SHADES AND SHADOWS

PART II

SHADES AND SHADOWS

WITH APPLICATIONS TO ARCHITECTURAL DETAILS
AND EXERCISES IN DRAWING THEM WITH THE BRUSH OR PEN

PART II

CURVILINEAR FIGURES

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NOTE

The promises of *Exercises in Draughtsmanship* held out in the Title Pages of these volumes have not been explicitly fulfilled. But almost any one of the Figures inserted in the Text, and any one of the *Examples of Architectural Details* which follow, are well adapted to such uses, either as here presented, or altered, either in size, or in the proportions of their parts, at the discretion of the student.

In any case, the skill and judgment of the student will be exercised in substituting for the flat tints by which both the Shades and Shadows are here represented, a *graded tint*, the *Line of Shade and Shadow* disappearing, since along this line of meeting the Shade and the Shadow are of equal depth. This is exemplified in Figures 100, 110, 130, and 137, C and D, on Pages 118, 128, 145, and 149.

The chapter upon Perspective, in the Appendix to Part 1, presents in a condensed form a theory of perspective which is based principally upon the phenomena of Parallel Planes, and their Horizons. This theory was first set forth in a volume entitled *Modern Perspective*, published for me in 1882 by the Macmillan Company. It is by their courtesy that I am permitted here to present this matter in this form.

The application of these principles to the Perspective of Shadows is, however, here made I believe for the first time.

W. R. W.

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Chapter V

CIRCLES AND ELLIPSES

49. *Circles.*—The shadow of a circle is most easily found by getting first the shadow of a circumscribing Square [Fig. 39 (*a*)] or Octagon [Fig. 39 (*b*)]. The shadow of the circle is an ellipse, tangent to each side of the resulting parallelogram or polygon at its middle point.

50. Under the conditions shown in these figures the sides of the rectangular parallelogram circumscribing the ellipse measure 1 and $\sqrt{2}$, if the diameter of the circle is taken as the unit of measure (or 2 and $2\sqrt{2}$, if the radius is taken as the unit).

The Major Axis of the Ellipse of shadow is longer than the Minor Axis by just the length of the Diameter of the Circle. It measures 1.618 D, which is a little less than a Diameter and two-thirds, or nearly 1.666 D. The Minor Axis measures .618 D, or a little less than $\frac{2}{3}$ D. The Semimajor Axis measures accordingly .809 D, or 1.618 R, and the Semiminor Axis .309 D, or .618 R.

51. Graphically, the direction and length of these axes are easily determined. (Fig. 40.) In this figure the Radius R is taken as the unit of measure.

Let C be the shadow of the center of the Circle and consequently the center of the Ellipse of shadow, and let D be one of the upper corners of the circumscribing parallelogram. D is directly above C, at a distance equal to the radius of the circle (=1).

Then if the point E be taken at the distance of half the Radius of the Circle from the point D, the distance $E\,C$ will equal $\frac{1}{2}\sqrt{5}$ $(=\frac{1}{2}\ 2.236=1.118)$. If now this distance be laid off on each side of E to A and B, the Major Axis will fall in the direction $C\,A$, and the Minor Axis in the direction $C\,B$.

The length of the Semimajor Axis is equal to $A\,D$, or $\frac{1}{2}\sqrt{5}$ plus $\frac{1}{2}$ $(=1.118+.5=1.618)$, and that of the Semi-minor Axis is equal to $B\,D$, or $\frac{1}{2}\sqrt{5}$ minus $\frac{1}{2}$ $(=1.118-.5=.618)$, the radius of the Circle being taken as 1. The difference of length of the two Semi-Axes is thus equal to the length of the Radius: $(\frac{1}{2}\sqrt{5}+\frac{1}{2})-(\frac{1}{2}\sqrt{5}-\frac{1}{2})=1.$

52. This construction follows from the so-called "Method of Shadows," a device commonly used to obtain the principal axes of an ellipse when only two conjugate diameters are known. [Fig. 41 (*a*).] This figure illustrates the general case, the Conjugate Diameters making any angle whatever with each other.

Let $a\,a$ and $b\,b$ [Fig. 41 (*a*)] be the given Conjugate Diameters, and let a tangent line $G\,L$ be drawn at a, parallel to $b\,b$. If a circle whose diameter equals $b\,b$ is drawn tangent to the Ellipse at this point it may be regarded as lying in a vertical plane; the Ellipse may be regarded as the shadow of this circle, cast upon a horizontal plane, and tangent to the Ground Line.

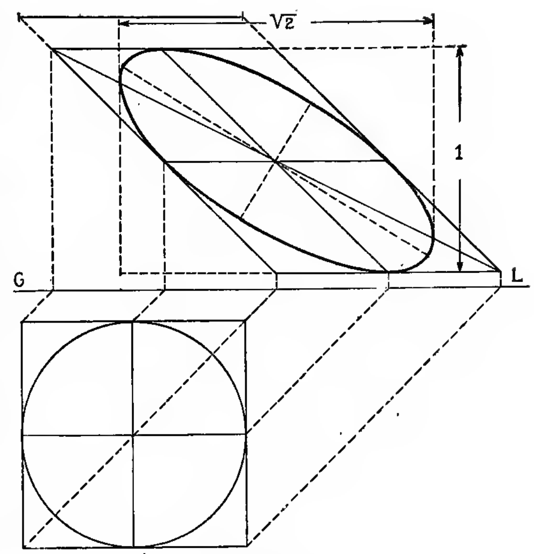


Fig. 39 (*a*)

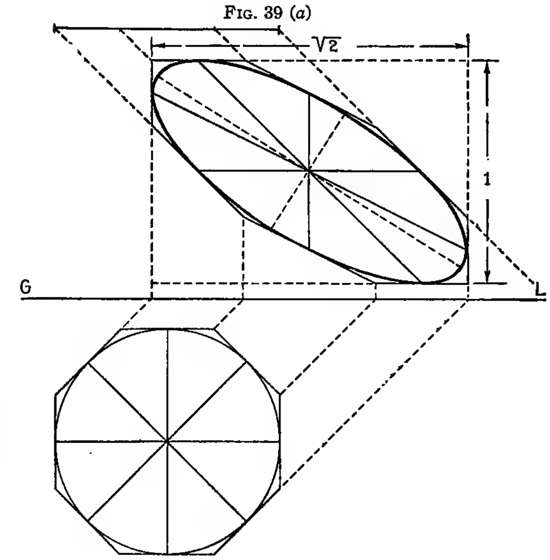


Fig. 39 (*b*)

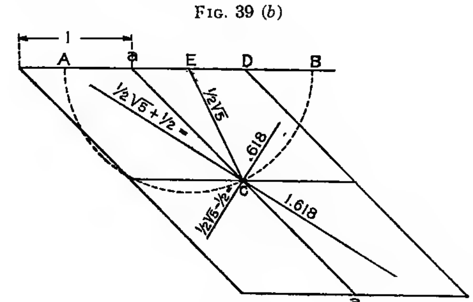


Fig. 40

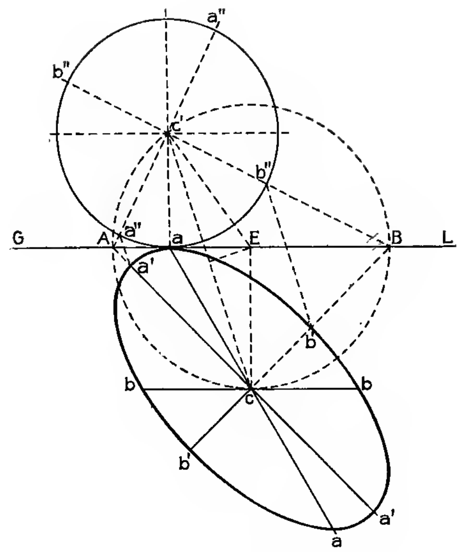


FIG. 41 (*a*)

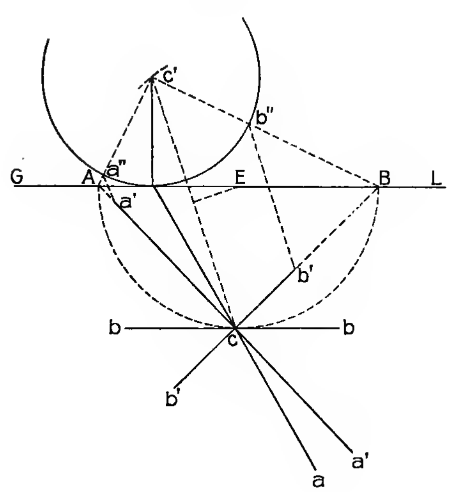


FIG. 41 (*b*)

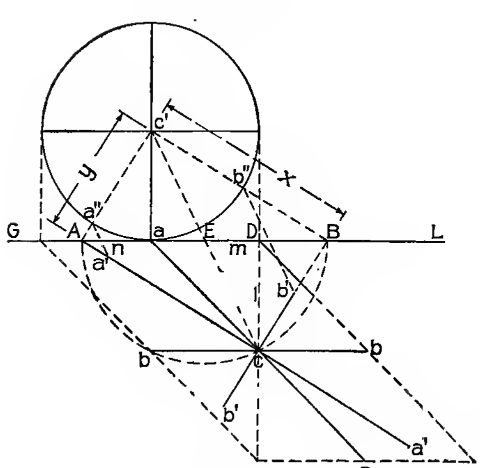


FIG. 42

The center of the Ellipse, C, will then be the shadow of the point C', the center of the Circle; the given Conjugate Diameters will be the shadows of the vertical and horizontal diameters of the Circle, and the required Axes will be the shadows of other diameters of the Circle. But since, in any pair of Conjugate Diameters, each is parallel to the tangents drawn at the extremities of the other, the same must be true of the Diameters of the Circle whose shadows they are. Each must be parallel to the tangents at the extremities of the other. The two Axes of the Ellipse being Conjugate Diameters, those Diameters of the Circle whose shadows are these Axes must be at right angles to one another.

The problem then is to find two Diameters of the Circle, at right angles to each other, whose shadows are also at right angles to one another.

Let $a'' a''$ and $b'' b''$, in Fig. 41 (*a*), be the required Diameters, casting their shadows in the lines $a' a'$ and $b' b'$, the axes of the Ellipse. If the Diameters $a'' a''$ and $b'' b''$ are produced until they meet the Ground Line at A and B, their shadows $a' a'$ and $b' b'$ will meet them at these points. The problem is to find the points A and B.

Now since $A\,C\,B$ and $A\,C'\,B$ are both right-angled triangles, a circle constructed upon the line $A\,B$ as a diameter will pass through the two centers C and C'. The Center of the circle is then a point E upon the line $A\,B$, equidistant from C' and C, a point found by drawing the line $C\,C'$, and erecting a perpendicular upon its middle point, as in the figure. The points in which a circle described about the point E as a center, with a radius equal to $E\,C$ or $E\,C'$, cuts the line $G\,L$, will be the required points A and B. The Diameters $a' a'$ and $b' b'$ can then be directed toward them from C.

Since C' casts its shadow upon C, the shadow of a'' will fall at a', and that of b'' at b'. Lines drawn from a'' and b'', parallel to $C' C$, will then determine the length of the Axes.

Everything that is essential in these operations is shown in Fig. 41 (*b*).

53. Fig. 42 shows that when, as in Fig. 40, the Conjugate Diameters of the given Ellipse make an angle of 45 degrees with one another, the point E is distant from the corner D by half the radius of the circle, as stated in paragraph 51. The point E, midway between C' and C, is now also midway between a and D, the triangles $C'\,a\,E$ and $C\,D\,E$ being equal. $E\,D$ and $E\,a$ will then be equal to each other, and will be as long as half the radius of the circle, that is to say, $E\,D = E\,a = \frac{1}{2}\,R$.

54. The lengths of the Semimajor and Semiminor Axes, $C\,a'$ and $C\,b'$, are not to be read directly from the diagram, but follow from the following considerations:

If in Fig. 42 the radius of the circle be taken as 1, then $a\,D$ will also equal 1, $E\,D$ will equal $\frac{1}{2}$, and $E\,C$, $A\,E$, and $E\,B$ will all equal $\sqrt{\frac{5}{4}}$, or $\frac{1}{2}\sqrt{5}$.

The equal dimensions $A\,a$ and $D\,B$ will equal $\frac{1}{2}\sqrt{5} - \frac{1}{2}$, i. e., $\frac{1}{2}\,(2.236) - .5$, or $1.118 - .5 = .618$.

Since $A\,a$ is equal to $D\,B$, and $C'\,a$, $C\,D$, and $a\,D$ are equal to 1; and since the right triangles $A\,C\,C'$, $A\,a'\,a''$, $B\,C\,C'$, $B\,b'\,b''$, $A\,C\,B'$, $A\,C'\,B$, $A\,C'\,a'$, and $B\,C'\,a$ are all similar, we have, (short side : long side) $\quad A\,a : 1 = 1 : a\,B \;(= A\,a + 1)$.

Hence $A\,a$ and $A\,a+1$ are reciprocals, and $\frac{1}{A\,a}-A\,a+1$.

We have, then, ($h\,y\,p$: short side) $=\frac{A\,C}{1}=\frac{A\,C'}{A\,a}$; i. e., $A\,C=\frac{A\,C'}{A\,a}$, and $A\,C.\ \ A\,a=A\,C'$.

I. To find the Semi-Major Axis, $C\,a'$: (long side : short side)

$$A\,a' : A\,a'' \ (=A\,C'-1)=A\,a' : A\,C'-1=1 : A\,a'.$$

Hence, $\quad Aa'=\frac{A\,C'}{A\,a}-\frac{1}{A\,a}=A\,C-\frac{1}{A\,a}$;

But, $\quad A\,a'=A\,C-C\,a'$;

Therefore, $\quad C\,a'=\frac{1}{A\,a}=1+A\,a=1.618.$

II. To find the Semiminor Axis, $C\,b'$: (short side : long side)

$$B\,b' : B\,b''\,(=A\,C-1)=B\,b' : A\,C-1=A\,a : 1.$$

Hence, $\quad B\,b'=A\,C.\ \ A\,a-A\,a=A\,C'-A\,a$;

But, $\quad B\,b'=A\,C'-C\,b'$.

Therefore, $\quad C\,b'=A\,a=.618.$

The Axes of this ellipse of shadow can most easily be drawn, as is shown in Fig. 43, by first describing the semi-circle $A\,C\,B$ with the point E as a center, with the Radius $C\,E$ $(=\frac{1}{2}\sqrt{5}+\frac{1}{2})$, and thus determining the points A and B, as above. Then, from the same center with the radius $E\,D$ $(=\frac{1}{2})$ describing the smaller semi-circle $F\,D\,G$, determining the points F and G, upon the line $C\,E$. The line $C\,F$ then measures $\frac{1}{2}\sqrt{5}+\frac{1}{2}$, which is the length of the Semi-Major Axis, $C\,a'$, and the line $C\,G$ measures $\frac{1}{2}\sqrt{5}-\frac{1}{2}$, which is the length of the Semi-Minor Axis, $c\,b'$. Arcs struck from C, as a center, with these dimensions as radii, give the points a' and b', upon lines drawn through C to A and B.

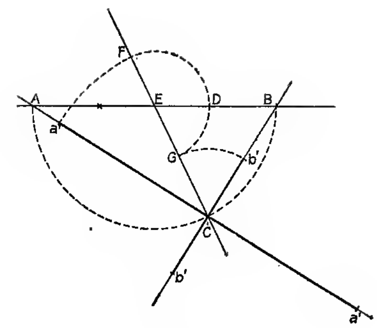


Fig. 43

55. *Approximations.*—Both the direction and the length of the axes of the ellipse of shadow can be obtained with sufficient accuracy for most purposes by simpler methods. (Fig. 44.)

The angle which the Major Axis of the Ellipse makes with the Ground Line is 31° 44′ (the tangent of which is .618). It is a little steeper than a line at 30 degrees (whose tangent is .577), and not quite so steep as a slope of $\frac{2}{3}$ (the tangent of which is .666). Fig. 44 shows these three angles. The Major and Minor Axes can accordingly be drawn at 30 degrees and 60 degrees, respectively, without obvious error, though a sufficient correction should be made. If then the lengths are taken as $1\frac{2}{3}$ D and $\frac{2}{3}$ D, respectively, that is, as 1.666 D and as .666 D (instead of 1.618 and .618), the Major Axis comes out only about one-fortieth too long, and the Minor Axis only about one-fourteenth too long.

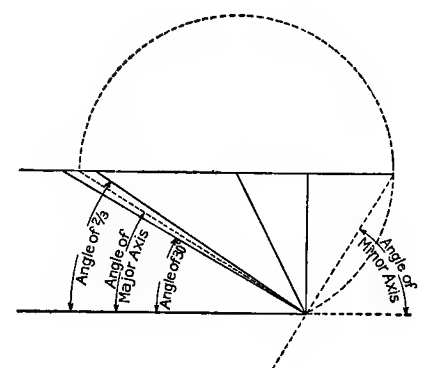


Fig. 44

56. *Circles on Centers, and Semiellipses of Shadow.* It frequently happens that the position of the elliptical shadow of a Circle is given, not by a circumscribing parallelogram, or Octagon, but by the shadow of the center of the Circle and the length of its radius. From these are easily obtained a number of points of the Ellipse, with the tangents at those points; the circumscribing parallelogram; and the direction and length of the Axes.

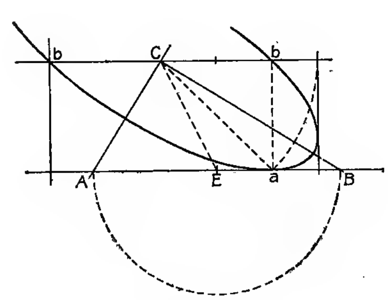


Fig. 45

Fig. 45 shows how these procedures are applied to the Semiellipse of the shadow cast by a Semicircle.

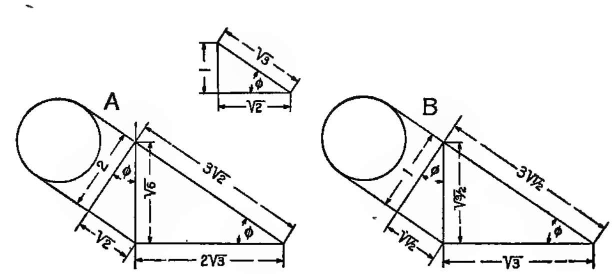


Fig. 46

57. *Circles Edgewise to the Light.* (Fig. 46.)

If a circle stands edgewise to the light its shadow upon a plane is a right line. If the light falls in a direction that makes the angle ϕ with the horizontal plane, the shadow cast upon a Vertical plane measures $\sqrt{6} = \sqrt{\frac{1}{2}}\ \sqrt{\frac{1}{3}}$, and upon a horizontal plane, $2\sqrt{3}$, the radius of the circle being taken as 1. (Fig. 46 *A*.) If the Diameter of the circle is taken as 1, the length of these shadows is $\sqrt{\frac{3}{2}}$ and $\sqrt{3}$, respectively. (Fig. 46 *B*.)

58. *The Auxiliary Plane at 45 Degrees.*—The only shadow cast by a Circle upon a Vertical Plane lying at 45 degrees with the Vertical Plane of Projection which is of any practical importance is that shown in Fig. 47, in which the shadow of a horizontal circle is projected as a smaller circle, the two diameters being in the ratio of $\sqrt{2}$ to 1, or 1 to $\frac{1}{2}\sqrt{2}$, and the areas as 2 to 1. (See Fig. 47 *A*.)

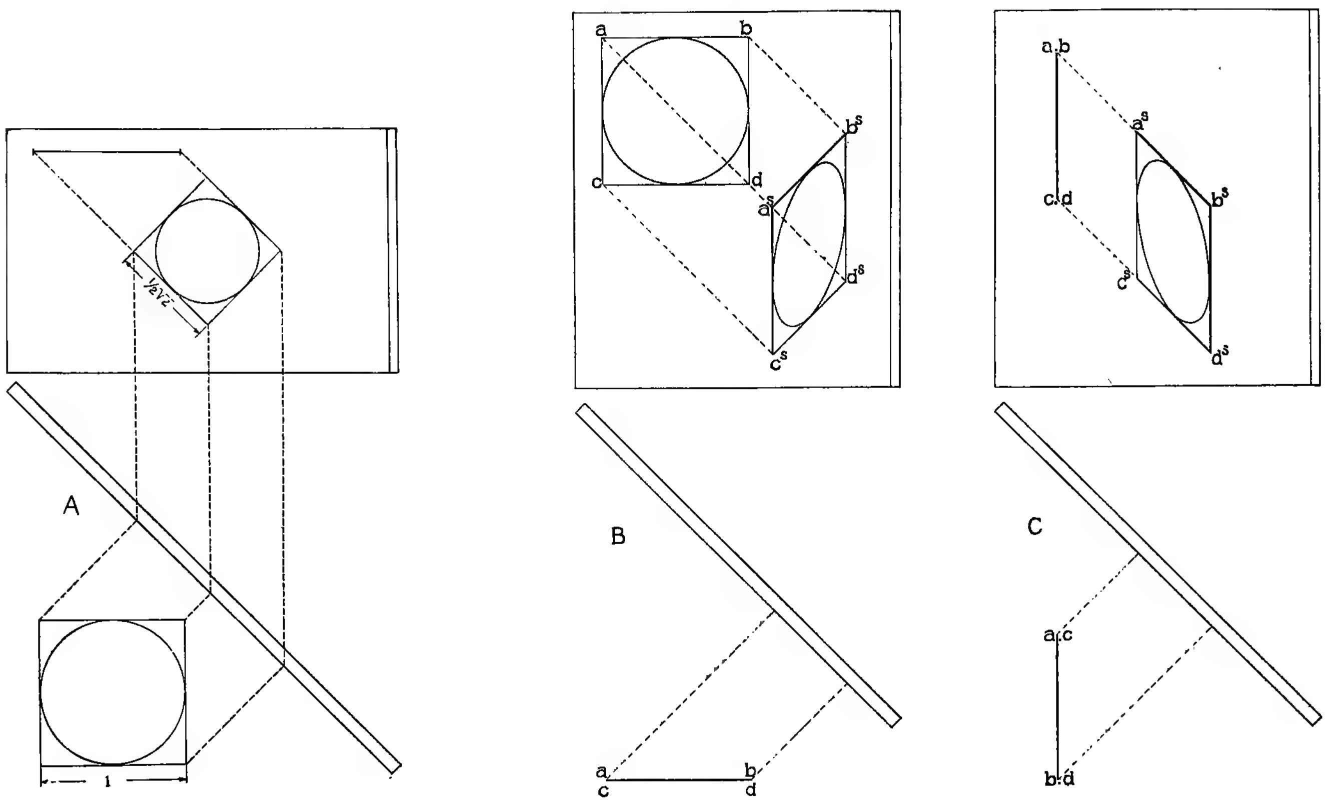


FIG. 47

This is the only case in which the shadow of a circle is easy to draw, except, of course, the case in which the circle is parallel to the plane upon which the shadow falls, when also the shadow is a circle. As has been said in the Introduction, M. Pillet has taken advantage of this circumstance to find the shades and shadows of Surfaces of Revolution whose axis is a vertical line parallel to the Plane of Projection. Such surfaces are composed of horizontal circles normal to that plane, and the vertical shadows of these circles cast upon a 45-degree plane are projected upon it as circles.

As is to be seen in the Figure, the 45-degree rays of light, as projected upon the Vertical Plane of Projection, give at once the diameter of the small circle in which the Elliptical shadow is projected.

Figure 47, *B* and *C*, shows the vertical projections of the shadows of Vertical Circles cast upon a 45° Vertical Plane. They somewhat resemble the shadows of such circles cast upon the interior of hollow cylinders, as shown in Fig. 59 *A* and *B*, at the end of Chapter VII.

59. *Shadows of Ellipses.*—Parallelograms may be drawn about ellipses and the shadows of the ellipses can then be drawn within the shadows of the parallelograms, just as the shadows of circles are drawn within the shadows of squares. Also oblong Octagons can be made to circumscribe Ellipses by laying off on each side of the parallelogram half the diagonal of that side, and drawing the four corner sides of the Octagon

between the points thus obtained. The Ellipse can then be drawn within the parallelogram, and one shadow within the other.

The only case which it is worth while to consider is that shown in Fig. 48, (*A*), (*B*), and (*C*), where the Ellipse lies in a vertical plane, set at 45 degrees with the Vertical Plane of Projection, with its Minor Axis vertical, and its Major Axis horizontal and equal in length to the diagonal of the Minor Axis. The horizontal projection of such an ellipse is a straight line making an angle of 45 degrees with the Ground Line, and the Vertical Projection is a circle whose diameter is equal to the Minor Axis of the ellipse.

At *A* is the circular projection of the Ellipse and of the rectangle that circumscribes it. The rectangle is foreshortened into a square, and the ellipse is foreshortened into a circle. At *B* the ellipse is shown in its true shape. At *C* are the shadows of the Ellipse and of the circumscribing parallelogram.

At *D*, *E*, *F*, and *G* are the shadows of the four quadrants of the Ellipse, the shadows of two of which, *E* and *F*, are themselves inscribed within squares. These are all subdivided so as to show the shape of the shadows of the half quadrants.

