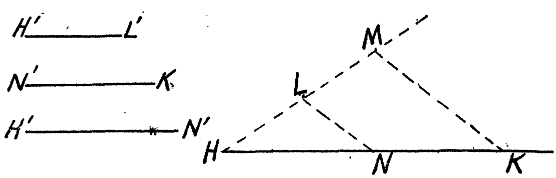
Problem II.

182. *To divide a line in a given ratio.*



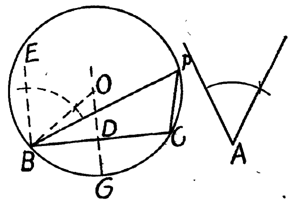
Problem III.

183. *To find a fourth proportional to three given lines.*



PROBLEM IV.

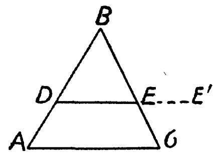
184. *On a given line construct a segment containing a given angle.*



Apply in construction §§49, 51: 1, 34; in proof, §§67: 1, 122: 1, 177, 178: 1.

THEOREM XII.

185. *A straight line which divides two sides of a triangle proportionally is parallel to the third side.*



Let AD : . . . . ., . . . || . .

1. For draw DE' || . .
2. . . : . . $= CB : CE'$. Why?
3. But $AB : BD = CB : CE$. Why? §180: 2
4. $CE = CE'$ each being = . . .
5. . . . .

Q. E. D.

Ex. 298. Of what preceding proposition is §185 the converse?

186. DEF. Two ***polygons*** are ***similar,*** when each angle of the one is equal to the homologous angle of the other and the sides joining the equal angles are proportional.

Ex. 299. If in fig. 2, §180, $AD = 9$, $DB = 12$, $EC = 12$, find EB.

Ex. 300. If $AD = 6$, $DB = 9$, and $BC = 20$; find CE and EB.

Ex. 301. The angles of a triangle are $2x$, $3x$, and $4x$. Find each angle.

Ex. 302. The angles of a quadrilateral are $2x$, $3x$, $4x$, and $5x$. Find each angle.

Ex. 303. The angles of a pentagon are $2x$, $3x$, $4x$, $5x$, and $6x$. Find each angle.

THEOREM XIII.

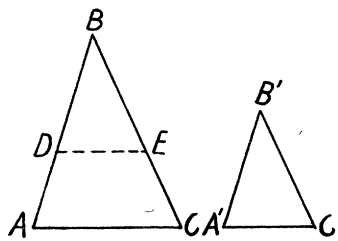
187. *Two triangles are similar:*

1. If they are equiangular.

2. If two angles of the one are equal to two angles of the other, each to each.

3. If their homologous sides are proportional.

4. If an angle of the one is equal to an angle of the other and the sides about this angle are proportional.



I. In $\triangle$s . . . .

1. For place $\angle B'$ . . . . $B'A'$ along . . ., join . . .

2. $\triangle BDE = \triangle$ . . . Why? §38

3. $\angle BDE = \angle A$. Why?

4. $\therefore$ . . . $\parallel$ . . Why? §66: 2

5. $\therefore AB : DB$ . . . . . . Why? §180: 2

6. Sim. $AB : \ldots = AC : \ldots$

7. $\therefore$ . . . . Why? §186

II. Apply §72: 1.

III. In $\triangle$s . . . .

1. For from BA cut off . . . = . . ., from BC . . . ., join . . .

2. $AB : BD = \ldots$ Why?

3. $DE \parallel \ldots$ Why? §185

4. $\angle \ldots = \angle \ldots$ and $\angle \ldots =$ Why?

5. $\triangle \ldots \sim \ldots$ Why?

6. $\ldots : \ldots = AC : DE$. Why?

7. $DE = A'C'$. Why?

8. $\triangle \ldots = \triangle \ldots$ Why? §48

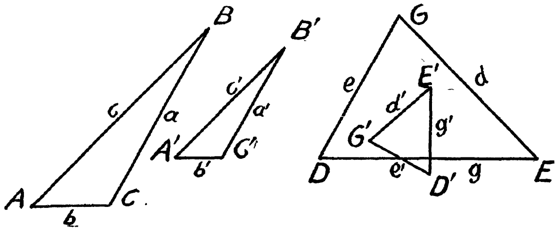
9. . . . .

IV. The first five steps may be the same as in §187: 3.

Theorem XIV.

188. *Triangles whose sides are parallel are equiangular and similar.*

2. Triangles whose sides are mutually perpendicular are equiangular and similar.

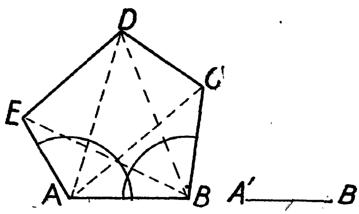


I. Apply §§74: 1 and 187: 1.

II. Apply §§74: 2 and 187: 1.

Problem V.

189. *On a given line construct a polygon similar to a given polygon.*



190. Def. The ***ratio of similitude*** of two polygons is the ratio of any two homologous sides.

Ex. 304. The angle between a chord and a tangent is 70°. Find the two arcs into which the chord divides the circle.

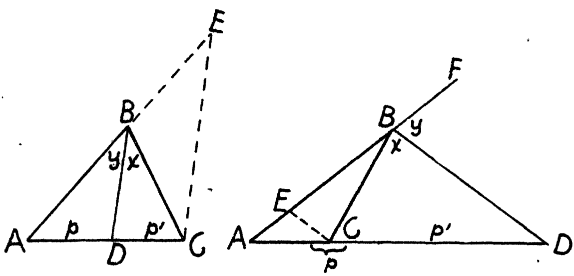
Ex. 305. A chord divides a circle so that the greater arc is three times the smaller. Find the angle between the chord and a tangent at one of its extremities. Ans. 45°.

Ex. 306. The angle between two secants is 40°, and the greater of the intercepted arcs, 120°. Find the smaller intercepted arc.

THEOREM XV.

191. *1. The bisector of the vertical angle of a triangle divides the base into segments proportional to the adjacent sides of the triangle.*

2. The bisector of the exterior vertical angle of a triangle divides the base produced into segments proportional to the adjacent sides of the triangle.



I. Let BD bisect . . . . . .

1. For draw $CE \parallel$ . . and . . . .
2. $\angle x =$ . . . . — Why? §67: 1
3. $\angle y =$ . . . . — Why? §67: 2
4. . . . . — Why? Hypoth. and §17: 1
5. $BE = BC$. — Why? §46: 1
6. . . . . — Why? §180: 2
7. . . . . — Why? Q. E. D.

II. The proof is closely similar to the above.

Ex. 307. The sides of a triangle are 4, 6, and 8. Find the segments of each sides made by the bisector of the opposite angle.

Ex. 308. In a $\triangle$, $a = 9$, $b = 8$, $c = 12$. Find the segments of b by the bisector of the ext. $\angle$ at B. Ans. 24, 32.

Ex. 309. The segments of the base of a triangle made by the bisector of the vertical angle are 2 and 3, and the sum of the other sides is 15. Find the other sides. Ans. 6, 9.

THEOREM XVI.

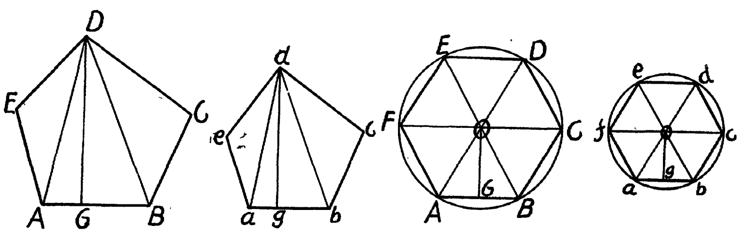
192. *1. The diagonals from homologous angles of similar polygons divide them into the same number of similar triangles similarly placed.*

2. Homologous diagonals and altitudes of similar polygons have their ratio of similitude. Formulas: a) $d : d' = a : a'$; b) $h : h' = a : a'$.

3. The perimeters of similar polygons have their ratio of similitude. Formula: $p : p' = a : a'$.

4. The perimeters of regular polygons have the same ratio as the radii of their inscribed or circumscribed circles. Formula: $p : p' = r : r' = R : R'$.

5. The circumferences of circles have the same ratio as their radii or diameters. Formula: $C : C' = r : r' = d : d'$.



I. Let DA . . . .

1. For in $\triangle$s ADE . . . . . .
$\angle E =$ . . and $AE :$ . . $=$ . . . . Why? §186

2. $\therefore \triangle$ . . $\sim$ . . . Why? §187: 4

3. $AD : ad =$ . . . . and $AD : ad = AB :$ . .

4—6. Prove $\angle DAB := \angle dab$.

7. $\therefore \triangle$ . . $\sim$ . . .

II. Use fig. 1.

III. Let $p \equiv$ the perimeter . . . and p' . . . .; then . . .

1. For $AB : ab =$ . . . . $=$ . . . . $=$ . . . $=$ . . . Why?

2. $\because AB +$ . . . . . . $= ab +$ . . . . . Why? §161

3. $\therefore$ . . . .

Q. E. D.

IV. Use fig. 2.

V. Apply §§173 and 192: 2.

193. REMARK. In applying theorems like the above to triangles, or pentagons, etc., it is advantageous to substitute the word triangle, or pentagon, for polygon. Making such changes mentally gives the student a more thorough mastery of the principles involved.

194. REMARK. By methods similar to those of §192 it may be shown, that:

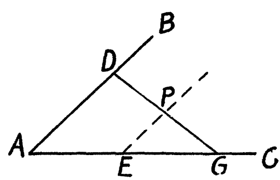
1. *All possible homologous lines and parts of lines of similar polygons have their ratio of similitude.*

2. *All homologous arcs and chords of similar curves are proportional.*

Ex. 310. Similar polygons whose ratio of similitude is unity are equal.

PROBLEM VII.

195. *Through a given point in a given angle draw a line terminated by the sides of the angle which shall be bisected at that point.*



Apply §64; in proof, §180: 2.

Ex. 311. Construct a segment containing an angle of 45°. Apply §§51, 42, and 184.

Ex. 312. Construct a segment containing an angle of 30°.

Ex. 313. Construct an equilateral triangle of which each side is 2 inches and the locus of a point half an inch from its perimeter.

Ex. 314. The sides of a triangle are 6, 8, and 10. Find the segments of the longest side made by the bisector of the opposite angle.

SUPPLEMENTARY EXERCISES.

Ex. 315. State and prove the converse of §191, beginning: 1) The line which divides . . . .

Apply §§185, 67: 1, and 67: 2.

Ex. 316. The sine of an arc is equal in length to the sine of its supplement. ($BE = GK$).

Ex. 317. (Fig 1.) The tangent of an arc is equal in length to the tangent of its supplement.

Ex. 318. The secant of an arc is equal in length to the secant of its supplement.

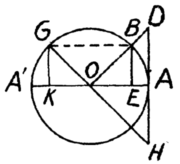


Fig. 1.

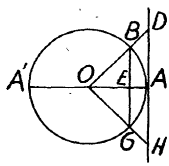


Fig. 2.

Ex. 319. (Fig. 2). The sine, tangent, and secant of an arc are respectively equal in length to the corresponding functions of its conjugate.

Ex. 320. Given the hypotenuse of a right-angled triangle; find the locus of its vertex. See §121.

Ex. 321. Given a side of a triangle and the magnitude of the opposite angle; find the locus of its vertex. See §184.

Ex. 322. Given a side of a triangle and the length of the altitude on it; find the locus of the vertex of the opposite angle.

Ex. 323. Draw a line ½ inch long, and the opposite angle of a triangle = 30°. Construct the locus of its vertex.

Ex. 324. Construct a circle 1 inch in diameter and the locus of a point from which a tangent 1 inch long can be drawn to it.

Ex. 325. Construct the locus of a point at which tangents to a given circle make an angle of 30°.

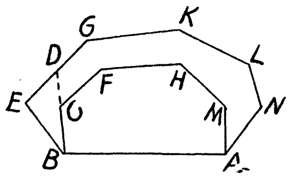
Ex. 326. The arcs intercepted by two intersecting chords are 60° and 90°. Find the angle between the chords.

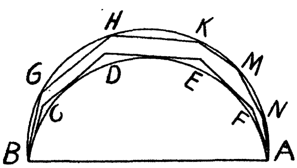
Ex. 327. The angle between two intersecting chords is 40°, and the ratio of the intersepted arcs is 2 : 3. Find the arcs.

THEOREM XVII.

196. *1. An enveloping system of lines is greater than the enveloped system, both being convex.*

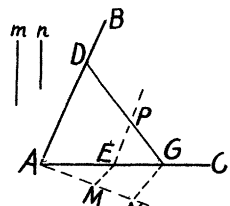
2. An enveloping curve or system of straight lines and curves is greater than the enveloped system, both systems being convex.





I. Prove by a method similar to that of §57.

II. Show that the curves are the limits of polygons as the number of sides is indefinitely increased and their length diminished by bisection and apply §173.



Ex. 328. Through a given point within a given angle draw a line terminated by the sides of the angle and divided by the point in a given ratio.

Ex. 329. Construct a triangle, its perimeter and two angles being given.

Ex. 330. Construct a right-angled triangle, the length of its altitude and its hypotenuse being given.

Ex. 331. Given a side of a triangle the length of the altitude on it and the magnitude of the opposite angle; construct a triangle.

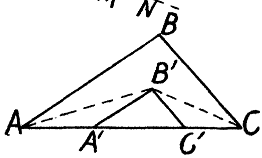


Fig. Ex. 329.

Ex. 332. Given one side of a triangle, the magnitude of the opposite angle, and the ratio of the other two sides; construct a triangle. Apply §§182 and 184; in proof, §191: 1.

Ex. 333. With a given radius draw a circle touching two intersecting lines. How many such circles can be drawn?

Ex. 334. With a given radius construct a circle which has its center in one given line and touches another given line.

Ex. 335. With a given radius construct a circle which touches a given circle and has its center in a given line.

Ex. 336. With a given radius construct a circle which touches one given circle and has its center in the circumference of another. In what cases is this impossible?

Ex. 337. With a given radius construct a circle which shall touch two given circles. In what cases is this impossible?

Ex. 338. With a given radius construct a circle which shall touch a given line and a given circle. When are eight solutions possible? when six? when four? when two? and when only one?

Ex. 339. Construct three equal circles and two other circles tangent to each of these. Suggestion: Construct an equilateral triangle, etc.

Ex. 340. Construct a circle which shall touch a given circle at a given point and also touch a given line.

Ex. 341. Construct a circle tangent to two other circles, the point of tangency to one circle being given.

Ex. 342. Construct a triangle whose sides are ½ in., ¾ in., and 1 in., and three circles escribed to the triangle.

Ex. 343. In a $\triangle$, given: $A = 30°$, $c = 6$, $a = 3$; find B, b, C by construction.

Ex. 344. In a $\triangle$, given: $a = 5$, $b = 12$, $c = 13$; find A, B, and C by construction.

Ex. 345. Given the base of an isosceles $\triangle$ 12 and its altitude 8; find the other sides by construction.

Ex. 346. Construct a regular octagon whose diagonal joining opposite angles is 1 inch.

Ex. 347. Construct a regular hexagon whose diagonal joining opposite angles is 1 inch.

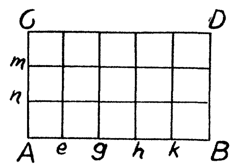
Ex. 348. The sides of a triangle are 6, 8 and 12. Find the segments of the third side made by the bisector of the angle included by the two first, also the segments made by the bisector of the exterior angle.

BOOK IV.

AREA, PROPORTIONAL FIGURES.

197. Def. ***Multiplication*** was, as first defined in Arithmetic, limited to integral numbers. The definition was first extended to include multiplying by a fraction as: $12 \times \frac{1}{3} = 4$. It was further extended in Algebra so that incommensurable, negative, or imaginary numbers could be used as factors. We now further extend the definition.

198. The rectangle AD is the surface generated by moving the line AB vertically the distance AC or by moving AC vertically the distance AB.



Def. The ***product*** of two lines is as many square units of surface as the product of the number of units of length in the lines.

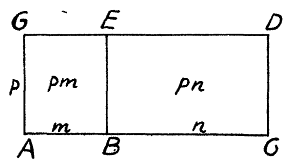
How many units of length in AB? How many units of length in AC? How many units of surface in the rectangle AD?

199. Since moving the line AB the distance AC generates the same rectangle as moving AC the distance AB, the commutative law of multiplication is valid in multiplying lines by lines.

200. $p \times m = pm.$

$p \times n = pn.$

$p\ (m + n) = pm + pn.$



Hence the distributive law of multiplication is valid in multiplying lines by lines.

201. Since 5 in. $\times$ 3 in. $=$ 15 sq. in., 15 sq. in. $\div$ 3 in. $=$ 5 in. Hence surfaces divided by lines give lines as quotients.

202. Show to what extent the applicability of §§154 and 155 is extended by §198, and illustrate each case by an example.

203. The expression *the product of two lines* is used by many who explain it as an abbreviated method of saying: A surface containing as many units of surface as the product of the numbers expressing the number of linear units in one line by the number of linear units in the other.

Ex. 349. Find the product of: 3 in. $\times$ 8 in.; 7 ft. $\times$ 8 ft.; 9 m $\times$ 8 m; 11 cm $\times$ 12 cm.

Ex. 350. Multiply $2\frac{1}{2}$ ft. by 7 in. What must first be done? State the general principle.

Ex. 351. Find the products of the following lines: 3 in. $\times$ $3\frac{1}{3}$ in.; 4 in. $\times$ $6\frac{1}{4}$ in.; 8 cm $\times$ $6\frac{1}{4}$ cm; 8 m $\times$ $12\frac{1}{2}$ m; 3 m $\times$ $33\frac{1}{3}$ m.; 6 m $\times$ $16\frac{2}{3}$ m; 12 dm $\times$ $8\frac{1}{3}$ dm; 8 mm $\times$ $38\frac{1}{2}$ mm.

Ex. 352. Find the quotients of surfaces divided by lines: 6 sq. yds $\div$ 2 yds; 100 sq. yds $\div$ 8 yds; 500 sq. ft. $\div$ 8 ft.; 2 acres $\div$ 8 rods.

Ex. 353. The area of a rectangular field is 24 square rods, and the ratio of its sides 2 : 3. Find its sides.

Ex. 354. The area of a rectangular field is 15 acres and the greater side 6 times the smaller. Find its sides.

Ex. 355. The bisectors of the base $\angle$s of an isosceles $\triangle$ form an $\angle$ of 100°. Find the $\angle$s of the $\triangle$.

Ex. 356. Find the number of sides of regular polygons whose exterior angles are: 30°; 90°; 60°; 45°; 36°; 72°; 120°; $51\frac{3}{7}$°.

Ex. 357. Find the number of sides of regular polygons whose interior angles are: 108°; 135°; 144°, 60.

Ex. 358. The arcs intercepted between two tangents to a circle are as, 4 : 5. Find the arcs and the angle between the tangents.

Ex. 359. The angle between two secants is 40°, and the smaller of the intercepted arcs is 50°. Find the greater intercepted arc.

Ex. 360. The sides of a $\triangle$ are 6, 9, and 4. Find the segments of the side 4 produced made by the bisector of the exterior vertical angle.

THEOREM I.

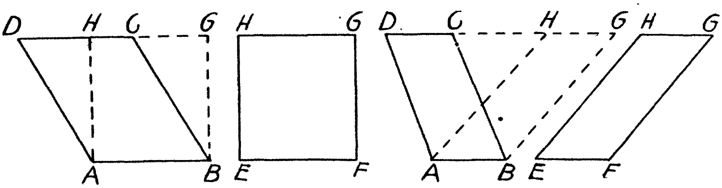
204. *1. Parallelograms having equal bases and altitudes are equivalent.*

2. The area of a parallelogram is equivalent to the product of its base by its altitude. Formula: $A \bumpeq bh$.

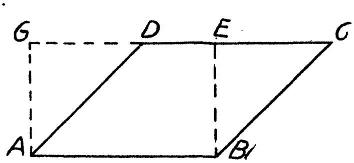
3. Any two parallelograms are to each other as the products of their bases and altitudes. Formula: $A : A' = bh : b'h'$.

4. Parallelograms having equal altitudes are to each other as their bases. Formula: If $h = h'$; $A : A' = b : b'$.

5. Parallelograms having equal bases are to each other as their altitudes. Formula: If $b = b'$; $A : A' = h : h'$.



I. In the ▱s . . . .
1. For place ▱ EG on . . . .
2. The upper base HG falls . . . . . .
3.—6. Prove $\triangle DAH =$ . . . .
Complete the proof.



II. Let $AC \equiv$ . . . . . .
1. For ▱ $AC \bumpeq$ □ . . Why? §204: 1
2. If AG and AB are cmbl. . . . . Why? §198
3. This is also true when AF and AB are incmbl. Why? §168: 3

4. . . . .

III. Let P and $P' \equiv 2$ parallelograms, b and b' . . ., h and h', then . . . .

1. $P \bumpeq$ . . . Why? §204: 2
2. Sim. $P' \bumpeq$ . . .
3. . . . . Why? §17: 2

IV. and V. Deduce from §204: 2.

THEOREM II.

205. *1. The area of a triangle is half the product of its base and altitude. Formula:* $T \bumpeq \frac{1}{2}bh$.

2. The area of a triangle is half the product of its perimeter, by the radius of the inscribed circle. Formula: $T \bumpeq \frac{1}{2}(a + b + c)\,r \bumpeq \frac{1}{2}pr \bumpeq sr$.

3. Any two triangles are to each other as the products of their bases and altitudes. Formula: $T : T' = bh : b'h'$.

4. Triangles having equal altitudes are to each other as their bases. Formula: If $h = h'$; $T : T' = b : b'$.

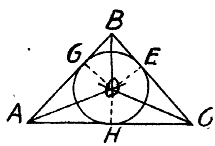
5. Triangles having equal bases are to each other as their altitudes. Formula: If $b = b'$; $T : T' = h : h'$.

6. Triangles having equal bases and altitudes are equivalent.

I. Complete a ▱ and apply §§87 and 204: 2.

II. Join center of O to the vertices of $\angle$s and apply §205: 1.

III., IV., and V. may be proved by the method used in §204.



Ex. 361. Find the areas of parallelograms having the common altitude, 12 cm, and the bases: 3 cm, 5 cm, 8 cm, 9 cm, 11 cm.

Ex. 362. Find the areas of parallelograms whose common base is 11m, and whose altitudes are: 2 m, 4 m, 6 m, 12 m.

THEOREM III.

206. *1. The square on the sum of two lines is equivalent to the sum of their squares increased by twice their product. Formula:* $(a+b)^2 \bumpeq a^2 + 2\,ab + b^2$.

2. The square on a line is four times the square on its half.

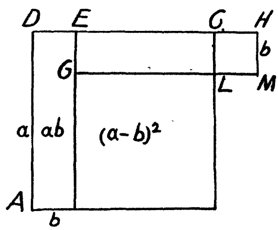
I. Let a and $b \equiv$ the |s; $a + b \equiv \ldots .$; then $(a + b)^2 \bumpeq a^2 + \ldots \ldots$

Prove this, either algebraically by multiplication, or by drawing the squares, etc.

THEOREM IV.

207. *The square on the difference of two lines is equivalent to the sum of their squares diminished by twice their product. Formula:* $(a-b)^2 \bumpeq a^2 + b^2 - 2\,ab$.

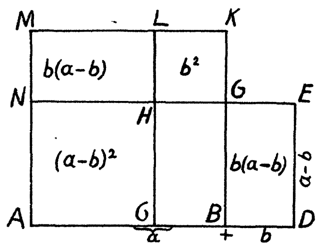
Prove this by using the adjacent figure, or algebraically.



THEOREM V.

208. *The product of the sum and the difference of two lines is equivalent to the difference of their squares. Formula:* $(a+b)(a-b) \bumpeq a^2 - b^2$.

Prove this by using the adjacent figure, or algebraically.



Ex. 363. Prove by a figure, without applying §206: 1, that the square on the whole of a line is four times the square on its half. $[a^2 = 4\,(a \div 2)^2.]$

Ex. 364. Draw a ▱ whose base is 4 in., area 8 sq. in. and oblique sides each 4 in. Suggestion: First find its altitude.

Ex. 365. In a $\triangle$, $s = 24$, $r = 3$; find its area by §205: 2.

209. DEF. The ***medial of a trapezoid*** is the line joining the mid-points of its non-parallel sides.

THEOREM VI.

210. *1. The area of a trapezoid is equivalent to the product of its altitude by half the sum of its parallel sides. Formula:* $Tr \triangleq \frac{1}{2} h (b + b')$.

2. The medial of a trapezoid is half the sum of its parallel sides. Formula: $m = \frac{1}{2}(b + b')$.

3. The area of a trapezoid is equivalent to the product of its altitude and medial. Formula: $Tr \triangleq hm$.

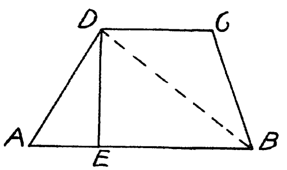


Fig. 1.

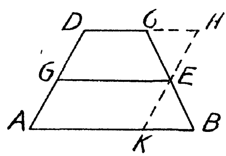


Fig. 2.

I. (Fig. 1.) Apply §205: 1.

II. (Fig. 2.) Apply §§67: 1, 72: 2, and 87: 3, 2.

III. (Fig. 2.). Apply §210: 1, 2.

Ex. 366. In a trapezoid: $b = 4$, $b' = 7$, $h = 12$. Find the area.

Ex. 367. In a trapezoid: $Tr = 120$, $b = 14$, $b' = 10$. Find h.

Ex. 368. The bases of a trapezoid are 12 and 18. Find its medial.

Ex. 369. Find the areas of triangles having bases and altitudes equal to those of the parallelograms in Ex. 361 and 362.

Ex. 370. Illustrate theorems III. to V. by numerical examples in which $a = 5$ cm., $b = 3$ cm.

Ex. 371. In ▱s in which $b = 9$, find h if $P = 27$; 36; 72; 108; 144.

Ex. 372. Find b in $\triangle$s in which $h = 9$ m, $T = 45$ m; 63 m; 99 m.

Ex. 373. In a $\triangle$, $a = 8$, $b = 15$, $c = 17$, $T = 60$. Find r.

211. Def. The ***projection*** of a point on a line is the foot of the perpendicular let fall from the point on the line. See fig. 1.

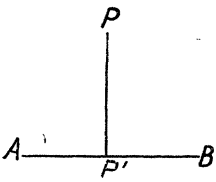


Fig. 1.

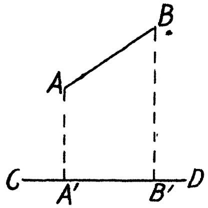


Fig. 2.

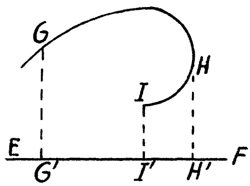


Fig. 3.

212. Def. The ***projection*** of a line on a line is the part of the latter which includes the projections of all the points of the former, as in figures 2 and 3 above.

213. There are many other kinds of projection besides orthogonal projection above defined, but the word projection is generally understood as meaning orthogonal projection, unless otherwise specified or implied.

Ex. 374. What are the projections of AD and BD on AB, Fig. 1 §210; those of AD and BD on DE, Fig. 1 of §210.

Ex. 375. What are the projections of OD and HO on HD, figure of ex. 319?

Ex. 376. What are the projections of the arcs AD, GC, EB, and GA on the line AA' in the first figure of §142.

Ex. 377. What are the projections of OG on KA and KG in the second figure of §142?

Ex. 378. If the mid-points of the adjacent sides of a square be joined a second square will be formed whose area is half the area of the original square.

Ex. 379. If equal distances be measured along the sides of a square from each angle, so that the long segment of each side shall be adjacent to the short side of the next and these points be joined, the original square is so divided as to consist of a new square and four equal right-angled triangles.

Ex. 380. The parallel sides of a trapezoid are 8 and 14 and its altitude 6. Find its medial and area.

Theorem VII.

214. *If the altitude on the hypotenuse of a right-angled triangle be drawn:*

1. The triangles formed are similar to the whole triangle and to each other.

2. The perpendicular is a mean proportional between the segments of the hypotenuse. Formulas: a) $a_c : h = h : b_c$; *b)* $h^2 \stackrel{\wedge}{=} a_c \times b_c$.

3. Each side is a mean proportional between the hypotenuse and the adjacent segment. Formulas: a) $c : a = a : a_c$; *b)* $a^2 \stackrel{\wedge}{=} c \times a_c$.

4. The squares of the legs are proportional to the segments of the hypotenuse. Formula: $a^2 : b^2 = a_c : b_c$.

I. Apply §187: 2.

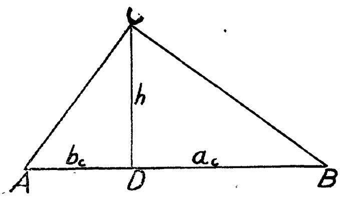
II. and III. Apply §§214: and 186.

IV. Let . . . . . .

1. For $a^2 =$ . . . . .
Why? §211: 3.

2. Sim. b^2 . . . . .

3. . . . . §17: 2



Q. E. D.

Ex. 381. (Fig. to §214) $b_c = 4$, $a_c = 9$. Find h.

Ex. 382. The segments of the hypotenuse of a rt△ made by the altitude on it are 4 and 16. Find the altitude.

Ex. 383. $c = 25$, $b = 10$. Find b_c.

Ex. 384. The legs of a rt△ are 5 and 12. Find the ratio of the segments of the hypotenuse made by the altitude on it.

Ex. 385. In a △, $a = 9$, $b = 41$, $r = 4$, $T = 180$. Find p and c. See §205: 2.

Ex. 386. In a △, $T = 18$. Find all possible pairs of integral bases and altitudes the bases being even integers.

Ex. 387. In a △, $T = 10$. Find all possible pairs of integral bases and altitudes.

Theorem VIII.

215. *The square of the hypotenuse of a right-angled triangle is equivalent to the sum of the squares on the legs. Formulas: 1)* $c^2 \doteq a^2 + b^2$; *2)* $c \doteq \sqrt{a^2 + b^2}$; *3)* $a^2 \doteq c^2 - b^2$; *4)* $a \doteq \sqrt{c^2 - b^2}$; *5)* $a \doteq \sqrt{(c + b)(c - b.)}$

I. (Fig. of §211.) Let ACB . . . . . .

1. For . . $\doteq$ . . . . . . Why? §210: 3
2. Sim. . . . .
3. $a^2 + b^2 \doteq c \times a_c + c \times b_c$ Why?
4. . . . . Q. E. D.

Prove formulas 2—5, and state them in words.

Ex. 388. Illustrate §215 by constructing squares on the three sides of a right-angled triangle and using these squares in stating the theorem.

216. §215, called the Pythagorean theorem from its discoverer Pythagoras (540—510 B. C.), is the most celebrated theorem in geometry. It is the 47th (Book I) in Euclid, and the 11th (Book IV) in Legendre. The method of proof used in these works is outlined in the supplementary exercises.

Ex. 389. If $a_c = 9$ and $b_c = 4$; find h; find a to 2 figures.

Ex. 390. If $h = 8$ and $a_c = 4$: find b_c; if $a = 15$ and $b = 8$; find c.

Ex. 391. If $c = 41$ and $a = 40$; find b without squaring 41.

Ex. 392. If $c = 65$ and $a = 63$: find b. If $c = 37$ and $a = 35$; find b.

Ex. 393. Rewrite the general enunciation of 214: 2, 3, 4, using the word projection and avoiding the use of the word segment.

Ex. 394. The hypotenuse of a right-angled triangle is 17 and one leg is 15. Find the other leg without squaring 17 or 15.

Ex. 395. The projections of the legs on the hypotenuse of a right-angled triangle are 4 and 9. Find the altitude on the hypotenuse, the area and the legs to one place of decimals.

Ex. 396. The difference of the hypotenuse and base of a right-angled triangle is 1 and the perpendicular is 3. Find the area.

THEOREM IX.

217. *1. The square on the side opposite an acute angle of a triangle is less than the sum of the squares on the other two sides by twice the product of one of these sides and the projection of the other on it. Formulas:* a) $a^2 \bumpeq c^2 + b^2 - 2\,c \times b_c$; b) $b_c \bumpeq (c^2 + b^2 - a^2) \div 2\,c$.

2. The square on the side opposite the obtuse angle of an obtuse-angled triangle exceeds the sum of the squares of the other two sides by twice the product of one of these sides and the projection of the other on it. Formula: $a^2 = c^2 + b^2 + 2\,c \times b_c$.

3. The product of the sum and difference of any two sides of a triangle is equivalent to the product of the sum and difference of their projections on the third side. Formulas: a) $(a + b)\ (a - b) \bumpeq (a_c + b_c)\ (a_c - b_c)$; b) $a_c - b_c = (a + b)\ (a - b) \div c$.