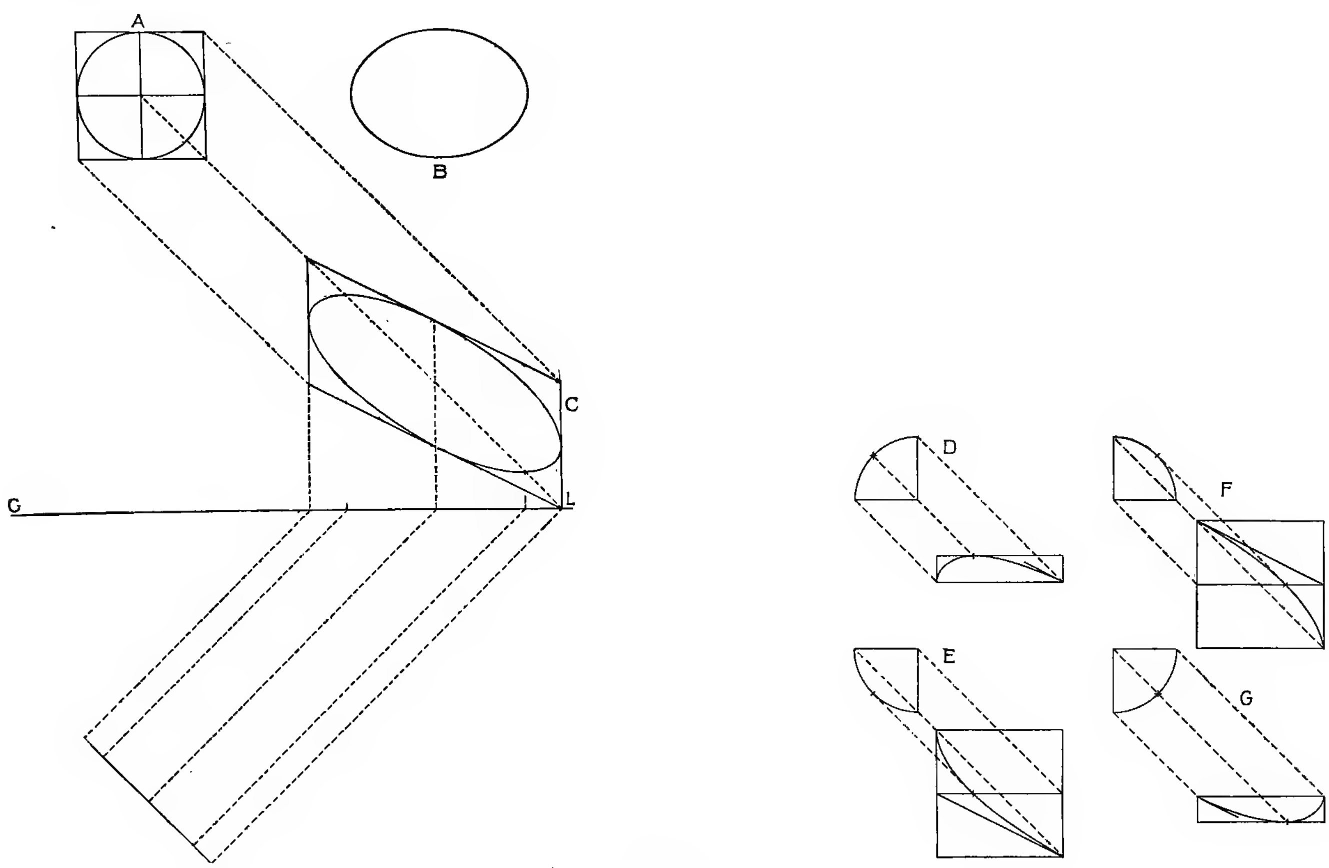


Fig. 48

Chapter VI

CIRCULAR CYLINDERS

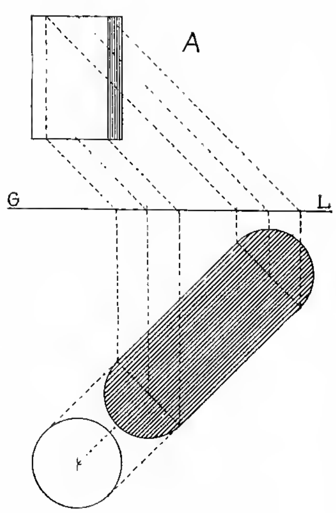


FIG. 49 (*a*)

60. *A Right Cylinder With a Circular Base.*—[Fig. 49 (*a*) and (*b*).]—If the axis of a Right Cylinder with a Circular base is a Principal line, perpendicular to one of the Principal planes of projection, both the axis and the elements of the cylinder are parallel to the two other planes, and the cylinder is projected upon them as a rectangular parallelogram as long and wide as the cylinder itself. On the plane parallel to the base it is projected as a circle.

61. *Shade.*—The illuminated half of such a cylinder comprises one end of the cylinder and half the cylindrical surface, the other half and the other end being in shade. The Line of Shade is in four parts, two of which are right lines which are elements of the cylinder and are on opposite sides of the axis, and two are semicircles, one at each end.

If the axis is parallel to the Plane of Projection, so that the projection of the cylinder upon that plane is a rectangular parallelogram, part of the Line of Shade is seen as an element of the cylinder, situated at the "corner" away from the light, as in the figure.

62. *Shadow.*—The shadow of such a cylinder on a plane perpendicular to the axis and parallel to the base is a rectangular parallelogram terminated by two semi-circles which are the shadows of the semicircular portions of the Line of Shade. The length of the parallelogram is the diagonal of the length of the cylinder, and its width is the diameter of the cylinder. [Fig. 49 (*a*).]

The shadow of such a cylinder upon a plane to which the cylinder is parallel is an oblique parallelogram terminated by oblique semiellipses. The sides of the parallelogram are parallel to the axis of the cylinder and are equal to it in length. Each is the shadow of one of the Lines of Shade which are on the corners of the cylinder, and it is as far from the projection of that Line of Shade as the Line of Shade itself is from the plane upon which the shadow falls. The lines which form the ends of this parallelogram are inclined to the Ground Line at the angle of 17° 33′, the angle whose Tangent is $\frac{1}{2}$. Each semiellipse is included in, and is tangent to, the shadow of the square that circumscribes the base of the cylinder. *The width of the shadow is the diagonal of the diameter of the cylinder.* The shadow of the Cylinder is as wide as that of the circumscribing octagonal prism, that is to say, it is equal to $D\sqrt{2}$, or the Diagonal of the base of the cylinder. [Fig. 49 (*b*).]

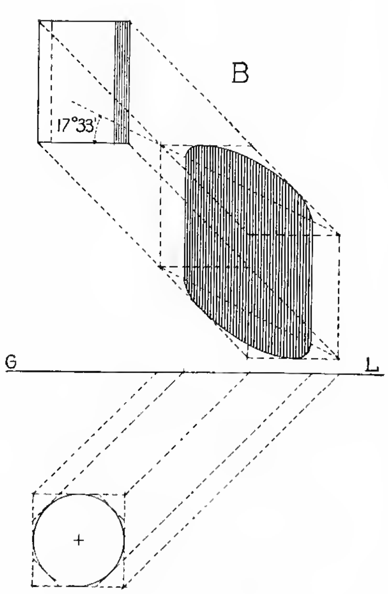


FIG. 49 (*b*)

63. When, as often happens, the plane upon which the shadow falls cuts the cylinder, or is so near it that the shadow is partly concealed by the cylinder itself, then, in the projection, the visible edge of the shadow is always as far from the corner of the cylinder that casts the shadow as this corner is from the plane upon which it falls. (Fig. 50 *A*, *B*, *C*, *D*, and *E*.)

The width in which the Shade upon the Cylinder is projected equals $1 - \frac{1}{2}\sqrt{2}$, the radius of the cylinder being taken as 1.

It follows, as may be seen from the figure, that if a half cylinder, or half column, or half of an octagonal prism, is set against a plane surface, as at A, it will have a shadow upon that surface as wide as half the side of the octagon, or $\sqrt{2}-1$, and the edge of the shadow will, in the projection, be as far from the corner that casts the shadow as is the axis of the cylinder. The visible shadow of the three-quarter column will be half the Radius wider, as at B, or $\sqrt{2}-\frac{1}{2}$.

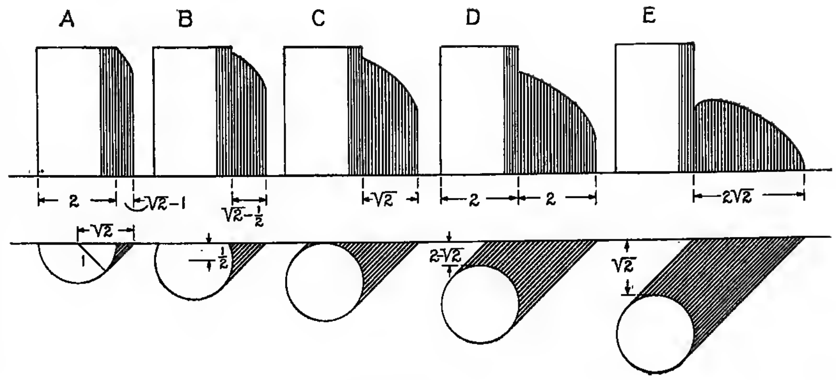


FIG. 50

A whole column, just touching a wall, as at C, will show just half its shadow, the width of the visible portion being $\sqrt{2}$, or the diagonal of the Radius.

If the distance of the column from the wall is $2\sqrt{2}$, i. e., is equal to the projection of the oblique side of the octagon, as at D, the shadow is just as wide as the column.

If the distance is $\sqrt{2}$, i. e., equal to the diagonal of the Radius, the whole of the shadow is seen, as at E.

64. If the axis of the cylinder is inclined so as to be at right angles to the rays of light, the shadow cast upon a plane surface is a parallelogram, the two ends of the cylinder being planes of Light and Shade, and their shadows right lines.

If the axis is parallel to the rays of light, so that the circle which is in light at one end casts its shadow directly upon the circle which is in shade at the other end, either circle may be regarded as the Line of Shade, the cylindrical surface is a surface of Light and Shade, and the shadow cast upon any plane surface is a circle, or ellipse.

65. *The Shadow of a Principal Line Upon a Principal Cylinder* [Fig. 51 (*a*), (*b*).]—If a principal line is parallel to the plane of projection, and casts its shadow upon a cylinder parallel to itself, its shadow will fall upon an element of the cylinder. Its position can be obtained by means of a second plane of projection normal to the first. [Fig. 51 (*a*).]

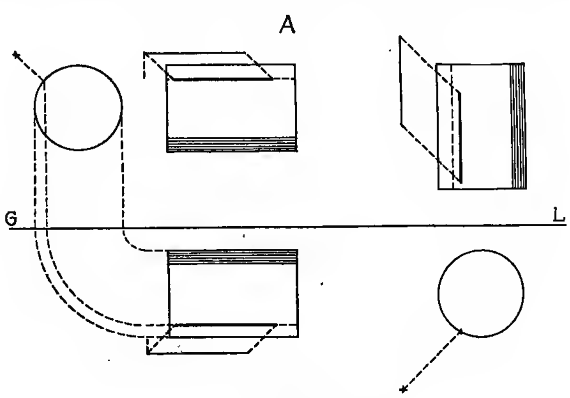


FIG. 51 (*a*)

If a principal line is normal to the given plane of projection [Fig. 51 (*b*)], and casts a shadow upon a principal cylinder parallel to that plane, the plane of the invisible shadow of the line cuts the cylinder at an angle of 45 degrees to its axis. The line of intersection is an ellipse of which the minor axis is equal in length to the diameter of the cylinder and the major axis is equal in length to its diagonal. The two axes are in the ratio of 1 to the square root of 2.

The projection of this ellipse on the given plane of projection is a right line, lying at 45 degrees, in the direction of the light, and equal in length to the diagonal of the diameter of the cylinder, that is to say, to the major axis of the ellipse. Upon each of the other two planes of projection, the ellipse is foreshortened into a circle, equal to the cross-section of the cylinder. [Fig. 51 (*b*).]

The projection of the shadow cast by a principal line upon a principal cylinder is accordingly either a right line parallel to the axis, or a right line at 45 degrees, or an arc of the circle which is the base of the cylinder.

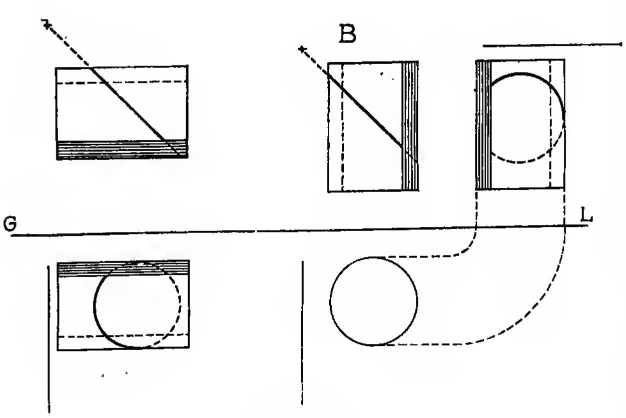


FIG. 51 (*b*)

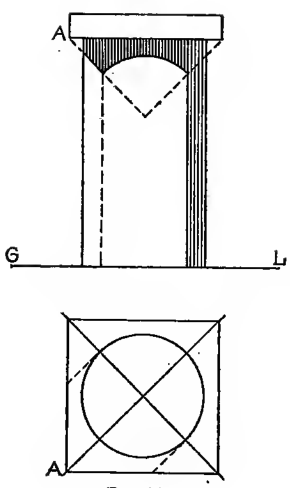


FIG. 52

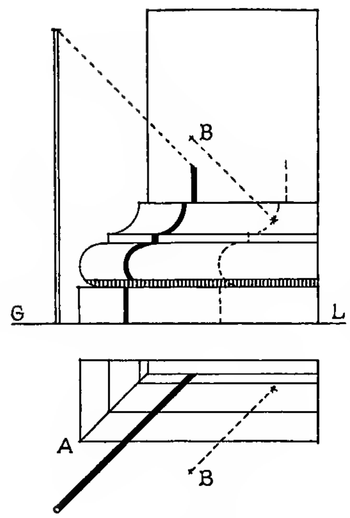


FIG. 53

66. If in this last case the line which casts the shadow stops short opposite the corner of the cylinder, as do the sides of the square in Fig. 52, at A (and as happens when it is the edge of an abacus resting on a column), then the projection of the shadow is an arc of 90 degrees, whose radius is equal to the radius of the cylinder. This arc lies between the corner of the cylinder which is nearest the light and the corner away from the light, which is occupied by the Line of Shade. The cylindrical surface beyond this Line of Shade is in shade and cannot receive the shadow of the abacus.

67. *The Shadow of a Principal Line Upon Any Cylindrical Surface, Whose Elements Are Parallel to Two of the Principal Planes of Projection.*—It appears from these examples that: (a) if the line is parallel to the Plane of Projection and also parallel to the elements of the cylindrical surface, its shadow on that surface, and the projection of this shadow are equal and parallel to the line itself, as in Fig. 51 (a); (b) if the line is perpendicular to the Plane of Projection, the shadow cast upon any cylindrical surface is projected upon that plane as a right line, lying at 45 degrees, as in Figs. 51 and 52, without regard to the nature of the surface; (c) if the line is parallel to the Plane of Projection and at right angles to the elements of the cylindrical surface the projection of its shadow is a true section of the cylindrical surface, as in Fig. 51 (b), and in Fig. 52.

Such a line of shadow is the line of intersection of the cylindrical surface by a plane of invisible shadow cutting across it at an angle of 45 degrees.

This line is parallel to, and exactly resembles, the line of intersection, or miter line, which occurs when a cylindrical surface returns upon itself at right angles, as appears in Fig. 53 at A.

68. It happens accordingly that the shadow cast upon a molding under these conditions, *shows the true contour of the molding*, and that the line of shadow and the outline of the object, as defined by the miter line away from the light, often show a symmetrical figure, as they do in Fig. 54.

In like manner the projection of the shadow cast by such a line upon an inclined plane *shows the true slope of the plane*, as in the case of the sloping surface above the moldings in Fig. 54, and as is the case with the shadow of the chimney in Example XIX.

69. *The Shadow of a Point Upon a Cylindrical Surface.*—The Shadow of a Point on a cylindrical surface is found by passing through the point an auxiliary line parallel to the Plane of Projection and finding its shadow, as explained above. The shadow of the given point will be a point on this line of shadow, as is that of point B in Fig. 53.

70. *The Shadow of a Surface Upon a Cylindrical Surface.*—The shadow of an irregular line, or of the outline of a plane surface, lying parallel to the Plane of Projection, such as the circle in Fig. 55 A, may be found by passing through it an auxiliary line, and erecting upon the shadow of this line ordinates of the given line, or of the outline of the given surface, taken parallel to the elements of the cylindrical surface. The shadows of these ordinates will be equal and parallel to the ordinates themselves, and may be laid off upon the cylindrical surface from the shadow of the auxiliary line, as in the Fig. 55, at A.

71. *The Shadow of a Solid Upon a Cylindrical Surface.*—The shadow of a solid object, such as the Cube in Fig. 55 B, upon a cylinder, may be treated as if it were the shadow of a plane figure of the same shape and size as is the shadow of the solid object when cast upon the given plane of projection.

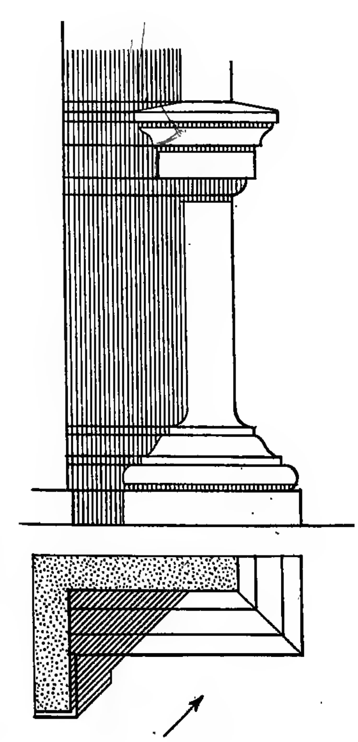


FIG. 54

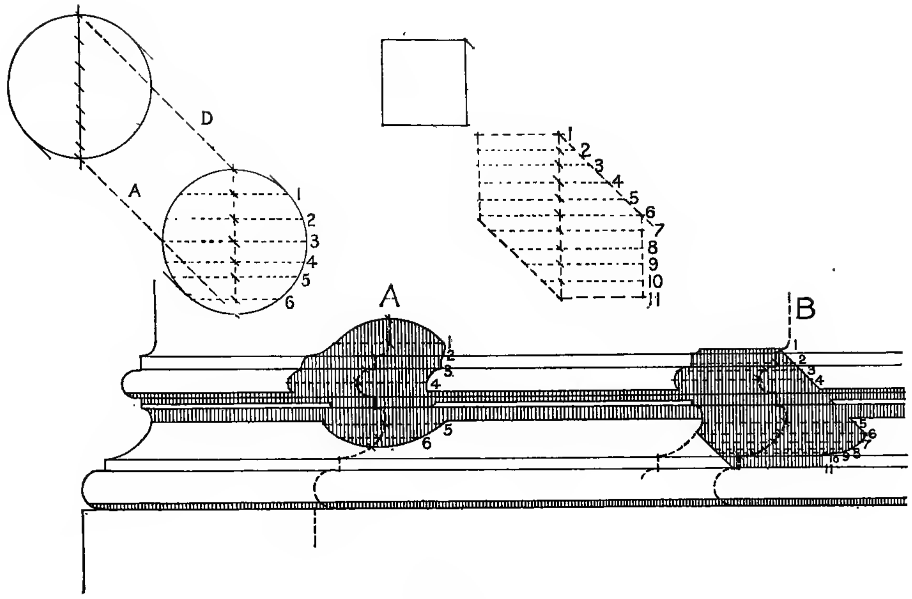


FIG. 55 *A* and *B*

Thus in this figure the shadow of the Cube upon the molding is the same as the shadow upon the molding of the irregular hexagon, which is the shadow of the Cube.

The shadow of a solid upon such a cylindrical surface may accordingly be found by first finding its shadow upon the Plane of Projection, as in the figure, and then finding (as in the previous problem, Fig. 55 *A*) the shadow which would be cast by a plane figure of the same shape and size as this shadow.

The shadow cast by a plane figure, parallel to a principal plane of projection, upon a cylindrical surface whose elements are parallel to the plane and are perpendicular to one of the other principal planes, may also be obtained by the method of slicing, as in Fig. 55 *C*, where the shadow is like that in Fig. 55 *A*.

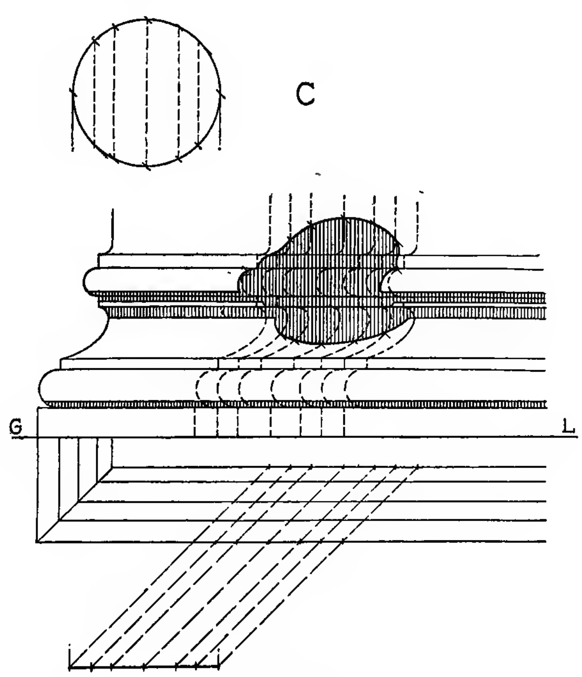


FIG. 55 *C*

Chapter VII

HOLLOW CYLINDERS

72. *Shade and Shadow.*—In a hollow Half Cylinder, like those shown in Fig. 56 *A*, *B*, and *C*, the Quarter Cylinder which is farthest from the Sun is in light, while the Quarter Cylinder which is nearest the Sun is half in shade and half in shadow, the element of the Cylinder at the corner nearest the light being a Line of Shade and Shadow, separating the portion of the concave surface which is in shade from the portion which is in shadow. The element which is on the edge of the half cylinder nearest the light is a Line of Shade and the element of the cylinder which, in the projection, coincides with that of the Axis and receives the Shadow of Line of Shade, is a Line of Shadow.

73. *The Shadows of Principal Right Lines Cast Upon a Hollow Semi-cylinder.* (1) The shadow of a Right Line lying in the Axis of the Cylinder is a Right Line parallel to the line that casts it and of the same length. It falls upon the element of the cylinder which occupies the corner farthest from the light. (Fig. 56 *A*.)

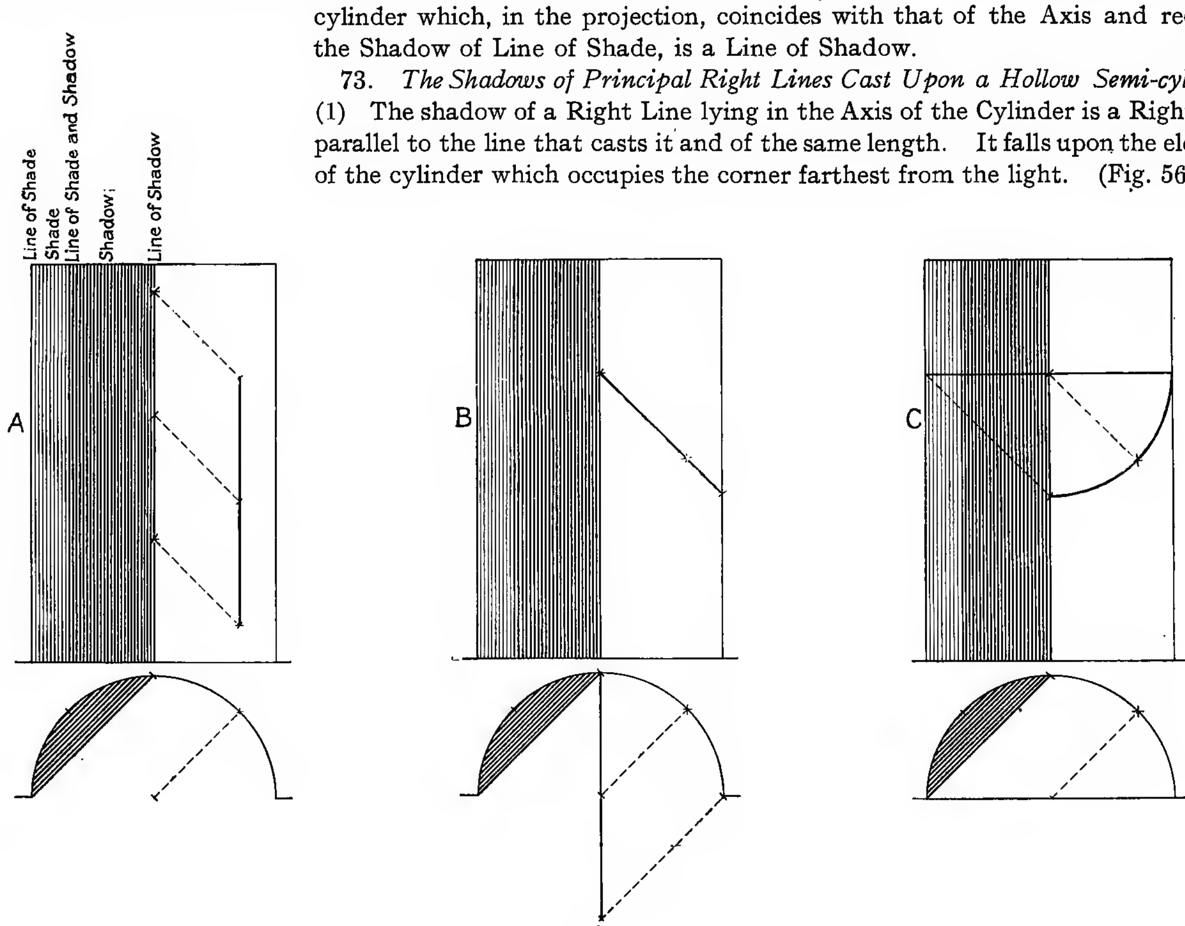


Fig. 56

(2) The shadow of a diameter of the cylinder which is normal to the Plane of Projection is a quarter of an ellipse. Its projection is a Right Line, lying at 45 degrees, parallel to the projection of the rays of light, and extending from the Axis of the cylinder to the edge farthest from the light. (Fig. 56 *B*.) See Paragraph 65.

(3) The shadow of a diameter parallel to the Plane of Projection is also a quarter of an ellipse. Its projection is a circular arc of 90 degrees, of the same radius as the cylinder, concave toward the light, and extending from the shadow of the edge nearest the Sun, on the Axis of the cylinder, to the other edge. (Fig. 56 *C*.) See Paragraph 65.

These figures show also the projections of the shadows of the middle points of these diameters. They illustrate the propositions in regard to the shadows cast upon hollow cylindrical surfaces by principal lines which are contained in Paragraph 67, namely, that one is parallel to the Axis of the cylinder, one lies at 45 degrees, and one gives a true section of the surface on which it falls.

74. *The Shadows of Circles Cast Upon a Hollow Semicylinder.*—A Principal Circle of the same diameter as a Vertical Hollow Cylinder, and with its center on the Axis of the Cylinder, may lie in a plane parallel to either of the three Planes of Projection. It may be either (1) horizontal, and normal to the vertical plane of projection, forming a cross-section of the cylinder; or (2) vertical and normal to the vertical plane of projection; or (3) vertical and parallel to that plane.

75. (1) In the first case, as shown in Fig. 57 *a*, the shadow cast upon the interior surface of a semicylinder is the shadow of a portion of the horizontal semicircle *2*, *3*, *4*, *5*, *6*, which projects in front of the semicylinder.

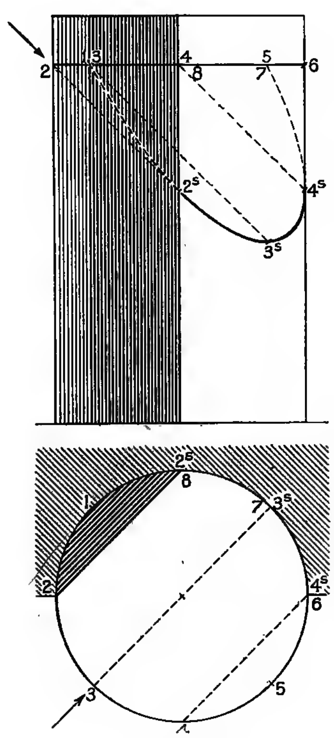


Fig. 57 (*a*)

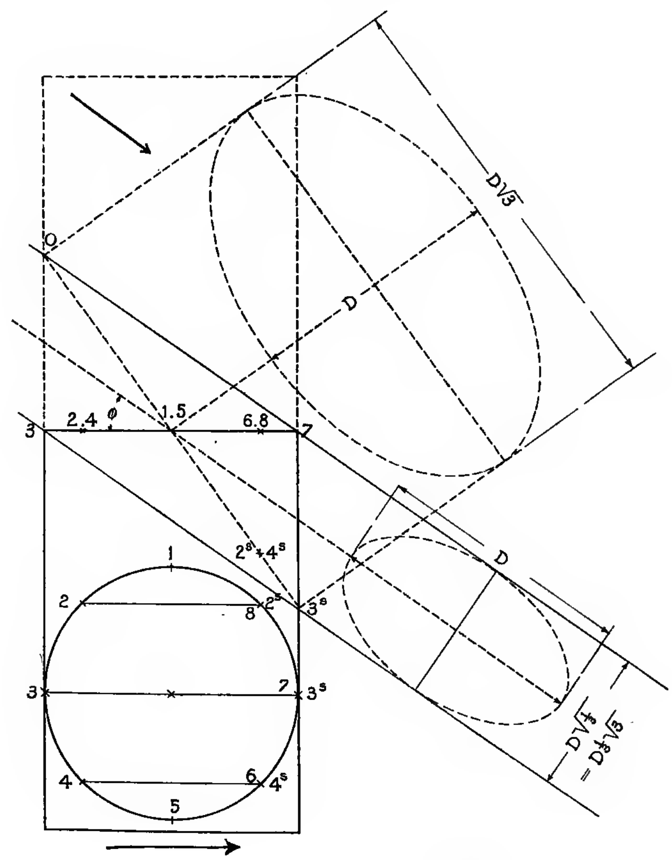


Fig. 57 (*b*)

The portion that casts the shadow is the 90-degree arc *2*, *3*, *4*, which lies between the point *2* and the point *4*; for the arc *1*, *2*, whose shadow would fall in the dotted arc *1*, 2^s, is not a Line of Light and Shade, being itself in shade, and the shadow of the arc *4*, *5*, *6* does not fall upon the semicylinder, since it would come upon the portion of the cylinder that has been removed. The Line of Shadow cast by the quarter circle *2*, *3*, *4* is a plane figure, being a quarter of an ellipse, and its projection, as shown in the figure at 2^s, 3^s, and 4^s is also a quarter of an ellipse.

For it is a maxim of Geometry that if a Cylinder, Cone, Sphere, or other Conic Section of Revolution is cut by a plane, the line of intersection is some one of the Conic Sections, that is to say it is a Right Line, a Hyperbola, a Parabola, or an Ellipse or Circle; and also that the projection of this line of intersection upon the concave Surface of Revolution is also a Conic Section, and is consequently a plane curve. If, moreover, any Conic Section is projected upon a plane, this projection also is either a right line or a Conic Section of the same kind.

Now a Hollow Cylinder is such a Surface of Revolution, the circular arc that casts the shadow is such a line of intersection, and its shadow upon the concave surface is such a projection. This shadow must then be a plane curve, and since the ellipse is the only plane curve that can lie in a cylindrical surface, the Line of Shadow must be an ellipse, and its projection upon the Vertical Plane of projection must be an ellipse also.

Fig. 57 (*b*), in which the cylinder is turned 45 degrees upon its axis, so as to bring the rays of light parallel to the Vertical Plane of Projection, shows the same thing. For the Invisible Shadow, or Shadow in Space, of the horizontal circle *1, 2, 3, 4, 5, 6, 7, 8* is an elliptical cylinder whose axis is parallel to the direction of the light, making the angle ϕ with the Ground Plane. Its cross-section is an ellipse whose Major Axis is D, the diameter of the given circle, and whose Minor Axis, as appears from the figure, equals $D\sqrt{\frac{1}{3}}$. This inclined elliptical cylinder intersects the vertical circular cylinder in the two lines *3, 7*, and $0, 3^s$, both of which lie in planes normal to the Vertical Plane of Projection and are consequently projected upon it in right lines. One of these lines of intersection is the horizontal circle which casts the shadow, and which is projected in the horizontal line *3, 7*. The other, which is an ellipse, being an oblique section of the elliptical cylinder, is projected in the oblique line $0, 3^s$. The cast shadow $1, 3^s, 5$, which occupies part of this inclined line, is a plane figure lying in the surface of the cylinder, and is accordingly an arc of an ellipse, as stated in the preceding paragraph.

The Minor Axis of this ellipse of intersection, *1, 5*, equals the diameter of the vertical circular cylinder D, as appears from the plan, and the Major Axis, $0, 3^s$, as appears from the elevation, equals $D\sqrt{3}$. This elliptical line of intersection, and the cross-section of the elliptical cylinder of shadow are accordingly similar in shape (since $D : \sqrt{\frac{1}{3}} :: D : \sqrt{3}$). But the large ellipse has three times the area of the other.

Both are shown, in the figure, revolved parallel to the vertical plane of projection.

76. Of the other two vertical circles, shown in Figs. 58 and 59, the shadows are not plane figures, but are curves of three dimensions, lying on the interior surface of the cylinder, and their projections upon the Vertical Plane of Projection are irregular ovals. If the light is horizontal, lying at 45 degrees with the vertical plane, as in Fig. 58 A and B, the shadows in space of the two circles are similar horizontal elliptical cylinders, and the shadows cast upon the hollow cylindrical surface by the circles and their horizontal diameters are exactly alike. But if the rays of light are inclined, making the usual angle with the horizontal plane, as in Fig. 59 A and B,

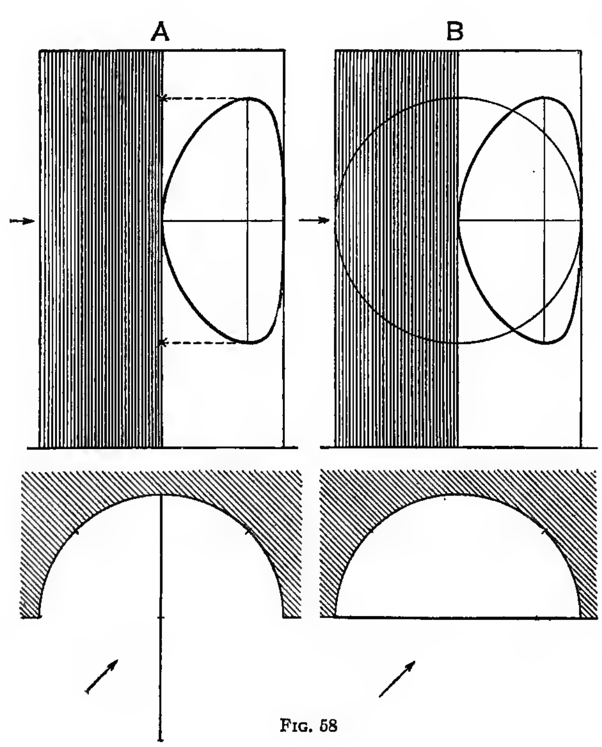


Fig. 58

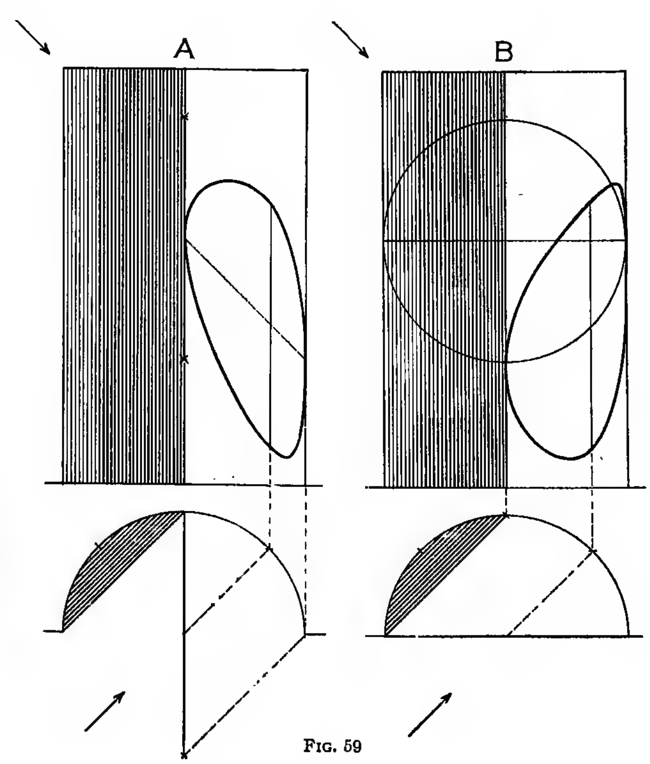


Fig. 59

the shadows of the horizontal diameters are different, being a 45-degree right line in one case, as at A, and a 90-degree circular arc in the other, as at B, where the projection of this shadow coincides with the projection of one quadrant of the circle. The shadows of the two circles, though still symmetrical about the shadows of their diameters, differ accordingly, as appears in that figure.

The shadows of the horizontal diameters are really quarter ellipses lying in the concave surface, as in Fig. 56 B and C, and have the same projections as in that figure.

77. The most common case of a circle of the same diameter as a cylinder, and perpendicular to its axis, thus throwing its shadow upon the concave inner surface of the cylinder, is presented by the Half Cylinder shown in Fig. 60. Here the shadow of the upper rim of a vertical cylinder, *1, 2, 3, 4, 5*, open at the end nearest the light, is shown as passing through the points *1*, 2^s, 3^s, 4^s, and *5*, as in Fig. 57 (*a*), just as if the nearer half of the cylinder had not been removed. The points *1* and *5* are, so to speak, their own shadows. Point *2* throws its shadow at 2^s; Point *3*, which is on the front corner of the cylinder, throws its shadow back to 3^s, upon the element situated at the opposite corner. Since this shadow has the farthest to go of any, the point 3^s is the lowest point in the curve. The shadow of Point *4* falls at 4^s, on the edge of the cylinder, at the same level as 2^s.

The shadows of the horizontal lines tangent to the circle at the points *2, 3,* and *4*, which are drawn as if cast upon planes tangent to the cylinder at the points 2^s, 3^s, and 4^s, show that the projection of the curve is vertical at 4^s, and horizontal at 3^s, and that at 2^s it lies at an angle of 45 degrees.

78. In the case of a semi-cylinder, open in front and at the end, the shadow on the interior surface is limited to the arc *1*, 2^s, cast by the circular arc *1, 2.* Just as, in Fig. 57 (*a*), it was limited to the shadow 2^s, 3^s, 4^s, cast by the arc *2, 3, 4.*

79. *The Ellipse of Shadow Projected as a Right Line.*—The shadow cast by the rim of a hollow cylinder upon its interior surface cannot of course be seen unless the spectator is looking in, as when he is at the open mouth of a well, or tunnel, or is actually inside it. If, being outside of it, he places his eye in the plane of the ellipse of shadow, the shadow will appear as a straight line.

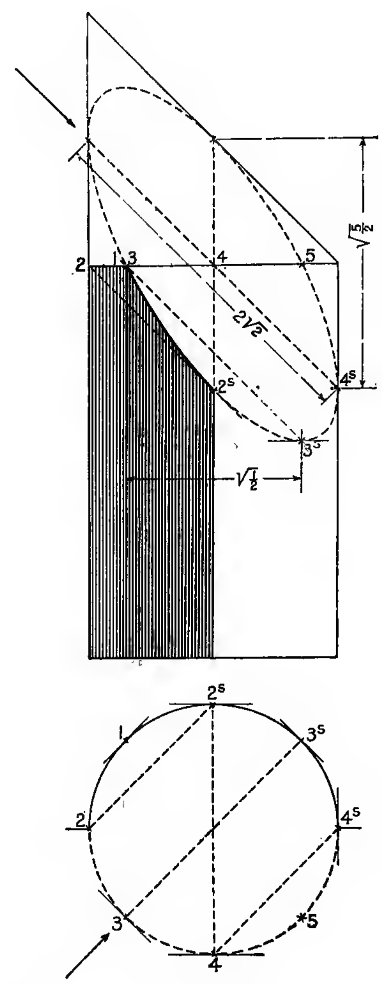


Fig. 60

This is illustrated in Figs. 61 and 62. At 61 A is the vertical section of a short horizontal circular tunnel. This is shown in plan at B, and in elevation at C. In all three the Line of Shade, *1, 2, 3, 4, 5*, casts its shadow at *1*, 2^s, 3^s, 4^s, and *5* (*1* and *5* being, as before, their own shadows), and the semi-elliptical Line of Shadow passes through these 5 points.

The Point *0*, on the axis of the cylinder, is the center both of the circular Line of Shade and of the elliptical Line of Shadow. The point *2*, at the top of the circle, casts its shadow at 2^s, on the side of the tunnel, at the same level at the center *0*. The line 2^s *0*, is then horizontal, and is a horizontal semi-diameter of the ellipse of shadow. The plan shows that this line makes an angle of 45 degrees with the Vertical Plane of Projection.

80. If now the cylinder is projected upon an auxiliary Vertical Plane of Projection, which makes an angle of 45 degrees with the principal one, it appears as shown in the figure at D. The circle at the end of the tunnel is now projected as an ellipse, whose horizontal and vertical axes are in the ratio of 1 to $\sqrt{2}$; the horizontal line 2^s *0* is seen endwise, and is projected as a point, and, since all the chords of the ellipse of shadow which are parallel with 2^s *0* are also projected as points, the Ellipse is projected as a line passing through them. As the ellipse is a plane figure this line is a straight line, and the two points *0* and *1* suffice to determine it.

The plane in which this ellipse lies is, like these chords, normal to this vertical plan of projection, and all the lines and figures in it, as well as the ellipse itself, are projected in the same straight line. The line *1, 5*, in Fig. 61 D is therefore the projection (1) of the normal plane in question; (2) of the elliptical Line of Shadow; (3) of that diameter of the ellipse which is parallel to the auxiliary Plane of Projection; and (4) of one of the 45-degree diameters of the vertical circle which cast the shadow.

Fig. 61 D shows also that the angle which the plane of Invisible Shadow makes with the horizontal plane of projection is 90 degrees $- \Phi$, the complement of the angle Φ; for, in Fig. 61 D the line $0\ 6$ equals $\frac{1}{2} D$ and $6\ 4^s$ equals $\frac{1}{2}\ D \sqrt{\frac{1}{2}}$. The angle $6\ 0\ 4^s$ is then the angle Φ.

81. In Fig. 61 C the ellipse of the Line of Shadow, lying in the surface of the hollow cylinder, is projected as a circle, coinciding with the circle in which the surface of the cylinder itself is projected, and the diameter $1, 5^s$ coincides with that diameter of the circle which lies at 45 degrees with the Ground Line $G\,L$, as has just been said.

The rays of light in Fig. 61 D, which make the angle Φ with the Ground Plane, lie in planes perpendicular to both planes of projection, as is seen in the plan at B, and are projected at D as vertical lines perpendicular to the auxiliary Ground Line $G'\,L'$.

It appears then from Fig. 61 D that if a horizontal cylinder with a circular base has its axis at 45 degrees with the Vertical Plane of Projection, so that the rays of light, falling at the angle Φ, lie in vertical planes normal to that plane, then the inclined plane in which the ellipse of shadow lies will be normal to the vertical Plane of Projection and will make the angle $90° - \Phi$ with the Horizontal Plane.

82. *Perspective.*—Fig. 62 A and B shows the same subject in Perspective, in two positions. In both, the inclined plane, in which lie the elliptical line of Shadow and the 45-degree diameter of the circle at the base of the cylinder, is seen edgewise, and its perspective coincides with the perspectives of both these lines.

The line $0,\ 2^s$, which is a semi-diameter of the ellipse of shadow, and is really horizontal, and the chords of the ellipse which are parallel to it, are normal to the Plane of the Picture, and accordingly have their Vanishing Point at the Center, V^C.

The rays of light $2,\ 2^s$; $3,\ 3^s$; $4,\ 4^s$; etc., have their Vanishing Point at V^S, the Vanishing Point of Shadows.

The diameter $1,\ 5$, lies in a vertical plane inclined at 45 degrees to the Plane of the Picture and itself makes an angle of 45 degrees with the Ground Plane. Its Vanishing Point is accordingly to be found at V^{45}, at a distance above V^L equal to the diagonal of the distance $V^L\,V^C$, as has been explained in the Appendix, in paragraph 38. These lines, having this ratio, one to the other, the angle $V^L\,V^{45}\,V^C$, at the vertex of the triangle, is the angle Φ.

Since the Perspective of a plane, when it is seen edgewise, coincides with the Horizon of the plane, and the Horizon of a plane passes through the Vanishing Point of all the lines that lie in the plane, this Horizon (which is the common perspective of the inclined plane in question, of the elliptical Line of Shadow, and of the 45 degrees diameter of the circular base) may be drawn through V^{45} and V^C, as in the figure at both A and B.

83. *The Half Cylinder.*—It is only when shown in section that the inside of a Hollow Cylinder can be seen in Orthographic Projection, and a Half Cylinder, as has already been pointed out, shows only so much of the shadow as is cast by one-eighth of a circle. This shadow comprises accordingly only an eighth part of the ellipse under discussion, and that the flattest part of the curve, and this is all that generally occurs in architectural drawings.

84. The most frequent case is that of a horizontal cylinder, such as an archway, or barrel vault (Fig. 63). The curve of shadow starts from the corner of the arch, as in Fig. 61 A, at a slope of one-half, and terminates on the projection of the axis of the cylinder, with a slope of 45 degrees, at a point determined by the line drawn at that angle from the top of the arch. The distance of this point from the end of the cylinder is thus equal to the radius of the cylinder.

85. *The Square Niche.*—If the end of a Half Cylinder is closed, as in Fig. 64, the line of shadow cast by the line $1,\ 2$, extends from the point 1^s to 2, and its projection is a true section of the surface on which it falls, namely, an arc of 90 degrees, as appeared in Fig. 56 C.

86. *Oblique Hollow Semicylinders.*—If the vertical secant plane, which divides the vertical cylinder in two, is not parallel to the plane of projection, but is inclined to it, as in Fig. 65 A, B, C, D, E, F, G, and H, the shadows of circles lying in that plane take the shapes shown in these figures. It is to be noticed that the three points on the axis of the cylinder which are at the center and two extremities of the vertical diameter are common to all the circles, though their projections occur in different positions in the several figures. The shadows of the circles all, of course, pass through the shadows of the two extreme points, both

which lie in the corner element of the cylinder, and, unlike the points which cast them, are projected in the same places in all the figures.

The irregular ovals which constitute the lines of shadow are found, like those in Fig. 59 *B*, by first finding the semiellipse which is the shadow of the horizontal diameter of the circle, and then drawing vertical ordinates, above and below it, equal and parallel to vertical chords of the circle.

87. *The Oblique Semicylinder at 45 Degrees.*—When, as in Fig. 66, and in Fig. 65 *C*, the secant plane is taken at -45 degrees, the semicylinder faces the light, and neither of the vertical edges casts any shadow upon the concave surface.

The shadow of the vertical diameter of the circle, $d\,c\,d_1$, falls at $d^s\,c^s\,d_1^s$. The projection of this shadow coincides with that of one edge of the semi-cylinder. The horizontal diameter $a\,c\,b$ casts its shadow at $a\,c^s\,b$.