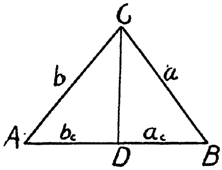


Fig. 1.

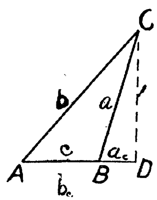


Fig. 2.

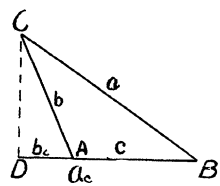


Fig. 3.

I. Let . . . .

1. For either: $a_c = c - b_c$ or $a_c = b_c - c$.
2. $\therefore\ a_c{}^2 \bumpeq$ . . . .
3. $h^2 \bumpeq b^2 -$ Why? §212: 3
4. $a^2 \bumpeq$ . . . . Why? §212: 1
5. . . . .

Substituting in equation 4 values of $a_c{}^2$ and h^2 . . . .

Q. E. D.

II. The proof differs from the above only in the signs.

III. Apply §212, subtract and factor.

THEOREM X.

218. *1. Formula*:* $c^2 \stackrel{\frown}{=} a^2 + b^2 \pm 2\,a \times b_a$.

2. An angle of a triangle is acute, right, or obtuse, according as the square on the side opposite to it is greater than, equal to, or less than, the sum of the squares on the other two sides.

I. Apply §217. State the theorem in words.

II. The method of §54: 2 may be used.

PROBLEM I.

219. *1. To find a square equivalent to the sum of two given squares.*

2. To find a square equivalent to the difference of two given squares.

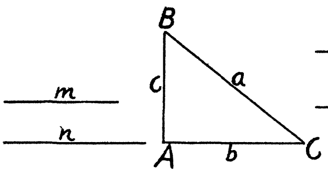


Fig. 1.

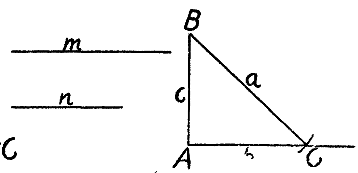


Fig. 2.

I. (Fig. 1.) Apply §215: 1.

II. (Fig. 2.) Apply §215: 2.

Ex. 397. Construct a square equivalent to the sum of three given squares.

Ex. 398. The sides of a triangle are 160, 200 and 240. Find, 1) the segments into which each side is divided by the altitude on it; 2) the altitude on the side 240; 3) the area. Ans. 25, 135; 60, 140; 90, 150; 132.29; 15873. +

Ex. 399. Show by §217: 1 that $a_c = \dfrac{a^2 + b^2 - c^2}{2a}$.

Ex. 400. State ex. 399 in words as a theorem.

Ex. 401. If $a = 10$, $b = 15$, c 20; find b_c, h, and T.

* This formula is called the generalized Pythagorean proposition.

THEOREM XI.

220. *1. Triangles of which the one has an angle equal to an angle of the other are proportional to the products of the sides including the equal angles.*

2. Similar triangles are to each other as the squares of any two homologous sides.

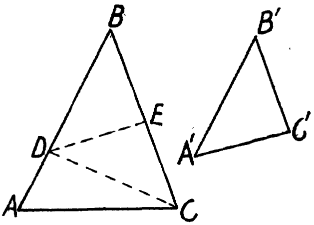


Fig. 1.

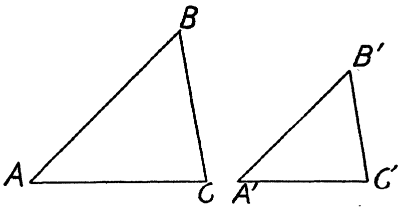


Fig. 2.

I. In $\triangle$s . . . . . .
1. For place $\angle B'$ . . . . . .
2. Join . . . . . .
3. $\triangle BDE = \triangle$ . . . Why? §38
4. $\triangle$ . . . : $\triangle$ . . $= BE : EC$. Why? §205: 4
5. . . . . . . $= BD : DA$. Why?
6. . . . .

Multiplying (4) by (5) and cancelling. Q. E. D.

II. (Fig. 2.) Let ABC . . . .
1. For $\triangle$ . . . : $\triangle$ . . . $= a \times b : a' \times b'$. Why? §220: 1
2.—5. Complete the proof.

Ex. 402. In $\triangle$s ABC and $A'B'C'$; $A = A'$; $b = 4$, $c = 6$; $b' = 12$, $c' =$; area $ABC = 9$. Find the area of $A'B'C'$.

Ex. 403. Similar triangles are to each other as the squares of homologous angle-bisectors. Apply §194.

Ex. 404. Similar triangles are to each other as the squares of homologous altitudes.

Ex. 405. Homologous sides of similar triangles are 5, 10, 15, 20, and 25. The area of the first is 2 sq. in. Find the areas of each of the others.

Theorem XII.

221. *Two similar polygons are to each other as the squares of two homologous sides, diagonals, or altitudes. Formula:* $P : P' = a^2 : (a')^2 = d^2 : (d')^2 = h^2 : (h')^2$.

2. Regular polygons are to each other as the squares of the radii of their inscribed or circumscribed circles. Formulas: a) $P : P' = r^2 : (r')^2 = R:^2 (R')^2$; *b)* $P : P' = d^2 : (d')^2$; *c)* $P : P' = c^2 : (c')^2$.

3. The areas of circles are to each other as the squares of their radii, diameters, or circumferences. Formula: $C : C' = r^2 : (r')^2 = d^2 : (d')^2 = c^2 : (c')^2$.

4. Polygons whose sides and angles are equal in the same order are equal.

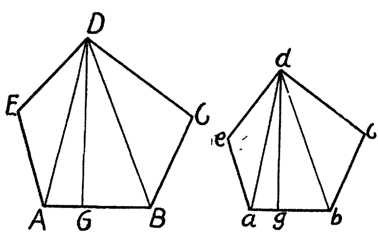


Fig. 1.

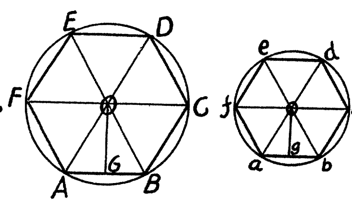


Fig. 2.

I. Let $P \equiv$ pol. . . ., $p \equiv$ . . . . and $P \sim p$; then $P : p$ . . . . . .

1. For $\triangle ADE \sim \triangle$ . .; $\triangle ADB \sim \triangle$ . . and . . . . . Why? §192: 1

2. $\triangle ADE : \triangle\ ade = \triangle$ . . : $\triangle$ . . $= \triangle$ . . . . . $= \overline{AB}^2 : \overline{ab}^2$. Why?

3. $P : p = \overline{AB}^2 : \overline{ab}^2$. Why? §161

Complete the proof.

II. (Fig. 2.) Apply §§221: 1 and 192: 2 ($\triangle$s).

III. (Fig. 2.) Apply § 173.

Theorem XIII.

222. *1. If two chords intersect in a circle the product of the segments of the one is equivalent to the product of the segments of the other.*

2. An ordinate to a diameter of a circle is a mean proportional between the segments of the diameter.

I. Let the chord . . .

1. For join . . . . . .

2. Now in $\triangle$s AEC . . . $\angle A = \angle$ . . and $\angle B$ . . Why? §178: 1

3. $\therefore \triangle$ . . $\sim$ . . . Why? §187: 2

4. $\therefore b : c =$ . . . . Why? §186

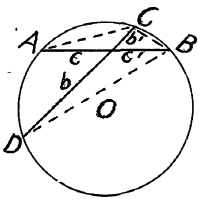


Fig. 1.

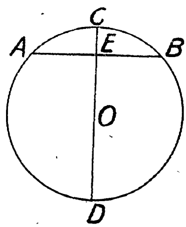


Fig. 2.

5. . . . . Why? §155: 9

Q. E. D.

II. (Fig. 2.) Apply §222: 1.

Ex. 406. (Fig. 1 above) $b = 8$, $b' = 3$, $c = 6$; find c'.

Ex. 407. An ordinate cuts a diameter of a circle into segments 2 cm and 18 cm in length. Find the length of the ordinate.

Ex. 408. A chord 12 cm long cuts another chord into segments of 4 cm and 9 cm in length. Find the segments of the first.

Ex. 409. The diameter of a circle is 20 in. Find the segments into which it is divided by an ordinate 8 in. in length.

Ex. 410. In the sim. $\triangle$s ABC and $A'B'C'$; $a = 3$, $a' = 5$, and the area of $ABC = 18$; find the area of $A'B'C'$.

Ex. 411. $P = 12$ sq. yds, P' sim. to P, $a = 2$ ft, $a' = 3$ yds; find P'. Apply §221.

Ex. 412. $P = 48$, $P' = 75$, $a = 4$; find a'.

Ex. 413. 3 and 5 are homologous sides of similar polygons; the area of the smaller is 18. Find the area of the larger.

Ex. 414. Find the ratio of similitude of two similar polygons whose areas are 45 and 80.

THEOREM XIV.

223. *1. If from a point without a circle two secants be drawn the product of the one by its external segment is equivalent to be product of the other by its external segment.*

2. If from a point without a circle a secant and a tangent be drawn, the product of the secant by its external segment is equivalent to the square of the tangent.

3. If from a point without a circle two tangents be drawn, the squares of the tangents are equal.

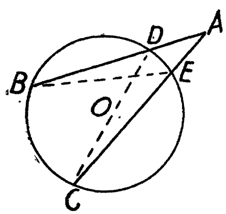


Fig. 1.

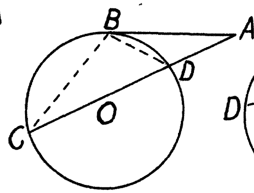


Fig 2.

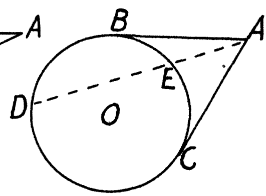


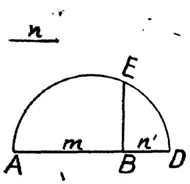
Fig. 3.

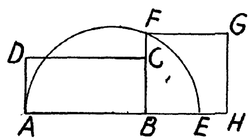
I. Prove by the aid of the sim. $\triangle$s ABE and ADC.
II. Prove by the aid of the sim. $\triangle$s ABC and ABD.
III. (Fig. 3.) Apply §223: 2.

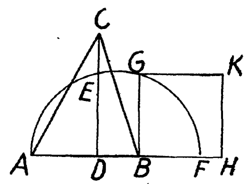
Ex. 415. (Fig. 1) $AD = 4$, AB, 10, and AC, 16; find AE.
Ex. 416. (Fig. 1) $AD = 4$, $BD = 8$, $AE = 3$; find EC.
Ex. 417. (Fig. 2) $AB = 6$, $AD = 4$; find AC.
Ex. 418. (Fig. 2) $AB = 8$, $CD = 12$; find AD and AC.
Ex. 419. (Fig. 1) $AD = 5$, BD 15, $CE = 21$; find AC.
Ex. 420. A secant 10 in. long is bisected by the circumferences. The segments of a second secant are $3x$ and $5x$. Find the segments.
Ex. 421. If $A < 90°$, $b = 8$, $c = 10$, $b_c = 1$; find a. See §217: 1.
Ex. 422. If $A < 90°$, $b = 12$, $c = 10$, $b_c = 1$; find a.
Ex. 423. If $A > 90°$, $b = 8$, $c = 6$, $b_c = 4$; find a. See §217: 2.
Ex. 424. Classify the following triangles by applying 218: 2: 1) $a = 6, b = 8, c = 9$; 2) $a = 6, b = 8, c = 10$; 3) $a = 6, b = 8, c = 12$; 4) $a = 8, b = 12, c = 17$; 5) $a = 8, b = 16, c = 17$; 6) $a = 9, b = 40, c = 41$.

Problem II.

224. *1. Find a mean proportional between two given lines.*
2. Construct a square equivalent to a given rectangle.
3. Construct a square equivalent to a given triangle.



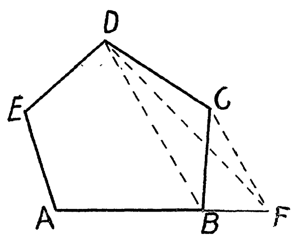




I. Apply §222: 2.
II. Construct $BE = BC$ etc.
III. Construct $BF = \frac{1}{2}\ DC$ etc.

Problem III.

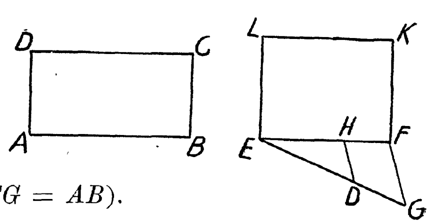
225. *Construct a square equivalent to a given polygon.*



The figure shows only the first operation. Prove by §205: 6 that the polygon $ABCDE \triangleq$ polygon $AFDE$. Finally apply §224: 3.

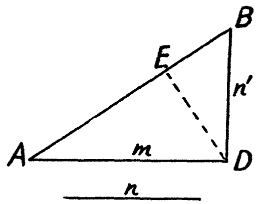
Problem IV.

226. *On a given line construct a rectangle equivalent to a given rectangle.*



Apply §181: 1.
$(EH = AD$ and $EG = AB)$.

PROBLEM V.

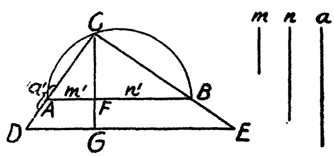


227. *Find two lines having the ratio of two given polygons.*

Find by §225 m and n, the |s of □s ⩯ the given polygons; then apply §214: 4.

PROBLEM VI.

228. *Find a square which shall be to a given square in the ratio of two given lines.*



Let m and n ≡ given |s; a the | of the □.

1. Lay off in a |, $AF = m$ and $FB = n$.

2. On AB const. a $\frac{1}{2}$ ○, also $FC \perp AB$.

3. Join CA and CB. If necessary produce CA and lay off $CD = a$. Const. $DE \parallel AB$ meeting CB or CB produced in E, CE is a side of the required square.

In proof apply §214: 4.

229. DEF. A line is said to be divided in the ***extreme and mean ratio*** when the greater segment is a mean proportional between the whole line and the smaller segment.

Formula: $a : x = x : a - x$.

Ex. 425. Inscribe a square in a given quadrant.

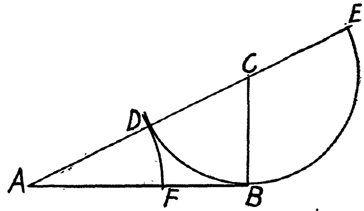
Ex. 426. Inscribe a square in a given semi-circle. Suggestion. Construct another square. Join the mid-point of one side to the vertex of an opposite angle, etc.

Ex. 427. Inscribe a square in a given segment.

Ex. 428. Inscribe a circle in a given semi-circle.

PROBLEM VII.

230. *Divide a line in the extreme and mean ratio.*



Find a pt. F in AB, such that $AB : AF = AF : FB$.

1. At B erect $\perp$ $BC = \frac{1}{2} AB$.
2. With C as a center . . . . .
3. Join AC and produce it cutting the ○ in D and E.
4. From AB cut off $AF = AD$; F is the required pt.
5. For . . . $\triangle$ . . . Why? §223: 2
6. $AE : AB = AB : AD$. Why?
7. $\therefore AE - AB : AB = AB - AD : AD$. Why? §155: 1
8. $AE - AB = AE - DE = AD = AF$. Why?
9. $AF : AB = BF : AF$. Why?
10. AB : . . . . Why? §155: 7

Q. E. F.

Ex. 429. Prove algebraically that $AF = \frac{1}{2} AB (\sqrt{5} - 1)$ and $FB = \frac{1}{2} AB (3 - \sqrt{5})$. Find AF to 3 decimal places if $AB = 10$.

Ex. 430. Find the segments of a line 4 inches long divided in the extreme and mean ratio. Ans. 2.472, 1.528.

Ex. 431. Draw three radii trisecting a given circle.

Ex. 432. Construct three equal circles touching each other, and also two other circles, each touching each of the first three.

Ex. 433. Construct six equal circles touching a given circle internally and each also touching two of the others. Prove that a seventh circle, concentric with the given circle may be constructed touching each of the circles constructed and equal to each of them.

Ex. 434. Construct eight equal circles each touching internally a given circle and also two of the other circles.

Ex. 435. Construct ten equal circles touching a given circle internally and each touching two others.

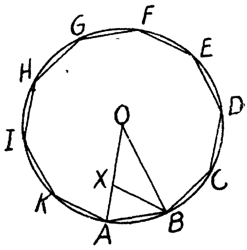
Ex. 436. An ordinate to a diameter of a circle is 8 and the diameter 20. Find its segments.

PROBLEM VIII.

231. *1. Inscribe a regular decagon in a circle.*

2. Inscribe a regular pentagon in a circle.

3. Inscribe a polygon of 5×2^n sides in a circle.



I. Inscribe a decagon in $\bigcirc O$.

1. Join OA; divide OA by X so that $OA : OX = OX : XA$. §230. OX is a side of the decagon.

2. For const. $AB = OX$; join OB and BX.

3. Now in $\triangle$s OAB, OBX, $OA : AB = AB : AX$. Why?

4. $\triangle OAB \sim \triangle \ldots$ and $\triangle \ldots$ is isosc. Why? §187: 4

5. $BX = OX$. Why?

6. $\angle O = \angle XBO$. Why? §45: 1

7. But $\angle AXB = \angle O + \angle XBO$. Why?

8. $\angle \ldots = 2 \angle \ldots$

9. $\angle OAB = 2 \angle \ldots$

10. $\angle \ldots = 2 \angle \ldots$

11. Sum $\angle$s $\triangle OAB = 5 \angle O$.

12. .. rt $\angle$s $= 5 \angle O$.

13. $\therefore \angle O = \frac{1}{5}$ (2 rt $\angle$s) $= \frac{1}{10}$ 4 rt $\angle$s.

14. $\therefore \smile AB = \frac{1}{10}$ circumference. Q. E. D.

II. Apply §231: 1.

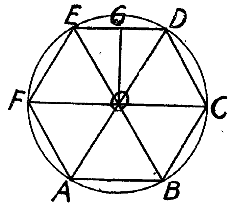
III. Apply §231: 1 and §42.

232. REMARK. We have now constructed polygons of 2×2^n sides, 3×2^n sides and 5×2^n sides, using the compass and straight edge only. In 1796, Gauss, the great German mathematician, then only 19 years old, proved by the aid of Algebraic analysis that all polygons of 2^n $(2^m + 1)$ sides could be constructed, m and n being integers and $2^m + 1$ a prime number. By the principles of factoring, $2^m + 1$ cannot be prime unless m is unity or a power of 2. If $m = 1$, $2^m + 1 = 3$; if $m = 2$, $2^m + 1 = 5$; if $m = 4$, $2^m + 1 = 17$ which is prime; if $m = 8$, $2^m + 1 = 257$ which is prime; if $m = 16$, $2^m + 1$ is not prime being divisible by 13. The construction of polygons of 17 and 257 sides by Gauss's method is however too tedious to be of practical value.

Theorem XV.

233. *1. The area of a regular polygon is half the product of its perimeter and apothem. Formula:* $P \cong p \times r$.

2. The area of a circle is half the product of its radius and circumference. Formula: $C \cong \frac{1}{2} rc$.



I. Apply 205: 1.

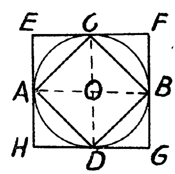
II. Apply §§233: 1 and 173.

234. Def. A ***unit circle*** is one whose radius is unity.

Theorem XVI.

235. *1. The area of a square circumscribed about a circle is* $4r^2$.

2. The area of an inscribed square is $2r^2$.



Ex. 437. If in the figure of §233. EO is 1, find ED, OC, and the area of the polygon. Ans. 1; $\frac{1}{2}\sqrt{3}$; $\frac{1}{2}(3\sqrt{3})$.

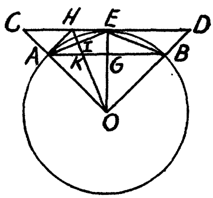
Ex. 438. Find the areas of the squares inscribed in and circumscribed about a circle whose radius is 6.

Ex. 439. The radius of a circle is 25. Find the distance between two parallel chords one 14 inches in length, one 30 inches (two cases). Apply §215. Ans. 4; 44.

Ex. 440. The lengths of two parallel chords 8 m. apart are 48 m. and 40 m. Find the diameter of the circle. Ans. 50.

PROBLEM IX.

236. *Given the areas of inscribed and circumscribed polygons of n sides, find the areas of inscribed and circumscribed polygons of 2 n sides.*



Let $P \equiv$ the area of the circumscr. .

$p \equiv$ . . . .

$P' \equiv$ the area of the circumscr. pol. of $2\,n$ |s.

$p' \equiv$ . . . .

$CD \equiv$ a | of P, $AB \equiv$ . ., AE. ., AH and $HE \equiv \frac{1}{2}$ |s of P'.

Evidently COE. . . . . are like parts of P, p, P', and p'.

I. Given P and p, to find p'.

1. $p : p' \equiv AOG : AOE = OG :$ . . Why? §205:4
2. $p' : P \equiv AOE :$ . . $=$ . . : . . Why? §205:4
3. But $OA : OC = OG :$ . . . Why? §180:2
4. $p' : P =$ . . . . . Why? §17:1
5. $p : p' = p' : P.$ Why? §17:1
6. $\therefore p' = \sqrt{p\,P}.$ Why?

II. Given P, p, and p', to find P'.

1. $\triangle\, OAH = \triangle\, OHE$, and $\angle\, HOE = \angle\, HOC.$ Why?
2. $\triangle\, OHC : \triangle\, OHE = CH : HE = CO : OE = p' : p.$
3. $\triangle\, OCE : \triangle\, OHE = p + p' : p.$ Why? §155:11

   Remembering that $\triangle$ . . $+ \triangle$ . . $= \triangle\, OCE$.
4. $\triangle\, OCE : 2\,\triangle\, OHE = p + p' : 2p.$ Why?
5. . . . . Why?
6. $P' = \dfrac{2\,p\,P}{p + p'}$ Why? §155:10

Ex. 441. The area of an equilateral triangle inscribed in a unit circle is 1.299, and the area of the circumscribed equilateral triangle is 5.196. Find the areas of the regular inscribed and circumscribed hexagons to 2 places of decimals.

PROBLEM X.

237. *Find the area of a unit circle.*

Use P, p, P', and p' as in §236, and $r = 1$:

I. Let $n = 4$.

1. $P = 4$, $p = 2$. Why? §235

2. $p' = \sqrt{2 \times 4 = 8} = 2.82834$.

3. $P' = \dfrac{2\ (2 \times 4)}{2 + 2.82834} = 3.31371$.

II. Let $n = 8$; then $P = 3.31371$, $p = 2.82834$.

1. $p' = \sqrt{2.82834 \times 3.31371} = 3.06147$.

2. $P' = \dfrac{(2\ 2.82834 \times 3.31371)}{2.82834 + 3.06147} = 3.18260$.

III. Continuing the operation 8 times, we find $P = 3.14159$, $p = 3.14159$.

It is evident that the area of the circle $< P$ and $> p$. Hence the area of a unit circle is 3.14159 correct to five decimal places. The exact area is represented by the Greek letter π.

Unless otherwise stated use $\pi = 3.1416$; $1 \div \pi = .3183$; more exactly, $1 \div \pi = .31830989$.

THEOREM XVII.

238. *Let $c \equiv$ the circumference of a circle, r its radius, d its diameter, and C its area.*

1. $C = \pi r^2 = \pi d^2 \div 4$.

2. $c = 2 \pi r = \pi d$.*

I. Apply §§221: 3 and 237.

II. Apply §§233: 2 and 238: 1.

* Since the time of Euler π has represented the ratio of the circumference of a circle to its diameter. The Jews, Babylonians, Chinese, and probably the most ancient Greeks, used the ratio 3 : 1. At one time the Egyptians used 3.16: 1. Archimedes proved that $\pi < 22:7$ and that $\pi > 223:71$. Little progress was made till the 16th century, when Vieta proved a more convenient formula, see Encyclopedia Britannica. Ludolph, in 1596, published the value correct to 16 places. Though this

Theorem XVIII.

239. *1. The area of a sector whose angle is A, is $\pi\ r^2\ \theta$, where $\theta = A \div 360°$.*

2. Similar sectors in unequal circles are to each other as the squares of their radii. Formula: $S : S' = r^2 : (r')^2$.

I. Apply §§175: 2 and 238: 1.

II. Apply §§221: 3 and 175: 2.

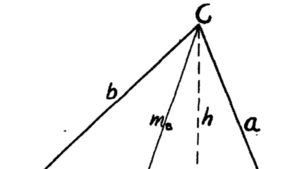
Ex. 442. Find the area of a sector whose angle is 36° in a circle whose radius is 5.

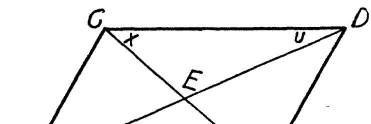
Ex. 443. The radius of a circle is 10, and the area of a sector of it is 26.18. Find the angle of the sector.

Theorem XIX.

240. *1. Twice the square on the median on any side of a triangle, increased by twice the square on half that side, is equivalent to the sum of the squares on the other two sides. Formula:* $2\ m_c^2 + 2\ (\frac{1}{2}\ c)^2 = a^2 + b^2$.

2. The sum of the squares on the sides of a parallelogram is equivalent to the sum of the squares on the diagonals.





I. Apply §217: 1, 2.

II. Apply §§240: 1 and 206: 2.

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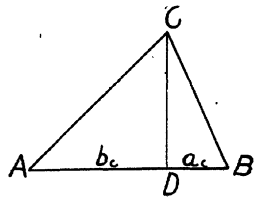
was amply sufficient for practical purposes, he continued the computation till his death in 1610, proving that $\pi = 3.141\ 592\ 653\ 589\ 793\ 238\ 462\ 643\ 383\ 279\ 502\ 88$. The Calculus enables us to find many series by which the computation is shortened. By the aid of these, the value has been found by a number of computers to 500 places of decimals, and by an Englishman named Schanks to 707 places of decimals.

Theorem XX.

241. *The area of a triangle is* $\sqrt{s(s-a)(s-b)(s-c)}$.

Let $T \equiv$ the area of a $\triangle$; then

. . . .



1. For $T \doteq$ . . . Why? §205: 1
2. $h = \sqrt{(b + b_c) \ldots}$. Why? §215: 5
3. $b_c = (b^2 + c^2 - a^2) \div 2c$. Why? §217: 1
4. $b + b_c = \ldots = \dfrac{(a + b + c)(c + b - a)}{2c}$.
5. $b - b_c = \ldots = \dfrac{(a - b + c)(a + b - c)}{2c}$.
6. $T \doteq \ldots\ldots \doteq \ldots \sqrt{s(s-a)(s-b)(s-c)}$.

Q. E. D.

Ex. 444. Find the area of a triangle whose sides are 8, 10, and 12.

Ex. 445. Find the area of a circle whose radius is 6.

Ex. 446. Find the circumference of a circle whose radius is 5.

Ex. 447. Find the radius and circumference of a circle whose area is 100.

Ex. 448. $c = 336.62$; find r and C. (See §238).

Ex. 449. $d = 12$; find C.

Ex. 450. $C = 286.479$; find r and d.

Ex. 451. The sides of a triangle are $3x$, $4x$, and $5x$ and its area 150. Find each side.

Ex. 452. Two sides of a triangle are 6 and 10 and its area is 24. Find the third side.

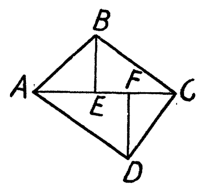
Ex. 453. The sum of the base and perpendicular of a right-angled triangle is 11, and its area 30. Find each side.

Ex. 454. If m and $m' \equiv$ the medians on the legs of the rt $\triangle$ ABC; $a = 2\sqrt{4(m')^2 - m^2} \div \sqrt{15}$, $b = 2\sqrt{4m^2 - (m')^2} \div \sqrt{15}$, $c = 2\sqrt{3\{m^2 + (m')^2\}} \div \sqrt{15}$. Find the sides if $m = 8$, and $m' = 10$. Ans. 12.22 +, 5.16 +, 13.26 +.

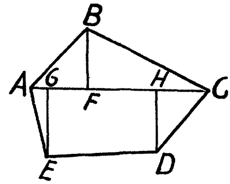
FIELD WORK.

242. The preceding principles enable us to determine the area of any piece of land, using only the simple apparatus described in §101. The exercises given are merely suggestive. It is better to measure pieces of land.

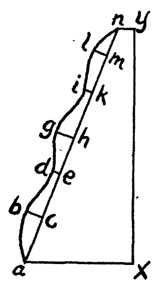
Ex. 455. $AC = 12$ ch, $BE = 4$ ch, $DF = 6$ ch, BE and $DF \perp AC$. Find the area in acres. Ans. 6 A.



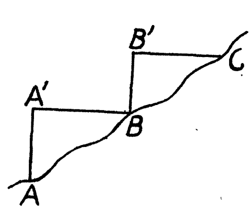
Ex. 456. $AC = 20$ ch, $BF = 5$ ch, $HD = 6$ ch, $GE = 8$ ch, $AG = 2$ ch, $HC = 6$ ch, BF, HD, and $EG \perp AC$. Find the area. See second figure. Ans. 16 A.



Ex. 457. $XY \perp ny$ and aX; bc, de etc. $\perp an$; $ac = 7$ ch, $ce = 8$ ch, $eh = 6$ ch, $hk = 10$ ch, $km = 7$ ch, $mn = 3$ ch, $bc = 3$ ch, $de = 1$ ch, $gh = 3$ ch, $ik = 2$ ch, $lm = 3$ ch, $nY = 3$ ch, $aX = 12$ ch. Ans. 38A. 88 P. Here the curved line represents the bank of a river, pond or brook. To secure a close approximation to accuracy the points b, d, etc. must be judiciously selected, straight lines between them being partly within and partly without the field if possible. The lines bc, de, etc. are called offsets and the method, the method of offsets.



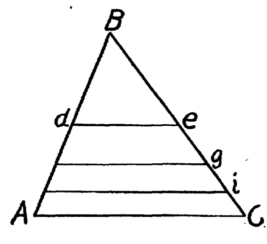
243. The area of a field is not its actual surface, but what its surface would be if graded to a level. When it is necessary to measure distances, the measuring line should be held level. For purposes of finding area, the distance from A to B is not the actual distance AB but the horizontal distance $A'B$, A' being vertically above A, etc.



Ex. 458. From a square field of ten acres cut off two acres by a line parallel to one side.

Ex. 459. From a square field of ten acres cut off two acres by a line from one of the corners.

Ex. 460. The triangular field ABC is divided into four equal parts by lines parallel to the base, determine Ci, ig, etc. in terms of a. Ans. $\frac{1}{2}a(2-\sqrt{3})$; $\frac{1}{2}a(\sqrt{3}-\sqrt{2})$; $\frac{1}{2}a(\sqrt{2}-1)$; $\frac{1}{2}a$.



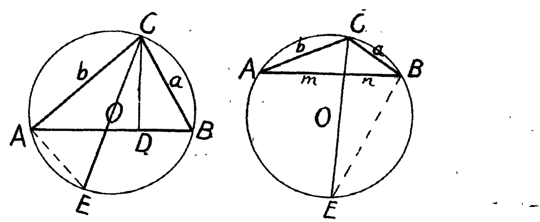
SUPPLEMENTARY THEOREMS AND EXERCISES.

Theorem XXI.

244. *In any triangle the product of two sides is equivalent:*

1. To the product of the altitude on the third side by the diameter of the circumscribed circle. Formula: $ab \bumpeq h_cD \bumpeq 2\,h_cR$.

2. To the square of the bisector of the angle formed by these sides increased by the product of the segments into which it divides the third side. Formula: $a \times b = mn + (c')^2$.



I. Constr. $\angle ACE = \angle BCD$; prove $\triangle\, ACE \sim \triangle BCD$; that $CE : a = b : CD$, etc.

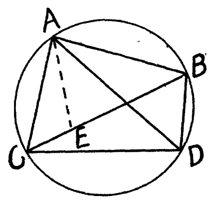
II. Prove $\triangle\ CAD \sim \triangle\ CEB$; that $a \times b \bumpeq CE \times CD \bumpeq CD\,(CD + DE) \bumpeq CD^2 + m \times n$.

Ex. 461. Given: $a = 12$, $b = 10$, $h_c = 8$; find R by §244.

Ex. 462. Given: $a = 8$, $b = 10$, $R = 5$; find h by §244.

Theorem XXII.