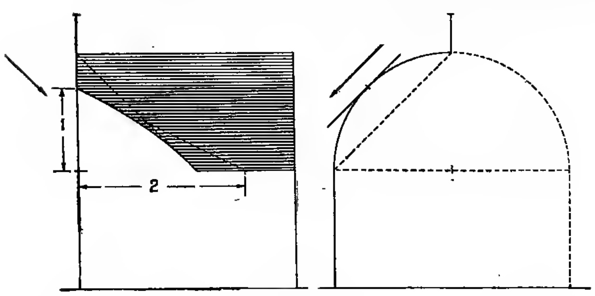


Fig. 63

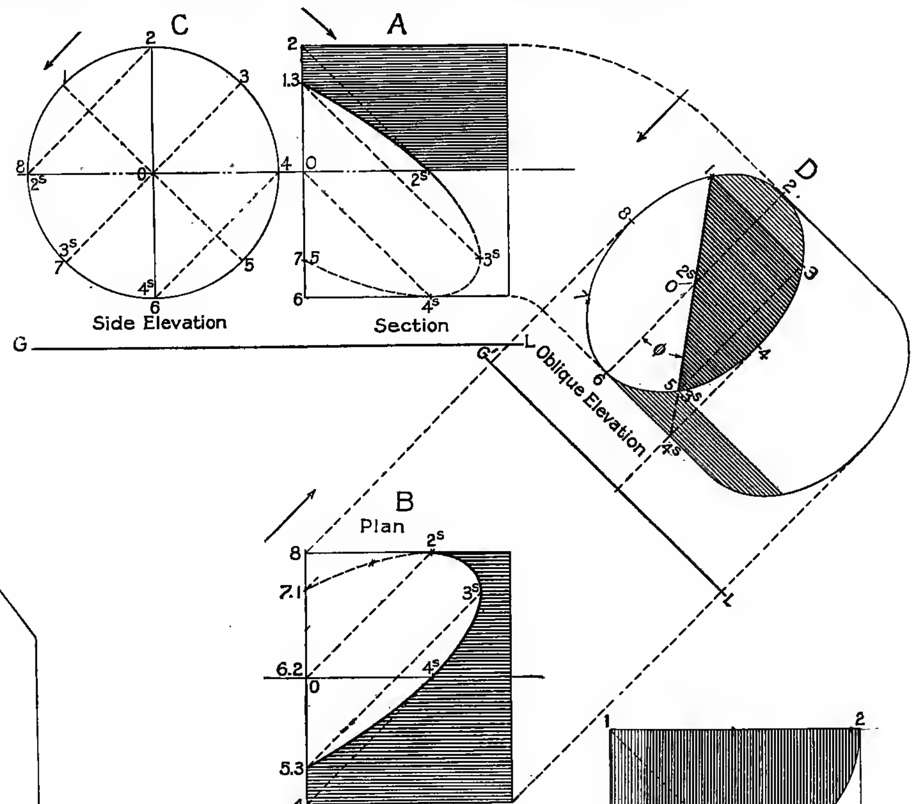


Fig. 61

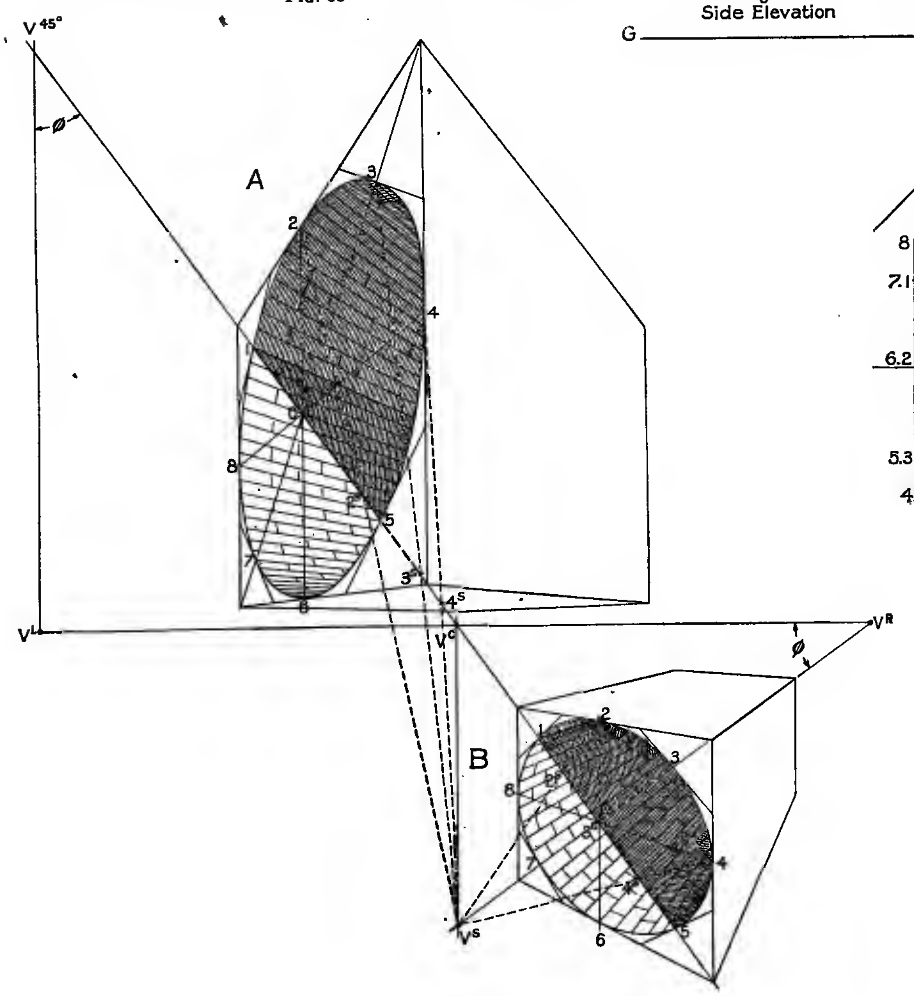


Fig. 62

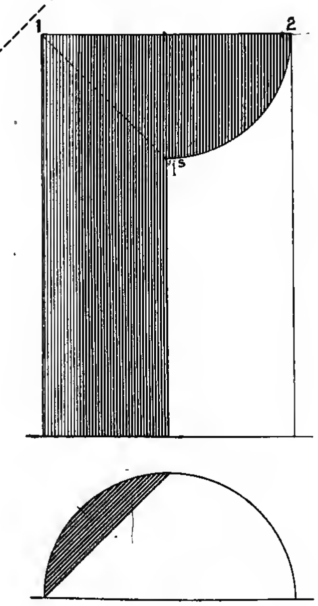


Fig. 64

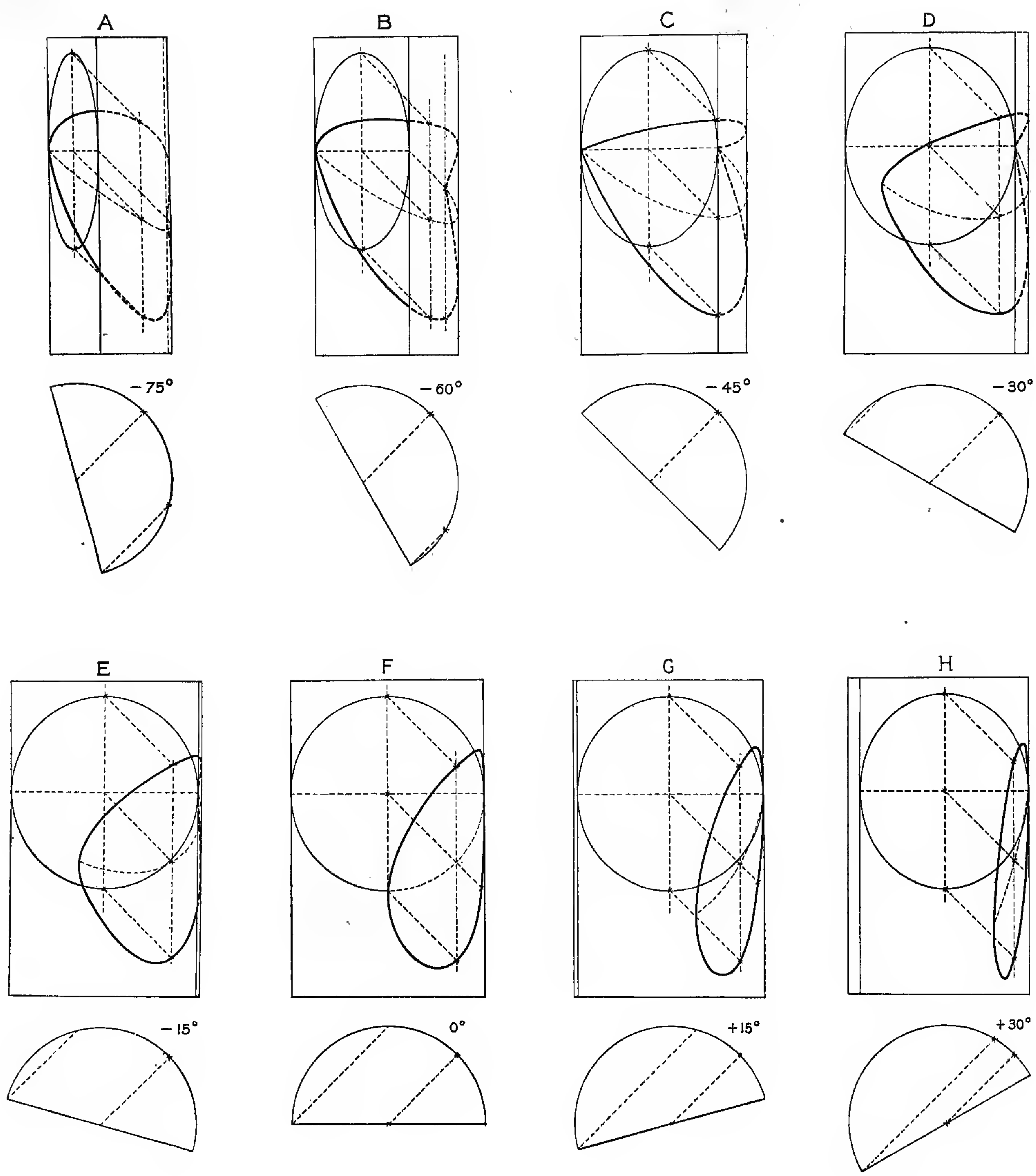


Fig 65

The shadow of the upper semicircle $a\,d\,b$ falls at $a\,d^s\,b$, and that of the lower one $a\,d_1\,b$ at $a\,d_1^s\,b$. The line $a\,c^s\,b$, which is the shadow of the diameter $a\,c\,b$, is a true semiellipse, and the shadows of the semicircles which are symmetrical above and below this diameter, are also symmetrical above and below its shadow, and are also true semiellipses, as are also their projections.

The Invisible Shadow of the circle is an elliptical cylinder, the Major Axis of which, the line $a\,b$, is horizontal, and is equal in length to the diameter of the cylinder. (Fig. 66 *B* and *D*.)

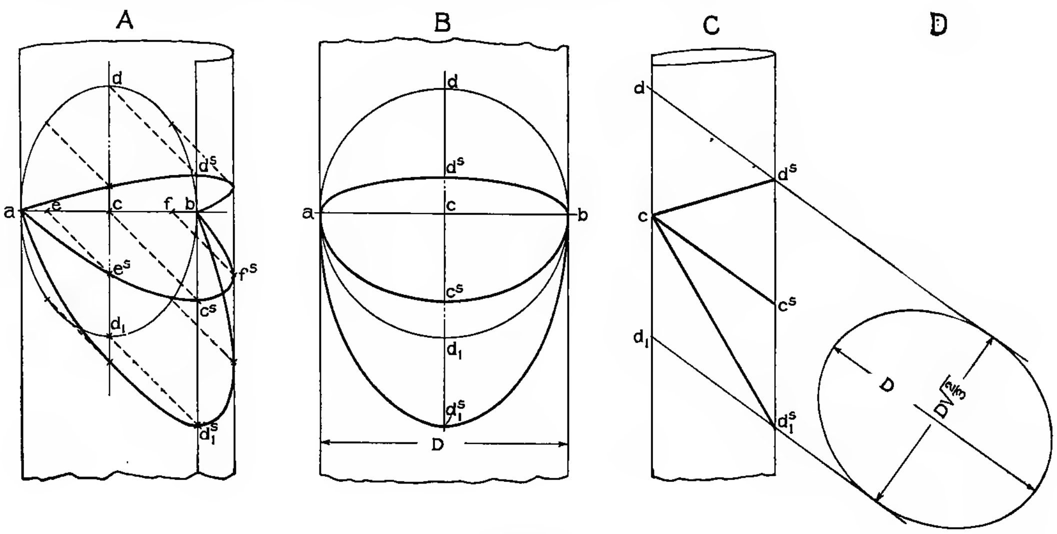


Fig. 66

At *C* are sections of both cylinders taken parallel to the light, which falls at the angle ϕ. The top of the circle, at d, throws its shadow at d^s, the bottom d_1 at d_1^s, and the center c at c^s.

88. When the secant plane is inclined to the Vertical Plane of Projection more than -45 degrees, as at *A* and *B*, in Fig. 65, the shadow cast by the edge of the cylinder is not visible in elevation. When the inclination is 45 degrees, or more, the hollow surface is altogether in shade or shadow, the visible portion being in shadow.

89. The shadows cast upon oblique cylinders by vertical circles which are normal to the secant plane, like that in Fig. 58 A, are not here considered. They are not of practical importance.

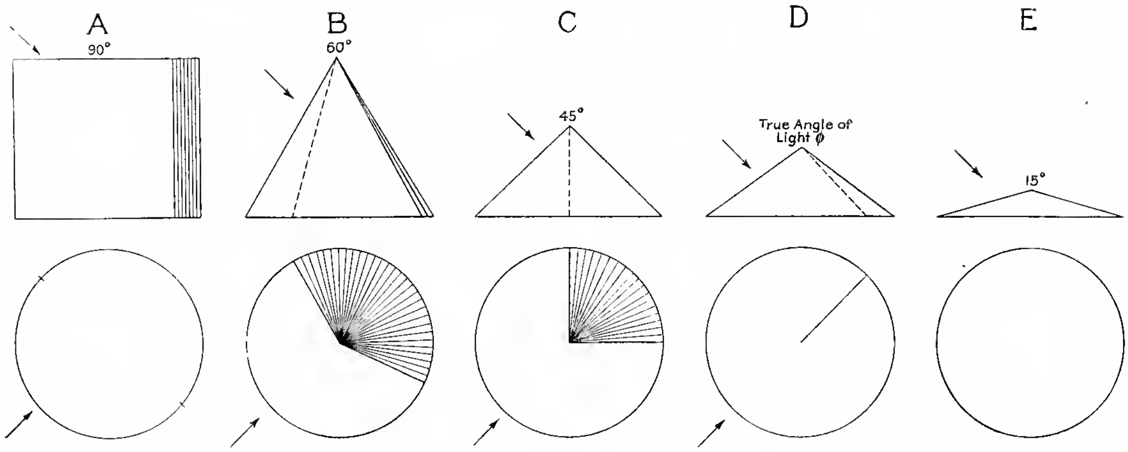


Fig. 67

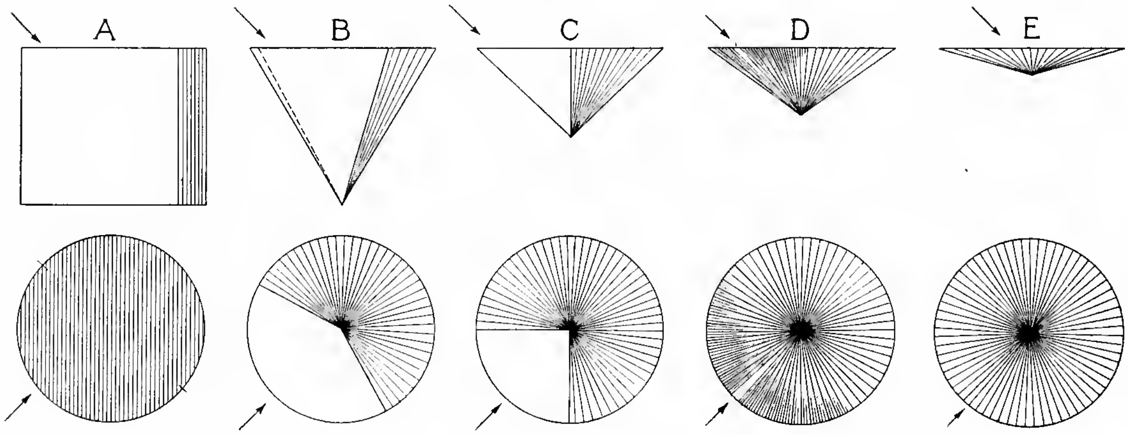


Fig. 68

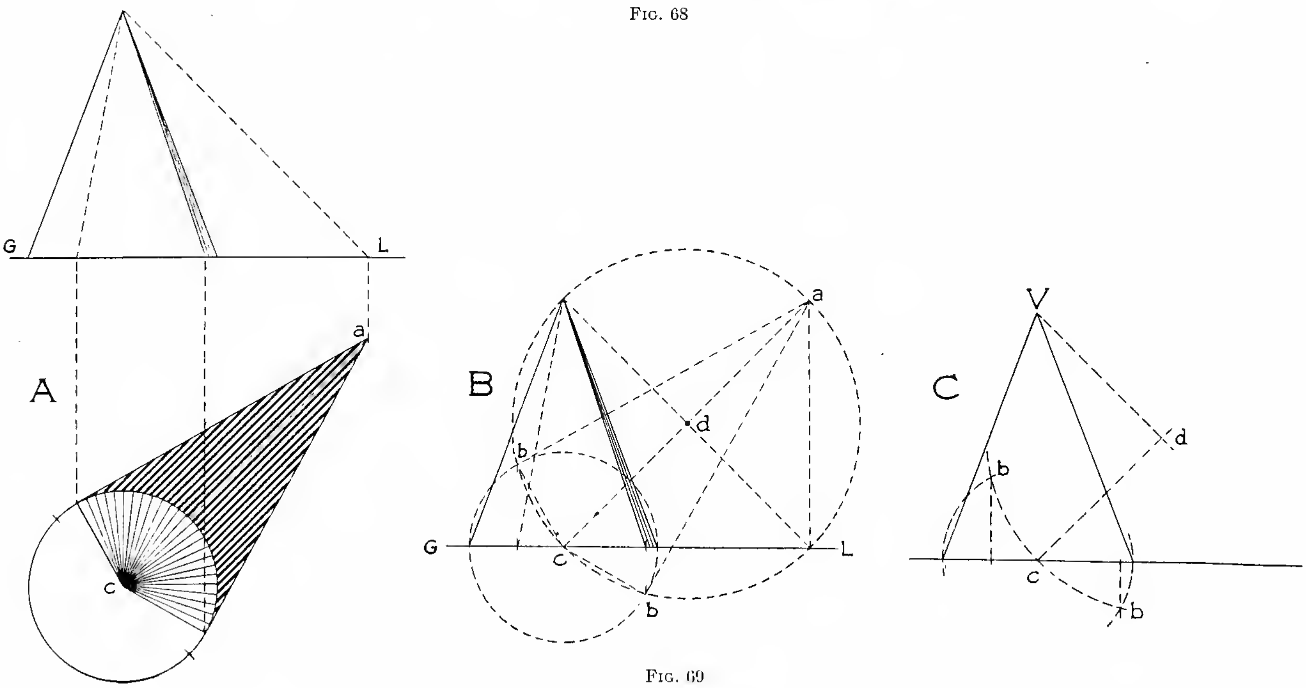


Fig. 69

Chapter VIII

CONES AND HOLLOW CONES

90. *Cones.*—The only cone that need be considered is the right cone of which the base is a circle and the axis is a principal line, that is to say, a line perpendicular to one of the planes of projection.

91. *The Line of Shade.*—If the base is turned away from the light, as in Fig. 67, it is in Shade, and from a half to the whole of the conical surface is in light, according as the angle at the base is more or less steep. If the base is turned toward the light, as in Fig. 68, from a half to the whole of the conical surface is in shade. The Line of Shade is made up, in either case, of an arc of the base, which is more than 180 degrees, and of the two elements of the cone at the extremities of this arc. But if the conical surface is wholly in light, as in Fig. 67 *E*, or wholly in shade, as in Fig. 68 *E*, the Line of Shade comprises the whole circle of 360 degrees, and does not include any element of the conical surface.

92. These figures show that the projection of such a cone upon the plane which is perpendicular to its axis is a circle, the projection of the Line of Shade appearing as two radii and the segment of the circumference included between them. Upon the other two planes of projection the projection of the cone is a triangle, and the Line of Shade appears, in the projection, as comprising a segment of the base of the triangle and the two elements of the cone which go from the extremities of this segment to the vertex, as the whole base.

93. *The Line of Shade.*—It is plain that if the slope of the elements of the cone is less than that of the Sun's rays the whole conical surface will be in light and the whole base in shade (as in Fig. 67 *E*), or *vice versa* (as in Fig. 68 *E*), according as the base is turned away from the light or toward it.

If the slope of the elements of the cone is Φ, or that of the sun's rays, as in Fig. 67 *D*, and the base is turned away from the light, the surface of the cone will all be in light. The shade upon its surface is then reduced to a line, its projection appearing in plan as that of a single element of the cone, lying at an angle of 45 degrees, as in Fig. 67 *D*. If the base is turned toward the light and the conical surface turned away, the illuminated portion of this surface will in like manner be reduced to a single line, as is approximately shown in Fig. 68 *D*. These lines start from one "corner" of the base, and lie endwise to the light, being neither in light nor in shade.

94. When the slope of the conical elements is just 45 degrees, as in Fig. 67 *C*, and 68 *C*, a quarter of the first conical surface is in Shade, and a quarter of the other in Light, and in the vertical projections the two elements of the cone that enter into the Line of Shade comes on the outline and on the axis.

When the slope is 90 degrees, as at *A* (that is to say, when the cone becomes a cylinder), half the surface is in light and half in shade, and the elements of the cylinder, which are Lines of Shade, start from the "corners" of the base. Intermediate slopes show the Line of Shade in intermediate positions between the corner and the outline, as at 67 *B* and 68 *B*.

95. *Critical Cones.*—The three cones which have the slope of 90 degrees, 45 degrees, and 35° 15′ 50″ (which is the angle Φ), are called the three "Critical" cones, though the first of them is really a cylinder.

The Line of Shade upon a Cone is used to obtain the Line of Shade upon Surfaces of Revolution. Cones are constructed tangent to such surfaces, as has been shown in Fig. P in the Introduction. The points at which the Lines of Shade upon the cones are tangent to the Surface of Revolution are points in the Line of Shade of that surface. The three Critical Cones are generally sufficient for this purpose.

96. *The Line of Shade Upon a Cone* (Fig. 69 *A*, *B*, and *C*).—The Line of Shade upon a Cone does not start, as might be supposed, as does that upon a Cylinder, from the "corner" of the base. Its position is found, as is that upon a Pyramid (see Fig. 35, Chapter IV), by first getting the shadow Cast by the vertex of the Cone upon the plane of its base, and then drawing the shadows of the two elements of the cone that form the Line of Shade. These proceed from the shadow of the vertex and are tangent to the base. The points of tangency are points in the required Line of Shade and their shadows are points in its shadow. The Line of Shade in the Elevation can then be obtained from that in the Plan, as in the Fig. 69 *A*. This process may be abbreviated by combining

the Plan and Elevation in the same figure, as is done in Fig. 69 *B*, where a circle described about the point *d* with the radius *d*, determines the points *b* and *b*. This procedure is still further abbreviated in Fig. 69 *C*, in which the point *d* is at the intersection of 45-degree lines drawn through *V* and *c*.

97. *The Shadow of a Cone.*—The shadow of a cone upon any plane is obtained by drawing from the shadow of its vertex lines tangent to the shadow of its base, as has been already illustrated in Figs. 69 *A* and 71. These lines are the shadows of the elements of the cone that enter into its Line of Shade.

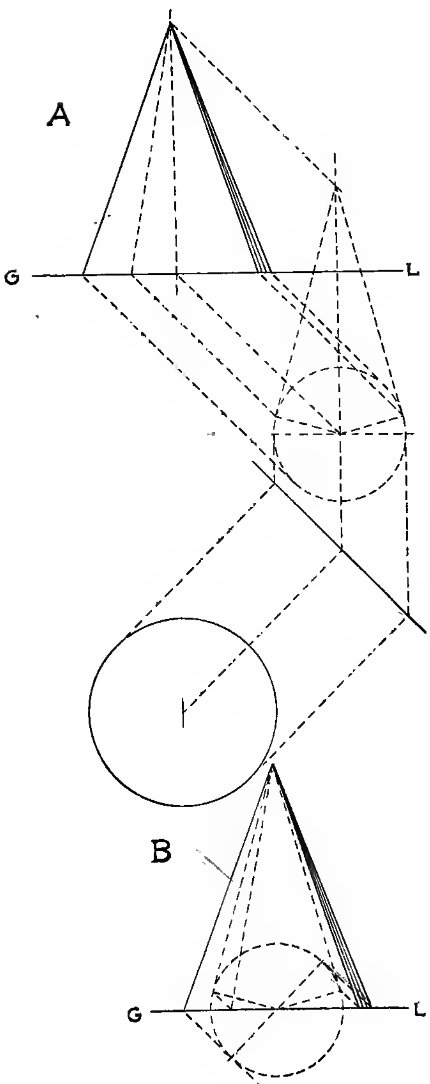


Fig. 70

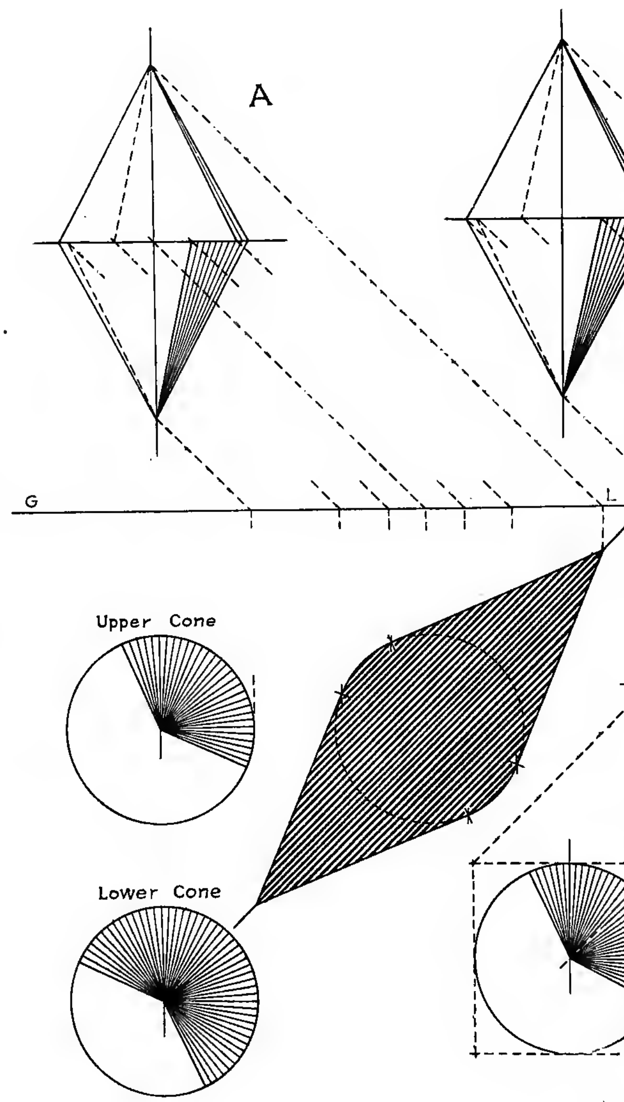


Fig. 71

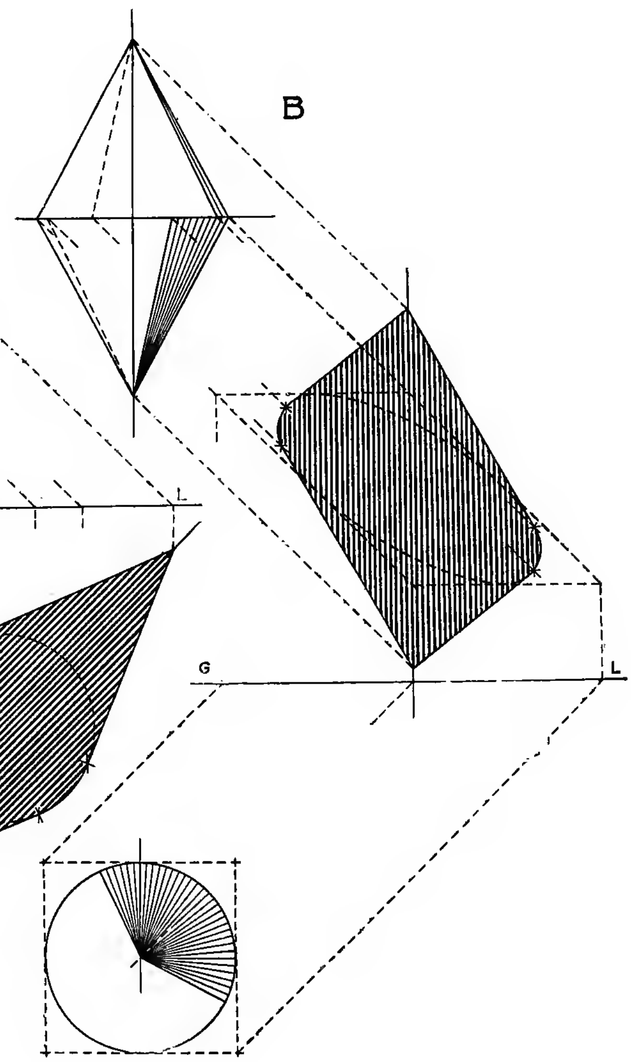


Fig. 72

98. Fig. 70 *A* and *B* shows how the same result is obtained by M. Pillet by means of the auxiliary 45-degree Plane of Projection, described in Chapters III and IV, and illustrated in Figs. 24 and 47.

99. The shadow of the base cast upon the auxiliary oblique plane is projected as a circle, as shown in these figures, the diameter of which can be obtained directly from the Elevation by drawing 45-degree lines from the extremities of the base. The Axis casts a shadow of its own length. Lines drawn from the shadow of the vertex of the cone tangent to this circular shadow of the base give, as at *A*, the shadows of the elements of the cone that form part of the Line of Shade, and the Line of Shade itself can be obtained from them, by tracing back, to points upon the projection of the base, the rays that cast their shadows. This process may again be

abbreviated, as at B, by combining the Plan and Elevation. This is virtually passing the auxiliary 45-degree plane through the Axis of the cone.

100. Figs. 71 and 72 illustrate the case of a Double Cone. In Fig. 71 the shadow is cast upon a horizontal plane parallel to the base of the Cone. In Fig. 72 it falls upon a vertical plane parallel to the Axis, and the rectilinear elements of the shadow are tangent to the elliptical shadow of the base. But it is not easy to draw this ellipse with sufficient precision to determine the point of tangency with exactness, and the methods of finding the Line of Shade shown in Figs. 69 and 70 A are preferable.

101. *Hollow Cones.*—A Cone, like a Polyhedron, Cylinder, or Sphere, may be either Solid or Hollow, that is to say, it may be either the outer surface of a solid body which it surrounds or the inner surface of a solid by which it is surrounded.

If two such cones, one solid and one hollow, of the same shape and size, are exposed to light falling in the same direction, as large a portion of the conical surface as is turned toward the light in one will be turned away from it in the other, and *vice versa.* The portion of the convex surface of the Solid Cone which is turned toward the light, and is consequently illuminated, will thus be exactly like the portion of the concave surface which is turned away from the light and is consequently in shade; and the portion of the solid cone which is in shade, being turned away from the light, will be just like the portion of the concave surface which is turned toward

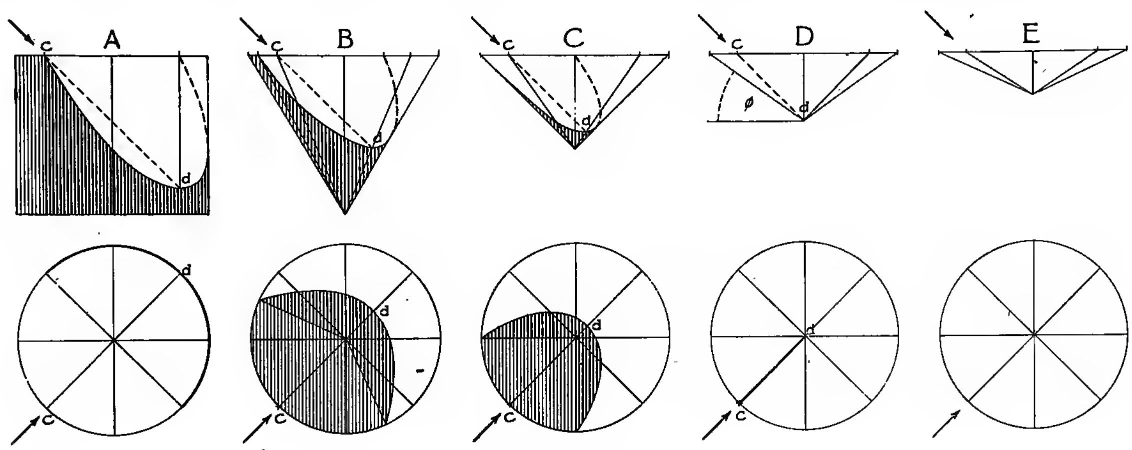


Fig. 73

the light, and is consequently either in light or in the shadow of the other part. The elements of the conical surface which constitute part of the Line of Shade upon the Solid Cone will correspond to the elements which constitute the Line of Shade and Shadow upon the concave surface of the Hollow Cone.

This becomes apparent on a comparison of the five Hollow Cones shown in Fig. 73, with the five Solid Cones shown in Fig. 68.

Fig. 73 D shows also that if the base of a Hollow Cone is turned toward the light, and its elements make with the base the angle ϕ, one element of the cone will be parallel to the light, and will be neither in light nor in shade, while all the other elements of the cone, that is to say, all the rest of the concave surface, will be in light.

102. If the angle is less than ϕ, as at E, the whole inside surface will be illuminated. The Line of Shade will be a complete circle, but it will cast no shadow upon the inner surface of the cone. If the angle is greater than ϕ, as at B and C, the Line of Shade will be made up of an arc of the circle less than 180 degrees, and of two of the elements of the cone which constitute the Line of Shade and Shadow. The portion of the inner surface included between these two elements will be turned from the light and be in shade, and will cast a shadow upon the portion of the interior surface that faces the light.

103. If the angle in question is 45 degrees (Fig. 73 C), the angle at the vertex of the cone will be 90 degrees, and a quarter part of the interior surface will be in shade. Part of the remaining surface facing the light will be occupied by a shadow, which will be bounded on the side toward the light by the two elements of the cone which form part of the Line of Shade and Shadow, and on the other side by the shadow of the 90-degree arc which forms the Line of Shade.

104. If the angle is 90 degrees, as at *A*, the Hollow Cone becomes a Hollow Cylinder. Half the rim is part of the Line of Shade, and half the interior surface is in shade, as appears in Figs. 57 and 60, Chapter VII. The Lines of Shade and Shadow separating the Shade from the Shadow are elements of the Cylinder.

The lowest point of the Shadow is, in every case, at the shadow of *c*, the corner of the base nearest the light.

105. In Fig. 74 *A*, *B*, *C*, the Hollow Cone shown in Fig. 73 *B* is shown at a somewhat larger scale, projected upon three different planes. Its axis is horizontal, and perpendicular to the vertical Plane of Projection.

At *A*, the Cone is projected upon the Vertical Plane of Projection, to which the base of the cone is parallel. The lines *a v* and *a′ v* are the elements of the cone which constitute the Line of Shade and Shadow. The segment *a′ c a v* is in shade, being turned away from the light, and the rest of the concave surface, being turned toward the light, is partly in light and partly in shadow. The surface in shadow lies between the Lines of Shade and Shadow *a v* and *v a′*, and the Line of Shadow *a b d b′ a′*, which is the shadow of the line *a c a′*, the shadow of the point *c* falling at *d*, as appears in Fig. 74 *C*.

106. The Line of Shadow cast by the base of the Hollow Cone upon its interior surface, is, like that cast by the rim of a Hollow Cylinder, as shown in Fig. 60, a true ellipse, and for the same reason. For, as has been said in Paragraph 75, Chapter VII, it is a maxim of Geometry that if a Conic Section of Revolution, such as a Cylinder or Cone, is cut by a plane, both the line of intersection and its projection upon the concave surface are Conic Sections. The Line of Shadow in question is such a projection and since it lies across the cone, must be an ellipse. It is accordingly a plane figure, and its projection upon a plane at right angles to the plane in which it lies must be a right line, as is shown at *a′ d*, in Fig. 74 *C*. The Projection of this Elliptical Line of Shadow upon the horizontal plane, as shown in Fig. 74 *B*, is of course also an ellipse.

107. Fig. 74 shows at *B* the Horizontal Projection of a Hollow Cone, and at *C* a Projection upon an Auxiliary Plane parallel to

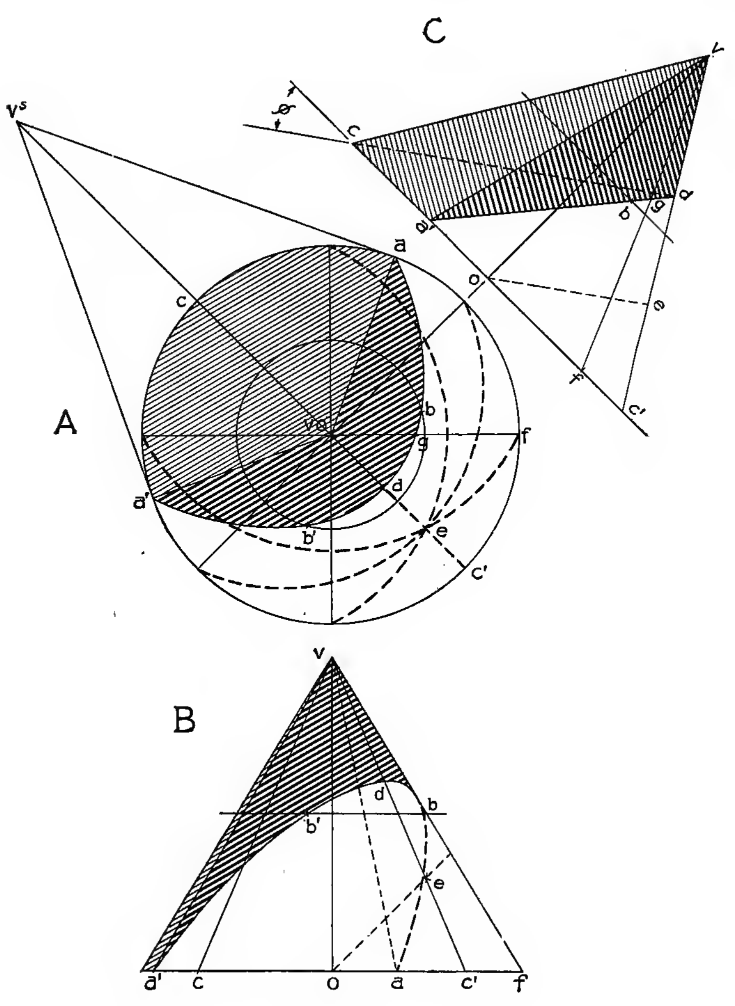


Fig. 74

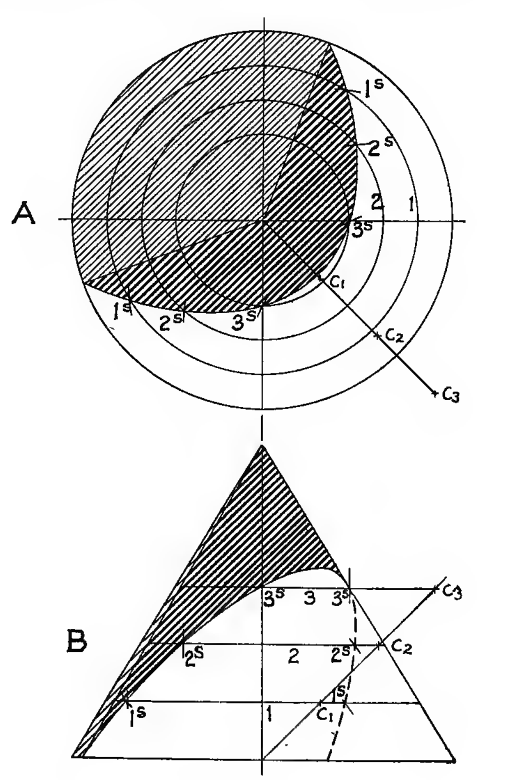


Fig. 75

the direction of the light. The points a and a' are found, in the Vertical Projection, by laying off upon the 45-degree line the distance $v\,v^s$, which is equal to the diagonal of the height of the Cone just as was done in the case of the inverted cones shown in Fig. 68, and then drawing lines $v^s\,a$ and $v^s\,a'$, tangent to the circumference of the base. Elements of the cone drawn from these points to the vertex v, give, in all three projections, the Lines of Shade and Shadow. The ray of light $c\,d$ in Fig. C, gives the lowest point of the ellipse of shadow at d. As this ellipse lies in a plane which contains the point d and the two points a and a', its projection at C must be a right line, as shown.

Through the point b, taken at random upon the line $a\,d$, is passed a horizontal plane which cuts the Cone in a circle that is projected as a right line in C and B, and as a circle at A. The point b is then transferred to A and B at b and b', by simple projection. The elliptical Line of Shadow passes through the five points a, b, d, b', and a'.

The shadow of the center of the base o falls at the point e, and the four principal diameters of the base, which meet and cross at the center, cast shadows that meet and cross at that point. One of these shadows is a right line. The others are ellipses, being the lines in which the Planes of the Invisible Shadows of the three diameters cut the conical surface. They all pass through this point e, and also through the extremities of the diameters, which are, so to speak, their own shadows.

107. The Line of Shadow in a Hollow Cone may also be found by the Method of Parallel Planes, as is illustrated in Fig. 75 A and B.

Three vertical secant planes *1*, *2*, and *3* being passed through the Hollow Cone parallel to the Vertical Plane of Projection and to the base of the cone, the lines of intersection appear projected as straight lines at B, and as circles lying in those planes at A. The shadows of the base of the cone cast upon these three planes have their centers at c_1, c_2, and c_3, and their circumferences intersect the circles *1*, *2*, and *3* at the points 1^s, 2^s, and 3^s. These are points in the required ellipse of shadow, which is the same ellipse as that found in Fig. 74. These points are found also at $75B$, by simple projection from the others.

107. *The Paradox of the Line of Shade* (Figs. 76 and 77).—If a number of Cylinders, gradually diminishing in size, are piled up in a conical manner, the Line of Shade on each will be on the "corner," and the total Line of Shade will be a broken line, or series of steps, at the corner of the pile, irrespective of its steepness, and irrespective of the size of the steps. Even if they are so small that the surface looks almost smooth, and the pile of cylinders looks like a cone, the Line of Shade will not change, though that upon the cone which it simulates may lie anywhere between the "corner" and the outline, according to the slope.

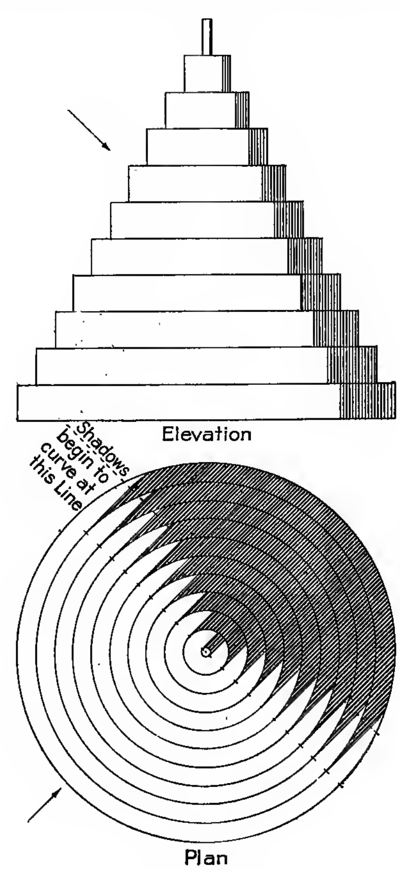


Fig. 76

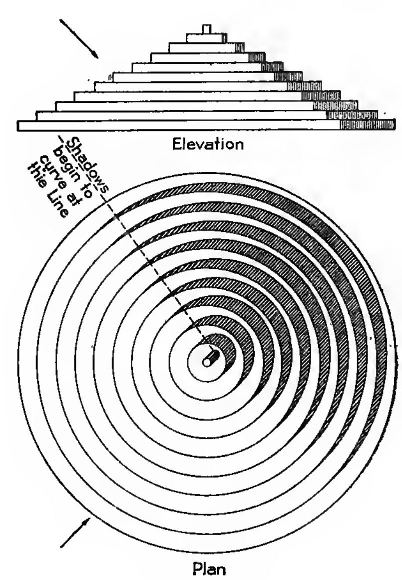


Fig. 77

Chapter IX

SPHERES

110. A Sphere is a solid generated by the revolution of a circle about one of its diameters, as an axis. Every point of its surface is at the same distance from the center, this distance being the length of the radius of the generating circle, and if the surface is cut by a plane the line of intersection is a circle. If the secant plane passes through the center of the sphere this circle is called a Great Circle. It has the same radius as the Sphere and is of the same size as the generating circle. Other sections are called Small Circles.

The projection of a sphere upon a plane which is perpendicular to the line of projection is a circle equal to a Great Circle of the sphere (Fig. 78, *A*, *B*, *C*); upon any other plane it is an ellipse whose Minor Axis is equal to the diameter of the sphere (Fig. 78 *D*).

The shadow of a sphere as cast upon a plane is either such an ellipse or, if the plane is perpendicular to the light, a Great Circle.

The Line of Shade Upon a Sphere.—As was said in the Introduction, there are five ways of finding the Line of Shade upon a Surface of Revolution: (1) The Method of Projected Tangent Rays and (2) the Method of Tangent Cylinders, both of which give six or eight points of the Elliptical Line of Shade, and (3) the Method of Tangent Cones; (4) the Method of Slicing, and (5) the Method of Revolved Tangent Rays, all three of which give any desired number of points. The Method of Projected Tangent Rays gives also the Major and Minor Axes of the ellipse of shadow.

111. *The Method of Projected Tangent Rays* (Fig. 78 *A*, *B*, and *C*).—When a Sphere is exposed to the sunlight, half of its surface is in light and half in shade, and its Line of Shade is a great circle of the Sphere, lying in a plane perpendicular to the direction of the light. All the diameters of this circle terminate in the Line of Shade, and that one of them which is parallel to the plane of projection terminates also in the circle in which the Sphere is projected. This diameter is projected of its true length and is at right angles to the projection of the rays of light. A diameter of the circle in which a Sphere is projected, drawn perpendicular to the projection of the rays of light, will then give, upon the circumference of this circle, two points in the Elliptical Line of Shade. The rays of light are tangent to the sphere at these two points. They are at the extremities of the Major Axis of the Ellipse.

112. Fig. 78 shows a Sphere projected upon three different Planes of Projection, in each of which a diameter taken at right angles to the light terminates in the Line of Shade and determines two points upon it. The Vertical Projection at *A* gives the points *a* and *a*, and the horizontal projection at *C* gives the points *c* and *c*. At *B* the sphere is projected upon a plane parallel to the rays of light, and the Great Circle which constitutes the Line of Shade is seen edgewise, as a right line. The diameter drawn perpendicular to the rays of light is here the projection of the Line of Shade. It terminates in the points *b* and *b*, which are two more points of the Line of Shade.

At *A* the four points thus found in Figs. 78 *B* and *C* are projected from the figures at *b* and *b*, *c* and *c*.

113. Fig. 78 A thus gives the Major Axis of the ellipse in which the Line of Shade is projected at *a a*, the Minor Axis at *b b*, two points upon the ellipse at *c c*, and, from the symmetry of the figure, two additional points at *c′* and *c′*. The ellipse is found to be described about two squares, whose diagonals constitute its Major Axis, and also to pass through the points *b* and *b*.

The Method of Tangent Rays thus gives eight points in the Line of Shade, five of which are visible.

The points *b*, *b*, *c*, *c*, and *c′*, *c′*, in Fig. 78 *A*, instead of being taken from Figs. B and C, may be found directly, as shown at A, by inscribing within the circle an equilateral triangle. The sides of this triangle cut the 45-degree diameter parallel to the projection of the rays of light, at *b* and *b*, and its base cuts the vertical and horizontal diameters at *c* and *c′*, points with which the other *c* and *c′* are symmetrical. Fig. 78 C, in which the ellipse is like that in Fig. 78 A (the projection of the rays of light again falling at 45 degrees), shows that the ellipse of the Line of Shade may be circumscribed about two small equilateral triangles, whose common base constitutes the Minor Axis of the ellipse.

114. The line ob (Fig. 78 A and B) measures $\sqrt{\frac{1}{3}}$, since it is the short side of a right triangle, the hypotenuse of which is 1, the radius of the sphere, and the smaller angle is ϕ (see Fig. 78 E).

At C the smaller angle measures 30 degrees, and the long side of the triangle equals 1. The short side then here also measures $\sqrt{\frac{1}{3}}$ (see Fig. 78 F).

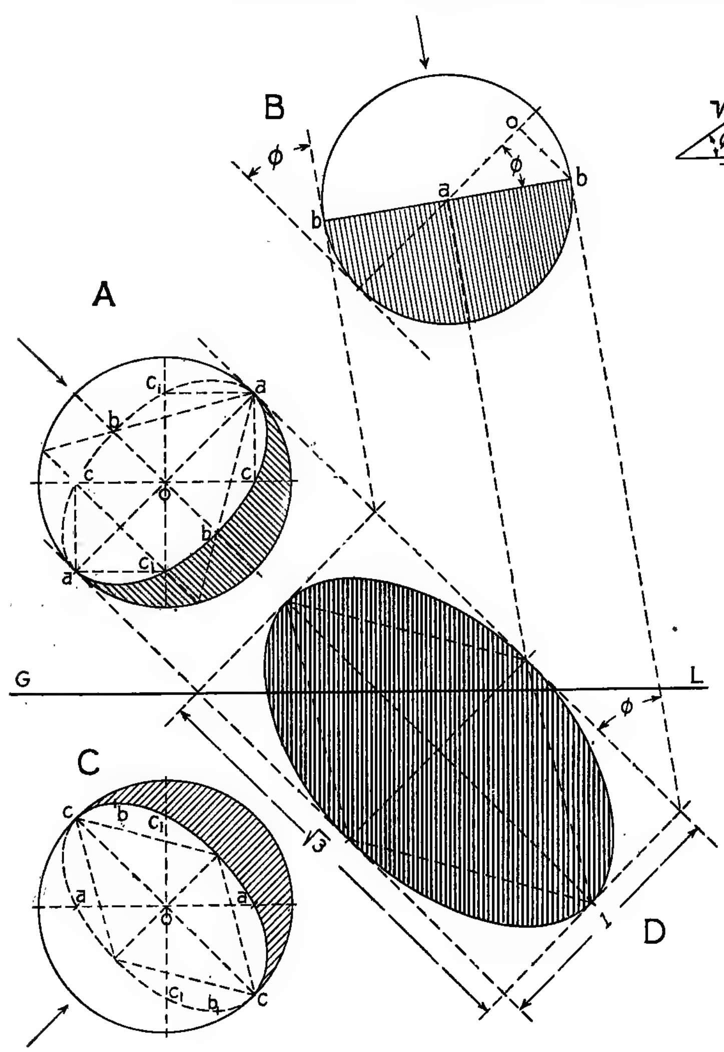
The Semiminor Axis of the ellipse is then $\sqrt{\frac{1}{3}}$, $(=\frac{1}{3}\sqrt{3})$, when the radius of the sphere is taken at 1.

115. *The Method of Tangent Cylinders* (Fig. 79 A and B).—As was shown in Fig. O in the Introduction, if a cylinder is tangent to a Surface of Revolution, both having the same axes, the two elements of the cylinder that form a part of its Line of Shade will cut the line of tangency at two points, and those points will be points in the Line of Shade of the Surface of Revolution. The figure shows at A that the Line of Shade upon the vertical cylinder gives the points c and c, that of the horizontal cylinder gives the points c' and c', and, at B, that of the Normal cylinder, also horizontal, and perpendicular to the Vertical Plane of Projection, gives the points a and a upon the Line of Shade of the Sphere.

The points c, c; c' and c' are most easily obtained by projecting the points a and a, thus determined in the vertical projection, upon the vertical and horizontal axes, as is done at A.

The three principal cylinders thus give six points upon the Line of Shade of the Sphere, four of which are visible in each projection.