245. *The product of the diagonals of a quadrilateral inscribed in a circle is equivalent to the sum of the products of the opposite sides.*

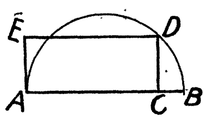


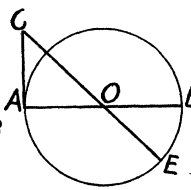
Let . . . . . .
1. For constr. $\angle CAE = \angle DAB$.
2—4. Prove $\triangle CAE \sim \triangle DAB$.
5—6. Prove $CA \times BD = CE \times AD$.
7, 8, 9. Prove $\triangle CAD \sim \triangle BAE$.
10, 11. Prove $CD \times AB = EB \times AD$.
12. . . . . Q. E. D.

Problem XI.

246. *1. Construct a rectangle, its area and the sum of two adjacent sides being given.*

2. Construct a rectangle, its area and the difference of two adjacent sides being given.





I. Let $AB \equiv$ the sum of |s, $AE \equiv$ | of □ containing given area.* Apply §222 : 2.
II. Let $AB \equiv$ the difference of |s, $AC \equiv$ | of □ . . . . . .*
Apply §223: 2.

Problem XII.

247. *Construct a polygon equivalent to one given polygon and similar to another.*

Let $P \equiv$ the first polygon; $P' \equiv$ the second. Constr. a 4th prop. to | of □ ≏ P, | of □ ≏ P', and any | of P by §§225, and 183, and apply §221.

* Observe that only a part of the construction is given.

THEOREM XXIII.

248. *Let r, R, and $R_a \equiv$ radii of circles inscribed in a triangle, circumscribed about it, and escribed on the side a:*

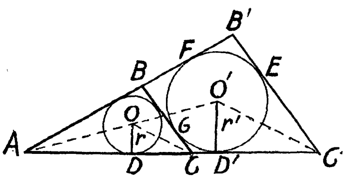
1. $r = T \div s = \sqrt{(s-a)(s-b)(s-c)} \div \sqrt{s}$.

2. $R = abc \div 4T = abc \div 4\sqrt{s(s-a)(s-b)(s-c)}$.

3. $R_a = T \div (s-a) = \sqrt{s(s-b)(s-c)} \div \sqrt{s-a}$.

I. Apply §205: 2.

II. Apply §§244: 1, 205: 1, and 241.



III. Prove by §§205: 2 and 128, that $R_a = T' \div s'$; that $a' = b' + c' - b - c - a$, and hence that $s' - a' = s$; prove, also, that $s' = a's \div a$; that $a's \div a - a' = s$; that $a' = as \div (s-a)$; that $R_a \equiv r' = ra' \div a = r^s \div (s-a) = T \div (s-a)$, etc.

Ex. 463. Find the area of a sector of 45°, if $r = 8$.

Ex. 464. Find the area of a sector of 60°, if $r = 12$.

Ex. 465. (See §244: 2.) Given $a = 8$, $b = 12$, $c = 10$; find m, n, and c'. Apply also §191: 1.

Ex. 466. Find the medians on the sides of a triangle whose sides are 12, 16, and 20. (§244: 1.)

Ex. 467. The sum of the sides of a rectangle is 20 and its area 24. Find its sides.

Ex. 468. Two sides of a parallelogram are 6 and 10 and one diagonal 11. Find the other diagonal.

Ex. 469. The area of a sector of 40° is 100. Find the circumference of the circle.

Ex. 470. Find the radii of the circles inscribed in and circumscribed about the triangle whose sides are 3, 5, and 6, also those of the three escribed circles.

Ex. 471. Find the radii of circles escribed about a triangle whose sides are 8, 10, and 12.

Ex. 472. Find in terms of π the areas of circles inscribed in and circumscribed about the triangle of ex. 471. Ans. 8 π; 3403125 π.

Ex. 473. $a + b = 17$, $c = 13$, $C = 90°$. Find a, b, and c.

THEOREM XXIV.

249. *The radii of circles inscribed in, and circumscribed about regular polygons, one of whose sides is a, and the areas of said polygons are as follows:*

Number of sides.	Radius inscr. circle.	Radius circumscr. circle.	Area polygon.
3	1) $\frac{a}{6}\sqrt{3}$	2) $\frac{a}{3}\sqrt{3}$	11) $\frac{a}{4}\sqrt{3}$
4	3) $\frac{1}{2}a$	4) $\frac{1}{2}a\sqrt{2}$	12) a^2
6	5) $\frac{1}{2}a\sqrt{3}$	6) a	13) $\frac{1}{2}3a^2\sqrt{3}$
8	7) $\frac{1}{2}a(\sqrt[2]{2}+1)$	8) $\frac{1}{2}a\sqrt{4+2\sqrt{2}}$	14) $2a^2(\sqrt[4]{2}+1)$
10	9) $\frac{1}{2}a\sqrt{5+2\sqrt{5}}$	10) $\frac{1}{2}a(1+\sqrt{5})$	15) $\frac{5a^2}{2}\sqrt{5+2\sqrt{5}}$

Prove 10) before 9), from: $r : a = a : (r—a)$ (proved in §231).

Ex. 474. Illustrate each part of §249 by a numerical example, and solve it.

Ex. 475. Deduce from §249 the areas of regular polygons of 3, 4, 6, 8, and 10 sides inscribed in and circumscribed about a circle whose radius is r. Ans. $\frac{3r^2}{4}\sqrt{3}$, $3r^2\sqrt{3}$; $2r^2$, $4r^2$; $\frac{3r^2}{2}\sqrt{3}$, $2r^2\sqrt{3}$; $2r^2\sqrt{2}$, $8r^2(\sqrt{2}-1)$; $\frac{5r^2}{8}\sqrt{10+2\sqrt{5}}$, $10r^2\sqrt{5-\sqrt{2}}$

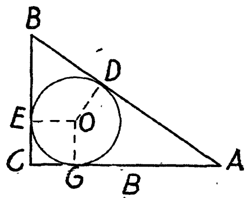
THEOREM XXV.

250. *1.* $C=\frac{\pi d^2}{4}=\frac{c^2}{4\pi}$. *3.* $d=\frac{c}{\pi}=2\sqrt{\frac{C}{\pi}}$.

2. $r=\frac{c}{2\pi}=\sqrt{\frac{C}{\pi}}$. *4.* $c=\sqrt{4\pi C}=\frac{2C}{r}$.

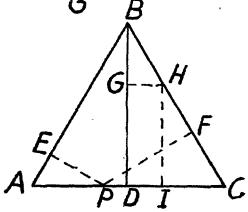
Ex. 476. The angle bisector of the angle C of the triangle ABC $=2\sqrt{abs(s—c)}\div(a+b)$.

Ex. 477. The sides of a triangle are 8, 10, and 12. Find the angle bisector on the side 12, and the segments into which it divides this side.



Ex. 478. The sum of the legs of a right-angled triangle is equal to the sum of the hypotenuse and the diameter of the inscribed circle.

Ex. 479. The sum of the perpendiculars from any point in one side of an equilateral triangle to the other sides is equal to the altitude of the triangle.



Ex. 480. The sum of the perpendiculars to the three sides from any point in an equilateral triangle is equal to the altitude of the triangle.

Ex. 481. In any right-angled triangle the sum of the legs is equal to the sum of the diameters of the inscribed and circumscribed circles.

Ex. 482. If $c = 12$, find r. Use $1 \div \pi = .31831$.

Ex. 483. If $d = 12$; find C; if $c = 8$; find C.

Ex. 484. If $C = 400$; find r and d.

Ex. 485. $b + c = 32$, $a = 8$, $C = 90°$. Find a, b, and c.

Ex. 486. $c - a = a - b = 1$ $C = 90°$. Find a, b, and c.

Ex. 487. The medians on the legs of a right-angled triangle are 6 and 8 respectively. Find its 3 sides.

Ex. 488. The sides of a triangular field are $5x$, $6x$, and $7x$, and its area 80 acres. Find the length of each side.

Ex. 489. The area of a walk four feet wide outside the fence of a field in the form of an equilateral triangle is one acre. Find the area of the field.

Ex. 490. The total number of diagonals that can be drawn in a polygon of n sides is $\frac{1}{2} n (n - 3)$.

Ex. 491. The total number of lines joining n points no three in a straight line is $\frac{1}{2} n (n - 1)$.

Ex. 492. Two parallel chords on opposite sides of the center of a circle are 16 and 12 inches, and the distance between them is 14 inches. Find the area of the circle.

Ex. 493. The area of a circle is 314.16, and two parallel chords on the same side of the center are 16 and 12 inches respectively. Find their distance apart.

Ex. 494. Prove the Pythagorean theorem by each of the following figures.

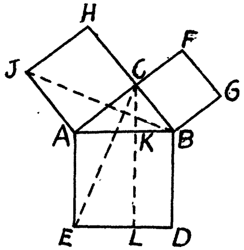


Fig. 1.

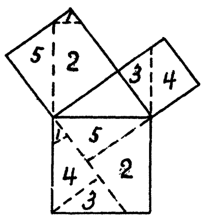


Fig 2.

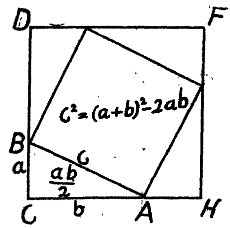


Fig. 3.

Fig. 1. is the method given in most text books.

251. Def. A ***radian*** is an arc whose length is equal to that of the radius.

Find the length of a radian in degrees; in minutes; in seconds. Value $1 \div \pi$ §114: 3.

Ans. 57.29578°, 3437.7468′, 206264.8″.

Ex. 495. Express in radians: 30°; 60°; 75°; 90°.

Ex. 496. Express in degrees: .75 radian; .3 radian; 1.65 radian.

Ex. 497. The surface of a ring $= \pi\,(R^2 - r^2)$; where R and r $\equiv$ the radii of the concentric circles forming the ring.

Ex. 498. Define a circular ring as explained in ex. 493, and draw one in which $R = 1$ inch, $r = \frac{3}{4}$ inch.

Ex. 499. Find the area of a circular ring in hectars, in which $R = 5$ kilometers, $r = 3$ kilometers.

252. Def. A line is divided ***externally*** in the ***extreme and mean ratio***, by a point in the line produced, when the smaller segment is a mean proportional between the line and the greater segment. Formula: $a : x = x : a + x$.

Ex. 500. Divide a line externally in the extreme and mean ratio. In fig. of §230 prod. AB and FD till they meet again, etc.

Ex. 501. The segments of a line a divided externally in the extreme and mean ratio are $1.11803a$ and $2.11803a$.

PROBLEM XIII.

253. *To find a method of computing sets of integral numbers representing the sides of a right-angled triangle.*

Let a and $b \equiv$ the legs of a rt$\triangle$ and c its hypotenuse. Let $n \equiv c - b$, $\therefore c = b + n$, $c^2 = b^2 + 2bn + n^2 = a^2 + b^2$ $\therefore$

A. $b = (a^2 - n^2) \div 2n = a^2 \div 2n - \frac{1}{2} n$.

If $2n$ is a square, formula A take the simpler form.

B. $b = (a \div \sqrt{2n})^2 - \frac{1}{2}n$, where a is any multiple of $\sqrt{n} > n$.

If n is an odd square, A takes the simpler form.

C. $b = \frac{1}{2} \{(a \div \sqrt{n})^2 - n\}$, where a is any multiple of $\sqrt{n} > n$.

Unless $2n$ is a square or n an odd square, it may be shown that $a^2 \div 2n$ and $\frac{1}{2}n$ have a common factor, and that the values of a, b, and c are multiples of those found by formulas B and C. We may also abridge the labor by omitting the computation in formula, B when $a \div \sqrt{2n}$ and $\frac{1}{2}n$ have a common factor, and in formula C, when $a \div \sqrt{n}$ and n have a common factor, these being multiples of lower sets, also of all values of a which make $b < a$, being identical with sets formed by lower values of n.

Substituting 1 for n in formula C we find 21 sets of values in which $c < 1000$, including multiples of these, 406 sets. Substituting 2 for n in formula B we find 134 sets in all not already included. $n = 8$ gives 91 additional sets; $n = 18, 35$; $n = 25, 30$; $n = 32, 28$; $n = 49, 24$; $n = 50, 12$; $n = 72, 9$; $n = 81, 9$; $n = 98, 8$; $n = 121, 11$; $n = 128, 6$; $n = 162, 3$; $n = 169, 4$; $n = 200, 2$; $n = 225, 2$; n 242, 1; $n = 289, 1$. Total 826 sets.

Ex. 502. Find by formula B, all sets up to $c = 100$, if $n = 1$. Ans. 3, 4, 5; 5, 12, 13; 9, 40, 41; 11, 60, 61; 13, 84, 85.

Ex. 503. Find by formula C, if $n = 2$. Ans. 4, 3, 5; 6, 8, 10; 8, 15, 17; 10, 24, 26; 12, 35, 37; 14, 48, 50; etc.

Ex. 504. The area of a circular field is 1200 hectares. Find its circumference in kilometers.

Ex. 505. The ratio of the areas of two circular fields is 4 : 9, and the difference of their radii 10 meters. Find the area of each.

Ex. 506. Three men buy a grindstone two feet in diameter, and agree to grind off equal portions not considering the part within 3 inches of the center. How many inches should each of the two first grind off.

GEOMETRY OF SPACE.

BOOK V.

PLANES AND POLYHEDRAL ANGLES.

254. To understand Geometry of Space we must clearly see in imagination the planes, points, lines, etc. not as pictured in the book, but in space. The pupil should always endeavor to place actual objects in similar positions. He should also remember that, unless he has shown that the figure presupposed by a demonstration can be constructed, the proof is incomplete, that it merely proves that the theorem is true, provided the construction assumed is possible. A sheet of cardboard may be used in constructing many of the figures.

255. Def. A ***dihedral angle*** is the inclination of two planes, measured by the angle between two lines, one in each plane, perpendicular to their common intersection at the same point.

Ex. 507. Raise one side of your book from the table so that this side of the book will form with the surface of the desk an acute angle increasing gradually to a right angle, then to an obtuse angle.

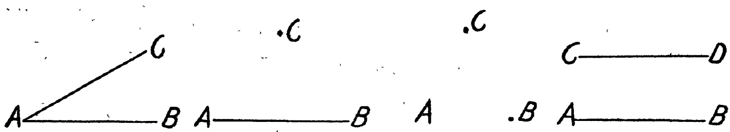
Ex. 508. How many dihedral angles has a chalk box? Are they acute angles, right angles, or obtuse angles?

Ex. 509. Place a piece of card flat on the table. Keep one edge in a fixed position and raise the other. Does it take one point only, or more than one besides the fixed edge to determine the position of the card-board? Would a four-legged stool stand with all its feet on the floor if you cut an inch off of one leg? Would a three-legged stool?

Theorem I.

256. *The position of a plane is determined:*

1. By two intersecting lines.
2. By a line and a point without the line.
3. By three points.
4. By two parallel lines.



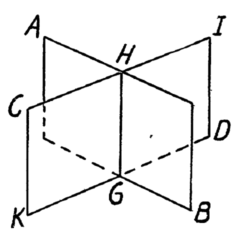
I. Two . . . .
1. For pass a plane through AB.
2. Revolve it till one pt. $\cong$ C.
3. AC . . . . . Why? §10
4. If the plane revolves either way . . . . .
5. Hence the |s . . . .

II. and III. May be proved similarly.

IV. Two || |s . . . .
1. For being || they lie in . . . . Why? §15
Complete the proof.

Theorem II.

257. *The intersection of two planes is a straight line.*

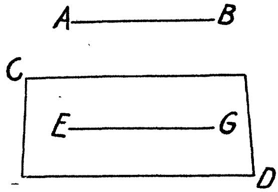


Let HG be . . . .
1. For join G and H by a |.
2. | GH . . . . §10
3. Sim. . . . .
4. | GH lies wholly in both planes . . . .

Q. E. D.

THEOREM III.

258. *A straight line without a plane parallel to a line in the plane is parallel to the plane.*



Let AB . . . .

1. For pass a plane through . . . . .
2. If AB meet the plane CD . . . . .
3. . . . absurd. Why? §15
4. AB cannot meet . . and || . . .

Q. E. D.

259. DEF. A straight line is ***perpendicular to a plane,*** when it is perpendicular to every line passing through its foot in that plane. The ***foot*** is the point where the perpendicular meets the plane.

Properly this definition should follow Theorem IV, since we have no right to assume without proof that such a line is possible. Observe that in proving the theorem we prove that such a line is possible before we make use of the definition.

Ex. 510. The sides of a triangle are 3, 4, and 5. Find the radii of the inscribed and circumscribed circles.

Ex. 511. Find in terms of π the area of a circle circumscribed about a triangle whose sides are 2, 3, and 4 cm. Ans. $\frac{32}{15}\pi$ cm.

Ex. 512. A driveway around a circular pond has a breadth of 6 rods and a surface of 10 acres. Find the diameter and area of the pond. Ans. to diameter, 84.88.

Ex. 513. How many planes may be determined by four points not in the same plane? See §256: 3.

Ex. 514. How many planes are determined by the parallel edges of a cube?

Ex. 515. How many planes are determined by one diagonal of a cube and the edges it intersects?

Ex. 516. How many planes may be determined by 3 points and 5 lines, no two lines being parallel or intersecting?

Theorem IV.

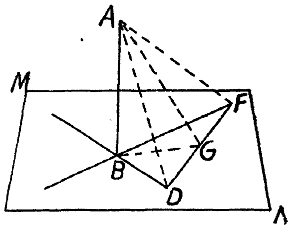
260. *1. A line perpendicular to two lines in a plane is perpendicular to the plane.*

2. Through a given point within or without a plane, only one perpendicular can be drawn to that plane.

I. Let AB be $\perp$ . . . .

1. For let $BG \equiv$ any other | . . . .

2. Through G, any pt. in BG constr. FD making $DG = GF$. §195



3. In $\triangle$ AFD, $\overline{AF}^2 +$ . . . . Why? §240: 1

4. Sim. in $\triangle$ FBD . . . . .

5. $\overline{AF}^2 - \overline{BF}^2 + \overline{AB}^2 - \overline{BD}^2 = 2\,\overline{AG}^2 - 2\,\overline{BG}^2$. §17: 2. Subtr.

6. But $\overline{AF}^2 - \overline{BF}^2 =$ . . . and $\overline{AB}^2 - \overline{BD}^2 =$ . . . . .

7. $2\,\overline{AB}^2 =$ . . .

Complete the proof applying §218: 2.

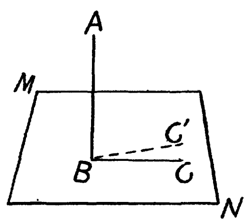
II. May be proved by a method similar to §59.

Theorem V.

261. *Every line perpendicular to another line, at a given point, is in the plane perpendicular to the line at that point.*

1. Let | $BC \perp$ . . . and the plane $MN \perp$ . . . then BC is . . . .

2. For pass a plane MN through AB and BC cutting plane MN in BC'.



3. $BC' \perp$ . . Why? §261

4. $BC' \cong$ . . . Why? §59

5. ∴ . . . .

Q. E. D.

THEOREM VI.

262. *If from a point without a plane, a perpendicular and oblique lines be drawn:*

1. The perpendicular is the shortest distance from the point to the plane, and conversely.

2. Oblique lines meeting the plane at equal distances from the foot of the perpendicular are equal, and conversely.

3. Of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular, the one cutting off the greater distance is the longer, and conversely.

4. Every point in the perpendicular is equidistant from all points of the circumference of any circle in the plane having the foot of the perpendicular for its center, and conversely.

I, II, and III. Review the corresponding theorems in plane geometry, illustrate by objects, draw figures, and prove.

IV. Apply parts II and III.

Ex. 517. A point is 4 inches from a plane. Find the length of the locus of a point 5 inches from it in this plane. Ans. 6 π inches.

Ex. 518. The locus of a point in a plane, 13 inches from a point without it, is 10 π inches in length. Find the distance of the point from the plane.

Ex. 519. Equal lines from a point to a plane make equal angles with the perpendicular to the plane from that point.

Ex. 520. Find the locus of points, in a plane, 5 inches from a point 3 inches above the plane.

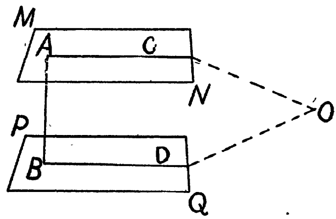
Ex. 521. Find the locus of a line perpendicular to a given line, at a given point.

Ex. 522. What is the locus of a point 1 inch from a given unlimited plane? Illustrate this, using pieces of cardboard or stiff paper.

263. Parallel planes are such as can never meet, however far produced.

THEOREM VII.

264. *Planes perpendicular to the same line are parallel.*



Apply §§259, 63: 1, 15, and 263.

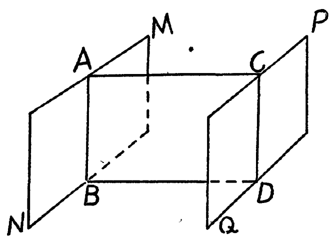
THEOREM VIII.

265. *If two parallel planes are cut by a third plane, their intersections are parallel.*

Let the planes . . . . . .

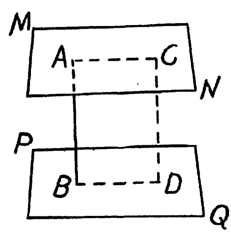
1. For if . . . . are not || . . . . Why? §15
2. If . . . the planes . . . which is absurd. §263
3. *AB* . . . . . ||

Q. E. D



THEOREM IX.

266. *A straight line perpendicular to one of two parallel planes is perpendicular to both.*



Apply §§265 and 63: 2.

THEOREM X.

267. *Two parallel lines included between two parallel planes are equal.*

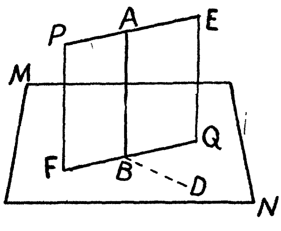
Apply §§265, 82, and 87: 2.

THEOREM XI.

268. *1. Every plane which passes through a line perpendicular to a plane is perpendicular to that plane.*

2. If two planes are perpendicular to each other, a straight line in one, perpendicular to their intersection, is perpendicular to the other.

3. A line perpendicular to a plane at a given point lies in every plane passing through that point perpendicular to the same plane.



I. Let $AB \perp$ . . . .

1. For draw | $BD \perp$ . . . .
2. $\angle ABD$ . . . Why? §259
3. Rt $\angle ABD$. . . Why? §255
   Complete the proof.

II. Apply §§255 and 260: 1.

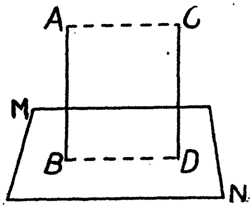
III. Apply §260: 2.

THEOREM XII.

269. *1. Two straight lines perpendicular to the same plane are parallel.*

2. A plane perpendicular to one of two parallel lines is perpendicular to the other also.

3. Lines parallel to the same line are parallel to each other.



I. Let AB . . . . . .

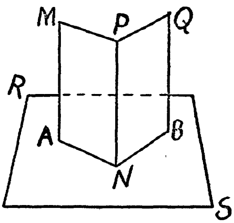
1. For through | AB and the pt. D . . . . .
2. The plane $AD \perp$ . . . §268: 1
3. The | CD . . . Why? §268: 3
4. . . . . . . | BD Why? §259
5. . . . . Why? §63: 2

II. Apply §§259, 63, 268: 1, and 268: 2.

III. Construct a figure and apply parts II and I.

THEOREM XIII.

270. *If two planes are perpendicular to a third plane, their intersection is perpendicular to that plane.*



Let . . . .

1. For at N constr. $PN \perp$ plane RS.

2. PM lies in . . . and in . . . and is their intersection.

Why? 268: 3

3. . . . .

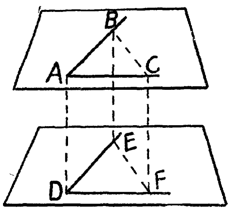
Q. E. D.

THEOREM XIV.

271. *If two angles, not in the same plane, have their sides parallel and lying in the same direction:*

1. The angles are equal.

2. Their planes are parallel.



I. Let the $\angle$s . . . .

1. For lay off $AB =$ . ., . . $=$ . ., and join . . .

Show by §87 that $AD =$ . ., and $\parallel$ . . and . . . and . . .

Complete the proof, applying §§269: 3, 17: 1, 87: 3, 87: 2, and 48.

II. Let the $\angle$s . . . .

1. For through A pass a plane $\parallel$ plane EDF.

2. This plane cuts from $|$s BE and CF parts $= AD$.

Why? §267

3. But BE and CF as above shown $= AD$.

Complete the proof applying §256: 3.

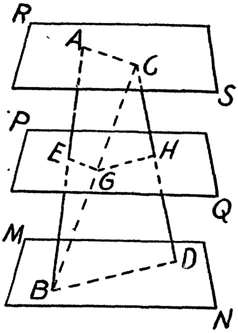
Ex. 523. Through a given point, one plane can be passed parallel to a given plane, and but one.

Ex. 524. Two parallel planes are everywhere equally distant.

THEOREM XV.

272. *1. If two straight lines are cut by three parallel planes, they are divided proportionally.*

2. If any number of lines diverging form a point are cut by two parallel planes, the segments cut off are proportional.



I. Let the |s . . . .

1. For join AC and pass planes . . . cutting . . . . . in EG and . . . also plane . . . .

2. EG || . . and GH . . . . Why? §265

3. $BE : EA =$ . . . Why? 180: 2

4. Sim. . . . . . .

5. . . . . §17: 1

Q. E. D.

II. Apply §§265 and 180: 2.

Ex. 525. All transversals cutting three parallel planes are divided proportionally.

Ex. 526. Two lines cutting any number of parallel planes are divided proportionally.

Ex. 527. The segments of one line made by three parallel planes are 15 and 20. Find the segments of a line whose length is 28 inches between the extreme planes.

Ex. 528. The segments of a line made by three parallel planes are $2a$, and $3a$. Find the segments of lines 15, 25, 35,and 45 inches long made by the same planes.

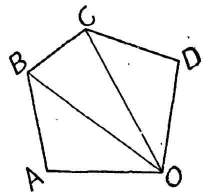
Ex. 529. Prove that §271: 1 holds good when the sides of the angle lie in opposite directions, and that §271: 2 holds good whenever the sides of the angles are parallel.

Ex. 530. The locus of a point equidistant from the planes forming a dihedral angle consists of two planes perpendicular to each other, bisecting the dihedral angle and its supplement.

Ex. 531. If three parallel lines not coplanar intercepted between two planes are equal, the planes are parallel.

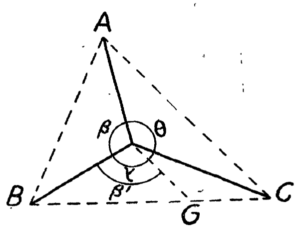
273. Def. A ***polyhedral angle,*** or ***solid angle,*** is the angle formed by three or more planes meeting in a common point called the ***vertex.*** Each of the plane angles is called *a* ***face angle.*** Polyhedral angles are classified as ***trihedral, quadrihedral, pentahedral,*** etc., according to the number of their face angles, and *are understood to be convex,* unless otherwise stated.

Ex. 532. Cut three plane angles out of cardboard and place together, forming a solid angle. The edges OB and OC may be cut half through and folded. OA and AD may be held together by pasting over a slip of paper. Figures made by the pupil himself aid him much more than the most expensive models.



Theorem XVI.

274. *In a trihedral angle, the sum of any two face angles is greater than the third.*



Let $O \equiv$ a solid $\angle$ contained by the plane $\angle$s, β, θ, and γ, of which γ is greatest; $\gamma < \beta + \theta$.

1. For, from γ, cut off $\beta' = \beta$. Lay off $OB = OA = OG$. Join BG and produce it to meet OC in C. Join . . . .

2, 3, and 4. Prove $\triangle\ ABO = \triangle\ BOG$.

5. $\therefore BG = AB$. Why?

6. But $BC <$ . . . . Why? §53

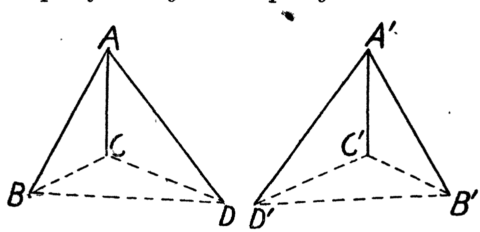
7. $GC < AC$. Why?

Complete the proof, applying §56.

Ex. 533. Two planes bisecting vertical dihedral angles form one and the same plane.

Theorem XVII.

275. *If the three face angles of one trihedral angle are equal to the three face angles of any other trihedral angle, each to each, the equal face angles are equally inclined to each other.*



* In the trihedral $\angle$s A and A' let . . . .

1. For lay off $AC = A'C'$.
   Through C pass the plane $BCD \perp AC$ . . . . .; through . . . . .

2, 3, and 4. Prove $\triangle ABC = \triangle$ . . .

5. Sim. $\triangle ACB = \triangle$ . . .

6, 7, and 8. Prove $\triangle BAD = \triangle B'A'D'$.

9, 10, and 11. Prove $\triangle BCD = \triangle B'C'D'$.

12. $\therefore \angle BCD = \angle B'C'D'$.

13. But $\angle BCD$ and $B'C'D'$ measure . . . . . §255

14. . . . .

15. Sim. the planes ABC and ABD . . . . . also . . . .

Q. E. D.

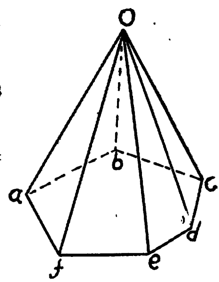
276. In the above figure, the trihedral angles A and A' are not equal though their parts are all equal. Such angles are called ***symmetrical solid angles,*** and similar solids, ***symmetrical solids.***

---

* The above proof was discovered by Miss Augusta Trumbull, now deceased, while studying geometry as a pupil of the author at Central College Academy in 1869. Whether she was the first discoverer, I do not know.

Theorem XVIII.

277. *The sum of all the plane angles containing a polyhedral angle is less than four right angles.*



Let $O \equiv$ a polyhedral $\angle$ of n faces; the sum of . . . . . .

1. For pass a plane cutting the edges in . . . . . .
2. $\angle baf < \angle baO + \dots$ Why? §274
3. Sim. $\angle afe < \dots$, etc.
4. Sum $\angle$s pol. $<$ sum base $\angle$ $\triangle$s.
5. But sum base $\angle$ $\triangle$s + sum vert. $\angle$s $\triangle$s $=$ 2n rt $\angle$'s. Why?
6. Sum $\angle$s polygon $= \dots$ Why? §98
7. Sum vert. $\angle$s $\triangle$s $\angle$. Why?
8. . . . .

Q. E. D.

278. Remark. Observe the limitation given in §273. The sum of the face angles of a polyhedral angle having reentrant angles may be increased without limit.

Ex. 534. How many kinds of polyhedral angles can be formed using angles of 60° as face angles?

Ex. 535. How many kinds of solid angles can be formed using angles of a regular pentagon as face angles?

Ex. 536. Can solid angles be formed using angles of regular polygons of six or more sides as face angles?

Ex. 537. If a transversal plane cut two parallel planes: 1. The alternate dihedral angles are equal. 2. The corresponding dihedral angles are equal. 3. The sum of the interior dihedral angles on the same side of the transversal is equal to two right angles.

Ex. 538. If a plane cut two parallel planes, the planes bisecting alternate dihedral angles are parallel, also the bisectors of corresponding angles.

Ex. 539. The distance between two parallel planes is 4 cm. Find the locus in one, of a point 5 cm. from a circumference 10 cm. in diameter, in the other.

279. DEF. The principles of projections as given in plane geometry, §§211 and 212, are valid with slight modifications in solid geometry. The ***projection*** of a ***point on a plane*** is the foot of the perpendicular let fall from the point to the plane. The ***projection of any figure on a plane*** is the figure formed by the projection of all its points on the plane.

Ex. 540. Hold a circle so that its projection on a plane is a straight line, hold it so that its projection is a circle. Hold a rectangle so that its projection is a rectangle; a square; a parallelogram; a straight line.

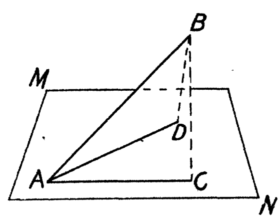
Ex. 541. Can you hold parallel lines so that their projections on a plane will intersect?

Ex. 542. Mark on a piece of paper the longest and shortest projections of a line three inches long on the plane of the paper.

SUPPLEMENTARY THEOREMS AND EXERCISES.

THEOREM XIX.

280. *A line inclined to a plane makes a smaller angle with its projection than with any other line in the plane.*



Apply §§263: 1 and 56.

Ex. 543. State and prove the converse of ex. 537.

Ex. 544. If a plane cut two parallel planes the planes bisecting the interior angles on the same side of the transversal plane are perpendicular.

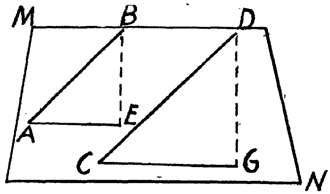
Ex. 545. Two planes perpendicular to the same line are parallel.

Ex. 546. Three parallel planes divide a line into segments of 4 and 8 inches; the smaller segment of another line is 5. Find the latter line.

THEOREM XX.

281. *1. The projections of lines equally inclined to a plane are proportional to the lines themselves.*

2. The projections of equal lines equally inclined to a plane are equal.



I. Apply §187: 2.

II. Apply §72: 2.

THEOREM XXI.

282. *1. The locus of a point equidistant from two given points is the plane perpendicular to the line joining them, at its mid-point.*

2. The locus of a point equidistant from three given points is the perpendicular to their plane at the center of the circle determined by them.

3. The locus of a point equidistant from four given points is a unique point.

I. Apply §§62 and 261.

II. Apply §§262: 4.

III. Find the intersection of the locus of a point equidistant from three of the points with the locus of a point equidistant from the fourth point and any one of the first three.

Ex. 547. The projection of a line 2 feet long, on a plane, is 16 inches. Find the projections of lines, 4, 5, and 6 feet long, parallel to the first line on the same plane.

Ex. 548. One side of a rectangle, 6 by 8 inches, is parallel to a plane inclined to the plane of the rectangle 60°. Find the area of the projection of the rectangle.

THEOREM XXII.

283. *1. The locus of a point equidistant from two intersecting planes consists of two planes bisecting the dihedral angles formed by the planes.*

2. The locus of a point equidistant from three given planes forming a solid angle is the intersection of the planes bisecting any two of the interior dihedral angles.

3. The locus of a point equidistant from four given planes of which three form a solid angle and the fourth cuts all of these, is a unique point, the intersection of the locus of a point equidistant from the three first planes with the plane bisecting the dihedral angle formed by any of these planes with the fourth plane.

I. Apply §75.

II. and III. Of what loci is the locus the intersection?

Ex. 549. The locus of a point in a plane, at a given distance from a point without the plane is a circle whose center is the projection of the point.

Ex. 550. Equal lines from a point to a plane make equal angles with their projections in the plane.

Ex. 551. Prove by the method of limits:

1. That the areas of the projections on the same plane of all figures whose planes are parallel are proportional to the areas of the figures themselves.

2. That the areas of these figures are to the areas of their projections, as a line in one perpendicular to their intersection is to its projection in the other.

Ex. 552. Find the length of the projection of a line 10 inches long on a plane to which it is inclined at an angle of 60°.

Ex. 553. Find the area of the projection of a triangle whose sides are 9, 40, and 41, on a plane to which the plane of the triangle is inclined at an angle of 60°.

Ex. 554. Three parallel planes cut two lines so that the sum of the parts intercepted between the extreme planes is 33. The difference of the greater segments of the two lines is 3. The sum of the shorter segments is 14. Find the lines and segments.

BOOK VI.

SURFACES AND VOLUMES OF SOLIDS.

284. DEF. A ***polyhedron*** is a solid bounded by planes called faces. Solid ***angles*** about the same face are said to be ***adjacent.***

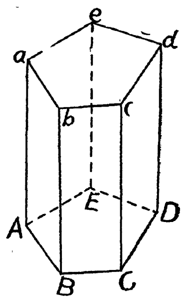
285. DEF. A polyhedron of four faces is called a ***tetrahedron,*** one of six faces a ***hexahedron,*** etc.

286. DEF. A ***diagonal*** of a polyhedron is a line joining two angles not adjacent.

287. DEF. A ***diagonal plane*** of a polyhedron passes through two edges not in the same face.

Ex. 555. How many diagonals has a chalk box? How many diagonal planes has it.

288. A ***prism*** is a polyhedron bounded by two equal polygons whose planes and sides are parallel, called ***bases,*** and by planes passing through the homologous sides of these polygons called ***lateral faces.*** The ***altitude*** of a prism is the distance between the bases. Specify in the figure the parts defined; also the ***lateral edges*** and ***basal edges.***



Ex. 556. Point out two faces of a chalk box that may be regarded as its bases as a prism. How many lateral edges has the chalk box? How many basal edges has it?

THEOREM I.

289. *1. The lateral faces of a prism are parallelograms.*
2. The lateral edges of a prism are equal.

I. Apply §§288 and 87: 3. Use fig. of §288.
II. Apply §87: 2.

290. DEF. A ***prism*** is called ***right*** when its edges are perpendicular to the base, ***oblique*** when they are inclined to the base.

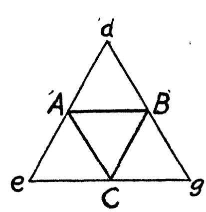
291. DEF. A ***regular prism*** is a right prism whose base is a regular polygon.

292. DEF. A ***pyramid*** is a polyhedron formed by a polygon called the ***base,*** and by triangles called ***lateral*** faces meeting in a point called the ***vertex.*** The ***basal edges*** are the intersections of the base with the lateral faces. All other edges are called ***lateral edges***.

293. DEF. A ***regular pyramid*** is one whose base is a regular polygon, the center of which is the projection of the vertex. Its lateral edges meeting the plane of the base at equal distances from the foot of the perpendicular are all equal, and its lateral faces are equal triangles, their homologous sides being equal.

294. DEF. Prisms and pyramids are called ***triangular, quadrangular, pentagonal, hexagonal***, etc., according to the number of sides in their bases.

295. DEF. Construct on paper an equilateral triangle *abc*, and triangles equal to it on each of its sides. Cut half through *ab*, *ac*, and *bc*, and fold over till *d*, *e*, and *f* meet. The solid thus constructed is called a ***regular tetrahedron.*** Define a regular tetrahedron.



296. Def. The ***altitude*** of a pyramid is the perpendicular from the vertex to the base. The ***slant height*** of a regular pyramid is the distance from the vertex to a basal edge.

Ex. 557. Construct a regular hexagonal pyramid each side of whose base is 2 cm. and its slant height 3 cm.

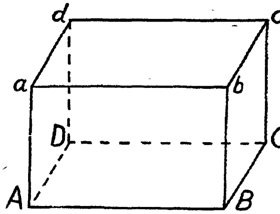
Ex. 558. Find the surface of a regular tetrahedron whose edge is 20 cm.

297. Def. A ***parallelopiped*** is a prism whose bases are parallelograms. If its lateral edges are perpendicular to its bases, it is called a ***right parallelopiped.*** If all its faces are rectangles it is called a ***rectangular parallelopiped.***

298. Def. A ***cube*** is a parallelopiped all of whose faces are squares.

Theorem II.

299. *The opposite faces of a parallelopiped are equal and parallel.*



Apply §§87: 2 and 271: 1, 2.

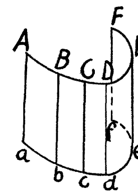
Ex. 559. A chalk box occupies the space of what geometrical solid? Measure its lateral edges and face diagonals. Compute its diagonal.

Ex. 560. Find the base diagonal and diagonal of a rectangular parallelopiped, the sides of whose base are 3 and 4, and altitude 12. Ans. 5; 13.

Ex. 561. The diagonal of the rectangular parallelopiped, whose edges are a, b, and c, is $\sqrt{a^2 + b^2 + c^2}$.

Ex. 562. Find the diagonal of a room 9×12, 8 ft high.

Ex. 563. Find the diagonal of the class room.



300. DEF. A ***cylindrical surface*** is one generated by a line moving constantly parallel to its first position, and also continually intersecting a curve called the ***directrix***. Any one position of the line is called an ***element***.

301. DEF. A ***cylinder*** is a solid bounded by a cylindrical surface whose directrix is a closed curve, and by two parallel plane figures, called bases. By §267 all elements of a cylinder are equal.

302. REMARK. By §§265 and 300, any line joining two points of one base is equal to the line joining the homologous points of the other base. By §87: 3 these lines are equal. The two bases may therefore be so placed as to coincide, and are equal. In a similar manner it may be shown that all sections of a cylinder or cylindrical surface made by parallel planes are equal.

303. DEF. A cylinder whose elements are perpendicular to its bases is called a ***right cylinder***. A cylinder whose bases are circles is called a ***circular cylinder***.

Ex. 564. Construct a right circular cylinder. Cut out two equal circles as bases. Make out of a rectangular piece of paper a tube whose diameter is the same as that of the circles, etc.

Ex. 565. Construct an oblique circular cylinder. Make the tube a little larger than in ex. 564, and flatten it till the circles fit at the ends obliquely, etc.

Ex. 566. Find the entire surface of the class room in meters.

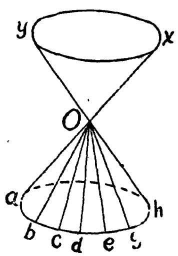
Ex. 567. The basal edges of a hexagonal pyramid are each 6, and its lateral edges, each 10. Find its altitude, also its slant height to two places of decimals. Find its entire surface.

Ex. 568. The diagonal of a cube whose edge is a, is $a\sqrt{3}$, and its face diagonal is $a\sqrt{2}$.

Ex. 569. Find the entire surface of a cube whose diagonal is 10.

Ex. 570. Find the entire surface of a cube whose face diagonal is 10.

Ex. 571. Find the diagonal of a cube whose face diagonal is 10.



304. DEF. A line which moves in such a way that it continually passes through a fixed point called the ***vertex*** and intersects a curve called the ***directrix,*** generates a ***conical surface.*** One position of the line is called an ***element.*** The part of the surface above the vertex is called the ***upper nappe,*** that below, the ***lower nappe.***

305. A ***cone*** is a solid bounded by a conical surface whose directrix is a closed curve and by a plane figure called the ***base.***

306. A ***circular cone*** is one which has a circle as its base. A ***right circular cone*** is one in which the center of the circular base is the projection of the vertex. Every element of a right circular cone is equal. Why? §262: 2.

Ex. 572. Construct a right circular cone. Make two circular pieces of cardboard, 1 inch and 3 inches in diameter, respectively. Cut from the larger a sector of 120° for the lateral surface and use the smaller circle as a base. Why do we use a sector of 120°? What is the circumference of the base?

Ex. 573. Find the slant height and altitude of the cone of ex. 572.

Ex. 574. What must be the diameter of the base, if we use the sector of 240° for the lateral surface?

Ex. 575. Find the angle required so that a sector may make the lateral surface of a cone whose slant height has to the radius of the base the ratio 2 : 1; 3 : 2; 5 : 2; 6 : 5; 8 : 3; 8 : 5; 12 : 7.

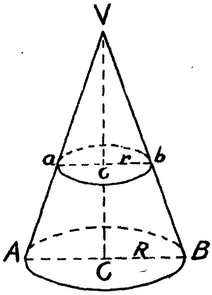
Ex. 576. The radius of the base of a right circular cone is 4, and its slant height, 5. Find its altitude.

Ex. 577. The altitude of a right circular cone is 8, and the radius of its base, 6. Find its slant height.

Ex. 578. The slant height of a right circular cone is 13, and its altitude, 12. Find the diameter, circumference, and area of its base. Ans. to area 78.54.

Ex. 579. The area of the base of a right circular cone is 81 π, and its slant height, 41. Find its altitude.

307. DEF. A ***cylinder of revolution*** is the solid generated by revolving a rectangle about one side as an axis. A ***cone of revolution*** is one generated by the revolution of a right-angled triangle about one leg as an axis. Prove that a cylinder of revolution is a right circular cylinder; a cone of revolution, a right circular cone; and conversely.



308. DEF. A ***frustum*** of a ***pyramid,*** or of a ***cone,*** is the part included between the base and a plane parallel to it.

309. DEF. A ***truncated*** prism, cylinder, pyramid, or cone is the part included between the base and a plane not parallel to it.

310. We will hereafter use letters to represent parts of solids as follows:

$V \equiv$ the volume.
$S \equiv$ the surface, S_l the lateral surface.
$B \equiv$ the larger base, $b \equiv$ the smaller base of a frustum.
R, $r \equiv$ the radii of B and b; also the radii of the circles circumscribed about and inscribed in the base of a prism.
P, $p \equiv$ the perimeters of B and b.
C, $c \equiv$ the circumferences of R and r.
$h \equiv$ the altitude of any solid.
$h_s \equiv$ the slant height of a right pyramid or cone.
L, $l \equiv$ the lateral edges of a solid.
A, $a \equiv$ the sides of B and b.

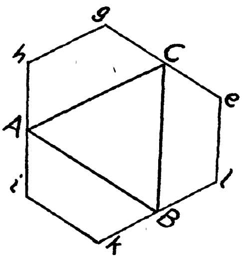
Ex. 580. The angle of the sector of a circle required as the lateral surface of a cone is $360° \, (R \div h_s)$. See ex. 572.

Ex. 581. Construct a frustum of a cone of revolution in which $R = 1$, $r = .5$, $h_s = 1.5$. (The lateral surface is a sector of 240° from a ring between two concentric circles, $1\frac{1}{2}$ and 3 inches radii.)

Ex. 582. The slant height of the entire cone of which the frustum is determined by R, r, and h_s is: $Rh_s \div (R - r)$. See fig. of §307, and prove by similar triangles.

311. Def. A prism is ***inscribed*** in a cylinder or ***circumscribed*** about it when its bases are inscribed in or circumscribed about the cylinder, and conversely. The pupil will define a ***pyramid inscribed in a cone***, a ***cone inscribed in a pyramid***, a ***frustum of a pyramid inscribed in the frustum of a cone***, etc.

Ex. 583. Construct a frustum of a regular triangular pyramid. Draw on cardboard the equilateral $\triangle$ ABC, on each side an equal isosc. trapezoid, also an equilateral $\triangle$ whose $| = ik$; etc.



Ex. 584. Construct a frustum of a regular hexagonal pyramid in which $A = 1$ in., $a = \frac{1}{2}$ in., $l = 1$ in.

Ex. 585. Find h_s, S_l, and S in the frustum of ex. 584.

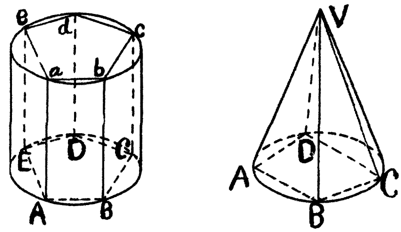
Ex. 586. In a frustum of a right circular cone, the area of the lower base is 4 times the area of the upper, the altitude is 1 inch more than the radius of the upper base, and the slant height is 1 inch greater than the altitude. Find the radii of the bases, altitude, and slant height. Ans. to last, 5.

312. Def. In applying the word ***limit*** to a polygon inscribed in a curve, a prism inscribed in a cylinder, a pyramid inscribed in a cone, etc., mathematical writers usually assume that the word limit, or any word or sign representing it, implies that the number of sides is indefinitely increased and their length diminished, so that the lateral faces, whether parallelograms or triangles, approach lines as limits. We will hereafter use the word limit as applied to such figures in this sense.

THEOREM III.

313. *1. The limit of a prism inscribed in a cylinder is the cylinder.*

2. The limit of a pyramid inscribed in a cone is the cone.



I. Let . . . .

1. For as the number of |s . . . and their length . . . . . polygon $ABCDE \doteq$ the curve . . . and pol. $abcde$ . . .

2. ▱ $Bc \doteq$ | Bb etc. and S_l prism $\doteq S_l$ cylinder.

3. The prism $AD \doteq$ cylinder Ad. Q. E. D.

II. The proof is similar to the preceding.

314. Observe that the proofs in §313 are valid as to all prism-like surfaces inscribed in cylindrical surfaces, whatever the form of the directrix, also to all pyramid-like surfaces inscribed in conical surfaces.

315. REMARK. Having proved that the cylinder is the limit of the prism, it follows that all properties of the prism are properties of the cylinder, excepting only the one distinguishing property that the prism has plane lateral surfaces. We do not, therefore, need to prove the properties of the cylinder and cone separately.

Ex. 587. Find the surface of the base of a triangular prism whose sides are 9, 40, and 41.

Ex. 588. Find the surface of the base of a regular triangular prism, one side of whose base is 3; a regular hexagonal prism, one side of whose base is 2. Ans. $2.25\sqrt{3}$; $6\sqrt{3}$.

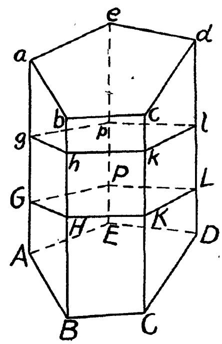
Ex. 589. Find the surface of the base of a circular cylinder, the radius of whose base is 6. Ans. 36π.

Ex. 590. Find the entire surface of a right triangular prism, the sides of whose base are 5, 6, and 7, and whose altitude is 10.

THEOREM IV.

316. *1. Sections of a prism, or cylinder, made by parallel planes which do not cut a base are equal polygons or curves.*

2. A section of a prism, or cylinder, parallel to the base is equal to the base.



II. Let . . . .

1. For $Gg \parallel Hh$ and $Gg = Hh$. §289: 1
2. $GH \parallel$ . .; $HK \parallel$ . .; etc. Why? §265
3. $\angle GHE = \angle$ . .; etc. Why? §271: 1
4. $GH =$ . .; $HK =$ . .; etc. Why? §87: 2
5. . . . . Why? §221: 4

II. Apply §316: 1. Q. E. D.

Ex. 591. The lateral surface of a right prism of 7 sides, each of which is a, whose altitude is h, is $7ah$. Apply §§204: 2 and 290.

Ex. 592. The lateral surface of a right prism of n sides, each of which is a, is nah.

Ex. 593. The lateral surface of a right prism is the product of the perimeter of its base by its altitude.

Ex. 594. The lateral surface of a right cylinder is the product of the circumference of its base by its altitude. Apply §313: 1.

Ex. 595. Find the lateral surface of a right cylinder whose altitude is h and the radius of its base r. Ans. $2\pi rh$. Find its entire surface. Ans. $2\pi r(h+r)$.

Ex. 596. Find the lateral surface of a pyramid of 8 sides, each side of the base being 2 and its slant height 20.

Ex. 597. Find the radius of the base of a right cone whose slant height is 10 and lateral surface 100. Find its altitude. Ans. 3.18; 9.42.

Ex. 598. Find a side of the base of a regular triangular pyramid whose slant height is 10 and lateral surface 100. Ans. 6.67.

THEOREM V.

317. *If a pyramid or cone be cut by a plane parallel to the base:*

1. The lateral edges (in a cone, the elements) are divided proportionately to each other, and to the altitude. Formula: $L : l = L' : l' = H : h$.

2. The section is a polygon (or curve) similar to the base.

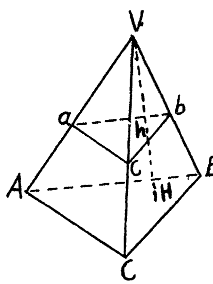
3. Homologous sides of the bases and the perimeters of the bases, are to each other as their distances from the vertex. Formula: $A : a' = P : p = H : h = L : l$.

4. The areas of the base and section are proportional to the squares of their distances from the vertex. Formula: $B : b = H^2 : h^2 = L^2 : l^2$.

I. Apply §272: 2.

II. Let . . . .

1. For AB || . ., . . || . ., and . . . Why? §265
2. $\angle ACB = \angle$ . .; . . . .; and . . . Why? §271: 1
3. $\triangle AVC \sim \triangle avc$; . . . $\sim$ . . .; and . . . Why? §187
4. $AC : ac = PC : Pc = BC :$ . ., etc. Why? §186
5. Pol. $ABC \sim$ pol. . . . Why?



Q. E. D.

III. and IV. State by the figure and prove.

Ex. 599. A pyramid, whose basal edges are 6, 9, 12, 15, and 18, and altitude 21, is cut by a plane parallel to the base and 7 inches from it. Find the edges of the upper base.

Ex. 600. A pyramid, whose lateral edges are 20, 25, 30, 35, and 40, and altitude 15, is cut by a plane parallel to the base and 9 inches from vertex. Find the sides of the upper base.

Ex. 601. Find the radius of the base of a cone whose slant height is 6 and entire surface 72.

THEOREM VI.

318. *1. The lateral surface of a prism is equivalent to the product of its lateral edge by the perimeter of a right section. Formulas:* a) $S_l \simeq Pl$; b) $S \simeq Pl + 2B$.

2. The lateral surface of a cylinder is equivalent to the product of the circumference of a right section by an element. Formulas: a) $S_l \simeq Pl \simeq 2\pi\ Rl$; b) $S \simeq 2\pi R\ (b + R.)$

3. The lateral surface of a regular pyramid or a right cone is equivalent to half the product of the perimeter of its base by its slant height. Formulas: a) $S_l \simeq \frac{1}{2}\ Ph$; b) $S_l \simeq \frac{1}{2}\ Ch = \pi\ Rh$; c) $S \simeq \pi R\ (h + R)$.

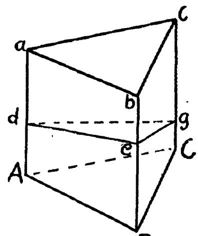
4. The lateral surface of the frustum of a pyramid or cone is equivalent to the product of the sum of the perimeters of its bases by half its slant height. Formulas: a) $S_l \simeq \frac{1}{2}\ (P + p)\ h_s$; b) $S_l \simeq \pi\ (R + r)\ h_s$; c) $S \simeq \pi\ \{(R + r)\ h_s + R^2 + r^2.\}$

I. Apply §§289 and 204: 2.

II. Apply §§318: 1 and 313: 1.

III. Draw a figure. Apply §§293 and 205: 1.

IV. Draw a figure. Apply §§265 and 210: 1.



Ex. 602. In a regular pentagonal prism $a = 2$, $b = 12$. Find S_l.

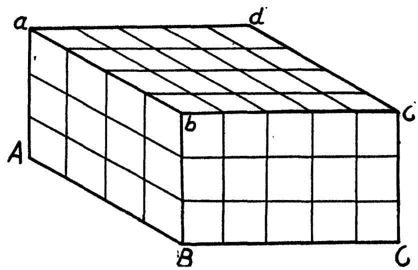
Ex. 603. Find in terms of π the entire surface of a cylinder in which $R = 5$, $h = 16$.

Ex. 604. In a cone $R = 15$, $h = 8$. Find S_l and S.

Ex. 605. If a pyramid be cut by any number of planes parallel to the base:

1. The edges and altitudes are divided proportionally.
2. The sections are similar figures.
3. The perimeters of the sections are proportional to their distances from the vertex, and their areas, to the squares of these distances.

319. Having defined the product of two lines as the surface generated by moving one line vertically the length of the other, we may define the ***product*** of a ***surface*** and a ***line*** as the solid generated by moving the surface vertically the length of the line. If a rectangle 5 inches long and one inch wide be moved vertically one inch, five cubical inches of volume will be generated. How many cubical inches of volume will be generated if it be raised two inches? 3 inches? 6 inches?



320. Evidently we cannot use the surface to multiply the line; that is, the commutative law of multiplication is not valid with the definition thus extended.

321. It is also evident that exactly the same solid is generated whether we raise the entire surface at once or raise 1, 2, 5, or 10 square inches at a time; so that the distributive law of multiplication is valid.

322. If l represents the length of the base; w, its width, and h, the altitude; it is evident that, so long as l, w, and h are commensurable, $V \triangleq l \times w \times h$. Since this is true, no matter how small the common measure, by the principles of limits, it holds when l, w, and h are incommensurable. It is also evident that exactly the same solid is generated whether we move the surface AC the distance Bb, the surface Ab the distance bc, or the surface BC the distance ab; that is, as regards the three line factors, the commutative law of multiplication is valid; that is:

The volume of a rectangular parallelopiped is the product of its three dimensions, length, breadth, and thickness, or the product of its base by its altitude.

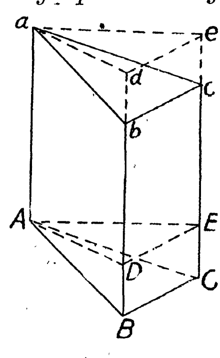
Ex. 606. In a rectangular parallelopiped: $l = 5$, $w = 6$, $h = 4$. Find V.

Ex. 607. Find the volume of a rectangular parallelopiped whose dimensions are $\sqrt{5}$, $\sqrt{10}$, and $\sqrt{2}$.

Ex. 608. Find the volume of a rectangular parallelopiped whose dimensions are: $\sqrt[3]{4}$, $\sqrt[3]{6}$, $\sqrt[3]{9}$.

THEOREM VII.

323. *An oblique prism is equivalent to a right prism having the same lateral edge and a right section of the oblique prism for its base.*



Let ABC-b be any . . . . and ADE-d . . . . having ADE a right section of . . . . . then . . . $\cong$ . . .

1. For $Bb = Dd$, etc. Why?
2. $\therefore$ $BD = bd$, etc. Why?

Prove the solid abc-d = ABC-D by superposition and complete the proof.

Ex. 609. Find the volume of an oblique prism, the area of whose right section is 15 and whose altitude is 8.

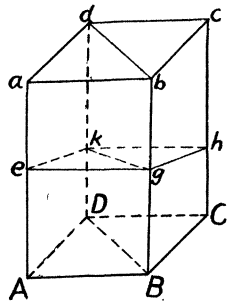
Ex. 610. Find the volume of an oblique cylinder, whose right section is a circle, its radius, 5, and an element, 12. Would the bases of this cylinder be circles?

Ex. 611. In a cone of revolution: $R = 40$, $h = 30$. It is cut by planes parallel to the base making the segments of the altitude, m, $2m$, $3m$, $4m$, and $5m$, beginning with the base. Find 1) each segment of the altitude; 2) each segment of the slant height; 3) the radius of each section; 4) the area of each section in terms of π. Ans. to areas: $\frac{12544\,\pi}{9}$, $1024\,\pi$, 576π, $\frac{1660\,\pi}{9}$.

THEOREM VIII.

324. *1. A diagonal plane divides a parallelopiped into two equivalent triangular prisms.*

2. Conversely: a triangular prism is half the volume of a parallelopiped having double its base.



I. Apply §323, and prove the right prisms equal by superposition.

II. Let ABD-$a \equiv$ the triangular prism. Complete the ▱ $ABCD$ as a base, then the parallelopiped AC, and apply §324: 1.

Ex. 612. Find a side of the base of a regular triangular pyramid whose entire surface is 100. Ans. 5.72.

Ex. 613. The edges of a rectangular parallelopiped are $2x$, $5x$, and $10x$, and its volum 2700. Find each edge.

Ex. 614. Find the edge of a cube whose volume is 100, to two places of decimals.

Ex. 615. The diagonal of a rectangular parallelopiped whose edges are a, b, and c is $\sqrt{a^2 + b^2 + c^2}$.

Ex. 616. The edges of a rectangular parallelopiped are 3, 4, and 12. Find its diagonal without finding any face diagonal.

Ex. 617. The diagonal of a rectangular parallelopiped is 34, the base diagonal is 30, and one side of its base is 24. Find its edges and volume.

Ex. 618. The difference between the diagonal and base-diagonal of a rectangular parallelopiped is 2, and the difference between the base diagonal and the shorter side of the base is 6. Find its edges and volumes.

Ex. 619. Find the radius of a circle whose surface is 100.

Ex. 620. Find the radius of a circle inscribed in an equilateral triangle whose area is 100.

Ex. 621. Find the radius of a circle inscribed in a hexagon whose area is 100.

Ex. 622. Find the surface of a cylinder in which $R = 5$, $h = 12$.

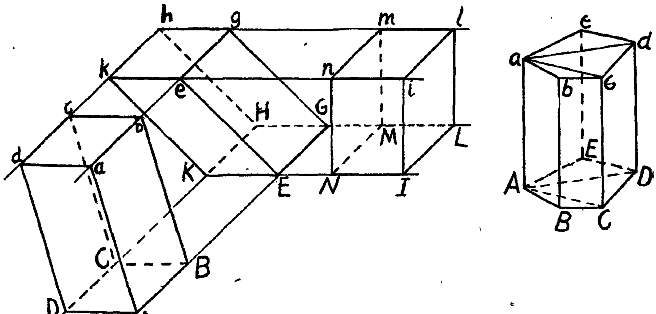
THEOREM IX.

325. *1. Any parallelopiped is equivalent to a rectangular parallelopiped, having an equivalent base and equal altitude.*

2. The volume of a prism is equivalent to the product of its base by its altitude, or to the product of its three dimensions. Formulas: a) $V = Bh$; *b)* $V = lbh$*.

3. The volume of a cylinder is equivalent to the product of its base by its altitude. Formulas: a) $V = Bh$; *b)* $V = \pi R^2 h$.

4. Prisms having equivalent bases and equal altitudes are equivalent.



I. Let $Ac \equiv$ . . . . and $Nl \equiv$ . . .; then $Ac \bumpeq$.

1. For produce the faces of Ac, let $Ek \equiv$ a right section of Ac, and $Eh \equiv$ a parallelopiped having Eh as its base and $EG = AB$ as its altitude.

2. $Eh \bumpeq Ac$. Why? §323

Produce the faces of Eh, let Mn be a right section and Mi . . . . . . Complete the proof.

II. Apply §§322 and 325: 1.

III. Apply §§324: 2 and 313: 1.

Apply 325: 2.

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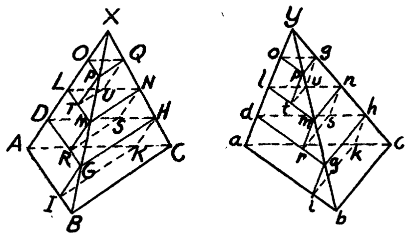
* $l \equiv$ the length, $b \equiv$ the breadth of the base.

THEOREM X.

326. *Triangular pyramids which have equivalent bases and equal altitudes are equivalent.*

Let Y and $Z \equiv$ . . .

1. For pass planes $\parallel ABC$ . . . . dividing AY and aZ into the same number of equal parts . . . . On DEG and deg as bases constr. the prisms . . . . . .



2. $\triangle ABC : \triangle DHG = \triangle abc$ . . . . Why? §317: 4
3. $\triangle DHG \doteq \triangle$ . . Why?
4. Prism DHG-$A \doteq$ . . . Why? §325: 4
5. Sim. prism . . $\doteq$ . ., etc.
6. Sum of . . . .
7. But indefinitely increasing the number of parts the sum of prisms in $Y \doteq Y$ and . . . . . $\doteq Z$.
8. . . . . §169: 3

Q. E. D.

Ex. 623. In a right prism: $B = 8$, $h = 10$. Find V.

Ex. 624. In a regular triangular prism: a (a side of the base) $= 8$, $h = 5$. Find V. Ans. $80\sqrt{3} = 13.856$.

Ex. 625. In a regular hexagonal prism: $A = 2$, $h = 10$. Find V, S', and S. Ans. $60\sqrt{3}$; 120; 140.784.

Ex. 626. The basal edges of a prism are 5, 6, and 7, $h = 10$; find V, S_l and S. Ans. 146.9; 180; 209.38.

Ex. 627. In a cylinder: $R = 5$, $h = 4$. Find V, S_l and S.

Ex. 628. In a square prism: $V = 80$, $h = 15$. Find A.

Ex. 629. In a square prism: $V = 100$, $A = 5$. Find h.

Ex. 630. In a regular hexagonal prism: $S_l, = 300$, $h = 10$. Find A, P, B, and V.

Ex. 631. In a cylinder: $V = 100$, $R = 5$. Find h and S.

Ex. 632. In a cylinder: $S = 200$, $S_l = 100$. Find V.

Ex. 633. In a cylinder: $S = 1000$, $h = 10$. Find V.

PROBLEM I.

327. *To construct two equal triangular pyramids and a third pyramid equivalent to each of these, which may be so joined together that their sum is a right triangular prism.*

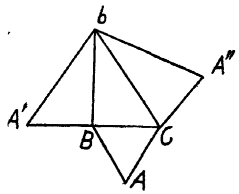


Fig. 1.

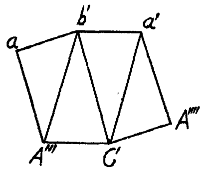


Fig. 2.

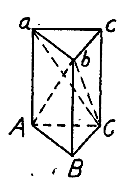


Fig. 3.

1. Construct the equilateral $\triangle$ ABC, produce CB making $BA' = CB$, constr. $Bb \perp BA'$ at B, join bA' and bC, constr. $\triangle$ bCA'' making $bA'' = bA'$ and $CA'' = CA$, and fold together forming a pyramid.

2. Constr. a second pyramid = the first.

3. Constr. (fig. 2.) $\triangle$ $ab'A''' = \triangle$ $A'Bb$ of fig. 1, $\triangle$s $b'C'A'''$ and $b'C'a' = \triangle$ bCA'' of fig 1, and $\triangle$ $a'C'A'''' = \triangle$ $A'Bb$ of fig. 1, and put together forming a third pyramid.