116. If now two more cylinders are constructed tangent to the sphere and still parallel to the Vertical Plane, but making 45 degrees with the Horizontal Plane, as in Fig. 79 C, the projection of the cylinder whose axis is



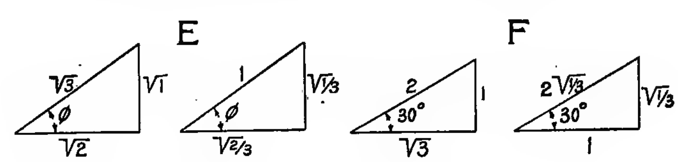


Fig. 78

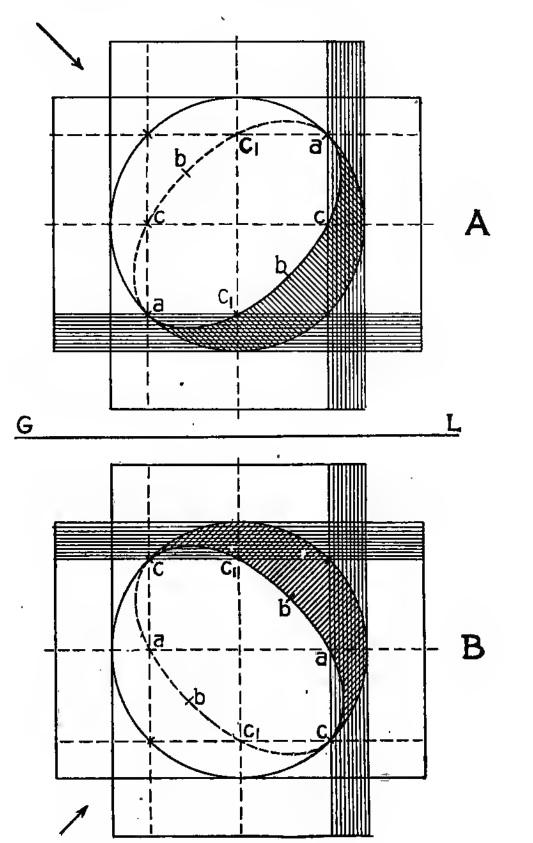


Fig. 79

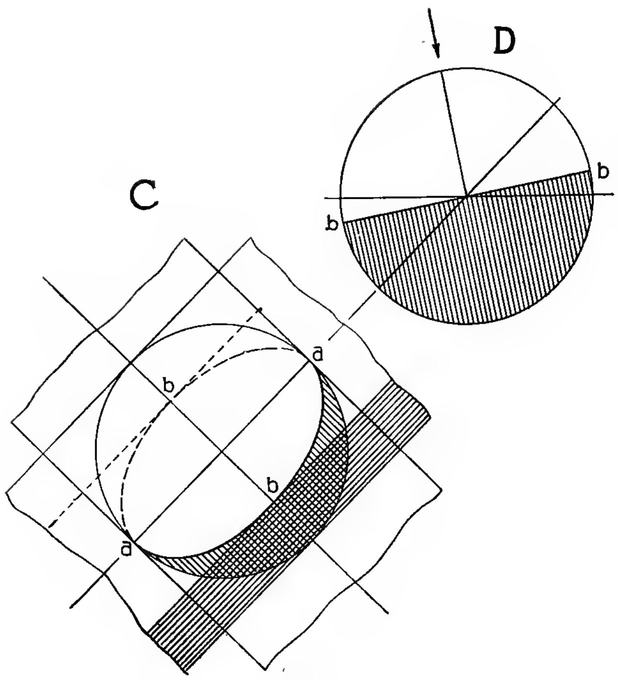


FIG. 79

parallel to the projection of the light, will have its Line of Shade upon its outline and will again give the points a and a on the Line of Shade of the Sphere. The tangent cylinder, the projection of whose axis is at right angles with the projection of the rays of light, as appears in C and D, will give two other points of the required Line of Shade, b and b.

The two 45-degree cylinders thus give two new points, b and b, of the required Line of Shade, making eight in all, five of which are visible in the Elevation at A, namely a, a; b; c, and c'

These eight points are the same as those given above by the method of projected Tangent Rays.

The Method of Tangent Cones.—As was shown in Fig. P in the Introduction, if a Cone which has the same axis as a Surface of Revolution is tangent to it, the two elements which form part of its Line of Shade will cut the circle of tangency at two points, and these points will be points in the Line of Shade upon the Surface of Revolution.

117. *The 45-Degree Cones.*—Fig. 80, A, B, C, and D shows how points in the Line of Shade upon a Sphere are thus found by means of tangent cones whose elements make an angle of 45 degrees with the base. All the spheres are shown in elevation.

Fig. 67 C has shown that when the axis of a 45-degree cone is a Principal Line, perpendicular to a Plane of Projection, and the base is turned away from the light, three-quarters of the conical surface, including the half which is visible, is in light; one of the elements of the cone which form part of the Line of Shade is on the outline which is farthest from the light, and the other is on the further side of the cone, and its projection coincides with that of the axis. This element is invisible, as is also the entire surface of shade. The visible portion of the cone is accordingly entirely in light. Fig. 68 C has shown that if the base of a 45-degree cone is turned toward the

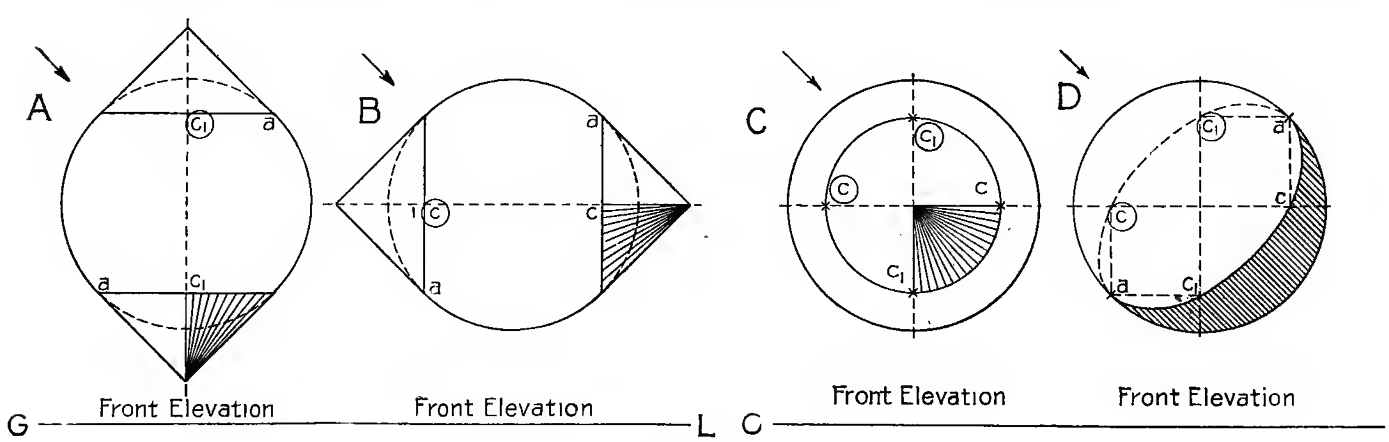


FIG. 80

light three-quarters of the conical surface is in shade, two of which are out of sight; one of the elements that constitute the Line of Shade is on the outline of the cone that is nearest the light, and the other is on the near side of the cone, its projection coinciding with that of the axis, and half of the visible surface is in shade.

Fig. 80 A, B, and C shows that six such cones can be tangent to a sphere, two upon each of the three principal axes. Three have their bases turned away from the light, and show no part of the Shade, and three have them turned toward the light and show one-third of the Shade, covering half the visible surface. At A, where the axis of the cones is vertical, the upper cone gives two points on the Line of Shade of the Sphere, a on the right, and c' behind. The lower cone gives a on the lift and c' in front. The horizontal cones at B give the four points a, a, c, and c; and the Normal cones at C give c, $c\ c'$ and c'. The six 45-degree tangent cones thus add no new points to the eight already determined by the tangent rays and cylinders.

Fig. 80 D shows that if either of these points in the Line of Shade of a Sphere is known, the other five can be determined from it.

In these figures, the letters c and c', which indicate points that are out of sight, being on the farther side of the sphere, are enclosed in circles.

118. *The ϕ Cones* (Fig. 81 A, B, C).—The six tangent cones whose elements make the angle ϕ with their base (Fig. 81 A, B, C), also give six points of the Line of Shade on the Sphere, four of which are new. Each cone gives one point, that, namely, at which the single element which constitutes its Line of Shade is tangent to the Sphere. The two vertical cones (Fig. 81 A), give d and d, the highest and lowest points of the Great Circle of the Line of Shade and of the ellipse in which it is projected; the horizontal cones at $b\ B$ give d' and d', the extreme right and left points of the ellipse; and the Normal cones at C give the points b and b. Since the line joining b and b lies at 45 degrees, at right angles with the Major Axis of the ellipse, these must be at the extremities of the Minor Axis.

These six points are all shown in Fig. 81 D.

119. Fig. 82 A shows that b and b are distant $\sqrt{\frac{1}{6}}$ from the vertical and horizontal diameters of the circle in which the Sphere is projected, since, in the equilateral triangle $a\ b\ b$, $o\ a = 1$, and $o\ b = \sqrt{\frac{1}{3}}$ and d and d are at the

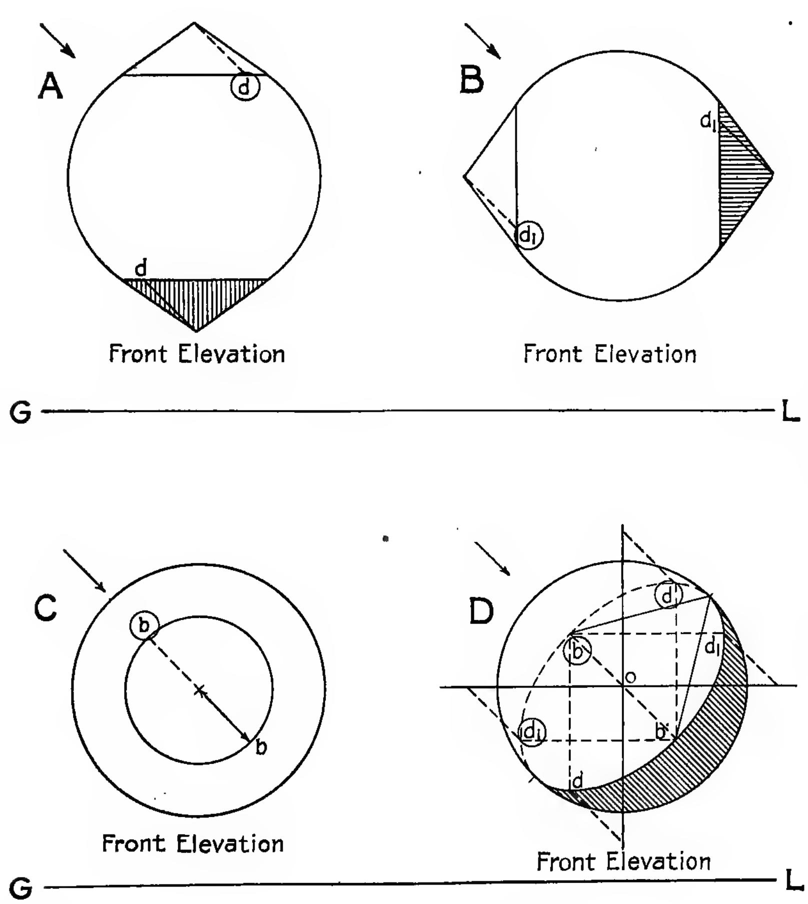


FIG. 81

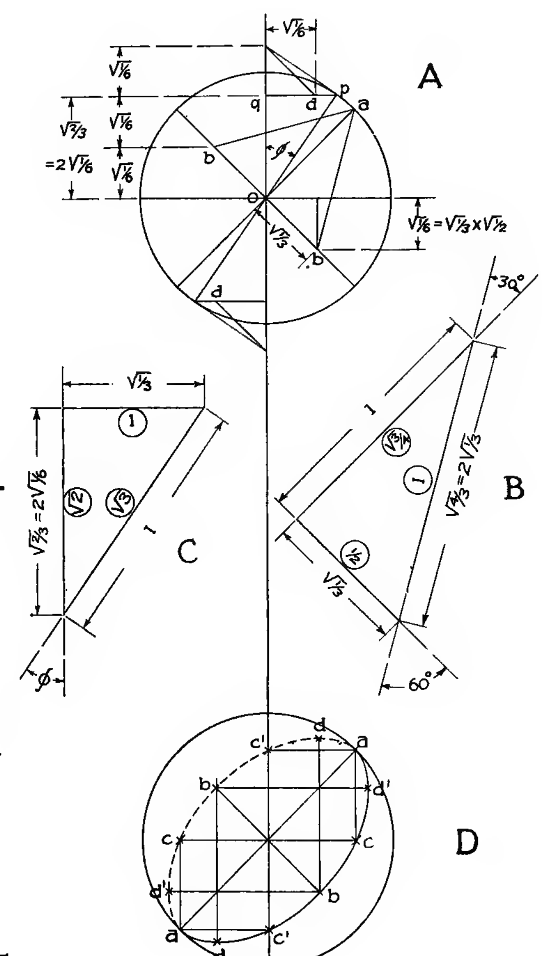


FIG. 82

same distance from one of these diameters and twice that distance from the other. The symmetry of the figure 81 *D* shows that this is true also of the points d' and d'.

Fig. 82 *B* shows that in a 30-degree and 60-degree triangle the long and short sides, which are in the ratio of $\sqrt{\frac{3}{4}}$ and $\frac{1}{2}$, are also in the ratio of 1 to $\sqrt{\frac{1}{3}}$, as in the triangle *a o b* (which is half of the equilateral triangle *a b b*), in Fig. 82 *A*. Fig. 82 *C* shows that in a ϕ triangle the hypotenuse and long side, which are in the ratio of $\sqrt{3}$ and $\sqrt{2}$, are also in the ratio of 1 and $\sqrt{\frac{2}{3}}$ (or 1 and $2\sqrt{\frac{1}{6}}$), as in the triangle *o p q*, in Fig. 82 *A*.

Fig. 82 *D* shows how, by a combination of 81 *D* and 80 *D*, twelve points of the ellipse of Shadow may be determined, seven of which are visible and five invisible, being on the opposite side of the sphere.

These abundantly suffice to determine the Line of Shade by points.

120. The six points furnished by the ϕ Cones may be obtained also as follows: (Fig. 83.)

Draw through the point *a* at the extremity of a 45-degree radius a horizontal chord, and take upon it the point (*1*) distant from the axis half the length of the radius. A radius drawn through this point gives, on the circumference, the point (*2*), the point of tangency of the ϕ cone. (This construction is justified by the consideration that, since the line *o b* equals $\frac{1}{2}\sqrt{2}$, and the line *b 1* equals $\frac{1}{2}$, the line *o 1* must equal $\frac{1}{2}\sqrt{3}$. The sides of the triangle being thus in the proportion of 1, $\sqrt{2}$, and $\sqrt{3}$, the angle at the center must be the angle ϕ.) The point *d* can then be got either by finding the corner of the horizontal circle taken through the point *2*, or as in the figure, by drawing the tangent line *2*, *3*, and from the point *3* drawing a 45-degree line back to the point *d*. This line represents a ray of light tangent to the sphere at the point *d*.

This line very nearly passes through the point (*1*) and the point *d* can accordingly be got without appreciable error by drawing a 45-degree line through the point *1*, as in Fig. 83 *B*. The error is less than $\frac{1}{24}$ of the radius.

121. Any number of additional points in the Line of Shade may be obtained by means of additional tangent cones, as for example, at *b* and *b* in Fig. 84 *A*.

A horizontal chord drawn across the circle at any point, such as the point *c* in the figure, may represent the base of a cone tangent to the sphere along the horizontal circle of which the chord is the projection. This cone has a vertex at *V*, *C V* being the height of the cone.

The circle drawn about the point *C* as a center, with this chord as a diameter, represents the base of the cone in its own plane, and the point V^s the shadow of the vertex in that plane, $C\,V^s$ being the diagonal of the line *C V*. Lines from V^s drawn tangent to the base at *a* and *a*, represent the shadows of the elements of the cone which are its lines of shade, and the lines *V b*, *V b*, are the vertical projections of the lines of shade. The points *b* and *b*

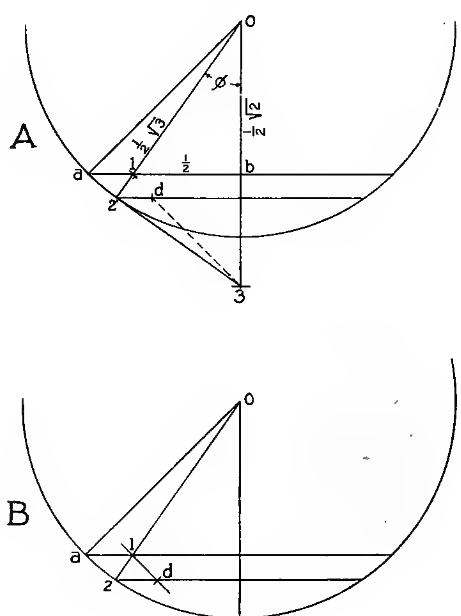


Fig. 83

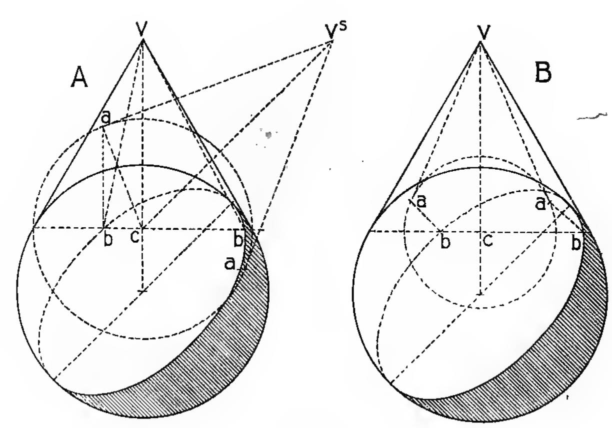


Fig. 84

are points of shade in the horizontal circle, and are points in the required line of shade on the surface of the sphere.

122. In Fig. 84 *B* the same two points are obtained by the Method of M. Pillet, shown in Fig. 71, Chapter VIII. The circle described about the point *C* has now, for its diameter, half the diagonal of the diameter of the base of the cone. This represents the shadow of the base upon a 45-degree plane. The tangent lines *V a* and *Va* represent the shadows of the lines of shade upon such a plane, and *b* and *b* the points which cast them.

123. *The Method of Slicing* (Fig. 85).—If through a sphere is passed a series of parallel planes, one element of which is parallel to the ray of light, the lines of intersection will be circles standing edgewise to the light. The planes will be surfaces of Light and Shade, and in each the half of the edge which is turned toward the sun will be illuminated, the other half will be in shade. Each circle will have two points of shade separating these semicircles, and these points will be points on the Line of Shade of the sphere.

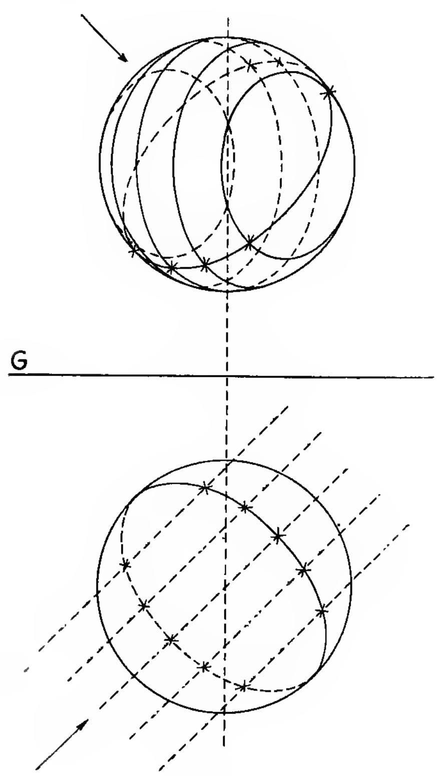


Fig. 85

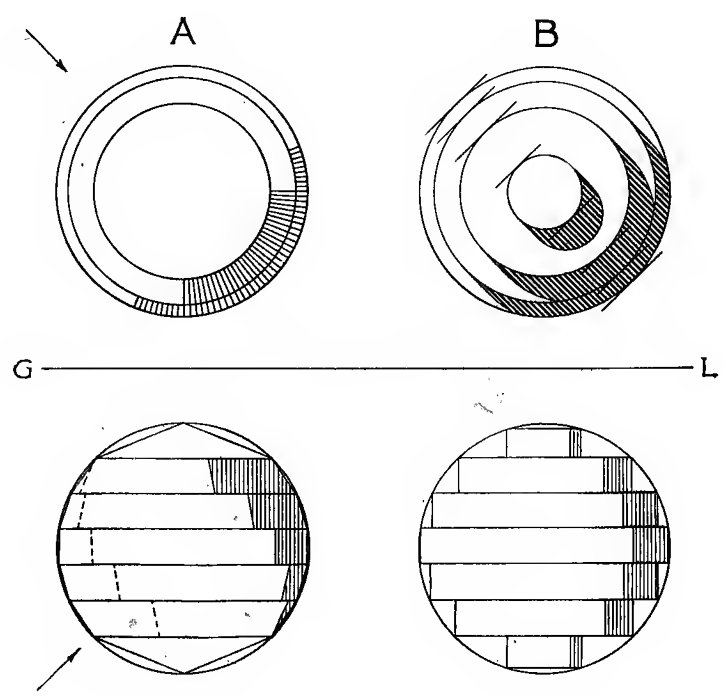


Fig. 86

If the secant plane is, as in the figure, at 45 degrees with the vertical plane of projection, and perpendicular to the horizontal plane, these circles will be projected upon the horizontal plane in right lines at 45 degrees, as in the figure, and upon the vertical plane as Ellipses. But these are difficult to draw and the position of the tangent rays difficult to determine.

124. If the secant planes are taken parallel to the vertical plane of projection, as in Fig. 86, the circles will be projected upon it as circles. But though these circles are easily drawn, it is not easy to find the points of shade upon them, for they must be treated as the bases of tangent Normal cones, and the line of shade upon each must be determined as in Fig. 70. This is somewhat coarsely illustrated in Fig. 86 *A*, in which the sphere is cut up into only seven slices.

However numerous the slices, they would still be segments of cones, and not cylinders. If they were cylinders, each of them would have its line of shade on the corner, as in Fig. 86 *B*, which is manifestly incorrect.

The Method of Slicing is accordingly of no practical advantage for finding the Line of Shade upon a Sphere.

125. *The Method of Revolved Tangent Rays.*—The Method of Revolved Tangent Rays, like that of Tangent Cones, gives any desired number of points on the Line of Shade. But while the Tangent Cones give them directly,

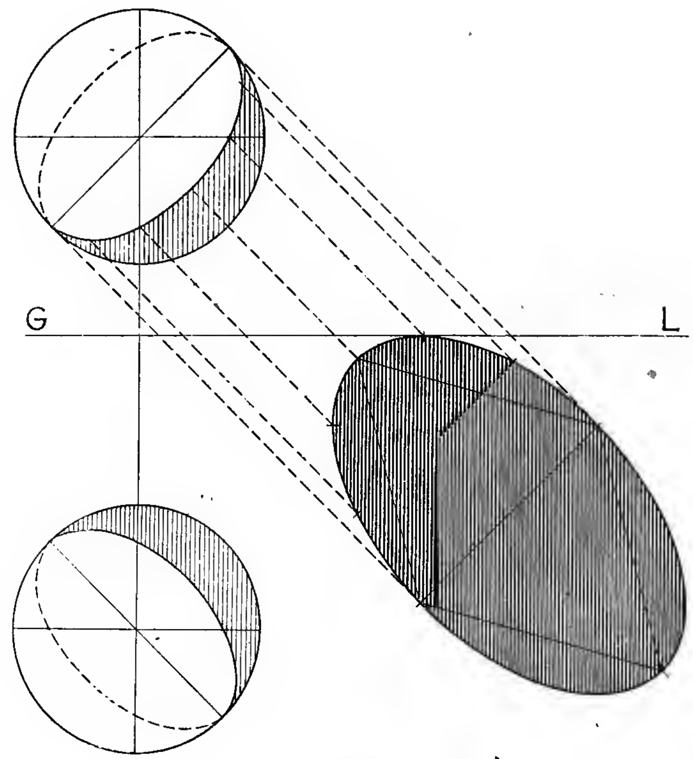


FIG. 88

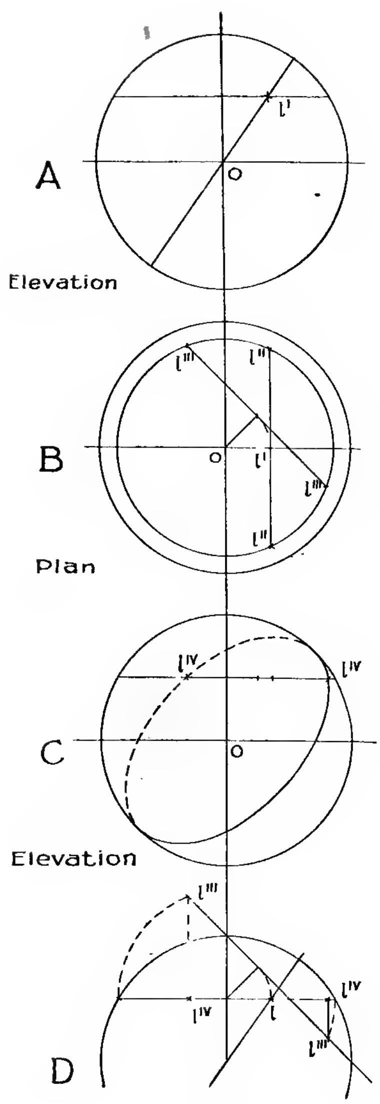


FIG. 87

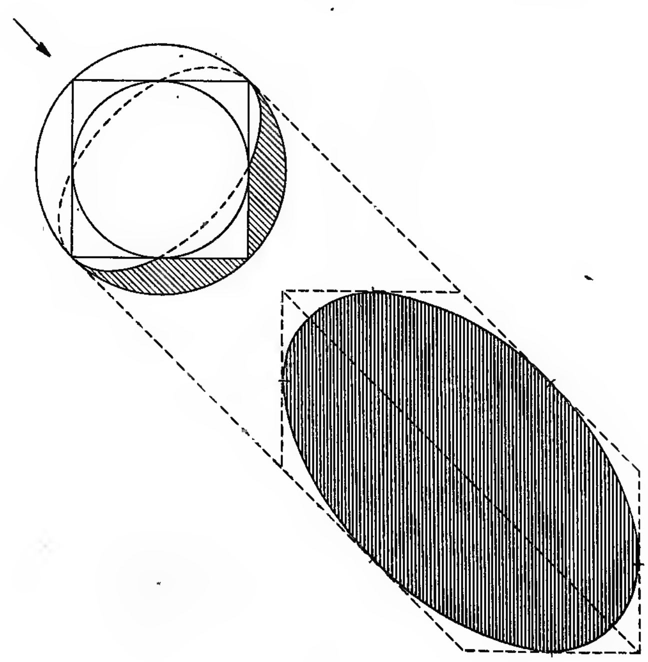


FIG. 89

the Tangent Rays first determine the projection of the Line of Shade when the Light falls parallel with the Vertical Plane of Projection, and then, by revolving the Surface of Revolution 45 degrees on its axis, determines its position when the light falls, as usual, at the angle ϕ so as to make, in both Planes of Projection, angles of 45 degrees with the Ground Line.