4. Place the three pyramids so as to form a prism (fig. 3), and prove that the pyramid $AaC\text{-}b \bumpeq$ the pyramid $ACB\text{-}b$.

THEOREM XI.

328. *1. The volume of a triangular pyramid is one third of the product of its base by its altitude. Formula:* $V \bumpeq \frac{1}{3} Bh$.

2. The volume of any pyramid or cone is one third of the product of its base by its altitude. Formulas: a) $V \bumpeq \frac{1}{3} Bh$; *b)* $V \bumpeq \frac{1}{3} \pi R^2 h$.

I. See §327 and give the proof formally using only fig. 3.

II. Construct a figure to represent any pyramid, and complete a prism having the same base and lateral edge. See the last figure of §325, and prove the theorem.

Theorem XII.

329. *A frustum of a pyramid, or cone, is equivalent to the sum of three pyramids, or cones, having the altitude of the frustum, as a common altitude, and its lower base, upper base, and a mean proportional between them as bases. Formulas: a)* $V \; \frac{1}{3} \bumpeq h \; (B + b + \sqrt{Bb})$; *b)* $V = \frac{1}{3} h \pi (R^2 + r^2 + Rr)$.

Let $V_c \equiv$ the volume of the cone of which the frustum is a part; $v_c \equiv$ the volume of the part cut off; $H \equiv$ the altitude of the entire cone, and $h \equiv$ the altitude of the frustum; whence $H - h \equiv$ the altitude of the part cut off.

1. $H : H - h = \sqrt{B} : \sqrt{b}$. Why? §317: 4
2. $\therefore H = h\sqrt{B} \div (\sqrt{B} - \sqrt{b})$.
3. $V \bumpeq V_c - v_c \bumpeq \frac{1}{3} HB - \frac{1}{3} (H - h) \, b \bumpeq \frac{1}{3} (HB - Hb + hb)$. Why? §328: 2
4. $HB - Hb \bumpeq H (B - b) \bumpeq h\sqrt{B} \, (B - b) \div (\sqrt{B} - \sqrt{b}) = h\sqrt{B} \, (\sqrt{B} + \sqrt{b}) = h \, (B + \sqrt{Bb})$.
5. $V \bumpeq \frac{1}{3} \{ h(B + \sqrt{Bb}) + hb \} \bumpeq \frac{1}{3} h \, (B + b + Bb)$.

Q. E. D.

Ex. 634. How many feet of lumber in a square piece of timber 30 feet long, the large end 12 in. square, and the small end 8?

Ex. 635. In the frustum of a regular triangular pyramid: an edge of the lower base is 6, an edge of the upper base 4, and the altitude 5. Find its volume. Ans. $\frac{1}{3} \times 95 \sqrt{3} = 54.85$.

Ex. 636. In a frustum of a regular hexagonal pyramid: $A = 6$, $a = 2$, $h = 6$. Find V. Ans. 270.2.

Ex. 637. In a frustum of a triangular pyramid: $h = 15$, the sides of the lower base are 10, 24, and 26, and the side of the upper base homologous to 26 is 13. Find V and S.

330. Def. A ***sphere*** is a solid bounded by a curved surface every point of which is equidistant from a point within called the ***center.***

331. Def. Let the pupil define the terms, ***radius, diameter,*** and ***tangent plane,*** as applied to a sphere. Consult §§12 and 112.

Theorem XIII.

332. *1. A plane tangent to a sphere is perpendicular to the diameter at the point of contact.*

2. Every section of a sphere made by a plane is a circle, whose center is in the diameter perpendicular to the plane of the circle.

I. May be proved similarly to §127.

II. Apply §262: 4.

333. Def. Sections of a sphere made by planes passing through the center are called ***great circles.*** All other sections are called ***small circles.*** Since the planes of all great circles include the center, their intersections are diameters and they bisect each other.

Ex. 638. In a square pyramid: A (one side of the base, see §310) $= 12$, $h = 30$. Find V.

Ex. 639. In a regular triangular pyramid: $A = 5\sqrt{3}$, $h = 12$. Find V.

Ex. 640. In a regular hexagonal pyramid: $A = 4\sqrt{3}$, $h = 15$. Find V.

Ex. 641. In a nonagonal pyramid: $B = 15$, $h = 5$. Find V.

Ex. 642. In a regular hexagonal pyramid: $A = 3$, $h = 5$. Find V.

Ex. 643. In a pyramid: $h = 15$, and the sides of the base are 7, 8, and 9. Find V.

Ex. 644. In a right cone: $R = 5$, $h = 24$. Find V.

Ex. 645. In a right cone: $C = 125{,}644$, $h = 10$. Find V.

THEOREM XIV.

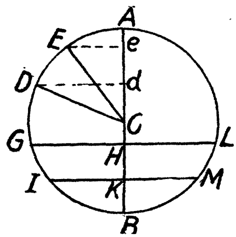
334. *1. Two points on the surface of a sphere, not in the same diameter, determine a great circle of the sphere.*

2. Any three points on the surface of a sphere, not coplanar with its center, determine a small circle of the sphere.

I. and II. Apply §§256: 3 and 332: 2.

335. DEF. The ***poles*** of a circle of a sphere are the extremities of the diameter perpendicular to the plane of the circle.

336. DEF. If a semicircle revolve about its diameter as an axis, it generates a sphere. Why? Every arc generates a ***zone.*** The perpendiculars from the extremities of the arc generate circles called the ***bases*** of ***segments*** bounded by them and the zones. The circumferences of these circles are the ***bases*** of the zones. The ***altitude*** of a segment or zone is the part of the axis intercepted by it.



337. DEF. Every sector of the generating semi-circle generates a ***spherical sector.*** The radii of the sector generate ***conical surfaces.*** See §304.

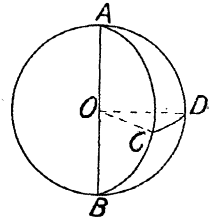
Ex. 646. Point out in the figure above the spherical segments and zones which have two bases, those which have but one base, the straight lines which generate plane surfaces, those which generate conical surfaces, and the one which does not generate a surface, the arc which generates each zone, and the arc and ordinates which generate each segment.

Ex. 647. In a right cone: $R = 7$, $h_s = 25$. Find h and V.

Ex. 648. In a frustum of a cone: $R = 6$, $r = 4$, $h = 9$. Find V.

Ex. 649. In a frustum of a cone: $R = 15$, $r = 5$, $h_s = 13$. Find h and V.

338. Def. As the semicircle ADB revolves about AB as an axis through the angle DOC it generates a ***spherical ungula***, or ***spherical wedge,*** and its circumference, a ***lune***. The angle of the wedge or lune, and in general the angle of any two arcs on the surface of a sphere, is the angle between the radii DO and OC perpendicular to the diameter. See §255.



Problem II.

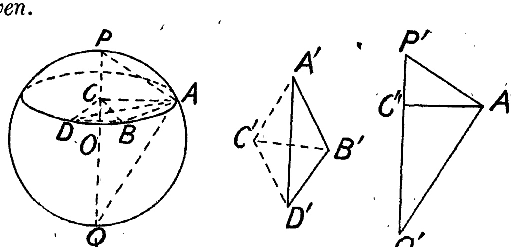
339. *1. Given its pole and one point on its circumference; construct a circle on a sphere.*

2. To find the diameter and circumference of a material sphere.

3. To construct a great circle of the sphere, its pole being given.

4. To construct a great circle of the sphere, two points being given.

5. To construct a small circle of the sphere three points being given.



I.* Given the pole P and the pt. A . . . .

Place one pt. of the dividers on P, one on A . . . .

* A spherical surface is an absolute necessity. For private study, a painted croquet ball is sufficient. In the absence of a slated globe, a cheap substitute may be provided by turning a sphere out of wood on a lathe. The following answers well for slating: ½ pt. alcohol, 1 oz. shellac, ½ oz. best lampblack. Apply several coats with a varnish brush.

Prove $\triangle PAC = \triangle PBC$, and hence that the $\perp$ from any pt. as B meets PP' in C, etc.

II. 1. Draw any small $\bigcirc$ as . . . ., take any three points as . . . ., measure the distances as AB, AD, BD, between . . . ., mark three points same distances on a plane surface and thus find the radius $A'C' = AC$.

2. Constr. (figure 3) the rt$\triangle$ $A'P'C''$ having $A'P' = AP$ and $A'C'' = AC$. Draw $A'Q' \perp A'P'$ meeting $P'C''$ produced in Q'; $P'Q'$ is the diameter of the sphere. Prove this.

III. Find the diameter and use the chord of 90° as the radius. Why? §262: 4.

IV. With the two given points as centers construct arcs of great $\bigcirc$s. Their intersections are the poles. Apply §262: 4.

V. Find the diameter of the sphere and the radius of the small $\bigcirc$ as above. From these find AP, etc.

Ex. 650. Draw on a spherical surface a great circle representing two opposite positions of the generating semicircle, and small circles to represent the paths of various points. Endeavor to imagine all of the figures in §§334—338.

Ex. 651. Point out the great circles, small circles, lines, zones, etc., on a geographical globe.

Ex. 652. Show that the zone generated by the arc ED, figure of §336, is the part of the surface of the sphere intercepted between planes perpendicular to the diameter AB in e and d.

Ex. 653. In a frustum of a cone: $R = 5$, $r = 3$, $V = 100$. Find h.

Ex. 654. In a frustum of a cone: $h = R = 2r$, $V = 504\pi$. Find h and r.

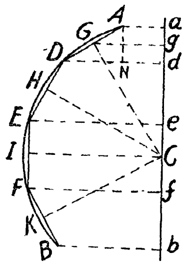
Ex. 655. The base of a triangle is 15, and the other sides, 6 and 12. Find the segments made by the bisector of the vertical angle.

Ex. 656. Find the projections, on the base, of the sides of the triangle of ex. 655; also, its altitude, its area, and the radii of the inscribed, circumscribed, and escribed circles.

THEOREM XV.

340. *1. The surface of a zone is equivalent to the product of its altitude by the circumference of a great circle. Formulas: a)* $S \doteq 2\,\pi\, Rh$; *b)* $S \doteq \pi\, Dh$; *c)* $S \doteq Ch$.

2. The surface of a sphere is equivalent to the product of its diameter by its circumference. Formulas: a) $S \doteq CD$; *b)* $S \doteq \pi\, D^2$; *c)* $S \doteq 4\,\pi\, R^2$.



I. Let $S \equiv$ the surf. zone generated by $\smile AB$ revolving about the diameter as an axis, $h \equiv ab$ the projection of $\smile AB$ on the diameter is the altitude. $S \doteq 2\,\pi\, Rh$.

1. For divide $\smile AB$ into 4 equal parts . . . . join . . . . . ., bisect AD in . ., join CG. Let fall $\perp$s . . . . .
2. $AN =$ . . Why? §87: 2
3. $CG =$ . ., $=$ . ., $=$., Why? §125: 1
4. Surf gen. by $AD \doteq$ . . . . . . Why? §318: 4b
5. But . . $+$ . . $= 2\, Gg$. Why? §210: 2
6. $\therefore$ Surf. gen. by $AD \doteq 2\pi\, Gg \times AD$.
7. $\triangle\, CGg \sim \triangle$ . . Why? §188: 2
8. $CG : AD =$ . . . . Why? §186
9. . . . $\doteq$ . . .
10. Surf. gen. by | $AD \doteq 2\pi\, CG \times ad$.
11. Sim. surf. gen. by | ED . . . . etc.
12. $\therefore$ Surf. gen. by $AD +$ . . . . . $\doteq 2\pi\, CG \times ab$.
13. Continually bisecting $\smile$s AD etc. the broken | $ADE \doteq \smile AB$, surf. gen. by broken | $AD \doteq$ zone gen. by AB, $CG \doteq R$.
14. $\therefore S \doteq 2\pi\, Rh$. Q. E. D.

Prove formulas b and c.

II. Since the surface of a sphere may be considered the surface of a zone in which $h = D$ . . . . . .

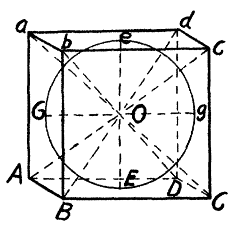
Ex. 657. The surface of a sphere is equivalent to the surface of four of its great circles.

Theorem XVI.

341. *1. The volume of a sphere is equivalent to the product of its surface by one third of its radius, or to one sixth of π times the cube of its diameter. Formulas: a)* $V = \frac{1}{3}SR$; *b)* $V \simeq \frac{1}{6}\pi D^3$; *c)* $V \simeq \frac{4}{3}\pi R^3$.

2. The volume of a spherical sector is equivalent to the product of its surface by one third the radius of the sphere. Formulas: a) $V \simeq \frac{1}{3}SR$; *b)* $\simeq \frac{2}{3}\pi R^2 h$.

3. The volume of a spherical segment of one base is equivalent to the sum of the volumes of a cylinder having the same base and half the altitude and a sphere whose diameter is the altitude of the segment. Formulas: a) $V \simeq \frac{1}{2}\pi R^2 h + \frac{1}{6}\pi h^3$; *b)* $V \simeq \frac{\pi h}{6}(3R^2 + h^2)$.



Let $O \equiv$ . . . .

1. For about O circumscribe . . ., pass planes Oab . . . . dividing the cube into . . . . . whose common altitude is R.

2. Sum bases pyramids is . . . . and sum vol. pyramids . . . . . .

3. Vol. cube $\simeq \frac{1}{3}SR$.

4. Continually increase the number of faces of the polyhedron by cutting off solid $\angle$s by tangent planes.

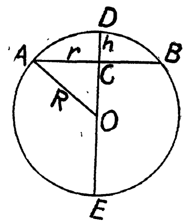
5. Surface of polyhedron $\doteq$ surface of sphere.

6. Volume . . . .

7. $V \simeq \frac{1}{3}SR$. Q. E. D.

Prove formulas b and c.

II. Circumscribe a polyhedron about the sector, etc.



III. Let $r \equiv$ the radius of the base of the sector . . . ., then $V \simeq$ . . . . + . . .

1. $(2R—h)\,h \simeq$ . . . Why? §222: 2

2. $\therefore$ . . . . . . $\simeq$ . . ; $R = \dfrac{r^2 + h^2}{2h.}$

3. $V \simeq$ vol. sector O-ADB—vol. cone O-$ACB \simeq \frac{2}{3}\pi R^2 h — \frac{1}{3}\pi r^2 (R—h) \simeq$ . . .

$$\simeq \frac{\pi}{3}\left\{\frac{r^4 + 2r^2 h^2 + h^4 — r^4 + r^2 h^2}{2h} \simeq \ldots\ldots\ldots\right\}$$

$$\simeq \tfrac{1}{2}\pi r^2 h + \tfrac{1}{6}\pi h^3 \simeq \frac{\pi h}{6}\left(3r^2 + h^2.\right)$$

Q. E. D.

Ex. 658. The surface of a sphere is equivalent to the lateral surface of the circumscribed cylinder.

Ex. 659. The surface of a sphere is two thirds of the entire surface of the circumscribed cylinder.

Ex. 660. Find the surface of a zone whose altitude is 4 in a sphere whose radius is 5.

Ex. 661. In a zone: $D = 25$, $h = 4$. Find S. See §340: 1.

Ex. 662. Find the surface of a sphere whose diameter is 5.

Ex. 663. In a sphere: $S = 78.54$. Find D and C.

Ex. 664. In a sphere: $C = 12$. Find D and S. (Use $1 : \pi =$ .31831.)

Ex. 665. In a sphere: $S = 100$. Find R.

Ex. 666. The radius of the base of a zone is 8 and its altitude is 2. Find R and S.

Ex. 667. The volume of a sphere is $\frac{2}{3}$ of the volume of the circumscribed cylinder.

Ex. 668. The volume of a cone inscribed in a hemisphere, its vertex being the pole of the hemisphere, is half the volume of the hemisphere.

Ex. 669. The volume of a cylinder circumscribed about a sphere is 18. Find the volume of the sphere.

Ex. 670. The volume of a cone inscribed in a hemisphere is 12. Find the volume of the hemisphere, also its convex surface and entire surface.

Ex. 671. The volume of a sphere is 318.31. Find the volume and surface of the inscribed cube.

THEOREM XVII.

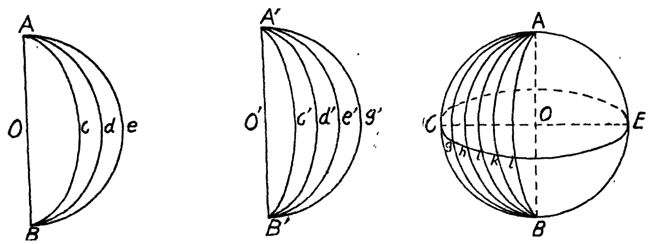
342. *1. On the same sphere, or on equal spheres, lunes and spherical wedges are equal when their angles are equal.*

2. On the same sphere, or on equal spheres, the surfaces of lunes and the volumes of spherical wedges have the same ratio as their angles. Formulas: a) $S_1 : S_2 : = \theta_1 : \theta_2$; *b)* $V_1 : V_2 := \theta_1 : \theta_2$, *in which* θ *is the numerical measure of the angle of the lune;* $\theta = A° \div 360°$.

3. A lune is the same part of the surface of a sphere that its angle is of four right angles. Formulas: a) $S : 4\pi R^2 = \theta : 4rt \angle s$; *b)* $S \triangleq 4\pi r_2 \theta$.

4. A spherical wedge is the same part of the sphere that its angle is of four right angles. Formula: $V \triangleq \frac{4}{3} \pi R^3 \theta$.

1. Construct a figure and prove by superposition.



II. Use method of §175.

III. and IV. Apply §342: 2.

Ex. 672. In a sphere: $S = 100\,\pi$. Find D and V.

Ex. 673. In a sphere: $V = 1000\pi$. Find D and S.

Ex. 674. In a spherical sector: $V = 288\,\pi$, $r = 6$. Find h.

Ex. 675. In a spherical sector: $V = 2000$, $h = 10$. Find R.

Ex. 676. Find the volume of a spherical wedge on a sphere whose radius is 3 if $\theta = 60°$; $45°$; $90°$; $120°$; 150; $210°$; $240°$.

Ex. 677. The angle between two intersecting arcs on the surface of a sphere is measured by the angle between the tangents to these arcs at the point of contact. Apply §§127 and 338.

Theorem XVIII.

343. *1. The surfaces of spheres and of similar zones and lunes have the same ratio as the squares of their radii. Formula:* $S_1 : S_2 = R_1^2 : R_2^2$.

2. The volumes of spheres and of similar sectors, segments, and spherical wedges have the same ratio as the cubes of their radii. Formula: $V_1 : V_2 = R_1^3 : R_2^3$.

Let $R_1 \equiv$ the radius of the first sphere;
$R_2 \equiv$ the radius of the second sphere;
$S_1 \equiv$ the surface of the first sphere;
$S_2 \equiv$ the surface of the second sphere; etc.

Both parts may be proved by finding the values of S_1, S_2, etc., in the preceding sections and dividing.

SUPPLEMENTARY THEOREMS AND EXERCISES.

Theorem XIX.

344. *There can be but five regular polyhedrons.*

I. 1. For not less than three equal plane $\angle$s can form . .

2. $3 \angle$s of equilateral $\triangle$s $=$ . .; $4 \angle$s . . . .; $5 \angle$s . . .

3. $6 \angle$s . . . $= 360°$ and cannot be used.

4. $\therefore$ not more than . . . . can be formed . . . . .

5. $3 \angle$s of $\square$s $=$ . . ., and $4 \angle$s . . . . $\therefore$ not more than one $\square$ . . . .

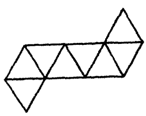
6. $3 \angle$s of a regular pentagon $=$. , . . and $4 \angle$s of a regular . . . . . . $\therefore$ not more than . . . .

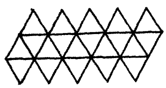
7. $3 \angle$s of a regular heptagon . . . .

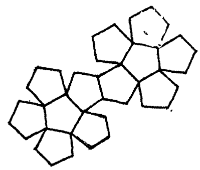
8. Not more . . . equilateral $\triangle$s, ..., or . . . ., and no regular polygon of more than 6 |s can be used . . . . .; $\therefore$ not more than five regular . . . . .

II. Five regular polyhedrons can be formed.

The pupil will prove this by actually constructing the three with which he is not yet familiar; viz: the ***octahedron,*** whose 8 faces are equilateral triangles; the ***icosahedron,*** whose 20 faces are equilateral triangles, and the ***dodecahedron,*** whose 12 faces are regular pentagons. Convenient groupings to save labor in constructing the regular polyhedrons and folding them together are suggested by the figures below. Edges constructed adjacent should be cut half through, the others fastened by pasting over them strips of thin but strong paper.





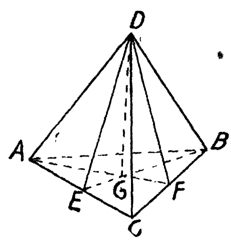


THEOREM XX.

345. *1. In a regular tetrahedron:* a) $h_s = \frac{1}{2} a\sqrt{3}$; b) $S \triangleq a^2\sqrt{3}$; c) $h = \frac{1}{3} a\sqrt{6}$; d) $V \triangleq a^3\sqrt{2} \div 12$.

2. In a regular octahedron; a) $d = a\sqrt{2}$*; $S = 2a^2\sqrt{3}$; $V = \frac{1}{3} a^3\sqrt{2}$.

I. a) Apply §215: 4.
b) Apply §205: 1.
c) Construct BE and $AF \perp$ . . and prove $DG \perp$ base by §§262: 4, 63, and 132.



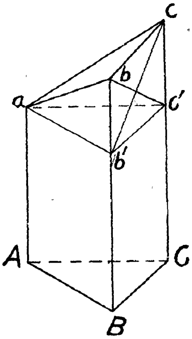
II. Examine the octahedron you have constructed. Observe that its diagonal is the diagonal of a square, etc.

* $d \equiv$ the diagonal.

Theorem XXI.

346. *The volume of a truncated triangular prism is equivalent to: 1) The product of its bases by one third of the sum of the altitudes of the vertices of the angles of its upper base. Formula:* $V \doteq \frac{1}{3}(h + h' + h'')B$.

2. The product of one third of the sum of its lateral edges by a right section. Formula: $V \doteq \frac{1}{3}(l + l' + l'')B_r$*.



I. Suggestions: In the figure, $abc \equiv$ the upper base and $ab'c'$ a section $\parallel$ the lower base. Show that the solid abc–$b'c' \doteq 2$ pyramids having the common base $ab'c'$, the vertex of one being c, of the other b, etc.

II. Suggestion: Divide the prism into two truncated right prisms by a right section.

Ex. 678. The volume of a truncated prism whose right section is a parallelogram is half the product of the sum of its edges by a right section.

Ex. 679. A prism whose right section is a parallelogram is a parallelopiped.

347. Def. A ***wedge*** is a solid bounded by a rectangular base, two trapezoids meeting in an ***edge*** parallel to the base, and two triangular ends. The ***altitude*** is the perpendicular from any point in the edge to the base.

348. Def. We may regard a truncated triangular prism as a ***wedgoid;*** its bases as the ends, one lateral face the base, and the edge without it the edge of the wedge. One of the parallel sides of the base may be reduced to zero, reducing the base to a triangle.

* $B_r \equiv$ a right section.

349. Def. A ***prismoid*** is a solid whose lateral faces are trapezoids, and bases polygons. The angles of the polygons are equal, and their planes are parallel since sides forming the angles are parallel.

350. A ***quadrangular prismoid*** has a quadrilateral base. If two sides of the base are parallel it is called ***trapezoidal,*** which includes those having parallelograms as bases. If the base is a rectangle it is called a ***rectangular prismoid.***

Theorem XXII.

351. *1. The volume of a wedgoid is one sixth of the product of the sum of the edge and the parallel sides of the base, by the width of the base, by the altitude. Formula:* $V \bumpeq \frac{1}{6}(L + L' + l)\,wh$.

2. In a wedge: $V \bumpeq \frac{1}{6}(2L + l)\,wh$. (*Express this in words.*)

3. The volume of a rectangular prismoid is one sixth of the product of the sum of its bases and four times a mean section between them by the altitude. Formula: $V \bumpeq \frac{1}{6}(B + b + 4M)\,h$.

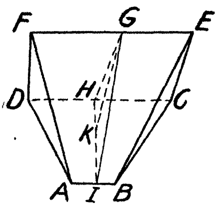


Fig. 1.

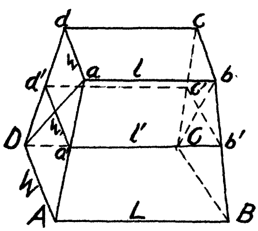


Fig. 2.

I. Apply §§346: 2 and 205: 1.

II. Apply §§347 and 351: 1.

III. Find the volume of the prismoid in terms of L, l, and h regarding it as two wedges. Show that $M \bumpeq \frac{1}{4}(L + l)(W + w)$, and substitute.

THEOREM XXIII.

352. *1. In any prismoid* $V \doteq \frac{1}{6}(B + b + 4M)h$.

2. In a frustum of a pyramid: $V \doteq \frac{1}{6}(B + b + 4M)h$.

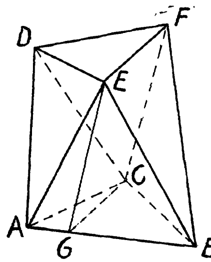
3. In any pyramid or wedge: $V \doteq \frac{1}{6}(B + 4M)h$.

I. Prove by the method of §351: 3.

II. For a frustum of a pyramid is the limit of a prismoid as a side of the lower base and the homologous side of the upper base $\doteq 0$.

III. Prove by the method of limits.

353. DEF. A ***prismatoid*** is a solid bounded by polygons whose faces are parallel, as bases, and by triangles or trapezoids, as lateral faces.



The adjacent figure represents a prismatoid both of whose bases are triangles, with six triangles as lateral faces. It may be divided by passing planes into four wedgoids and pyramids, the first wedgoid cut off being indicated in the figure. The sum of the bases of these solids is the same as the sum of the bases of the prismatoid, also the sum of their mean section and volumes; also the altitude of all is the same. In like manner it might be shown in any particular case that any particular prismatoid may be divided into pyramids and wedgoids, all having the same base and altitude. The general proof is too abstruse to be profitably studied by pupils of the grade for which this book is written.

Ex. 680. Make and solve an easy numerical exercise illustrating each formula given in Theorems XX—XXIII.

Ex. 681. Give a definition of the wedge as a wedgoid beginning: *A wedge is a wedgoid .....*

BOOK VII.

THE SPHERE.

354. DEF. A ***spherical polygon*** is a spherical surface bounded by arcs of great circles, usually limited to surfaces each of whose sides and angles is less than 180°, unless otherwise expressly stated.

355. DEF. Besides classes corresponding to those of plane triangles, spherical triangles may be ***birectangular***, having two right angles, ***trirectangular***, having three right angles, ***quadrantal***, having one side a quadrant, ***biquadrantal*** and ***triquadrantal***.

356. DEF. A ***spherical pyramid*** is a solid whose base is a spherical polygon, whose vertex is the center of the sphere, and whose lateral faces are planes determined by its sides. Since central angles are measured by the arcs intercepted by their sides, the sides of the polygon have the same measures as the face angles of the polyhedral angle. The angles of the polygon are evidently the inclination of the faces.

Ex. 682. Name ten classes of spherical polygons. See §92.

Ex. 683. Name six classes of spherical triangles corresponding to the six classes of plane triangles. See §§31 and 32.

Ex. 684. Define the base, vertical angle, altitude, median, and angle bisector of a spherical triangle, by the aid of the corresponding definition in Plane Geometry; also the hypotenuse and legs of a right spherical triangle.

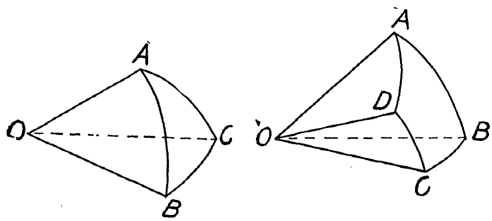
Ex. 685. Is it possible for a spherical triangle to have three obtuse angles? See §356, cut out three obtuse angles and see whether they can be put together to form a solid angle.

Ex. 686. Can a spherical $\triangle$ have three rt $\angle$s?

THEOREM I.

357. *1. Any side of a spherical triangle is less than the sum of the other two.*

2. The sum of the sides of a spherical polygon is less than 360°.

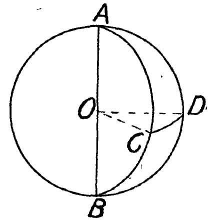


I. Apply §274.
II. Apply §277.

358. *Through two given points on the surface of a sphere construct a great circle of the sphere.*

Find the circumference of the sphere if not given, §339: 2. With the chord of a quadrant as radius and the points as centers . . . .

THEOREM II.



359. *The angle between two arcs of great circles is measured by the arc of a great circle intercepted between them, which has their intersection as its pole.*

Apply §§177 and 255.

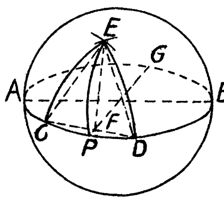
Ex. 687. Find the surface of a lune whose angle is 72° on a sphere whose radius is 4; also the volume of the ungula of which the lune is the base.

Problem I.

360. *Construct on the surface of a sphere, at a given point in the circumference of a great circle, an angle equal to a given angle.*

Problem II.

361. *Construct a great circle perpendicular to any circle of a sphere at a given point.*



Through P . . . .

1. From A lay off equal distances PD and PC on . . . . .
2. With C and D as centers . . . .
3. Through E and P . . . . . .

In proof apply §§123: 3, 122: 2, 48, 260: 1, and 268: 1.

Ex. 688. Construct a spherical polygon of each class named in exercises 682 and 683.

Ex. 689. Construct a spherical triangle of each class named in §355. See §§339 and 361.

Ex. 690. Find the volume of a cone in which the surface of the base = 31.831 in. and the lateral surface 62.162, to two places of decimals.

Ex. 691. The upper base of a frustum of a cone is one fourth of the lower base, its altitude is 10, and its volume, 1000. Find the radii of its bases, its slant height, and its entire surface.

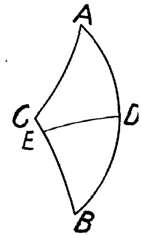
Ex. 692. The sides of the base of a triangular pyramid are $3a$, $4a$, and $5a$, and its altitude, $6a$. Find its volume in terms of a.

Ex. 693. Find the value of a in the pyramid of exercise 692 if its volume is 1000, also each side of its base, the surface of its base, the surface of each lateral face, and its entire surface.

Ex. 694. A pyramid whose altitude is 100 is divided into three equal parts by planes parallel to its base. Find the altitude of each part.

Theorem III.

362. *The shortest distance on the surface of a sphere between two points is the smaller of the two arcs of the great circle determined by them.*



Let AB be the less . . . .

1. For let C be any pt. . . . .
2. Join AC and CB by arcs of great ○s.
3. From AB cut off $AD = AC$.
4, 5. Prove that $\smile DB < \smile BC$.
6. Since $\smile AC = \smile AD$ they may be so placed as to coincide and the distances, whatever the shortest path, must be equal.
7. With B as a pole construct a small ○ through D. Since $\smile BD < \smile BC$, $\smile DE$ cuts BC. Why? §17: 4
8. ∴ The shortest path from B to $C >$ the shortest path from B to D and the sum . . . . . .
9. ∴ The shortest path from A to B passing through any point C without $\smile AB >$ . . . . Q. E. D.

Ex. 695. If two parallel planes are cut by a third plane: 1) the alternate dihedral angles formed are equal; 2) the corresponding dihedral angles are equal; 3) the sum of the interior dihedral angles on the same side of the transversal plane are equal.

Ex. 696. State and prove the converse of exercise 695.

Ex. 697. The part of a line intercepted between two parallel planes is 50 cm and its projection on one of these planes is 48 cm. Find the distance between the planes.

Ex. 698. Supposing the earth to be a sphere, in what direction should a flying machine capable of moving over every part of the earth start from New York to reach by the shortest route a point 180° east of New York on the same parallel of latitude. See §§362 and 332: 2.

Ex. 699. Construct a great circle of a sphere and find its pole (see §339: 2), then construct a great circle perpendicular to the first (§361). Prove that the second circle passes through the poles of the first.

THEOREM IV.

363. *1. Every great circle bisects a sphere and its surface.*

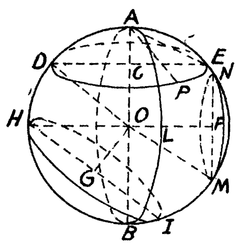
2. Any two great circles bisect each other.

3. Equal circles are equally distant from the center, and conversely.

4. Of two unequal circles the smaller is the more remote from the center, and conversely.

5. A point a quadrant distant from each of two points on the circumference of a great circle is a pole of that circle.

6. A pole of any circle of a sphere is equally distant from every point on its circumference.



I. Prove by superposition.

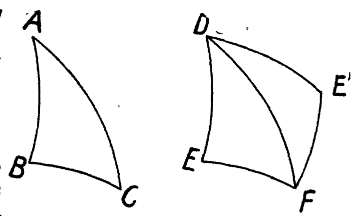
II. Their common section passes through what point?

III. and IV. Pass a plane through the centers of the circles and sphere, and apply §125.

V. Apply §332: 2 and 262: 4.

VI. Apply §§333, 335, and 262: 4.

364. DEF. ***Symmetrical spherical triangles*** have all their parts equal, but differently arranged.



Ex. 700. The radii of two spheres are to each other as 2 to 5, and the volume of the first is 64 cu. cm. Find the volume of the second.

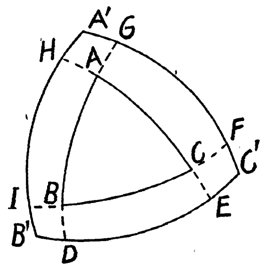
Ex. 701. Find the surface of a lune of 45° on each sphere of ex. 700.

Ex. 702. Find the volume of an angula of 120° on each sphere of ex. 700.

THEOREM V.

365. *If, from the vertices of the angles of a spherical triangle, arcs of great circles be drawn intersecting each other, a second triangle will be formed whose vertices are poles of the given triangle.*

Apply §363: 2.



366. DEF. Two spherical triangles so related that the vertices of the angles of one are poles of the sides of the other are called ***polar triangles.***

The pupil should draw the arcs, complete the circumferences, note the number of spherical triangles formed, and remember that only the two triangles most central to each other are regarded as polar.

THEOREM VI.

367. *Each angle of a spherical triangle is the supplement of the opposite side of the polar triangle.*

Let ABC . . . .

1. For if necessary produce . . . . . .
2. $\angle A$ is measured . . . . Why? §359
3. $\smile B'E = 90°$ $\smile DC' =$ . . . . Why? §366

Complete the proof.

Ex. 703. The angles of a spherical triangle are 80°, 100°, and 120°. Find the sides of the polar triangle.

Ex. 704. The sides of a spherical triangle are 70°, 80°, and 90°. Find the angles of the polar triangles.

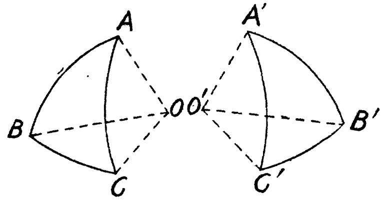
Ex. 705. If one side of a spherical triangle is a quadrant one angle of its polar triangle is a right angle.

Ex. 706. Name the triangles described in exercise 705.

THEOREM VII.

368. *If two spherical triangles have the three sides of the one equal to the three sides of the other, each to each, the angles are equal, and conversely.*

Apply §275, observing that $\smile AC$ measures $\angle AOC$, etc., and that $\angle B$ measures the inclination of the planes AOB and COB, etc.



THEOREM VIII.

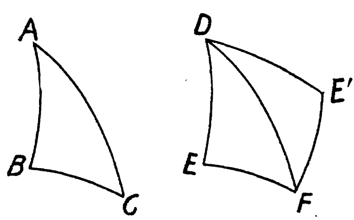
369. *1. If two spherical triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles are either equal or symmetrical.*

2. If two spherical triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles are either equal or symmetrical.

I. Let $\triangle$s ABC . . . . . .

A. Let the equal parts be similarly situated. Prove as in §38.

B. Let the equal parts be differently arranged as in $\triangle$s ABC and $DE'F$.



1. For constr. $\triangle DEF$ symmetrical with $DE'F$.

2.—5. Prove that $\triangle\ DEF = \triangle\ ABC$.

6. But $\triangle\ DEF$ is by constr. . . . . .

7. . . . . Q. E. D.

II. Prove in a manner similar to that used above.

Problem III.

370. *1. To bisect a spherical angle.*

2. To bisect an arc on a sphere.

Theorem IX.

371. *1. In an isosceles spherical triangle the angles opposite the equal sides are equal.*

2. Converse of the first part.

This may be proved by the method used in proving the like theorems in plane geometry.

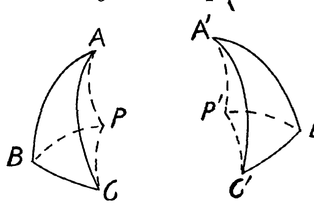
Theorem X.

372. *Two equilateral spherical triangles are equivalent.*

Let the spherical . . . . . .

1. Let P be the pole of the small circle determined by A, B, and C; and P' . . .

2. Join AP, ...



Prove that $\triangle\ ABP = \triangle\ A'B'P'$, $\triangle\ PBC =$ . . ., $\triangle\ PAC =$ . . .; and complete the proof.

Theorem XI.

373. *The greater side of a spherical triangle is opposite the greater angle, and conversely.*

This may be proved by the method used in §54.

Ex. 707. Are $\triangle$s ABC and $A'B'C'$ in the above figure equal or symmetrical?

Ex. 708. Two equiangular spherical triangles are equivalent.

Ex. 709. State the reciprocal of §371: 1.

THEOREM XII.

374. *1. The sum of the sides of a spherical polygon is less than the circumference of a great circle.*

2. The sum of the angles of a spherical triangle is more than two right angles and less than six right angles.

3. The sum of the angles of a spherical polygon of n sides is more than $2(n-2)$ right angles and less than $2n$ right angles.

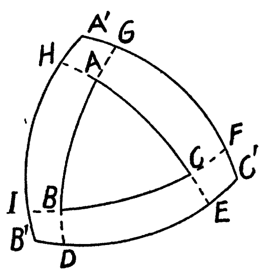
I. Apply §277.

II. Let A, B, and $C \equiv \angle$s . . .

1. For let a', b', and $c' \equiv$ the corresp. |s of the polar $\triangle$.

2. Then: $A + a' = 2$ quadrants, $B + b' = \ldots\ldots$, $C + c' \ldots\ldots\ldots$ Why? §367

3. $\therefore A + B + C + a' + b' + c' = 6$ quadrants.



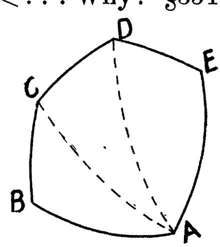
4. But $a' + b' + c' < 4 \ldots$ Why? §374: 1

5. $A' + B' + C' > 2$ quadrants.

6. $A < 2$ quadrants, $B < \ldots$ and $C < \ldots$ Why? §354

7. $\therefore \ldots\ldots\ldots\ldots$ Q. E. D.

III. Draw a polygon, divide it into triangles, and apply §§374: 2 and 354.



Ex. 710. If two spherical triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, they are equivalent.

Ex. 711. State and prove the reciprocal of ex. 710.

Ex. 712. Of what theorem is §371: 1 the reciprocal?

Ex. 713. Construct a trirectangular spherical triangle.

Ex. 714. Construct a triquadrantal spherical triangle.

Ex. 715. A trirectangular triangle is triquadrantal, and conversely.

THEOREM XIII.

375. *1. The surface of a trirectangular triangle is one eighth of the surface of the sphere. Formula:* $S = \frac{1}{2}\pi R^2$.

2. The surface of a lune is twice the product of the number of right angles in its angle by a trirectangular triangle. Formula: $S \triangleq \pi R^2 T$.

I. Construct two great ○s ⊥ each other, and a third ○ with their intersections as its pole by §§339 and 361, and prove the theorem.

II. Apply §§342: 3 and 375: 1.

376. DEF. The ***spherical excess of a spherical triangle*** is the number of right angles by which the sum of its angles exceeds two right angles. Observe that it is a number, not a magnitude, as here defined. Formula: $E = A + B + C - 2$.*

Ex. 716. Find the volume of a sphere whose surface is 100.

Ex. 717. Find the surface of a sphere whose volume is 1000.

Ex. 718. Find the circumference of a sphere whose surface is 100; whose volume is 1000.

Ex. 719. Find the surface and volume of a sphere whose circumference is 60.

Ex. 720. In a cylinder $S' = 100$, $h = 12$, find R; if $S' = 200$ and $R = 5$, find h; if $V = 300$ and $h = 12$, find R; if $V = 400$ and $R = 8$, find h.

Ex. 721. Prove the following formulas:

1. In a cylinder: $R = S' \div 2\pi h$, $h = S' \div 2\pi R$, $D = S' \div \pi h$, $h = S' \div \pi D$; $h = V \div \pi R^2$, $R = \sqrt{V \div \pi h}$, $h = 4\,V \div \pi\, D^2$

2. In a cone: $R = S' \div \pi h'$, $h' = S' \div \pi R$; $h = 3V \div \pi R^2$, $R = \sqrt{3V \div \pi h}$.

3. In a sphere: $R = \frac{1}{2}\sqrt{S \div \pi}$, $D = \sqrt{S \div \pi}$, $S \triangleq C^2 \div \pi$, $C = \sqrt{\pi S}$; $V \triangleq C^3 \div 6\pi^2$, $R = \sqrt[3]{3V \div 4\pi}$, $D = \sqrt[3]{6V \div \pi C}$, $= \sqrt[3]{6\pi^2 V}$.

---

* A, B, and C, here ≡ the number of rt ∠s in each ∠.

THEOREM XIV.

377. *1. If two arcs intersect in the surface of a hemisphere, the sum of the two opposite triangles thus formed is equivalent to the surface of a lune whose angle is the inclination of the arcs.*

2. The surface of a spherical triangle is equivalent to the product of its spherical excess by a trirectangular triangle. Formula: $S = ET$.

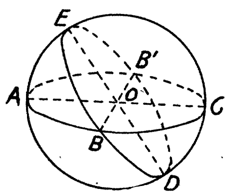


Fig. 1.

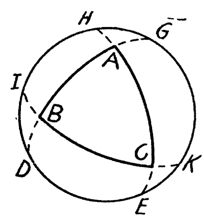


Fig. 2.

I. Let $\smile$s ABC and EBD . . . . ., then $\triangle\ ABE + \triangle\ BCD \bumpeq$ the lune $BEB'A$.

Prove that $\triangle\ BCD = \triangle\ AB'$, etc.

II. Let ABC be . . . . . .

1. For constr. the great $\bigcirc$ DHK so that $\triangle\ ABC$ may fall wholly in one hemisphere, prod. AB . . . ., AC . . . .
2. $\triangle\ HAG + \triangle\ DAE \bumpeq 2A \times T.$* Why?
3. $\triangle\ OBD + \triangle\ GBK \bumpeq 2B \times T.$* Why?
4. $\triangle\ KCE + \triangle\ HCI \bumpeq 2C \times T.$* Why?
5. But the sum of the 6 $\triangle$s $= 4\ T + 2\ \triangle\ ABC.$
6. $4\ T + 2\ \triangle\ ABC \bumpeq 2(A + B + C)T.$
7. . . . .
8. $ABC \bumpeq (A + B + C - 2)T = ET.$ Q. E. D.

Ex. 722. In a cone: $S' = 150$, h' 5, find R; $S' = 250$, $R = 25$, find h; $V = 500$, $R = 5$, find h; $V = 600$, $h = 12$, find R.

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* A, B, and $C \equiv$ the number of right angles in these angles.

378. DEF. The ***spherical excess of a polygon*** is the number of right angles found by subtracting from the sum of its angles two right angles taken as many times as the polygon has sides less two. Formula: $E = \Sigma - 2(n-2)$.*

THEOREM XV.

379. *The area of a spherical polygon is the product of its spherical excess by a trirectangular triangle. Formula:* $S \simeq ET \simeq \{\Sigma - 2(n-2.)\} T$.

Apply §377: 2.

SUPPLEMENTARY THEOREMS AND EXERCISES.

THEOREM XVI.

380. *1. The volyme of a spherical triangular pyramid is* $E\pi R^3 \div 18 = (A + B + C - 2)\pi R^3 \div 18$.

2. The volume of any spherical pyramid is $E\pi R^3 \div 18 = \{\Sigma - 2\,(n-2)\}\pi R^3 \div 18$.

Use the method of limits, applying §§328, 377: 2, and 379.

Ex. 723. Find the surface of a spherical triangle whose angles are 70°, 80°, and 90°, if R is 12.

Ex. 724. Find the surface of a spherical pentagon whose angles are 80°, 100°, 130°, 150°, and 170°.

Ex. 725. There can be no spherical pentagon having the angles 95°, 100°, 110°, 115°, and 120°.

Ex. 726. In a spherical △: $A = B = C = 75°$, $R = 90$; find V.

Ex. 727. Each angle of a quadrangular spherical pyramid is 105° and $R = 5$; find V.

Ex. 728. Find the volume of a hexagonal spherical pyramid each of whose angles is 125° on a unit sphere.

* $\Sigma \equiv$ the number of right angles in the sum of the angles of the polygon.

THEOREM XVII.

(Requires §§282 and 283.)

381. *1. Through any four points, not in the same plane and no three in a straight line, one surface of a sphere may be made to pass, and but one.*

2. A sphere may be circumscribed about any tetrahedron or inscribed in it.

I. Apply §282: 3.
II. Apply §§282: 3 and 283: 3.

Ex. 729. In a regular tetrahedron: $h_s = \frac{1}{2}a\sqrt{3}$; $a = 2\,h_s \div \sqrt{3} = \sqrt[3]{6V\sqrt{2}} = \sqrt{S \div \sqrt{3}} = \frac{1}{2}\,h\sqrt{6}$; $S = \frac{3}{2}\,h^2\sqrt{3}$; $V = h_s^3\sqrt{6} \div 27 = h^3\sqrt{3} \div 8 = \sqrt{S^3}\sqrt{2} \div (12 \times \sqrt[4]{3^3})$.

Ex. 730. Find formulas for the octahedron corresponding to those for the tetrahedron in ex. 729.

Ex. 731. Find the slant height, altitude, surface, and volume of a regular tetrahedron whose edge is 3.

Ex. 732. Find the edge, altitude, surface, and volume of a regular tetrahedron whose slant height is 2.

Ex. 733. In a regular tetrahedron $S = 100$; find a and h.

Ex. 734. In a regular octahedron $S = 240$; find a.

Ex. 735. In a regular icosahedron $S = 720$; find a.

Ex. 736. In a frustum of a pyramid the sides of B are 12, 16, and 20. The side of b homologous to 12 is 6, $h = 15$. Find V.

Ex. 737. The ends of a round log are 8 ft. and 5 ft. and its length 180 feet. How many feet of lumber can be made from it, allowing two fifths for waste?

Ex. 738. In a cylindrical cistern: $R = 10$ ft., $r = 6$ ft., $h = 12$ ft. 10 in. How many barrels of water does it contain when full?

Ex. 739. The weight of a cubic foot of water is about 62.5 lbs., the specific gravity of the earth is about 6*, and its volume about that of a sphere whose diameter is 7,918 miles. Find the volume of the earth in cubic miles, also its weight in tons.

* That is, each cubic foot of the earth is as heavy as 6 cubic feet of water.

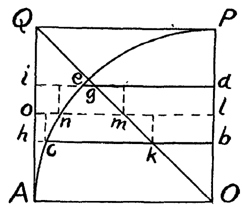
THEOREM XVIII.

382. *If a cone has the center of the sphere as its vertex and the same base as the circumscribed cylinder:*

1. Any section of the sphere by a plane parallel to the base is less than a great circle by the corresponding section of the cone.

2. The volume of any segment of the sphere made by planes parallel to the base is less than the corresponding segment of the circumscribed cylinder by the corresponding segment of the cone.

3. The volume of a spherical segment is equivalent to the product of one half the sum of its bases by its altitude increased by the volume of a sphere whose diameter is the altitude of the segment. Formulas: a) $V \doteq \frac{1}{2}\pi h\,(r_1^2 + r_2^2) + \pi h^3 \div 6$; *b)* $V \doteq \frac{1}{2}\pi h(r_1^2 + r_2^2 + \frac{1}{3}h^2)$.



I. Let the quadrant AP, ▭ . . ., △ . ., and the ordinate bc be revolved about . . . . ., then $\pi\,\overline{bc}^2 \doteq \pi R^2 - \pi \times \overline{bk}^2$.

1. For $Ob = bk$. Why?
2. $\overline{bc}^2 \doteq (R + Ob)\,(R - Ob)$. Why? §222: 2
3. $\pi\overline{bc}^2 \doteq \pi R^2 - \pi\overline{bk}^2$. Why?

II. Let $V \doteq$ the volume and h the altitude of the segment generated by the revolution of *bcned*, and frustum *gb* the volume of the frustum of a cone generated by trapezoid *gb*, then $V \doteq \pi R^2 h$ — frustum *gb*.

1. For bisect bd in l, and draw . . . . ., erect ⊥s to bc and di at . . . .
2. $\pi\,\overline{bc}^2 \doteq \pi R^2 - \pi bk^2$; $\pi de^2 \doteq \pi R^2 - \pi gd$. Why?
3. $\pi\overline{bc}^2 \times bl \doteq \pi R^2 \times bl$ . . .; $\pi de^2\ de \doteq$ . . . . Why?
4. Indefinitely increasing the number of divisions: sum cyl. $\pi\overline{bc}^2\ bl$ etc. $\doteq V$; sum cyl. bh etc. $\doteq$ frustum gb.

5. $\therefore V \Leftrightarrow \pi R^2 h -$ frustum bg. Why?

III. 1. $V \Leftrightarrow \pi R^2 h - \frac{1}{3}\pi h(bk^2 + dg^2 + bh \times dg) \Leftrightarrow \frac{1}{2}\pi h(R^2 - bk^2) + \frac{1}{2}\pi h(R^2 - dg^2) + \frac{1}{6}\pi h\,(bh^2 + dg^2$ $2bkxdg)$.

2. But $R^2 - bk^2 = bc^2 = r_1^2$; $R^2 - \ldots$; and $dg - bk = h$. Why?

3. $V \Leftrightarrow \frac{1}{2}\pi h(r_1^2 + r_2^2 + \frac{1}{3}h^2)$. Q. E. D.

383. Def. It is evident from the above demonstration that every section of the part of the cylinder without the sphere made by a plane parallel to the base is equivalent to the corresponding section of the cone. All solids of which this is true, whatever their form, are called ***Cavalieri bodies***, from Cavalieri (1598—1647), an Italian mathematician, who proved that the segments of all such bodies included between parallel planes are equivalent.

Ex. 740. Find the volume of a segment of one base if $r = 10$ and $h = 5$.

Ex. 741. Find the volume of a segment of two bases if $r_1 = 15$, $r_2 = 20$, $h = 6$.

Ex. 742. Find the altitude of a segment of the bases if $V = 1{,}000$, $r_1 = 12$, and $r_2 = 10$, also the diameter of the sphere.

Ex. 743. In a section of a railroad cut 100 feet long the slope of the natural surface is uniform, the width at the bottom is 30 feet, the depth at the first end is 10 feet and 20 feet at the other end, and the slope of the sides of the cut is one and one half to one, that is, where the depth is 10 feet the width of the cut at top is 60 feet. Find the number of cubic yards in the excavation. See §352. What would be the number of cubic yards if the depth at one side remained the same and the depth of the other side was reduced 40 per cent.* Ans. 2,962 cu. yds; 2,111 cu. yds.

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* Observe that it makes no difference whether the bottom is level or on a grade. By making the sections short enough to make errors owing to inequalities of surface of no consequence, the volume of any cut required in grading railroads, streets, and lots may be obtained, using only a level, a tape measure, and rods.

BOOK VIII.

SYMMETRY, MAXIMA and MINIMA, LOCI OF EQUATIONS.

I. Symmetry.

384. Def. Two points are ***symmetrical with reference to*** the ***mid-point*** of the line joining them (see fig. 1).

385. Def. Two figures are ***symmetrical with reference to*** a ***point*** as the ***center of symmetry*** when to every point in the one there corresponds a symmetrical point in the other.

386. Def. A figure is ***symmetrical with reference to*** a ***center of symmetry*** when it may be divided into two symmetrical figures, — by a line if a plane figure; by a plane if not a plane figure.

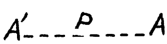


Fig. 1.

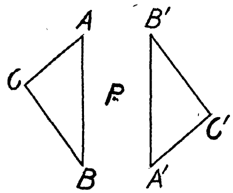


Fig. 2.

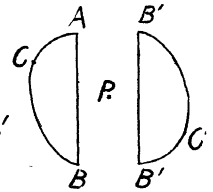


Fig. 3.

Ex. 744. Illustrate §§384, 385, and 386 by the above figures.

Ex. 745. Illustrate §386 by dividing a circle into two symmetrical figures.

Ex. 746. If, on revolving one of two figures about a point as a center through an angle of 180°, every point coincides with a corresponding point of the other, the figures are symmetrical.

387. Def. Two points are ***symmetrical with reference to*** a ***line*** as an ***axis of symmetry*** when it bisects at right angles the line joining them.

388. Def. Two points are ***symmetrical with reference to*** the ***plane*** which bisects at right angles the straight line joining them.

389. Def. Figures are ***symmetrical with reference to*** a ***line*** or ***plane*** as defined with reference to a point in §§385 and 386.

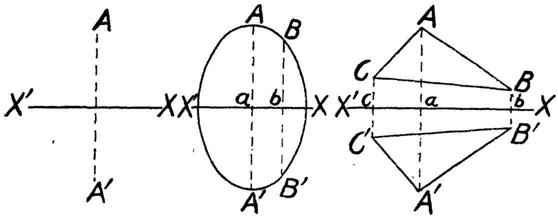


Fig 1. Fig 2. Fig. 3.

Ex. 747. A square is symmetrical with reference to the intersection of its diagonals.

Ex. 748. Illustrate symmetry with reference to an axis by the above figures.

Ex. 749. Illustrate by material objects: a) two points symmetrical with reference to a plane; 2) two lines symmetrical with reference to a plane; 3) two equal plane figures symmetrical with reference to a plane.

Ex. 750. A square is symmetrical with reference to either diagonal.

Ex. 751. A square is symmetrical with reference to what two other lines?

Ex. 752. A rectangle is symmetrical with reference to a line through the intersection of its diagonals perpendicular to its sides.

Ex. 753. An isosceles triangle is symmetrical with reference to its altitude.

390. Def. Every line through the center of symmetry of a symmetrical figure and terminated by it is called a ***diameter***. Hence by the definition of the *center of symmetry* this center is the *mid-point* of any diameter of every figure symmetrical with reference to a center, and conversely.

THEOREM I.

391. *1. Two figures symmetrical with reference to a point are equal.*

2. Two figures symmetrical with reference to an axis are equal.

3. A plane figure symmetrical with reference to a point is bisected by every diameter.

4. A plane figure symmetrical with reference to an axis is bisected by it.

5. Lines symmetrical with reference to a point are parallel.

6. Lines symmetrical with reference to an axis (if not parallel) intersect in the axis of symmetry, which is also their angle bisector, and conversely.

I. Make the figures coincide by rotating one about the center of symmetry. (See figures of §386.)

II. Make the figures coincide by revolving one about the axis. (See figures of §389.)

III. Apply §§386, 385, and 384.

IV. Apply §§383 and 391: 2.

V. Construct a suitable figure, draw two diameters forming two $\triangle$s, prove them equal, and apply §66: 1.

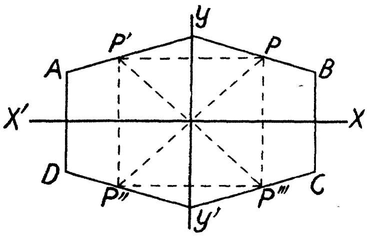
VI. Construct a suitable figure, join 2 symmetrical points, and prove by equal $\triangle$s.

392. REMARK. It may similarly be proved that plane figures symmetrical with reference to a plane are equal; and it may also be proved by the method of limits, inscribing plane surfaces and pyramids, that all surfaces and solids symmetrical with reference to a plane are equivalent. When symmetrical spherical triangles are so placed that two angles of the one are adjacent to the equal angles of the other, as in the figure to §364, they are symmetrical with reference to the plane of the common side.

Ex. 754. How may two symmetrical solid angles be so placed that they are symmetrical with reference to a plane?

THEOREM II.

393. *A figure symmetrical with reference to two axes intersecting at right angles is symmetrical with reference to their intersection, as its center of symmetry.*

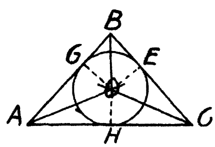


Let axis $XX' \perp$ . . . and . . . .

1. For let $P \equiv$ any pt. in . . . . .

2. Let P' be sym. to P with . . . YY', P''' . . . and P'' sym. to P' . . . .

Complete the proof showing that PP'' is ▭, that its diagonals meet at the intersection of the axes, etc.



Ex. 755. Hero's proof that the area of any triangle $= \sqrt{s\ (s-a)\ (s-b)\ (s-c)}$ * (§241): Inscribe a ⊙ and join the center to the points of tangency and to the vertices. On AC produced take $CM = GB$. Then $AM = s$, and area $\triangle = AM \times OH$ (Why?). From O draw a line $\perp$ to OA, and from C draw a line $\perp$ to AC; let these lines intersect in N, and let ON intersect AC in P. Join AN. Show that $\triangle$s AOH, AOP, OHP, and PCN are similar; also that $\triangle OPC \sim \triangle APN$, and $\triangle ACN \sim \triangle GOB$. $\therefore \dfrac{AC}{GB \text{ or } CM} = \left(\dfrac{CN}{GO \text{ or } OH}\right) = \dfrac{PC}{HP}$ $\therefore \dfrac{AM}{CM} = \dfrac{HC}{HP}$ (Why?). $\therefore \dfrac{\overline{AM}^2}{AM \cdot CM} = \left(\dfrac{HC \cdot AH}{HP \cdot AH}\right) = \dfrac{HC \cdot AH}{\overline{OH}^2}$ (Why?)

Complete the proof.

Ex. 756. Find by Hero's Formula the altitude of a triangle in terms of the sides; also the radius of the inscribed circle.

* This is known as Hero's Formula, from its discoverer, Hero or Heron of Alexandria, 284–221, B.C., or, according to some authorities, about a century later. Ball's History of Mathematics, p. 90.

SUPPLEMENTARY THEOREMS AND EXERCISES.

THEOREM III.

394. *A solid symmetrical with reference to three planes at right angles to each other is symmetrical with reference to their common intersection as its center of symmetry.*

THEOREM IV.

395. *1. A regular polygon of $2n + 1$* |s *is symmetrical with reference to any altitude as an axis.*

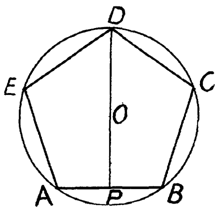


Fig. 1.

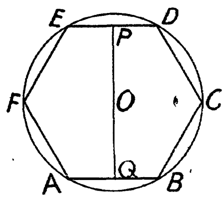


Fig. 2.

2. A regular polygon of $2n$ sides is symmetrical with reference to any diagonal, and also to any line joining the mid-points of opposite sides.

396. DEF. We will briefly note a few of the many kinds of symmetry besides ***twofold symmetry*** above explained, and understood where no other is indicated. If, on rotating a figure about a point through an angle of 120°, every point coincides with the former position of another point, it has ***threefold symmetry***; if through 72°, ***fivefold symmetry***; if through $\pi \div n$, n-fold symmetry. Similarly a solid may have n-fold symmetry with reference to an axis.

Ex. 757. A triangle symmetrical with reference to any altitude is isosceles, and conversely.

Ex. 758. A polygon of 6 sides has 2-fold, 3-fold, and 6-fold symmetry.

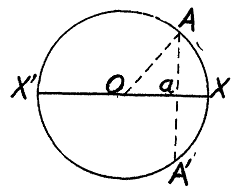
Theorem V.

397. *1. The circle is a figure of perfect symmetry.*

2. Conversely: A figure symmetrical with reference to every diameter is a circle, etc.

3. The sphere is a figure of perfect symmetry. *

I. a) The circle has evidently n-fold symmetry, whatever the value of n with reference to its center.



b) It is symmetrical with reference to every diameter.

Why? §§12 and 384

II. a) Let the figure XAX' . . . .

For if not . . . . §12

b) Let . . . . . .

For to every pt. A there corresponds . . . Why? §122

III. a) Rotating the sphere about its center . . . .

Why? §330

b) Revolving it about any diameter . . . Why? §119: 2

c) It is symmetrical with reference to every plane passing through its center.

Let $O \equiv$ a sphere, XX' any plane through its center and A any point on its surface, then to A . . . . . .

1. For through A pass a great $\odot \perp XX'$.

Complete the proof.

Ex. 759. a) A polygon of n sides has n-fold symmetry; b) a polygon of mn sides has m-fold, n-fold, and mn-fold symmetry; c) a polygon of mnr sides has m-fold, n-fold, r-fold, mn-fold, mr-fold, nr-fold, and mnr-fold symmetry.

Ex. 760. How many axes of symmetry have regular polygons of the following number of sides: 3, 5, 6, 7, 8, 9, 10, 11, 12?

Ex. 761. What kinds of symmetry has a dodecagon?

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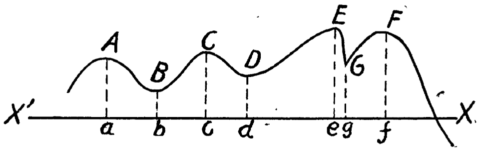
* It may be shown conversely that a figure symmetrical with reference to any plane through its center or possessing n-fold symmetry with reference to any diameter for all possible values of n, is a sphere.

II. MAXIMA and MINIMA.



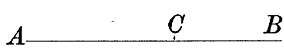
398. DEF. In the curve *ABCDEGF*, *A*, *C*, *E*, and *F* represent ***maximum values*** of the ***ordinate*** or distance from the axis *XX'*. *B*, *D*, and *G* represent ***minimum values*** of the ***ordinate.*** Observe that the minimum value *D* is greater than the maximum *A*, etc.

A ***maximum*** is greater than the values immediately preceding and succeeding it.

Let the pupil define a ***minimum.***

THEOREM VI.

399. *The product of the parts into which a line is divided is a maximum when the parts are equal.*



Let | *AB* . . . then $AC \times CB$ is a maximum . . .

1. For let $a + d \equiv AC$ the greater part.
2. $a - d \equiv$ . . . . .
3. $\therefore$ $a^2 - d^2 = AC \times CB$. Why? §208
4. Which is a maximum when $d = 0$ and $AC = CB$.

Q. E. D.

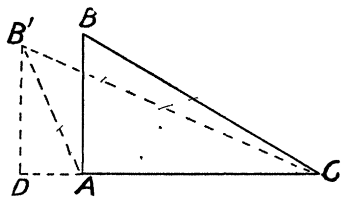
Ex. 762. Find the area of the maximum rectangle the sum of whose adjacent sides is 20.

Ex. 763. Let *ABCD* be any quadrilateral and *E* and *F* the mid-points of its diagonals *AC* and *BD*; $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\overline{EF}^2$ *. Apply §240: 1.

* This is known as Euler's Theorem, from its discoverer, Leonard Euler, 1707—1783, whose mathematical writings more than filled forty folio volumes.

THEOREM VII.

400. *The area of a triangle having two sides given is a maximum when these sides form a right angle.*



Let BAC be a $\triangle$ in which $BA \perp$ . . . and $AB'C$. . . .
Then $\triangle\ ABC >$ . . .

1. For constr. $B'D \perp$ . .
2. $B'A >$ . . Why? §60
3. $\therefore BD = B'D.$
4. $\therefore BAC >$ . . Why? 205: 5

Q. E. D.

401. DEF. ***Isoperimetric*** figures are such as have equal perimeters.

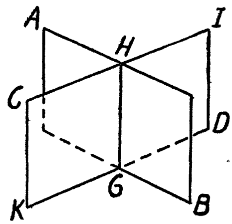
Ex. 764. Find the area and third side of the maximum triangle, two of whose sides are 5 and 12.

Ex. 765. Find the maximum area of a triangle, the sum of two of whose sides is 20; also the length of each side. (Apply §§400 and 399.)

Ex. 766. The rectangle of a given area whose perimeter is a minimum is a square.

Ex. 767. Find the sides of the quadrilateral whose area is 100 having a minimum perimeter.

Ex. 768. Prove as a limiting case of ex. 763 that the sum of the squares on the sides of a parallelogram is equivalent to the sum of the squares on the diagonals.