But the first of these steps, though serviceable with the Ovoid, as shown in the Introduction (Fig. M), is not needed for the Sphere, inasmuch as the Line of Shade, when the rays of light fall parallel to the Vertical Plane, is projected in a Right Line, which is the diameter of the Sphere at right angles to the projection of the rays, as is shown in Fig. 76 *B*, and Fig. 87 *A*. The plan, Fig. 87 *B*, shows that any point such as l', Fig. 87 *A*, taken upon this line is really the projection of the two points on the Line of Shade marked in the plan as l'' and l'', and that if the Sphere is revolved 45 degrees on its axis these points of the Line of Shade will be at l''' and l''', and will now be projected as points of the elliptical projection of this line, as appears in Fig. 87 *C*, at l^{iv} and l^{iv}.

Fig. 87 *D* shows how all these operations may be combined in a single comparatively simple operation.

126. *Summary.*—In choosing among these five ways of determining the ellipse which defines the Line of Shade upon a Sphere, the simplest way is first to obtain the points *a* and *a*, at the ends of the Major Axis, by drawing a 45-degree diameter at right angles to the direction of the projection of the rays of light, and then to obtain the points at the extremities of the Minor Axis by cutting the other 45-degree diameter by an equilateral triangle, as shown in Fig. 87 *D*.

The ellipse can then either be drawn as a continuous curve, by getting its foci, or can be determined by points. The points *a* and *a* give the four points c, c, c', and c', as is shown in Fig. 80 *D*, and the points *b* and *b*, as appears from Fig. 81 *D*, give the four points d, d, d', and d'. These two operations are combined in Fig. 82 *D*, giving twelve points of the curve, as has been already said, seven of which are visible. If more points are desired, they can be obtained either by the Method of Tangent Cylinders and Cones, or by the Method of Revolved Tangent Rays, which is more convenient for points near the Equator of the Sphere, as the work is more compact.

127. *The Shadow of a Sphere.*—The Shadow of a Sphere, as of any Solid of Revolution, may be obtained, as was explained in the Introduction, in five different ways, that is to say (1) by Points; (2) by Tangent Cylinders or Cones; (3) by Envelopes; (4) by Slicing; or (5) by the Method of Parallel Planes, though this last is not available for casting shadows upon a plane surface. But the special geometrical properties of the Sphere make it possible to obtain the Line of Shadow as well as the Line of Shade, more directly. This may be done both by direct projection and by the use of the Inscribed Square.

128. *The Method of Projections* (Fig. 88).—The shadow of a sphere upon the vertical plane of projection is an ellipse whose Major Axis lies at 45 degrees in the direction of the light, and whose Minor Axis, which is as long as the Diameter of the Sphere, also lies at 45 degrees, at right angles to it.

Fig. 78 has already shown that if the Minor Axis of the Ellipse of Shadow is taken as 1, the Major Axis equals the square root of 3.

This ellipse is accordingly of the same shape as the elliptical projection of the line of shade, but larger, the linear dimensions being in the ratio of $\sqrt{3}$ to 1, and the areas as 3 to 1. Like that ellipse it may be circumscribed about two equilateral triangles, set base to base, as in the figure.

129. *The Method of the Inscribed Square.*—If a square be inscribed in the circle which is the projection of a sphere, it may be regarded as the projection of a Cube. (Fig. 89.) But this cube does not lie wholly within the sphere. It is partly outside of it, partly inside. Each face of the cube cuts the sphere in a small circle. The portion of the cube inside these circles is inside the sphere, but the eight solid angles at the corners of the cube are outside the sphere. The eight small circles, and the eight points where the edges of the cube are tangent to these circles, are common to both surfaces. The Line of Light and Shade, the shadow of which is the outline of the shadow of the sphere, passes through six of these points, and their shadows are points of its shadow, and consequently of the shadow of the Sphere.

The shadow of the Sphere can then, as in the figure, be inscribed within the symmetrical hexagon which is the shadow of the cube, the projection of which has in the first place been inscribed within the projection of the sphere.

The projection of the Sphere circumscribes the projection of the cube, but the shadow of the cube circumscribes the shadow of the Sphere.

130. *The Method of Points.*—Since the shadow of a solid object is bounded by the shadow of its Line of Shade, the Line of Shadow being in fact the shadow of the Line of Shade, the Line of Shadow can be drawn through the shadows of any points of the Line of Shade whose positions are in any way ascertainable. Such are the twelve points shown in Fig. 82 D.

131. *The Method of Tangent Cylinders* (Fig. 90).—The three principal cylinders shown in Fig. 79 and again in Fig. 90, are tangent to the sphere in three great circles, one vertical, one horizontal, and the third vertical and normal to the plane of projection. The shadows of these cylinders are bounded by the shadows of their lines of shade, and as these are tangent to the line of shade of the sphere at six points, a and a, c and c, c' and c', the shadows of these six points lie in the outline of the shadow of the three cylinders. The intersections of their outlines form a hexagon which is identical with the hexagonal shadow cast by the "Inscribed Cube" (Fig. 89).

Since the width of the shadow of a right circular cylinder is the diagonal of its diameter, and the elliptical shadow of the sphere is tangent to the sides of the hexagon at their middle points, a horizontal projection of the sphere and its tangent cylinders is not needed. The circumscribing hexagon can be drawn independently.

132. *The Method of Tangent Cones* (Figs. 91 and 92).—Fig. 91 shows, as was shown in the Introduction, Fig. L, that if a Cone is tangent to a Surface of Revolution, the axis of the Cone coinciding with that of the Surface, the line of contact between the two surfaces is a circle, which is the base of the Cone, and that this circle of contact, the Line of Shade of the Cone, and the Line of Shade of the given surface, all pass through the same two points on that surface. Moreover, the ray of light that passes through either of these points is tangent to both surfaces, and lies in a plane which is also tangent to both surfaces at this point. This tangent plane, being parallel to the light, is a Plane of Light and Shade, standing edgewise to the light, and its shadow, cast upon any plane surface, is a right line, which is tangent both to the shadow of the Cone and to the Shadow of the Surface of Revolution, at the point which is the shadow of the common point of tangency, and the shadows of the two surfaces and of the circle of tangency are tangent to one another at this point of shadow.

This is illustrated in Fig. 91, in which the two Lines of Shade and the circle of tangency all three meet at the points a^s and a^s.

133. The figure shows the Lines of Shadow, (1) of the two lines which constitute the Line of Shade at the Cone, (2) of the ellipse that is the shadow of the circle of tangency at its base, and (3) the ellipse that is the Shadow of the Sphere (which coincides with the shadow of the Line of Shade on the Sphere), all passing through the points a^s and a^s, which are the shadow of the points of tangency, all three being tangent to one another at those points.

It follows that if a Cone is drawn tangent to a given Sphere, the two points, such as a^s and a^s in the shadow of the cone, where the shadow of the conical surface is tangent to the shadow of the base, will be points of the elliptical shadow of the Sphere, which will be tangent to the shadow of the Cone at these points.

134. Since any number of Cones can be drawn tangent to a Sphere, the shadow of a sphere can be thus determined at any desired number of points, the shadows of the elements of the cone giving the direction of the Line of Shadow at those points, as in the figure, since they are tangent to the ellipse of shadow. These tangent lines form then a polygonal figure in which the ellipse of shadow may be inscribed, just as was done within the hexagons furnished by the two previous methods.

In Fig. 92 the construction lines necessary for finding the Lines of Shade of the Cones and the shadows of the Lines of Shade are omitted.

135. *The Method of Envelopes* (Fig. 93).—If the sphere is cut by planes parallel to the plane surface upon which its shadow falls, the sections will be circles, and the shadows will be other circles just like them.

If the plane surface to which these secant planes are parallel is a plane of projection, as in the figure, both the projections of these circles and their shadows will be circles. The shadows will intersect so as to form a sort of polygon, each side of which is the arc of a circle, and the shadow of the sphere may be circumscribed about it, as shown.

136. *The Method of Slicing* (Fig. 94).—If the secant planes are taken parallel to the direction of the light, the line of shade of the sphere will pass through the points of shade upon the small circles, and the shadow of the sphere will pass through the shadows of these points, which will lie in the shadows of the small circles. But though the shadows of these small circles will be right lines, their projections will be ellipses, and difficult to

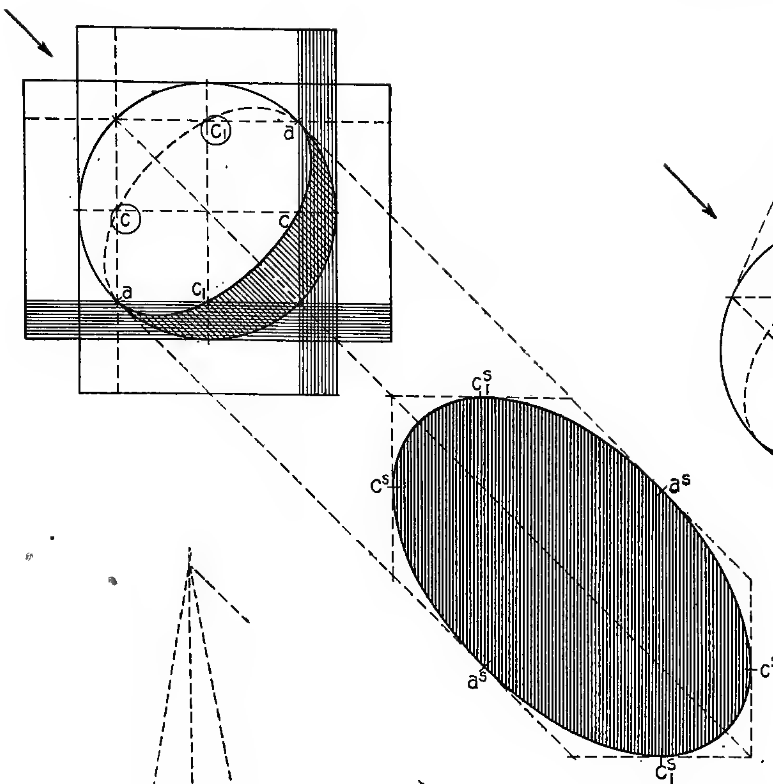


FIG. 90

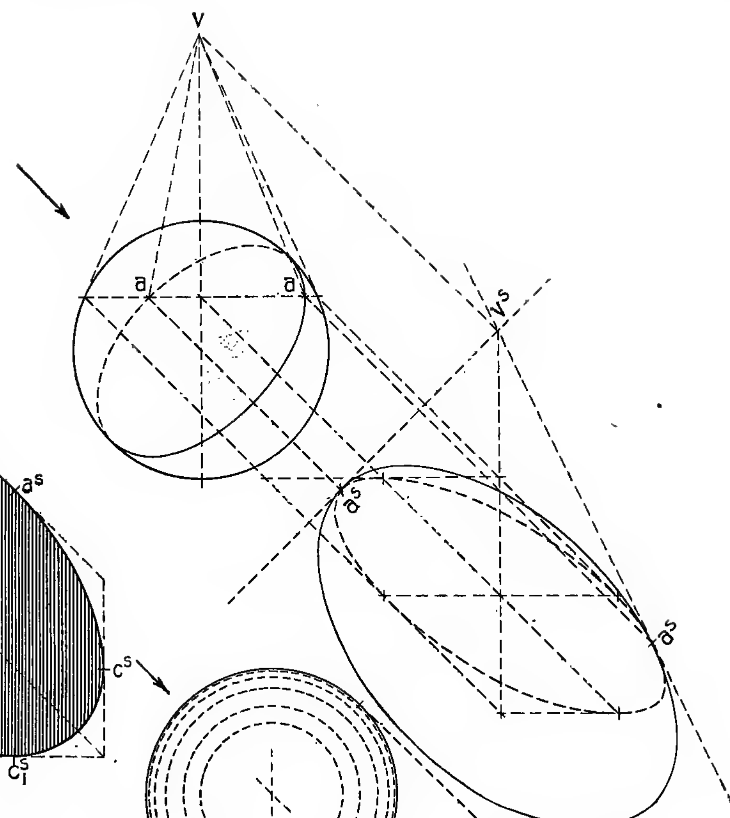


FIG. 91

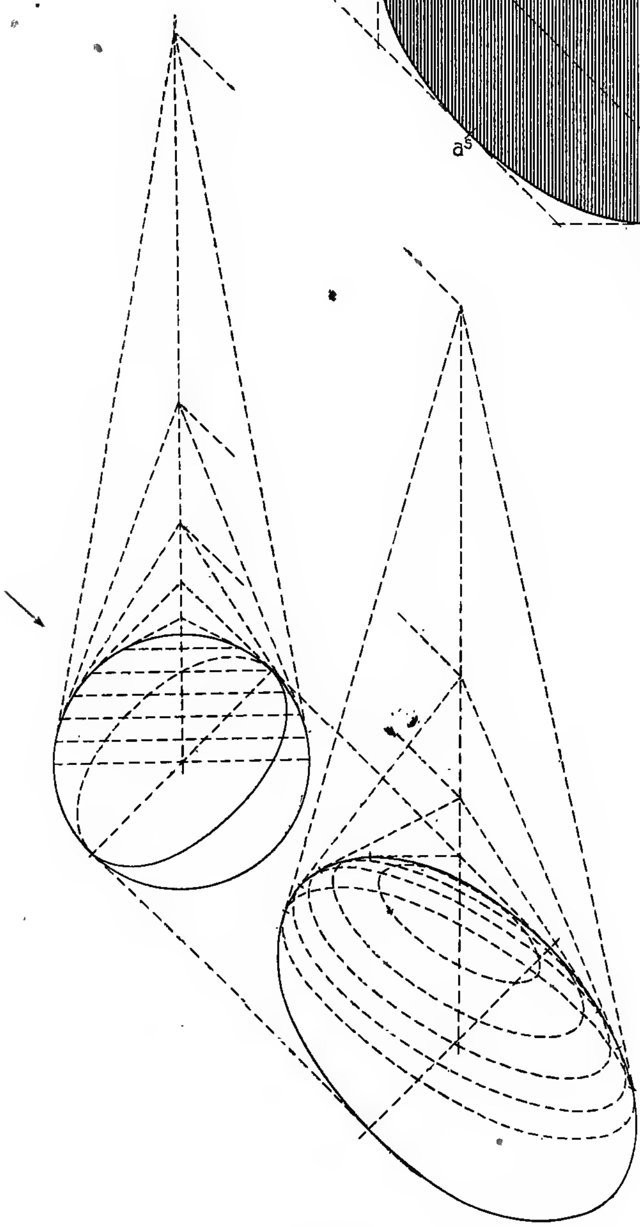


FIG. 92

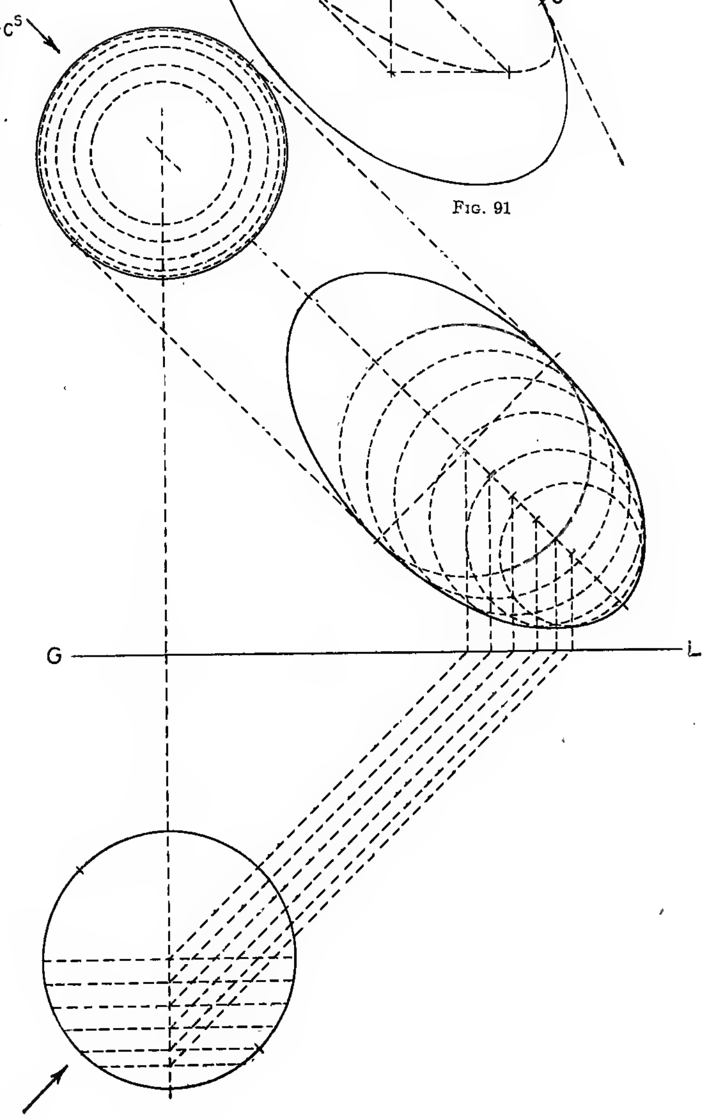


FIG. 93

draw, and the points of tangency are difficult to determine. The procedure illustrated in Fig. 94 *A* is thus of no practical value.

But since the shadows of the centers of these small circles are easily found, and the length of the right lines which are the shadows of such circles bears a constant ratio to the diameters of the circles (namely, that of the hypotenuse of the ϕ triangle to its long side, (which is the ratio of $\sqrt{3}$ to $\sqrt{2}$, or $\sqrt{\frac{3}{2}}$ to 1), any number of points in the Line of Shadow can be found without constructing the Line of Shade itself, or drawing the ellipses in which the circles are projected (Fig. 95).

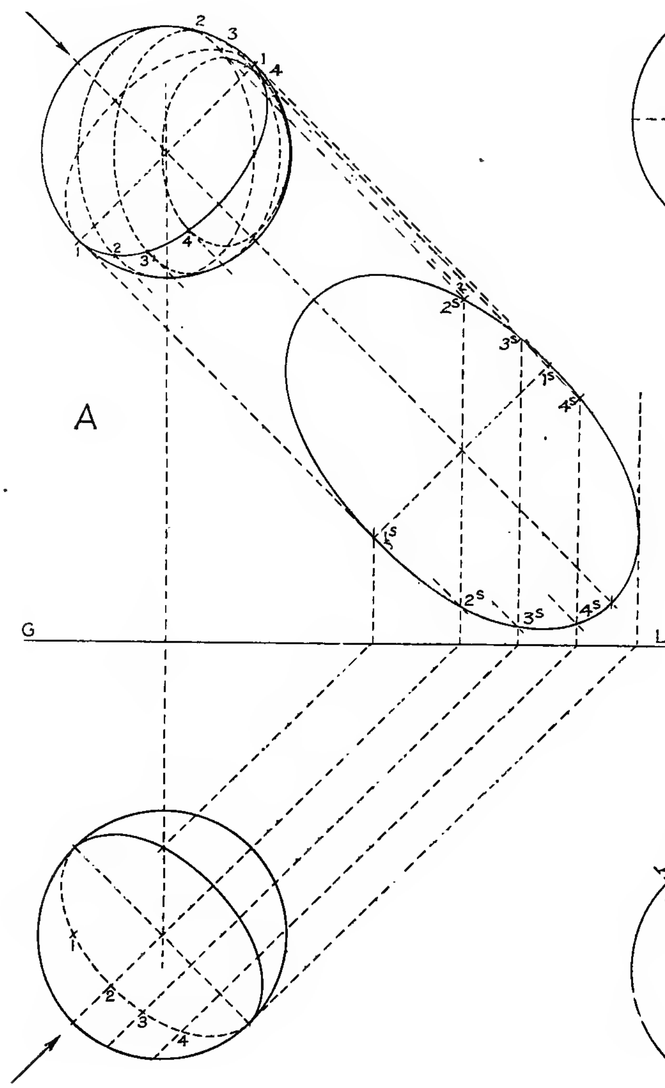


Fig. 94

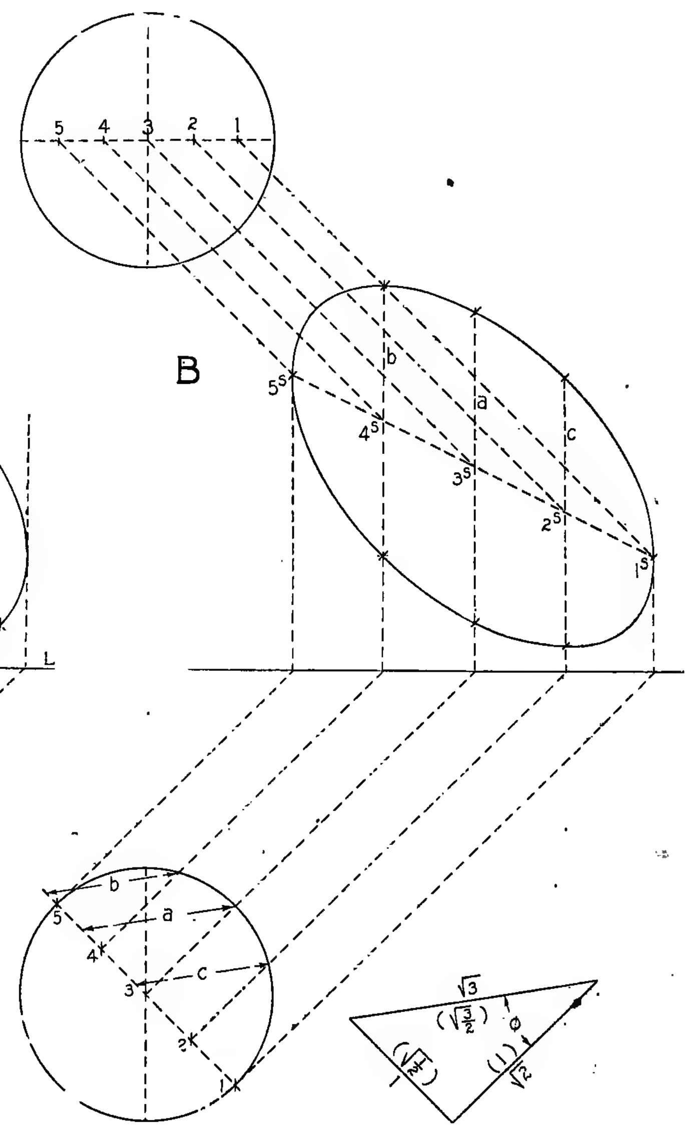


Fig. 95

137. *The Auxiliary Plane at 45 Degrees* (Fig. 96).—If the shadow of a sphere falls upon an auxiliary vertical plane, set at an angle of 45 degrees with the Vertical Plane of Projection, as in the figure at A, the shadow will be an ellipse whose Minor Axis is horizontal, and equal to 2, that is to say to the diameter of the sphere, and whose Major Axis is equal to $\sqrt{6}$, as shown at B. The projection of this shadow upon the Vertical Plane of Projection will be an ellipse whose vertical axis is also vertical and equal to $\sqrt{6}$ ($= \sqrt{3}\ \sqrt{2}$) and the Minor Axis equal to $\sqrt{2}$, as shown at C. The axes will then be in the ratio of $\sqrt{3}$ to 1 and the projection of the ellipse, like the Line of Shade upon the Sphere, and like the shadow of a sphere upon a Principal Plane, may be circumscribed about two equilateral triangles, as in the figure.