Ex. 769. Between two lines not coplanar one perpendicular can be drawn, and but one. Let AH and $KD \equiv$ . . . . . Through H, any point in AH, pass | CI $\parallel$ KD. Through KD pass a plane $\perp$ the plane of AH and CH which will cut AH in some point as L* (Why?). From L let fall $LM \perp KD$. Prove that LM is also $\perp$ AH, and that no other | can be $\perp$ to AH and KD.

* L, M, and the line LM are not shown in the cut.

Theorem VIII.

402. *1. Of all triangles having a given base and area the isosceles triangle has the minimum perimeter.*

2. Of all isoperimetric triangles having a given base the isosceles triangle has the maximum area.

3. The maximum isoperimetric polygon of a given number of sides must be equilateral.

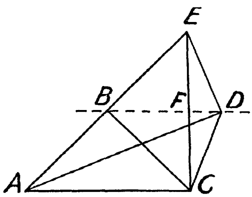


Fig. 1.

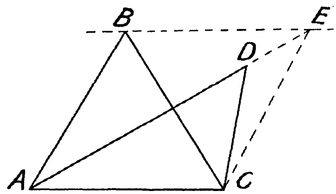


Fig. 2.

I. (Fig. 1.) Let ABC be an isosceles $\triangle$ and . . . ., then $AB + BC < AD + DC$.

1. For prod. AB making $BE = AB$; join . ., . ., . ., . .

Complete the proof showing that $BD \parallel AC$, $EC \perp AC$, that $CD = ED$, etc.

II. (Fig. 2.) Let . . . . .

If E is the intersection of AD produced (or if possible AD) with $BE \parallel AC$, prove that D falls between A and E (see §57 proof); hence that the altitude and area of $\triangle ADC <$ . . .

III. For, if any two sides of a polygon are unequal, its area may be increased by making these sides equal without making any change in their sum or in any other part of the figure. Let the pupil illustrate this by a figure.

Ex. 770. Find the area of the greatest triangle whose perimeter is 100.

Ex. 771. Find the minimum perimeter of a triangle whose base is 6 m. and area 12 sq. m. Ans. 16 m.

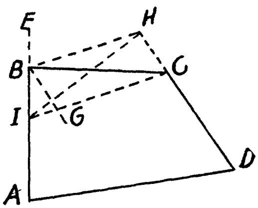
SUPPLEMENTARY THEOREM AND EXERCISES.

Theorem IX.

403. *1. Of two isoperimetric polygons of the same number of sides the regular polygon is the maximum.*

2. Of regular isoperimetric polygons the one which has the greater number of sides is the greater.

3. The circle is the maximum isoperimetric figure.



I. Let $P \equiv$ a maximum equilateral polygon of any number of |s, then P is equiangular and regular.

1. For, if possible, let two adjacent $\angle$s, B and C, be unequal. Let $ABCD \equiv$ the part cut off by a diagonal joining the vertices of the $\angle$s A and D adjacent to B and C. Through B, the vertex of the smaller $\angle$, construct $BG \parallel CD$, produce AB to E; bisect $\angle GBE$ by BH meeting DC produced in H; construct $CI \parallel BH$ . . . . .

2, 3, 4, 5, 6. Prove $BI = HC$. §§67: 1, 2, 46: 1, 87: 2, 17

7. $\triangle BHI \bumpeq \triangle BHC$. Why?

8. $ABCD \bumpeq AIHD$.

Subtracting equals above from $ABHD$.

9. Also $AI + IB = AB$.

10. $DH - CH = DC$.

11. $AI + DH = AB + DC$. Why?

$AIHD$ and $ABCD$ are both isoperimetric and equivalent, and P' is isoperimetric and equivalent to a polygon not equilateral, which is absurd. Why? §402: 3

12. $\therefore \angle B = \angle C$, and P is equiangular and regular.

II. Let $P \equiv$ a regular polygon of n |s, $Q \equiv$ a regular polygon of $n - 1$ |s; $P > Q$.

1. For any | of Q may be considered as 2 |s connected by a straight $\angle$, and Q an irregular polygon of n |s.

2. $\therefore P > Q$. Why? §403: 1

3. $\therefore$ Much more then $Q >$ a regular polygon of $n - 2$ |s, or a smaller number of sides.

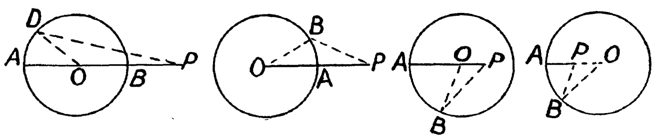
III. Apply §173.

Ex. 772. Find the minimum distance: a) from a point to a line, §60: 1; b) ditto to a plane, §262: 1; c) from the center to a chord through a given point in a circle, §§125 and 60; d) ditto to a circle through a given point in a sphere, §§363 and 262.

Ex. 773. Find the maximum chord in a circle.

Ex. 774. Find the area of the maximum triangle inscribed in a semicircle whose radius is 10.

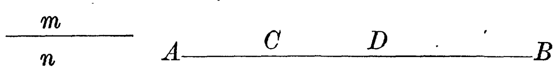
Ex. 775. Find the area of the largest field that can be inclosed by a fence 62,832 rods long.



Ex. 776. Find the maximum and minimum distances of a point from a given circumference: a) the point without the circle; b) the point within the circle.

III. CONSTRUCTION OF ALGEBRAIC EXPRESSIONS.

404. In constructing algebraic expressions the minus sign (—) signifies a direction opposite to that expressed by the plus sign (+). Usually + is to the right, or upwards. Construct $m + n$.



On the indefinite line AB lay off $AC = m$, and $DC = n$; then $AD = m + n$. Let the pupil construct $m - n$.

Ex. 777. Construct $x = \sqrt{a^2 + b^2}$. See §219: 1.

Ex. 778. Construct $z = \sqrt{a^2 + b^2 + c^2}$. Construct $a^2 + b^2 = v^2$; then $\sqrt{v^2 + c^2}$.

Ex. 779. Construct $y = \sqrt{a^2 - b^2}$, where $a > b$. See §219: 2.

Ex. 780. Construct $x = \frac{ab}{c}$; also $x = \frac{a^2}{b}$. See §183.

Ex. 781. Construct $x^2 = bc$. See §224: 1.

Ex. 782. Construct $x = abc^2 \div deg$. First construct $v = \frac{c^2}{g}$, then $y = bv \div e$, etc.

405. To construct the sum or difference of surfaces they must in general either be reduced to similar figures or to rectangles, parallelograms, or triangles, having equal altitudes or bases. Unless they are similar figures, it is usually most convenient to reduce them to squares.

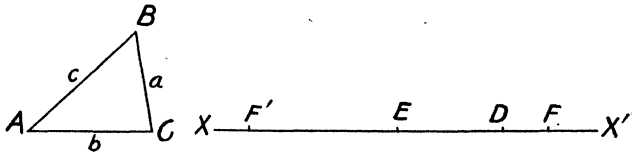
406. As we can only add and subtract like magnitudes, we must make algebraic expressions homogeneous to construct them, which is usually done by a linear unit factor which we will represent by u. $a + bc + cd^2 = au + bc + cd^2 \div u = a + bc \div u + cd^2 \div u^2$ when $u = 1$, the first consisting of areas, the latter of lines.

Ex. 783. Construct $z^2 = m^2 + n^2$. See §219: 1.

Ex. 784. Make the following expressions homogeneous by the use of the unit factor u, viz.: $a + b^2$, $a + bc + d^3$, $x + y3 + z^5$.

Ex. 785. Construct the four forms of the two roots of the equation, $x^2 \pm 2ax = \pm b^2$.

I. $x = -a \pm \sqrt{a^2 + b^2}$.



Construct $AB = \sqrt{a^2 + b^2}$. From D, any point in the indefinite line XX', lay off $DE = -a$ to the left. From E lay off $EF = AB$ to the right, and $EF' = -AB$ to the left. DF and $DF' \equiv$ the values of x.

II. $x = + a \pm \sqrt{a^2 + b^2}$. The pupil will construct this.

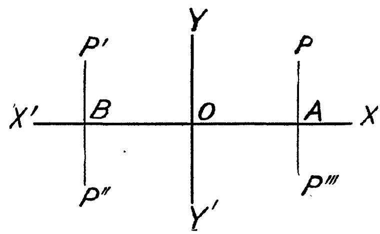
III. $x = - a \pm \sqrt{a^2 - b^2}$. Apply §219: 2, and complete as in I.

IV. $x = + a \pm \sqrt{a^2 - b^2}$.

The pupil will illustrate imaginary roots by a construction showing that $\sqrt{a^2 - b^2}$ is impossible when $b > a$.

IV. Loci of Equations.

407. The same equation may represent various loci according to the system used. The simplest is that of orthogonal coordinates, also called Cartesian coordinates, from its inventor, René Descartes.*



408. Def. Two lines called ***axes*** intersect at right angles at a point called the ***origin***. XX', usually horizontal, is called the ***axis of abscissas*** or axis of X. YY' is called the ***axis of ordinates*** or ***axis*** of Y.

409. Def. The position of a point P is determined by two ***coordinates:*** its ***ordinate*** y, the distance from the axis of X; and its ***abscissa*** x, the distance from its projection on the X axis to the origin. The pupil will specify the coordinates of P, P', P'', and P'''.

410. Def. When the point is to the right of the Y, axis x is +, when to the left, —; when above the X, axis y is +, when below, —. The pupil will specify the signs of x and y for the points P, P', P'', and P'''.

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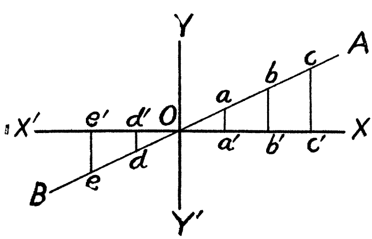
* Rene Descartes, 1596—1650, the greatest mathematician and philosopher of his age, and one of the most distinguished of any age, was born at La Haye, Touraine, France, wrote chiefly in Holland, and died in Sweden, where he went by invitation of Queen Christina.

411. DEF. The axes divide the angular magnitude about O into four ***quadrants***; the ***first quadrant*** containing P; the ***second***, P'; the ***third***, P''; the ***fourth***, P'''.

412. DEF. The ***equations of a point*** are $x = a$, $y = b$; if $x = 2$ and $y = 3$, the point is named 2, 3. To construct this point we measure to the right 2, upwards 3; if 2 is —, we measure to the left, etc.*

Ex. 786. Construct the points: 3, 4; —3, 4; —3, —4; 3, —4.

413. DEF. In the equations of a point, x and y can have only one value. In all other equations of loci, x and y are variables, having an indefinite number of values. In the equation $y = \frac{1}{2} x$, find the values of y corresponding to: $x = 0$, $x = 2$, $x = 4$, $x = 6$, $x = -2$, $x = -4$, and draw a line through these points, thus constructing the equation.



414. Known points may be represented by x', y'; x'', y''; etc. If $Oa' = x'$, $aa' = y'$; if $Oc' = x'''$, $cc' = y'''$; etc.

415. DEF. Prove that $aa' : Oa = bb' : Ob' = cc' : Oc'$, etc., thus proving that, when a line passes through the origin, the ratio of the ordinate to the abscissa is constant. This ratio is called the ***tangent*** ‡ of the angle XOA, and is *positive* or *negative* as y and x have like or unlike signs.

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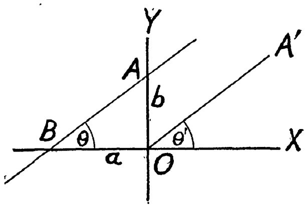
* Quadrille ruled paper with every fifth line heavy is the most convenient in constructing loci

‡ Similarly *sin* $\theta = y : d$; *cos* $\theta = x : d$; *cot* $\theta = x : y$; *sec* $\theta = d : x$; *csc* $\theta = d : x$; in which d is the distance from the origin to the point of which x and y are coordinates, and always positive, the sign of the function in every quadrant being determined by the signs of x and y.



416. Def. $\theta \equiv \angle XBA$; $m \equiv$ the tangent of θ; $a \equiv OB$ and $b \equiv OA$, the intercepts on the axes of X and Y respectively. $m \equiv b : a$, in this case positive since a is measured to the right, and b upwards, from B, the vertex of the angle. By §67 : 2, θ and m are the same for all parallel lines.

Ex. 787. Find m if $a = 6$ and $b = 3$; if $a = 2$ and $b = 4$; if $a = 9$, $b = 3$; if $a = 6$; $b = 9$; if $a = -2$, $b = -4$; if $a = 2$, $b = -6$; if $a = -3$, $b = 6$.

Theorem X.

417. *The equation of a straight line:*

1. through the origin is, a) $y = \dfrac{y'}{x'}\,x$; b) $y = mx$;

2. when m *and* b *are given,* $y + mx = b$;

3. through a given point, $y - y' = m\,(x - x')$;

4. through two given points, $y - y' = \dfrac{y'' - y'}{x'' - x'}\,(x - x')$.

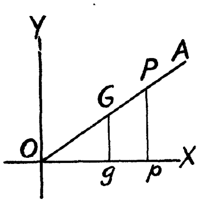


Fig. 1.

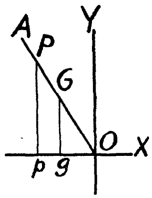


Fig. 2.

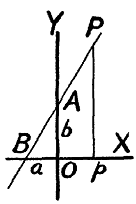


Fig. 3.

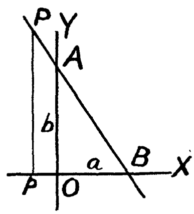


Fig. 4.

I. (Figures 1 or 2). Let $AO \equiv$ any | . . . ; then . . .

1. For $\triangle OGg \sim \triangle$ . . . Why?

2. $\therefore x' : y' = x : y$ $\therefore y = \dfrac{y'}{x'}\,x$. Why?

4. Also, since $y' : x' = m$, $y = mx$. Q. E. D.

II. (Figure 3 or 4). Let $AB \equiv \ldots$ and ..., then ...

1. For $\triangle \ldots \sim \triangle \ldots$ Why?
2. $\ldots : \ldots = \ldots : \ldots$ Why?
3. $a : b = x + a : y \therefore y = (b : a)\, x + b$.
4. But $b : a \equiv m$ (§416) $\therefore y = mx + b$.

III. Since the line passes through the point x', y', we have two equations containing b and m, viz: 1) $y = mx + b$; 2) $y' = mx' + b$. Eliminate b by subtraction.

IV. Since the points x', y', and x'', y'' are on the line, we have three simultaneous equations containing m and b, viz.: 1) $y = mx + b$; 2) $y' = mx' + b$; 3) $y'' = mx'' + b$.

Prove by equations 2) and 3) that $m = (y'' - y') \div (x'' - x')$, and substitute this value for m in the equation $y - y' = m\,(x - x')$, obtained as before.

Ex. 788. Prove that m is positive in the acute angle XBA (fig. 3), and negative in the obtuse angle XBA (fig. 4).

Ex. 789. Write the equations of lines passing through the following points: 1) 3, 6; 8, 4; 2) $-6, -8; +1, -1$; 3) $2, -4$; $-1, 3$.

Ex. 790. What is the value of m in the following equations: $y = 2x$; $y = -3x$; $y = -\frac{1}{3}x$; $3x + 4y = 0$, i.e., $y = -\frac{3x}{4}$; $2x + 5y = 0$; $4y - 16x = 0$; $6x - 3y = 0$?

Ex. 791. What are the values of m and b in the equations: $y = 2x + 5$; $y = -3x + 4$; $y = \frac{1}{3}x - 7$; $y = -\frac{1}{2}x - 8$; $2y + x = 10$ $x + y = 4$; $y - x = 6$; $x - y = 11$?

Ex. 792. Write the equations of lines in which: 1) $m = 2, b = 5$; 2) $m = -3, b = 6$; 3) $m = 2, b = 0$; 4) $m = 0, b = -4$.

Ex. 793. Compute mentally the value of m in the equations of lines passing through the following pairs of points: 1) 0, 0; 4, 4; 2) 0, 2; 4, 6; 3) 0, 2; $-4, -6$; 4) 0, -2; -4, 6; 5) $-10, -12$; $-15, -22$; 6) 20, 30; 40, 100.

Ex. 794. Reduce to the form $y = mx + b$ the following equations: $3x + 6y = 24$ $x = 2y - 10$; $\frac{x}{3} - \frac{y}{8} = \frac{1}{2}$; $\frac{5}{x} + \frac{6}{y} = \frac{60}{xy}$; $\frac{1}{3y + 7} + \frac{4}{2x + 30} = 0$.

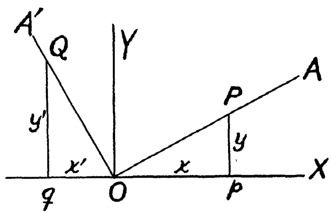
Ex. 795. Write the equations of the sides of a triangle, the vertices of whose angles are 0, 0; 0, 6; and 3, 4.

THEOREM XI.

418. *1. If two lines meet at right angles and m and m' represent the tangents of their angles with the axis of x, $m = -1 \div m'$.*

2. The equation of the line passing through the point x', y', perpendicular to the line $y = mx + b$, is: $y - y' = -\frac{1}{m}(x - x')$.

3. The equation of the line passing through the point x', y', parallel to the line $y = mx + b$, is: $y - y' = m(x - x')$.



I. Since, by §416, m is the same for all || |s, let $m \equiv$ tang. XOA and $m' \equiv$ tang. XOA' ⊥ each other.

Prove by ∼ △s, applying also §§416 and 410, that $x : y = y' : -x'$, etc.

II. Apply §§417: 3 and 418: 1.

III. Apply §§417: 3 and 416.

Ex. 796. Write the equations of lines through the origin perpendicular to the lines in which $m = 3; \frac{1}{2}; \frac{1}{3}; \frac{1}{4}; 2; -5$.

Ex. 797. Write the equations of lines through the origin parallel to the lines: $y = 3x - 10$; $y = -\frac{1}{2}x + 16$; $2y - 3x = 4$; $3x - 5y = -15$. Also write the equations of lines through the point 7, 12, perpendicular to these lines.

Ex. 798. Write the equations of lines parallel to the line $y = 2x + 20$, and passing through the points 3, 6; —4, 8; 5, —10; 12, —20; 6, 0.

Ex. 799. Find the value of m if a line passes through the points 3, 5 and 6, 12. See proof of §415: 4.

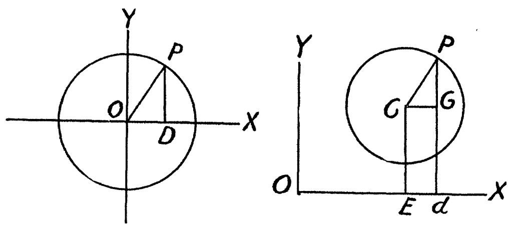
Ex. 800. Write the equations of lines passing through the origin perpendicular to each of the following lines: $y = x$; $2x + 3$; $3y = 6x + 7$; $x = 2y - 10$; $5x + 7y = 20$; $\frac{x}{2} + \frac{y}{3} = 1$; $\frac{2}{x + 3} - \frac{3}{2y + 7} = 0$.

SUPPLEMENTARY THEOREM AND EXERCISES.

THEOREM XII.

419. *1. The equation of a circle whose center is at the origin and whose radius is r, is:* $x^2 + y^2 = r^2$.

2. The equation of the circle whose center is at the point a, b, is: $(x-a)^2 + (y-b)^2 = r^2$.



I. Let $P \equiv$ any pt. in $\odot$ O, then . . .

1. For $Od = x$, $DP = y$, and $OP = r$.

2. $\therefore x^2 + y^2 = r^2$. Why? §216

II. Let $C \equiv$ . . . ., $O'e = a$, $Ce = b$; then . . . .

1. For $CG = x - a$. Why?

2. $PG = y - b$. Why?

3. $(x-a)^2 + (y-b)^2 = r^2$. Why? §216

Q. E. D.

420. Let the pupil construct the curve $x^2 - y^2 = \pm 24$. Let him then construct the curve $xy = 12$ on axes having same origin but bisecting the angles made by the first axes. He will find that the two loci exactly coincide, also that each curve has four separate parts called *branches*. The discussion of the remarkable changes in the form of the equation of a locus by changing the direction of the axes and the position of the origin is a most interesting part of analytic geometry.

Ex. 801. Find the points at which the circles $x^2 + y^2 = 25$ and $x^2 + y^2 - 6x - 4y = 12$ cut the axes.

Ex. 802. Find the maximum and minimum values of x and y in the equation of the circle $x^2 + y^2 = 100$. Ans. to minimum values: $x = -5$, $y = -5$.

421. If two loci have a common point, x and y must have the same values in both loci. The lines $y = 3x$ and $y = 2x + 5$ meet at the point $x = 5$, $y = 15$, since these values of x and y are true in both equations. They are found by solving the two equations considered as simultaneous.

Ex. 803. At what point do the lines $y + 2x = 11$ and $3y + 4x = 25$ intersect? Ans. 4, 3.

Ex. 804. Is the line $4x - 3y = 1$ concurrent with the two lines of ex. 803?

422. Since, at the axis of X, $y = 0$, the line $3x + 5y = 30$ cuts the axis of X at the point where $3x = 30$, $x = 10$, and $y = 0$, and cuts the axis of Y at the point where $x = 0$, $5y = 30$, and $y = 6$.

Ex. 805. Find the points where the line $2x + 3y = 18$ cuts the axes. What are the intercepts of this line on the axes? See §416.

Ex. 806. Write the equations of lines passing through the origin perpendicular to each of the lines specified in ex. 800.

Ex. 807. Find the distance from the origin to the point 3, 4.

Ex. 808. Find the points of intersection of the circle $x^2 + y^2 = 25$ with the straight line $3x + 4y = 12$; also the intersection of the above circle and straight line with each of the following curves: $y^2 = 10x$, $4x^2 + 36y^2 = 144$, $x^2 - y^2 = 16$.

In the four next exercises give any convenient value to a and c.

Ex. 809. Construct the cissoid of Diocles: $y^2 = x^3 \div (2a - x)$*.

Ex. 810. Construct the conchoid of Nicomedes: $x^2 = (y + c)^2 (a^2 - y^2) \div y^2$*.

Ex. 811. Construct the witch of Agnesi: $y^2 = 4a^2x \div (2a - x)$.†

Ex. 812. Construct the semi-cubical parabola or Neilian: $y = a\sqrt{x^3}$‡.

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* These curves, which may be mechanically constructed, were invented by the Greek mathematicians whose names they bear, who solved by their aid the celebrated problems of trisecting an angle and duplicating the cube, probably in the second century before Christ.

† Invented by Donna Maria Gaetana Agnesi, 1718—1799, an Italian lady, professor of mathematics in the University of Bologna.

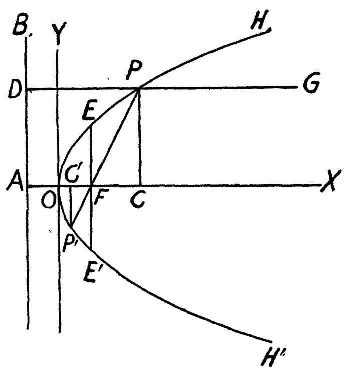
‡ This curve was rectified, i. e., the length of an arc found in linear units, by Wm. Neill, a pupil of Wallis, in 1657. It was the first curve rectified.

BOOK IX.

THE CONIC SECTIONS.*

423. DEF. The ***parabola*** is the locus of a point which is the same distance from a fixed point, called the ***focus,*** that it is from a fixed line, called the ***directrix.*** The point O nearest the directrix is called the ***vertex*** and is evidently on the perpendicular from F to AB, the directrix. All lines parallel to AF are called ***diameters.*** AF extended is called the ***principal diameter*** or ***axis.*** A chord through the focus PFP' is called a ***focal chord.*** EE', the double ordinate† to the axis through the focus, is called the ***latus rectum.*** $p \equiv OA = OF$. Why?

424. *The equation of the parabola is:* $y^2 = 4px$.



1. For $y^2 = \overline{PC}^2$.

2) $\overline{PC}^2 = \overline{PF}^2 - \overline{FC}^2$.

3) But $PF = PD = CA = OC + OA = x + p$.

4) $FC = OC - OF = x - p$.

5) $\overline{PF}^2 = \ldots$ Why?

6) $\overline{FC}^2 = \ldots\ldots$

7) $\therefore\ y^2 = 4px$. Why?

Q. E. D.

Ex. 813. Find p in the parabola $y^2 = 20x$. Ans. 5.

Ex. 814. Find the equation of the parabola in which $p = 2$. Ans. $y = 8x$.

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* The conic sections are so called since they are sections of a cone made by a plane as explained in works on conic sections.

† See §142.

425. $y = \pm\ 2\sqrt{px}$, hence for every positive value of y there is an equal negative value, and the curve is symmetrical with reference to the axis of X, and every ordinate is bisected by the axis. (See §389.)

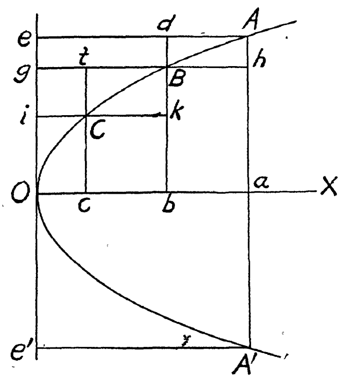
426. When $x = OF = p$, $y = EF$. $y^2 = 4px$ when $x = p$ becomes $y^2 = 4p^2$ $\therefore$ $y = \pm\ 2p$, and the latus rectum $= 4p$.

Ex. 815. What is the latus rectum of each of the following parabolas: $y^2 = 8x$; $y^2 = 12x$; $y^2 = 16x$; $y^2 = 20x$?

Ex. 816. Construct the parabola: $y^2 = 12x$. First find the values of x corresponding to $y = 0$; $y = +2$; $y = +4$; $y = +6$; $y = +8$; $y = +10$; $y = +12$; $y = +14$; and constr. these points.

Ex. 817. $(y')^2 : (y'')^2 = x' : x''$, that is, the squares of ordinates are proportional to the corresponding abscissas.

427. *The area included between the parabola and a double ordinate to the axis is two thirds of the circumscribed rectangle. Formula: Area $= 2xy \div 3$.*



Let . . . .

For let $A \equiv x'', y''$, $B \equiv x', y'$, etc., be points . . .

1. $\square\, bh \doteq y'\,(x'' - x')$, and $\square\, gd \doteq$ . . . Why?

2. $\therefore\ \square\, gd : \square\, bh = (y'' - y')\, x' : (x'' - x)\, y'$.

3. Since x'', y'', and x', y' are on the curve, $(y'')^2 = 4px''$, and $(y')^2 = 4px'$.

4. By subtraction and factoring $(y''-y')\,(y''+y') = 4p\,(x''-x')$ $\therefore$ $(y''-y') \div (x''-x') = 4p \div (y''+y')$.

5. As $y'' \doteq y'$, $4p \div (y'' + y') \doteq 2p \div y'$.

6. $\therefore$ $(y'' - y') \div (x'' - x') \doteq 2p \div y'$.

7. $(y'' - y')\,x' : (x'' - x')\,y' \doteq 2px' : (y')^2 = 2px' : 4px' = 1 : 2.$

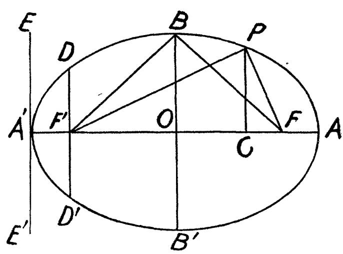
8. Now the sum of the inscribed ▭s differs from the part of the circumscribed rectangle within the parabola less than the sum of ▭s BA, CB, etc. The sum of the bases of these last ▭s is OA, and, as the number of points is indefinitely increased, their altitudes $\doteq O$.

9. $\therefore$ Sum ▭s bh, ck, etc., $\doteq$ the area of the parabola.

Complete the proof.

428. Def. An ***ellipse*** is the locus of a point the sum of the distances of which from two fixed points, called ***foci***, is constant.

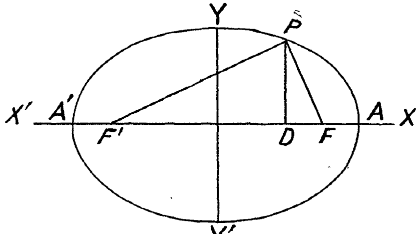
Ex. 818. Construct an ellipse. Fasten a pin to each end of a short piece of thread. Stick the pins through paper into a smooth board so that the thread will not be streched. Keep the ends of the thread down to the paper, and draw the curve with a pencil keeping the thread stretched by the point of the pencil.



429. Def. The distance from any point on the curve to either focus is called a ***radius vector***. By §428 the sum of the radii vectores is constant. This sum is represented by $2a$. The point O, half way between the foci, is called the ***center*** of the ellipse. Every line through the center is called a ***diameter***. The diameter AA' is called the ***major axis***, and the diameter BB' is called the ***minor axis.***

430. OB is represented by b, OF by c, FP by r, $F'P$ by r'. Let the pupil prove by §§428, 429, and 76 that $FB = F'B = OA = OA' = a$; also by §206 that $b^2 = a^2 - c^2$.

431. Find the equation of the ellipse, the major axis being the axis of x, and the center, the origin.



Let $P \equiv$ any point on the curve.

$FD = c - x$.

$F'D = c + x$.

1. $r^2 = y^2 + (c - x)^2$. Why?
2. $(r')^2 = y^2 + (c + x)^2$.
3. $(r')^2 - (r')^2 = 4cx$. Why?
4. $r' + r = 2a$. Why? §429
5. $r' - r = 2cx \div a$. By division.
6. $r = a - cx \div a$. Why?
7. $y^2 + c^2 - 2cx + x^2 = a^2 - 2cx + \frac{c^2x^2}{a^2}$. See 1 and 6.
8. $a^2 y^2 + (a^2 - c^2)\, x^2 = a^2\,(a^2 - c^2)$. Why?
9. $a^2 y^2 + b^2 x^2 = a^2 b^2$. Why? §430
10. Also $y^2 = \frac{b^2}{a^2}\,(a^2 - x^2)$.

Q. E. F.

432. Since $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$: 1) x cannot $> a$ (Why?), hence the curve lies between $+a$ and $-a$. 2) If $x = a$, $y = 0$, and y can be > 0 or < 0, only when $x < a$; hence the curve meets the lines perpendicular to the axis of X at A and A' in these points, and only in these points, and these perpendiculars are tangent to the curve. 3) Similarly the tangents at B and B' are perpendicular to the minor axis. 4) The ordinates to each axis are perpendicular to that axis. Why? §142.

433.* Since $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$ and $x = \pm\frac{a}{b}\sqrt{b^2 - x^2}$, the curve is symmetrical with reference to each axis, and:

* §§433 and 434 require a knowledge of §§384—391 and 393 in symmetry.

1. Each axis bisects all ordinates to it.
2. Tangents at the extremities of an ordinate to either axis meet this axis at the same point.
3. Normals at the extremities of an ordinate to either axis are also concurrent with the axis.

434. Since the curve is symmetrical with reference to two lines perpendicular to each other, it is symmetrical with reference to their intersection as a center, and:

1. Every diameter is bisected at the center.
2. Every diameter bisects the ellipse and its surface.
3. Tangents at the extremities of a diameter are parallel to each other.

Ex. 819. In the ellipse $25y^2 + 16x^2 = 400$, find y when $x = 3$, and when $x = 4$.

435. Def. The ***latus rectum*** is the double ordinate to the major axis at either focus, and is represented by $2l$.

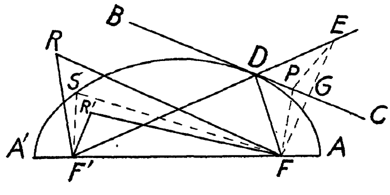
436. The latus rectum is a third proportional to the axes. Substitute c for x, and l for y, in the equation $y^2 = \frac{b^2}{a}(a^2 - x^2)$, and reduce, showing that $a : b = b : l$.

Ex. 820. Find the latus rectum of the ellipse of ex. 819.

437. *The squares of any two ordinates to the major axis of an ellipse have the same ratio as the products of the segments into which they divide it.*

1. $(y')^2$ . . . . §431 eq. 10
2. $(y'')^2$. . . . . . . §431 eq. 10
3. $(y')^2 : (y'')^2 = a^2 - (x'^2) : a^2 - (x'')^2 = (a + x')(a - x') : (a + x'')(a - x'')$. Why? §17: 2

Show that $a + x'$ and $a - x'$ are the segments into which y' divides the axis, and complete the proof.



438. *If a point is without an ellipse, the sum of its distances from the foci is greater than 2a; if it is within the ellipse, the sum of these distances is less than 2a; and conversely.*

Let R and $R' \equiv$ the points, and apply §57. The converse may be proved by the method of §54: 2.

439. *The bisector of the exterior angle made by the radii vectores at any point on an ellipse is a tangent to the curve.*

Fig. of §438. Let BC bisect EDF, the . . . .

1. For produce $F'D$ making $DE = DF$, and join EF.
2. $DG \perp EF$, and $FG = GE$. Why? §45: 2
3. Let $P \equiv$ any point in BC other than D; join . . . . .
4. $PE = PF$. Why?
5. $F'P + PE = F'P + PF$. Why?

Complete the proof.

440. From §439 it follows that $\angle CDF = \angle BDF'$. If a room were in the form of a *prolate* spheroid made by revolving a semi-ellipse on its major axis and its walls good reflectors of sound, a whisper in one focus would be plainly heard at the other, since, in the case of sound, the angles of incidence and reflection are equal. Let the pupil illustrate this property of the ellipse by heat and light, the law of reflection being the same.

441. Def. The circles whose diameters are the major and minor axes of the ellipse respectively are called the ***major*** and ***minor auxiliary circles.*** Let the pupil illustrate this definition by a figure.

Ex. 821. The squares of ordinates to the minor axis of an ellipse are proportional to the products of the segments into which they divide it.

442. *An ordinate to the major axis of an ellipse is to the corresponding ordinate to the major auxiliary circle as $b : a$.*

Let y and $y' \equiv$ corresponding ordinates to an ellipse and its major auxiliary circle.

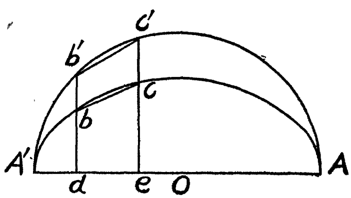
1. Since a circle is an ellipse in which $a = b = r$, $(y')^2 = a^2 - x^2$.
2. $y^2 = \ldots\ldots$ Why? §428:10
3. $\therefore y^2 : (y')^2 = b^2 : a^2$. Why?
4. $y : y' = a : b$. Why? Q. E. D.

Let the pupil illustrate this by a figure.

Ex. 822. An ordinate to the minor axis of an ellipse is to the corresponding ordinate to the minor auxiliary circle as $a : b$.

443. *The area of the ellipse is to the area of the major auxiliary circle as $b : a$.*

Let . . . .



1. For let $bc \equiv |$ of a polygon inscribed in the ellipse O; construct the ordinates bd and ce; extend them to meet $\odot O$ in b' and c', join bc and $b'c'$. Let $y' \equiv bd$; $y'' = ce$; $Y' \equiv b'd$, $Y'' \equiv c'e$. Prove that trapezoid dc: trap. $dc' = y' + y'' : Y' + Y'' = b : a$; that the inscribed polygons $\doteq$ the ellipse and $\odot$ respectively, and $\therefore$ that the area of the ellipse O : area of $\odot$ $O = b : a$, applying §§210, 155, and 442.

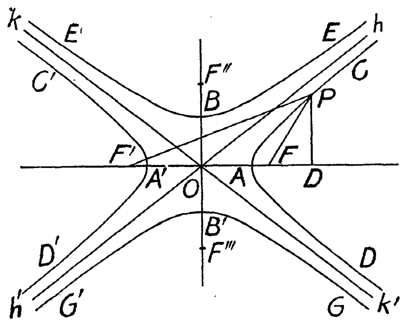
444. *The area of the ellipse is πab.* Why? §442

Ex. 823. Find the total area of the ellipse $9x^2 + 16y^2 = 144$.

445. Def. The ***eccentricity*** of an ellipse is the ratio of the distance between the foci to the major axis. Formula: $e = c : a = \sqrt{a^2 - b^2} : a$.

446. Def. The ***hyperbola*** is the locus of a point the difference of whose distances from two given points is constant.

447. The terms *foci*, *transverse axis*, *conjugate axis*, *diameter*, *radius vector*, *center*, etc., and the letters a, b, c, r, and r' are used as in the ellipse, but, by the above definition, $r' - r = 2a$ making the branch CAD, or $r - r' = 2a$, making the branch $C'A'D'$, and since $c > a$, we assume that $b^2 = c^2 - a^2$. The two branches EBE', $GB'E'$ constitute what is called the conjugate hyperbola in which F'' and F''' are the foci, and $r''' - r'' = 2b$ or $r'' - r''' = 2b$.



448. The equation of the hyperbola is: $b^2 x^2 - a^2 y^2 = a^2 b^2$. The proof differs from that of the equation of the ellipse only in the signs.

449. The equation of the conjugate hyperbola is similarly proved to be: $b^2 x^2 - a^2 y^2 = - a^2 b^2$.

450. The equation $b^2 x^2 - a^2 y^2 = \pm a^2 b^2$, by §§448 and 449, represents all four branches of the curve, and is called the complete hyperbola.

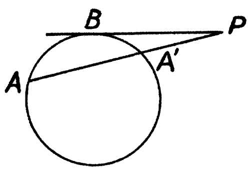
451. Since, by §§448, $y = \frac{b}{a} \sqrt{x^2 - a^2}$, no point in the hyperbola can lie between $x = + a$ and $x = - a$. Since, by §449, $x = \frac{a}{b} \sqrt{y^2 - b^2}$, no point of the conjugate hyperbola can lie between $y = + b$, and $y = - b$.

452. The pupil will prove that all the properties of the ellipse given in §§433 and 434 are properties of the hyperbola.

GENERAL SUPPLEMENT.

TANGENTS TO THE CONIC SECTIONS.

453. Let $x', y' \equiv$ the point A, and $x'', y'' \equiv$ the point A'. By §417 : 4 the equation of a straight line passing through these points is $y - y' = \dfrac{y'' - y'}{x'' - x'}(x - x')$.



By §170 the limit of a secant, as the points of intersection approach, is a tangent. Hence to find the equation of a tangent to any curve we need only to ascertain by the equation of the curve the limit of $y'' - y' : x'' - x'$ as $y'' \doteq y'$ and $x'' \doteq x'$, and substitute this value in the above equation.

454. In the parabola, by §427 eq. 6, as $y'' \doteq y'$ and $y'' \doteq y'$, $y'' - y' : x'' - x' \doteq 2p : y'$; $\therefore$ the equation of the tangent to the parabola is : $y - y' = \dfrac{2p}{y'}(x - x')$.

455. In the ellipse, by §431, the points x', y' and x'', y'' being on the curve, $b^2(x')^2 + a^2(y')^2 = a^2 b^2$ and $b^2(x'')^2 + a^2(y'')^2 = a^2 b^2$, hence by subtraction, factoring, and reducing: $y'' - y' : x'' - x' \doteq -(b^2 x' \div a^2 y')(x - x')$; $\therefore$ the equation of the tangent to the ellipse is :

$$y - y' = -\frac{b^2 x'}{a^2 y'}(x - x').$$

456. It may be shown in like manner that the equation of the tangent to the hyperbola is $y - y' = \frac{b^2 x'}{a^2 y'} (x - x')$.

457. Def. A ***normal*** to a curve is the line perpendicular to the tangent at the point of contact.

458. 1. The equation of the normal to the parabola is: $y - y' = -\frac{y'}{2p}(x - x')$. Why? §§418: 2, 454

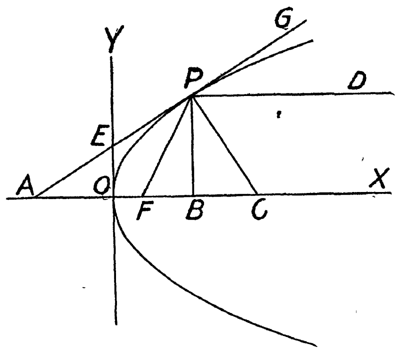
2. The equation of the normal to the ellipse is: $y - y' = \frac{a^2 y'}{b^2 x'}(x - x')$. Why? §§418: 2, 455

3. The equation of the normal to the hyperbola is: $y - y' = -\frac{a^2 y'}{b^2 x'}(x - x')$. Why? §§418: 2, 456

459. Clearing of fractions, transposing, and reducing, the equations of the tangents to the conic sections may be reduced to the following simpler forms:

1. Tangent to the parabola: $y' y = 2a (x + x')$.
2. Tangent to the ellipse: $a^2 y' y - b^2 x' x = a^2 b^2$.
3. Tangent to the hyperbola: $a^2 y' y - b^2 x' x = -a^2 b^2$.

460. Def. By the ***length of the tangent,*** or ***length of the normal,*** we understand that part of the line included between the point of contact and the axis of X. The ***subtangent*** and ***subnormal*** are the projections of the tangent and normal as above limited, on the axis of X.



Ex. 824. Find the equations of the tangent and normal to the parabola $y^2 = 12x$, at the points 3, 6 and 12, 12.

Ex. 825. Find the equations of the tangent and normal to the ellipse $9x^2 + 16y^2 = 144$, at $x' = 3$, $y' = \ldots$?; also the length of the latus rectum.

461. 1. Making $y = O$, in §459: 1, we find that $x = -x'$, and the length of the subtangent to the parabola is $2x'$.

2. In the rt$\triangle APB$, fig. of §460, $AP = \sqrt{PB^2 + AB^2}$; $\therefore$ the length of the tangent is $\sqrt{(y')^2 + 4(x')^2}$.

462. 1. Making $y = O$, in the formula of §458: 1, we find that $x = 2p + x'$, and hence that BC, the subnormal, is $2a$ or half the latus rectum for every point in the parabola.

2. The length of the normal is: $\sqrt{(y')^2 + 4a^2}$. Why?

463. Since $PF = x + p$, by §424, and $AF = x + p$, by §§423 and 409, the $\triangle APF$ is isosceles and $\angle APF = \angle PAF$. (Why?) Let PD be a diameter and therefore parallel to AX; $\angle GPD = \angle GAX$; $\angle GPD = \angle APF$; and the $\angle DPC = \angle CPF$; that is, the normal at any point of a parabola bisects the angle formed by the diameter and focal chord.

464. From §463 it is evident that a mirror, every section of which, passing through a common axis, is an equal parabola, reflects rays of light parallel to this axis to the focus.

Ex. 826. Find the points where the tangent and normal of the ellipse and hyperbola cut the axis of x.

Ex. 827. Find the lengths of the subtangent, tangent, subnormal and normal of the ellipse and hyperbola.

465. Def. From the equations $y = mx$ and $a^2y^2 - b^2x^2 = a^2b^2$, used as simultaneous, we find that $x = \pm\sqrt{a^2b^2 \div (a^2m^2 - b^2)}$. As a^2b^2 is constant, x can become indefinitely large, only when the denominator becomes indefinitely small, that is when $m \doteq b \div a$; that is, a line passing through the origin and making such an angle with the axis of X, that $m = \pm b : a$, meets the curve only at an infinite distance. Lines which a curve continually approaches but touches only at an infinite distance are called ***asymptotes***. Let the pupil point out the two asymptotes to the hyperbola in the figure of §447.

466. The equations of the tangent and normal are the same for the conjugate hyperbola as for the original, and the asymptotes are common, as may be shown by deducing them in the same way.

467. If $b = a$, the equation of the complete hyperbola becomes $y^2 - x^2 = \pm a^2$, differing from that of the circle, having the same center, only in the signs. Such an hyperbola is called an equilateral hyperbola. In many of its properties it remarkably resembles the circle.

468. DEF. The curve described by any point on the circumference of a wheel as it rolls on a flat surface is called a ***cycloid***, and the part included between two successive contacts of the point on the wheel with the surface on which it rolles is called an arch of the cycloid.

Ex. 828. Construct three arches of a cycloid on paper, using a spool, pencil, and straight edge.

469. It is proved in works on calculus that one arch of a cycloid is 4 times the diameter of the generating circle; that is, while the wheel makes one revolution, a point on its circumference moves four times the diameter of the wheel. Hence in any exact number of revolutions, a point on the circumference moves $4 \div \pi$ times the distance over which the wheel rolls. Formula: Length arch $= 8r$.

Ex. 829. Find the distance which a point on the circumference of a wheel of a railroad car moves while the care moves 100 miles. Ans. 127.324 miles.

470. If an arch of a cycloid be constructed so as to be convex downwards, a ball impelled by gravity would roll from any point to the lowest point in the same time as from the highest, not considering friction, also a body will roll or slide from one point to another quicker on the arc of a cycloid joining the points than on any other curve.

471. Def. The *catenary* is the curve representing the position of a rope, supposed to be perfectly flexible, suspended between two supports. The computation of the length involves the use of exponential equations.

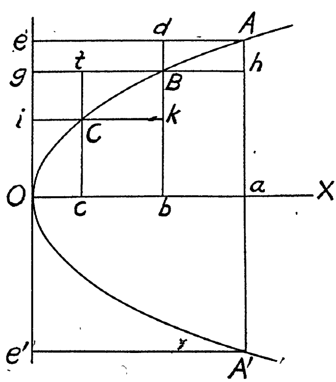
472. Def. Three or more lines passing through a common point called the ***ray center,*** taken together, are called a ***pencil of rays,*** and each line is called a ***ray.*** Two pairs of points are said to be radially situated when their distances from the ray center are proportional. Thus, in the figure of §137, the two pairs of points n, h; p, P; etc., are radially situated with reference to the center O.

473. Two figures are *radially situated* with reference to a center, when all their points are radially situated with reference to that center. Thus the two polygons of the figure of §137 are radially situated with reference to the center O. The circles O and P in the figure of ex. 248 are radially situated with reference to the intersections of both pairs of common tangents. In the figure of §326, four triangles are radially situated with reference to the center X.

474. Since two triangles having an angle in common and the sides about it proportional are similar, all homologous lines of figures radially situated are proportional, and the figures are similar, whether plane or solid. Conversely it may be similarly shown that all similar solids may be so placed as to be radially situated. If, in two similar plane figures, three pairs of homologous points, not colinear, are radially situated, the figures are radially situated. If, in two similar solids, four pairs of homologous points, not coplanar, and no three colinear, are radially situated, the solids are radially situated.

475. Def. A ***paraboloid*** is the solid generated by the revolution of the part of the surface of the parabola, included between an ordinate to the axis and the curve, about the axis.

476. *The volume of the paraboloid is half the volume of the circumscribed cylinder.* *Formula:* $V \bumpeq \frac{1}{2}\pi y^2 \mathrm{x}$.



Let the parabola OA revolve . . . . .

1. For let $A \equiv \mathrm{x}'', y''$

. . . .

2. Let solid $gd \equiv$ the solid generated by $\square\, gd$ as the parabola revolves about its axis, etc.

3. Solid $gd \bumpeq \pi\{(y'')^2 - (y')^2\}\ \mathrm{x}'$ and solid $bh \bumpeq \pi(\mathrm{x}'' - \mathrm{x}')\ (y')^2$. Why?

4. $\therefore$ Solid gd : solid $bh = \ldots = (y'' - y')\ (y'' + y')\ \mathrm{x}' : (\mathrm{x}'' - \mathrm{x}')\ (y')^2$.

5. But as $\mathrm{x}'' \doteq \mathrm{x}'$ and $y'' \doteq y'$, $y'' - y' : \mathrm{x}'' - \mathrm{x}' \doteq 2p : y'$. Why? §427: 6

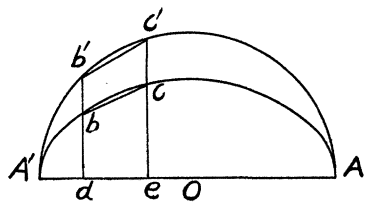
6. $\therefore$ Limit solid gd : solid bh = limit $2p\ (y'' + y')\ \mathrm{x}' : (y')^3 = 4p\mathrm{x}' : (y')^2 = 1 : 1$.

Complete the proof.

Ex. 830. Find the volume of the paraboloid in which $x = 8$, and $y = 10$.

477. DEF. The solid generated by an ellipse revolving about its major axis is called a ***prolate spheroid***, and the solid generated by an ellipse revolving about its minor axis is called an ***oblate spheroid.***

478. *The volume of a prolate spheroid is to the volume of the sphere whose diameter is its major axis as* $b^2 : a^2$.



Let $\odot O$ have the major axis . . . . .

1. For let bc* . . . .

2. Let vol. dc and vol. $dc' \equiv$ the volumes of the frustums of cones generated by the trapezoids dc and dc' as the figures revolve about the axis $A'A$.

3. Vol. $dc \stackrel{\frown}{=} \frac{1}{3}de \times \pi \left\{(y')^2 + (y'')^2 + y'y''\right\}$ and vol. $dc' \stackrel{\frown}{=}$ . . . Why?

4. $\therefore$ Vol. dc : vol. $dc' = (y')^2 + (y'')^2 + y'y'' : (Y')^2 + (Y'')^2 + Y'Y''$.

5. $\therefore$ As $y'' \doteq y'$ and $Y'' \doteq Y'$, vol. dc : vol. $dc' \doteq (y')^2 : (Y')^2 = b^2 : a^2$. Why? §439

6. As the number of |s increases indefinitely, the inscr. polygons $\doteq$ the ellipse and $\bigcirc$, as limits, and the sum of the frustums $\doteq$ the spheroid and sphere, and since all the frustums have the common ratio $b^2 : a^2$, vol. spheroid : vol. sphere $= b^2 : a^2$.

479. By methods similar to those of §443 and 478, it may be proved that *the area of an ellipse is to the area of the minor auxiliary circle as* $a : b$, *and the volume of an oblate spheroid to the volume of the sphere having its minor axis as a diameter as* $a^2 : b^2$.

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* See §443 for notation, etc.

480. *1. The volume of a prolate spheroid is* $\frac{4}{3}\pi ab^2$. Why?

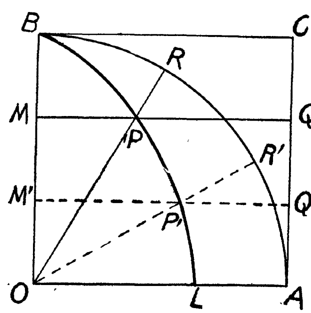
2. The volume of an oblate spheroid is $\frac{4}{3}\pi a^2 b$. Why?

Ex. 831. Find the volume of a paraboloid whose abscissa is 8 and ordinate 4.

481. Remark. By the proof of §440, the area of a segment of an ellipse formed by ordinates to its major axis is to the area of the corresponding segment of the major auxiliary circle as $b : a$; by proof of §474, a segment of a prolate spheroid is to the corresponding segment of the sphere as $b^2 : a^2$.

Ex. 832. Find the volumes of the prolate and oblate spheroids generated by an ellipse whose axes are 10 and 18.

Ex. 833. If the polar diameter of the earth be 7900 miles and its equatorial diameter 7926 miles, find its volume on the supposition that it is an oblate spheroid, also the diameter of the sphere having the same volume.



482. Let BOA be a quadrant; let the line MQ, parallel to OA, move at a uniform rate from the position OA to the position BC, and the radius OR move uniformly in the same time from the position OA to the position OB. BPL, the locus of the intersection of MQ and OR, is called a *quadratrix*.* Let MQ and $M'Q'$ be any two positions of MQ. Then by definition $OB : OM = \angle BOA : \angle AOP$ and $OB : OM' = \angle BOA : \angle BOP'$. $\therefore OM : OM' = \angle AOR : \angle AOR'$. Why?

Ex. 834. State the property of the quadratrix given in §482, and show that it is valid if the figure be extended to the quadrant below OA.

* The quadratrix was invented by Hippias of Athens about 462 B. C., for the purpose of solving the celebrated problem of the trisection of an angle.

Ex. 835. Show how any angle less than 180° may be trisected by means of the quadratrix; also how it may be divided into any number of equal parts.

Ex. 836. Show how a polygon of any number of sides may be inscribed in a circle by means of the quadratrix.

483. The trisection of an angle and the duplication of the cube, that is, finding the side of a cube whose volume is double that of a given cube, are two famous problems in geometry. It has long been known that the two problems are identical, each depending on the insertion of two geometric means between two lines, and also that they cannot be solved by the dividers and straight edge only. They may, however, be solved in many ways mechanically. Archimedes* solved them mechanically in a number of ways. They may be thus solved by the quadratrix, cissoid, and conchoid (see exercises 835, 809, 810), and by the parabola. See also Ball's *History of Mathematics*, and Klein's *Famous Problems in Elementary Geometry*.

484. Gauging or measuring the volume of a cask can be merely approximated by geometrical formulas. In case the cask is approximately the middle frustum of a prolate spheroid: $V = \frac{1}{3}\pi h\,(r_1 + r_2^2)$ approximately.† For proof and for other methods see Encyclopedia Britannica, "Mensuration".

* Archimedes of Syracuse (287—212 B. C.) was the most celebrated mathematician and physicist of ancient times.

† r_1 and r_2 represent the radii of the middle and ends of the cask respectively. Observe that the formula is identical with that for finding the volume of a prismatoid, §353.

FORMULAS.

1. Circle: area $= \frac{1}{2}Cr = \pi d^2 \div 4 = \pi r^2$. §§233, 238.
2. circumference: $C = \pi d = 2\pi r$.
   see $\triangle$s also §249 (1—14); ex. 475 (1—10).
3. Cycloid: length of arch $= 8r = 4 \times$ base $\div \pi$. §469
4. Cylinder: $S_l = Pl = 2\pi Rl$. §318
5. right: $S_l = Ph$; $S = 2\pi R\ (h + R)$. §318
   " $V = Bh = \pi R^2 h$. §325
6. Cone: $S_l = \frac{1}{2}Ch = \pi Rh$; $S = \pi R\ (h + R)$. §318
7. $V = \frac{1}{3}Bh = \frac{1}{3}\pi R^2 h$. §328
8. $L : l = L' : l' = H : h$; $C : c = L : l = H : h$
   $B : b = H^2\ h^2 : = L^2 = l^2 = R^2 : r^2$. §317
   See also ex. 721 (four formulas).
9. frustum of: $S_l = \pi (R + r)\ h_s$; $S = \pi\{h_s\ (R + r)R^2 + r^2\}$. §318
10. $V = \frac{1}{3}h\ (B + b + \sqrt{Bb}) = \frac{1}{3}\pi h\ (R^2 + r^2 + Rr)$. §329
11. Cycloid: length of arch $= 8r = 4 \times$ base $\div \pi$. §469
12. Ellipse: area $= \pi ab$. §444
13. eccentricity: $e = c : a = \sqrt{a^2 - b^2 : a}$. §445
14. ordinates: $(y')^2 : (y'')^2 = (a + x')\ (a - x') : (a + x'')\ (a - x'')$. §437
15. latus rectum: $l = b^2 \div a$. §436
16. Lune: $S = 4\pi R^2 \theta$. §342
17. Octahedron, regular: $d = a\sqrt{2}$; $S = 2a^2\sqrt{3}$; $V = \frac{1}{3}a^3\sqrt{2}$. §345
    See ex. 731.
18. Parabola: area $= \frac{2}{3}\ xy$. §427
19. Paraboloid: $V = \frac{1}{2}\pi y^2 x$. §476
20. Parallelogram: area, $A = bh$. §204

21. $A : A' = bh : b'h'$; if $h = h'$; $A : A' = b : b'$;
if $b = b'$: $A : A' = h : h'$. §204
22. Polygon, regular: area $= \frac{1}{4}pr$. §233
radii inscribed on circumscribed polygons, see §249 (15 formulas), ex. 475 (10 formulas). See also §§98 and 236.
23. Prism: $S_l = Pl$; $S = Pl + 2B$. §318
24. truncated triangular: $V = \frac{1}{3}(h + h' + h'')$
$B = \frac{1}{3}(l + l' + b'')\ Br$. §346
25. do base parallelogram: $V = \frac{1}{2}(l + l' + b'' + b''')\ Br$. Ex. 678
26. Prismatoid: $V = \frac{1}{6}(B + b + 4M)\ h$. §353
27. Prismoid: $V = \frac{1}{6}(B + b + 4m)\ h$. §353
28. Pyramid: $S_l = \frac{1}{2}\ Ph$, $S = \frac{1}{2}\ Ph + B$. §318
29. $V = \frac{1}{3}\ Bh$. §328
30. $L : l = L' : l' = H : h$; $A : a = P : p = H : h = L : l$. §317
31. $B : b = H^2 : h^2 = l^2 : l^2$. §317
32. $V = \frac{1}{6}(B + 4M)\ h$. §352
33. Frustum: $S_l = \frac{1}{2}(P + p)\ h_s$; $S = \frac{1}{2}(P + p)\ h_s + B + b$. §318
34. $V = \frac{1}{6}\ h_s\ (B + b + 4M)\ h$. §352
35. Sphere: surface $= CD = \pi D^2 = 4\pi R^2 = C^2 \div \pi$ §340
36. Volume $= \pi D^3 \div 6 = \frac{1}{3}\ SR = 4\pi R^3 \div 3$. §341
See also §342, 343, also ex. 721 (eight form.).
Miscellaneous, see ex. 721 (eight formulas).
37. Spherical polygon, area: $S = ET$. §379
38. spherical excess of: $E = \Sigma - 2\ (n - 2) = A + B + C + D$, etc., $-\ 2\ (n - 2)$. §379
39. Spherical pyramid: $V = e\pi R^3 \div 18$. §318
40. Spherical sector: $V = \frac{1}{3}\ SR = \frac{2}{3}\pi R^2 h$. §341
41. Spherical segment, one base: $V = \frac{1}{2}\pi R^2 h + \frac{1}{6}\pi h^3 = \pi h\ (3R^2 + h^2) \div 6$. §341
42. do two bases: $V = \frac{1}{2}\pi h\ (r_1^2 + r_2^2) + \pi h^3 \div 6 = \frac{1}{2}\pi h\ (r_1^2 + r_2^2 + \frac{1}{3}h^2)$. §382

43. Spherical triangle: area $= ET$. §377
44. spherical excess of: $E = A + B + C - 2$. §376
45. Spherical wedge: $V = \frac{1}{3}\pi R^3\theta$.. §342
46. Spherical zone: $S = 2\pi Rh = \pi Dh = Ch$. §340
47. Spheroid, oblate: $V = \frac{1}{3}\pi a^2 b$. §480
48. do prolate: $V = \frac{1}{3}\pi ab^2$. §480
49. Tetrahedron, regular: $h_s = \frac{1}{2}\ a\sqrt{3}$; $h = \frac{1}{3}\ a\sqrt{6}$; $S = a^2\sqrt{3}$. §345
50. $V = a^3\sqrt{2} \div 12$. §345
See also ex. 729 (9 formulas).
51. Trapezoid: area $= \frac{1}{2}h\ (b + b') = hm$. §210
52. medial: $m = \frac{1}{2}\ (b + b')$. §210
53. Triangle: area $= \frac{1}{2}bh = \frac{1}{2}\ (a + b + c)\ r = sr$. §205
54. area $= \sqrt{s\ (s - a)\ (s - b)\ (s - c)}$. §241
55. $T : T' = bh : b'h'$; if $h = h'$; $T : T' = b : b'$; if $b = b'$, $T : T'\ h : h'$. §205
56. angle bisector of angle $C = \sqrt{ab - mn}$. §244
57. do $= 2\sqrt{ab\ s\ (s - c)} \div (a + b)$. Ex. 476
58. acute: $c^2 = a^2 + b^2 - 2ab_a$. §217
59. obtuse: $c^2 = a^2 + b^2 + 2ab_a$. §217
60. right: $c = \sqrt{a^2 + b^2}$; $a = \sqrt{(c + b)\ (c - b)}$; $a_c : h = h : b_c$. §214
61. $c : a = a : a_c$; $a^2 : b^2 = a_c : b_c$. §214
62. median on side $c = \sqrt{\frac{1}{2}\ (a^2 + b^2) - (\frac{1}{2}\ c)^2}$. §240
63. $r = T \div s = \sqrt{s\ (s - a)\ (s - b)\ (s - c)} \div \sqrt{s}$. §248
64. $R = ab \div 2b_c$. §244
65. $R = abc \div 4T = abc \div \sqrt{s\ (s - a)\ (s - b)\ (s - c)}$. §248
66. $R_a = \dfrac{T}{(s - a)} = \sqrt{\dfrac{s\ (s - b)\ (s - c)}{s - a}}$. §248

Medians right triangles, see ex. 454.
Ungula, see spherical wedge.
67. Wedge; $V = \frac{1}{6}\ (2L + l)\ wh$. §351
68. Wedgoid: $V = \frac{1}{6}\ (L + L' + l)\ wh$. §357

FORMULAS OF ANALYTICAL GEOMETRY.

$m = (y' : x') = (b : a)$.

Equations of:

1. the straight line; a) $y = (y' \div x')\, x$; b) $y = mx$; c) $y = mx + b$; d) $y - y' = m\,(x - x')$; e) $y - y' = (y'' - y')\,(x - x') \div (x'' - x')$ §417
2. the line through x', y' $\perp$ the line $y = mx + b$ is $y - y' = m\,(x - x')$. §418
3. do $\|$, $y = mx + b$ is $y - y' = -\,(1 \div m)\,(x - x')$. §418
4. the circle: a) $x^2 + y^2 = r^2$; b) $(x - a)^2 + (y - b)^2 = r^2$. §419
5. the ellipse: $a^2y^2 + b^2x^2 = a^2b^2$. §431
6. the hyperbola: $a^2y^2 - b^2x^2 = \pm\, a^2b^2$. §§448, 449
7. the equilateral hyperbola: $x^2 - y^2 = \pm\, a^2$. §467
8. the parabola: $y^2 = 4px$. §424
9. the cissoid of Diocles: $y^2 = x^3 \div (2a - x)$. Ex. 809
10. the conchoid of Nicomedes: $x^2 = (y + c)^2\,(a^2 - y^2) \div y^2$. Ex. 810
11. the semicubical parabola or neilian; $y = a\sqrt{x^3}$. Ex. 812
12. the witch of Agnesi: $y^2 = 4a^2x \div (2a - x)$ Ex. 811
13. the tangent to a circle: $y - y' = -x'\,(x - x') \div y'$.
14. the tangent to a parabola: $y - y' = 2a\,(x - x') \div y'$ §454
15. the tangent to an ellipse: $y - y' = -\,b^2x'\,(x - x') \div a^2y'$. §455
16. the tangent to the hyperbola: $y - y' = b^2x'\,(x - x') \div a^2y'r$. §456
17. the normal to the circle: $y = y'\,x \div x'$.
18. the normal to the parabola: $y - y' = -\,y'\,(x - x') \div 2a$. §458
19. the normal to the ellipse: $y - y' = a^2y'\,(x - x') \div b^2x'$. §458
20. the normal to the hyperbola: $y - y' = -\,a^2\,y'\,(x - x') \div b^2x'$. §458

REGULAR POLYGONS.

Figure.	Side.	Apothem.	Area.	Apothem.	Side.	Area.
Triangle.........	1	.2887	.4331	1	3.4642	5.1963
Quadrilateral.	1	.5000	1.0000	1	2.0000	4.0000
Pentagon........	1	.6882	1.5210	1	1.4531	3.6328
Hexagon	1	.8660	2.5980	1	1.1547	3.4641
Heptagon	1	1.0383	3.6341	1	.9632	3.3712
Octagon	1	1.2071	4.8284	1	.8284	3.3134
Nonagon........	1	1.3737	6.1817	1	.7279	3.2756
Decagon.........	1	1.5388	7.6940	1	.6498	3.2490
Undecagon	1	1.7029	9.3660	1	.5872	3.2296
Dodecagon	1	1.8660	11.1960	1	.5359	3.2154
13-sided.........	1	2.0286	13.1859	1	.4930	3.2045
14- "	1	2.1906	15.3342	1	.4565	3.1955
15- "	1	2.3523	17.6423	1	.4251	3.1883
16- "	1	2.5137	20.1096	1	.3978	3.1824
17- "	1	2.6748	22.7358	1	.3739	3.1782
18- "	1	2.8357	25.5213	1	.3527	3.1743
19- "	1	2.9964	28.4658	1	.3337	3.1702
20- "	1	3.1569	31.5690	1	.3168	3.1680

CONSTANTS.

$\pi = 3.14159265$; $\log \pi = 0.49714987$; $1 \div \pi = .31830989$; $1 \div \pi^2 = .10132118$; $\sqrt{\pi} = 1.77245385$; $\sqrt{2} = 1.41421$; $\sqrt{3} = 1.73205$; $\sqrt{5} = 2.2361$; $\sqrt{6} = 2.4495$; $\sqrt{7} = 2.6458$; $\sqrt{10} = 3.1623$; $\sqrt[3]{2} = 1.25992$; $\sqrt[3]{3} = 1.44225$.

Equatorial semi-diameter of the earth = 3,963.3 miles.

Polar semi-diameter of the earth = 3,950.7 miles.

Mean distance of earth to sun = 92.8 millions of miles.

One cubic foot of water at 4° C. weighs 62.32 lbs.

INDEX.*

* Figures refer to sections except when preceded by ex., then to the number of the exercise.

Abbreviations used: def., definition; ex., exercise; th. ex., theorem exercise; n. ex., numerical exercise; pr., problem or construction; th., theorem.