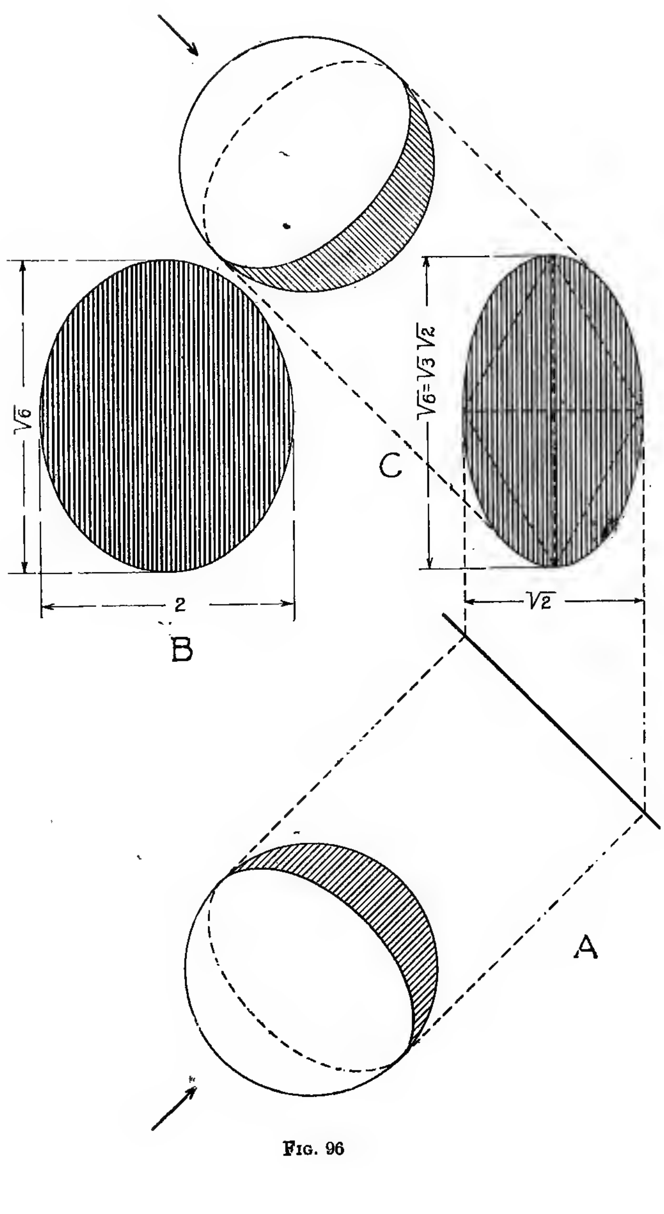


FIG. 96

138. *Ellipsoids.*—The projection of a Prolate or Oblate Ellipsoid upon the Plane perpendicular to its axis is a circle; upon a plane parallel to the axis it is an ellipse similar to the generatrix of the Ellipsoid.

The Line of Shade.—When the light falls parallel to the plane of projection, as in Fig. 97 *B* and in Fig. 98, the projection of the Line of Shade of an Ellipsoid upon that plane is found to be a right line, passing through the projection of the center of the ellipsoid. It is a diameter of the ellipse in which the ellipsoid is projected. The Line of Shade itself is then a plane figure, and this figure must be an ellipse. For it is a maxim of Geometry, as has already been said, that if a surface of revolution which has a Conic Section for a generatrix is cut by a plane, the line of intersection is also a Conic Section, and the ellipse and circle are the only Conic Sections that form a closed figure.

The center of the ellipse of projection being one point in the rectilinear projection of the Line of Shade, only one other point is needed, of course, in order to determine the line. This point may most conveniently be taken on the outline, as in the figure, where the ray of light tangent to the projection of the ellipsoid gives a point in the Line of Shade.

139. When, as usual, the light falls so that its projections make angles of 45 degrees with both planes of projection, the Line of Shade is an ellipse and is projected in an ellipse. As in the case of the Sphere, the highest points of the curve are at the corners of the Φ circle, and it is tangent to the outline at the 45-degree points. (Fig. 97 *A*.) The eight points, a, a'; b, b'; c, c'; and d, d', as shown in the figure, can be got without difficulty by means of Tangent Cylinders and Cones. Any number of additional points can then be got by taking points upon the rectilinear projection of the Line of Shade, as, for instance, at x and y, in the figure, and revolving them through an arc of 45 degrees, as shown for the Sphere in Fig. 87. This gives the points e, e', and f, f' in Fig. 97 *B*. These points are added in Fig. 97 *A*, and the symmetry of the figure allows them to be duplicated at e', e'; and f', f'.

140. Note.—That this projection of the Line of Shade of an Ellipsoid is really a right line appears on the application of the Differential Calculus to the Method of Revolved Tangent Rays (Fig. 98).

The general equation of the Upright Ellipse is $a^2 x^2 + b^2 y^2 = a^2 b^2$; a and b being semi-axes of the ellipse, and x and y the coordinates of any point, such as the point l.

$$\text{Hence: } x^2 = b^2 - \frac{b^2}{a^2}\cdot y^2; \qquad 2\,x\,d\,x = -\frac{b^2}{a^2}\cdot 2\,y\,d\,y; \qquad \frac{d\,x}{d\,y} = -\frac{b^2}{a^2}\cdot\frac{y}{x};$$

$$l\,n = x\cdot\tan\,\Phi = x\sqrt{\tfrac{1}{2}}; \qquad z = l\,n\cdot\frac{d\,x}{d\,y} = -\;x\sqrt{\tfrac{1}{2}}\cdot\frac{b^2}{a^2}\cdot\frac{y}{x} = -\;\sqrt{\tfrac{1}{2}}\cdot\frac{b^2}{a^2}\cdot y;$$

$$\text{But} \qquad z = y\cdot\tan\,\alpha. \quad \text{Hence: } \tan\,\alpha = \frac{z}{y} = -\;\sqrt{\tfrac{1}{2}}\cdot\frac{b^2}{a^2} = C.$$

Thus the angle α is a Constant, i. e., l_1 is always in the same direction from the point o, and always falls in the line $o\,d$. The Line of Shade is thus projected in a right line, and is therefore a plane curve.

The diameters $b\,b'$ and $d\,d'$ being conjugate diameters, the axes of the ellipse can be obtained from them by the Method of Shadows shown in Fig. 41, Paragraph 52, Chapter V.

141. *The Shadow of an Ellipsoid.*—The Line of Shadow of an Ellipsoid is the shadow of its elliptical Line of Shade, and when cast upon a plane surface is itself an ellipse. It may be found by taking points on the Line of Shade, and getting their shadows, or, more conveniently, by means of Tangent Cylinders and Cones, which give not only the position of any desired number of points, but also the direction of the curve at those points. The shadow upon a plane perpendicular to the axis of revolution can also be got by the method of Envelopes.

But the shadow of an ellipsoid can be got by direct projection, as is shown in Fig. 99 *A*. It is an ellipse whose Minor Axis is equal to the smallest diameter of the Ellipsoid, and whose Major Axis depends upon the shape of the ellipsoid. The greater the eccentricity of the generating ellipse, the greater is that of the ellipse of shadow.

142. All this applies equally to the Oblate Ellipsoid, Fig. 97 *B*. The only difference is that, in the algebraic work, the letters a and b change places, the equation of the horizontal ellipse being $a^2 y^2 \times b^2 x^2 = a^2 b^2$.

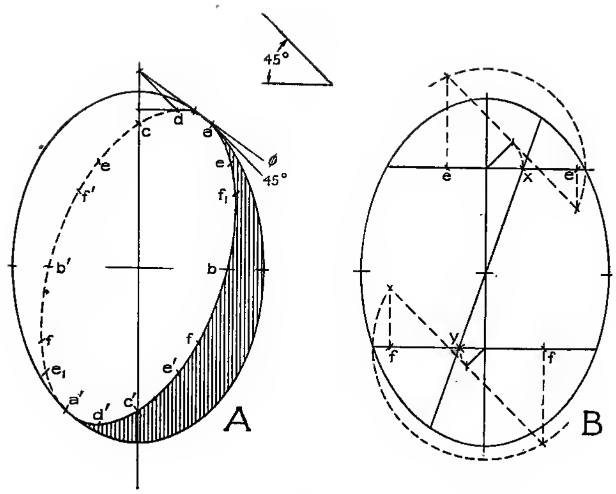


Fig. 97

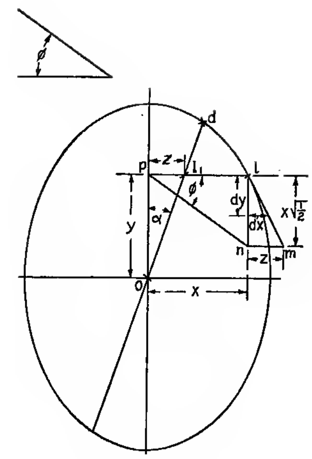


Fig. 98

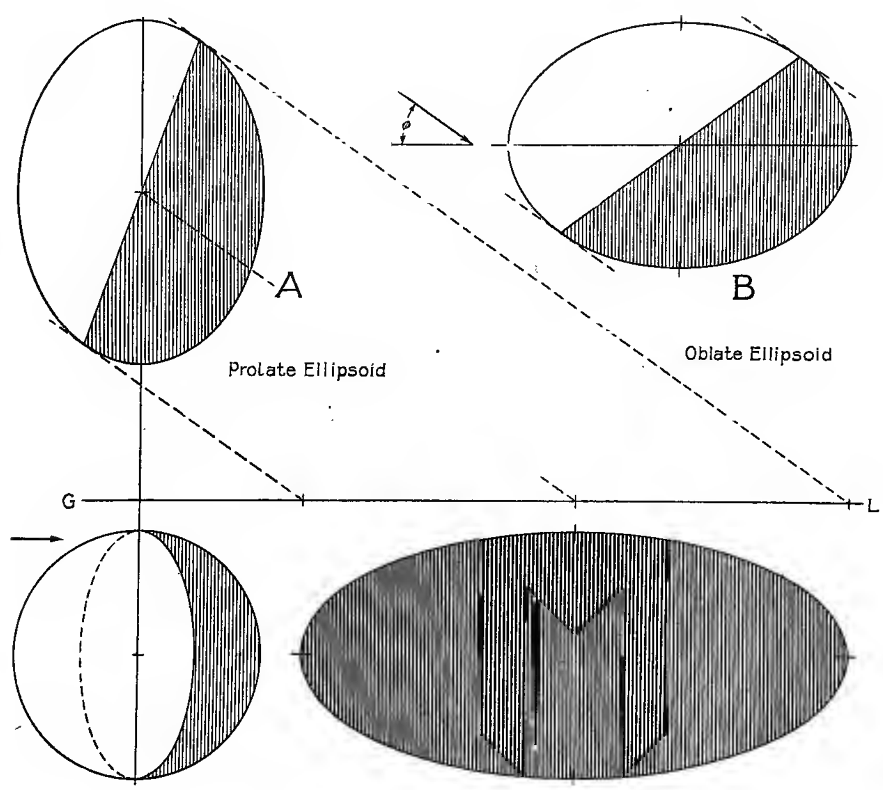


Fig. 99

Chapter X

HOLLOW SPHERES

143. *The Hollow Hemisphere.*—In the case of the Solid Sphere, the Line of Shade is on the external surface of the Sphere, and the shadow is cast on the ground, or elsewhere. But in the case of the Hollow Sphere, or Hemisphere (Fig. 100), as with the Hollow Cylinder and Hollow Cone, the shade and shadow both lie upon its interior concave surface, and the line *a b a*, which is the half of the rim, or edge, which is nearest the Sun, is the Line of Shade which casts the shadow.

The solid sphere shows upon its convex surface a single line, the Line of Shade, which is a Great Circle of the sphere, and divides the half of the surface which is in light from the half which is in shade.

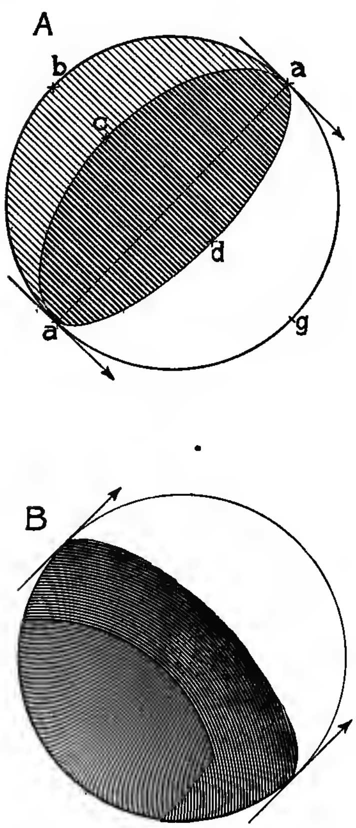


Fig. 100

The Hollow Hemisphere, on the other hand, has upon its concave surface three lines, all of which are Great Semi-Circles, namely, (1) the Line of Shade upon the rim, which casts the shadow; (2) the Line of Shade and Shadow, which divides the shade from the shadow; and (3) the Line of Shadow, which divides the shadow from the light.

144. The portion of the concave surface which is next to the edge which casts the shadow is in shade, being turned away from the light, Fig. 100 *A*. This shade extends as far as the Great Semicircle *a c a*, which, on the outside of the sphere, is the Line of Shade, being the line where the rays of light are tangent to the spherical surface. On the inside of the sphere it is the Line of Shade and Shadow, separating the surface in shade from that which is in the shadow cast by it upon the portion which faces the light. The shadow extends from the Line of Shade and Shadow *a c a* to the Line of Shadow *a d a*, which is the shadow of the Line of Shade *a b a*. The rest of the concave surface, extending from the Line of Shadow *a d a* to the other half of the rim at *a g a*, is in light.

Fig. 100 *B* shows a similar shade and shadow upon a horizontal hemisphere. But in this figure the effect of the reflected light is shown, and this is such as to obliterate the Line of Shade and Shadow.

145. *The Line of Shade and Shadow* (Fig. 100).—The Line of Shade and Shadow in a Hollow Hemisphere thus lies in the same Great Circle as the Line of Shade upon a solid sphere. It is projected in an ellipse whose Semimajor Axis is 1, and whose Semiminor Axis is $\sqrt{\frac{1}{3}}$. This may be obtained in any of the ways described in the previous chapter, but it is of less practical importance than the Line of Shade, for in the case of the Solid Sphere it is this line that casts the shadow, but in the case of the Hollow Hemisphere the line that casts the shadow is half the rim of the hemisphere. This is given as one of the data of the problem, and constitutes the semicircular Line of Shade *a b a*.

146. Moreover, in the case of the Solid Sphere the Line of Shade is a conspicuous line, and it needs to be exactly defined. It separates the illuminated half of the sphere from the shaded portion of the surface just where the shade is at its darkest, being there most turned away from the reflected light, and, if the surface is smooth, the Line of Shade often looks, as has been said, almost like a dark line. In the Hollow Sphere, on the contrary, the Line of Shade and Shadow separates the shade from the shadow, both of which are equally exposed to the reflected light along this line, and are, where they meet, of exactly the same degree of darkness, as is seen in the horizontal hemisphere at B. The line is in nature, as in this drawing, imperceptible. It exists only in

theory. In drawings, however, which, like Fig. 100 *A*, have the shades and shadows put in in flat tints, the Line of Shade and Shadow is visible, and needs to be carefully drawn. This is the case with most of the figures and illustrations in this book, and, generally, in architectural drawings made to a small scale.

147. The semicircular edge *a b a* which casts the shadow upon the hollow surface; the semicircular Line of Shade and Shadow *a c a*; the semicircular Line of Shadow *a d a*; the other semicircular edge *a g a*; and the rays of light that are tangent to the hemisphere at the extremities of these four semicircles, all meet at the points *a* and *a*, which are at the extremities of the diameter which, in the elevation, lies at 45 degrees.

148. *The Line of Shadow* (Fig. 101).—The rim of a Hollow Hemisphere is a great circle of the sphere, being the line where the sphere is cut by a plane taken parallel to the Plane of Projection and passing through its center. The half of the rim which is next to the sun casts its shadow, as in the previous figure, upon the concave surface. The Line of Shade is thus a semicircle. The Line of Shadow is a similar semicircle.

149. I. For, in the first place, as has already twice been said, it is a maxim of Geometry that if a Conic Section of Revolution is cut by a plane, and the line of intersection is then projected upon its interior surface, the projection is always a conic section, and is a plane curve. Now the sphere is such a surface of revolution, the Line of Shade is such a line of intersection, and the Line of Shadow is such a projection. It is accordingly a plane curve. But the only plane curve that can be drawn upon the surface of a sphere is a circle. The line of shadow is therefore a circle, and since its diameter is the diameter of the sphere, it is a Great Circle.

150. II. Fig. 101 *C* also shows that the shadow cast by the edge of the hemisphere upon the hollow spherical surface must needs be a Great Circle of the sphere, since it is just like the line that casts it.

For the invisible shadow of the Great Circle which forms the edge of the hemisphere is an elliptical cylinder, the axis of which passes through the center of the sphere at *o*. The symmetry of the figure about the ray through *o* shows that the line of intersection *o d* is just such a line as the line *b o*, and since *b o* is the projection of half of a Great Circle of the sphere, so must *o d* be.

Fig. 101 *D* shows the elliptical cross-section of this cylinder of invisible shadow. Its Major Axis is equal to the diameter of the sphere, or 2, and its Minor Axis equals $2\sqrt{\frac{1}{3}}$, or $\frac{2}{3}\sqrt{3}$. This ellipse is then of the same shape as the elliptical projection of the Line of Shade and Shadow, as shown in Fig. 78 *A* and *C*, and of the same shape and of the same size as the shadow cast by the sphere upon a principal plane of projection, as appears in Fig. 78 *D*. It circumscribes two equilateral triangles.

151. III. *The Method of Slicing*—The same result is reached by constructing the Line of Shadow graphically, employing the method of Slicing. This is seen also in Fig. 101 *C*. The hollow hemisphere is here shown as projected at *A* upon the vertical plane, at *B* upon the horizontal plane, and at *C* upon an auxiliary plane, parallel to the rays of light, just as was done in Fig. 78, to find the Line of Shade upon a sphere, and in Fig. 74 for the Line of Shade upon a Hollow Cone.

Planes parallel to the rays of light cut the hemispherical surface in semicircles which appear, when projected upon the auxiliary plane of projection, of their true size and shape. The points *b*, *1*, and *2*, on the edge nearest the Sun, throw their shadows on the points *d*, 1^s, and 2^s. The shadow of each point falls upon the semicircle that passes through it. Since, by construction, each point of shadow occurs in the same part of the semicircle, the line connecting them *o d*, which is the Line of Shadow, is just such a right line as is *o b*, and is also a radius of the semicircle. The Line of Shadow must accordingly be a plane curve, namely, a great Semicircle of the sphere, and its projection on the vertical and horizontal planes (as at *A* and *B* in Fig. 100, and at *A* in Fig. 101) must be a semiellipse.

152. Fig. 101 *A* shows also the shadows cast upon a hollow hemisphere by the four principal diameters of the Great Circle which is parallel to the plane of projection, just as was done for the Hollow Cone in Fig. 74. They all pass through the shadow of the center of the circle of *f*, touch the line of shadow cast by the semi-circumference, or rim, which is nearest the light, and terminate in the other semi-circumference. These Lines of Shadow are all arcs of Great Circles and are projected as arcs of ellipses.

153. *The Vertical Projection of the Line of Shadow* (Fig. 101 *A*).—The Major Axis of the ellipse in which the Line of Shadow is projected upon the vertical plane lies at 45 degrees, at right angles to the projection of the rays of light, and its length is that of the diameter of the sphere.

The point *d*, in Fig. 101 *A*, is distant from the center *o* one-third of the radius. The Minor Axis of the ellipse in which the Line of Shadow is projected is then one-third of the Major Axis, and the shade and shadow together cover two-thirds of the circular area.

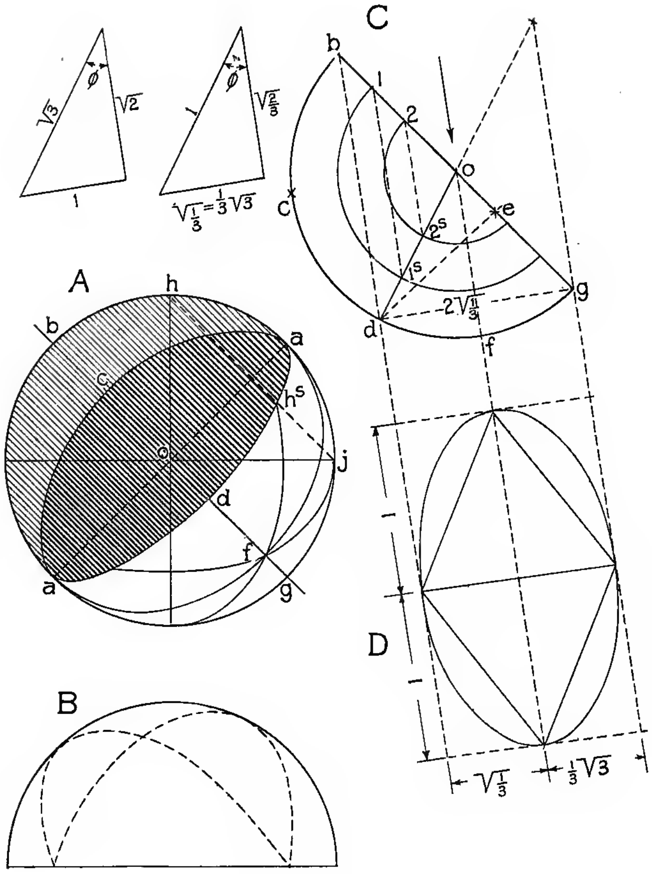


FIG. 101

154. These results are reached by the following computation. (Fig. 101 *C*.) Since the diameter $b\,g$, which is the hypotenuse of a ϕ triangle $b\,d\,g$, equals 2 (the radius being taken as 1), the chord $d\,g$, which is the short side of the same triangle, equals $2\sqrt{\frac{1}{3}}$, and the segment of the diameter $e\,g$, which is the short side of the triangle $d\,e\,g$, equals $d\,g\sqrt{\frac{1}{3}}=2\,(\frac{1}{3})=\frac{2}{3}$. The segment $o\,e$ then equals $\frac{1}{3}$, and so does the segment $o\,d$ at *A*, which is the Semiminor Axis of the ellipse. Hence, the Minor Axis equals $\frac{1}{3}$ of the diameter.

155. Fig. 101 *A* shows that 45-degree chords drawn across the vertical projection of the hemisphere parallel to the projection of the rays of light are also divided by the Line of Shadow in the ratio of 2 to 1, just as the diameter parallel to them is divided. The point h^s accordingly is at the distance of one-third of the length of the chord $h\,j$ from the point j, at the extremity of the horizontal diameter. The point h^s is the shadow of the point h, at the extremity of the vertical diameter.

156. *The Horizontal Projection of the Line of Shadow* (Fig. 102).—In Fig. 102 the vertical projection of the Semicircular Line of Shadow which has just been determined, is again shown at *A*. The problem now is to find the horizontal projection of the Great Circle of which it forms a part. This is shown at 102 *B*.

If it were required to find the shadow cast within a horizontal hemisphere by the semicircle of its own edge, which is parallel to the plane of projection, the result would be the same as that just obtained, and as shown in Fig. 100 *B*. The axes of the ellipse of shadow would lie at 45 degrees, and the shade and shadow would cover two-thirds of the circle. But the present problem is to find the shadow cast upon a horizontal hemisphere by a vertical circle, perpendicular to the plane of projection, the same circle $a\,b\,a$ which at *A* casts the shadow $a\,d\,a$.

157. The problem is, that is to say, to find the horizontal projection of the Great Circle which, in the vertical projection, appears as an ellipse whose axes lie at 45 degrees, and whose Minor Axis is one-third of the Major. The horizontal projection will, of course, also appear as an ellipse, the Major Axis of which will be a diameter

of the sphere. It is required to find the inclination of this Major Axis, and the length of the Minor Axis, at right angles to it.

This involves the determination, in Fig. 102 *A*, of the length of the half chord r', and of the line x; and in Fig. 102 *B*, of the half chord r'', of the lines y, z, and b, and of the angle θ. The line b is the required Semi-minor Axis.

The Minor Axis of this ellipse of shadow is just twice as long as that of the ellipse shown in the vertical projection, and the angle θ by which the Major Axis is inclined to the Normal Diameter of the sphere, is the angle of 26° 34′, the angle whose Tangent is $\frac{1}{2}$. These results are reached by the following computations:

In Fig. 102 *A* the Line of Shadow cuts off a third part of the half chord r', as has been explained in Paragraph 155, and illustrated in Fig. 101 *A*. It also cuts from the horizontal diameter the segment x, which is the hypotenuse of a right Isosceles triangle, the equal sides of which measure $\frac{1}{3}r'$.

We have then from Fig. 102 A: $r^2 = r'^2 + (r\,\frac{1}{3})^2 = \dfrac{10}{9}r'^2$. $\qquad x^2 = 2\,(\frac{1}{3}\,r')^2 = \frac{2}{9}\,r'^2$.

Hence, $\qquad r^2 = 5\,x^2;\; r = x\,\sqrt{5}; \qquad r : x = \sqrt{5} : 1$.

The Line of Shadow touches the projection of the outline of the sphere at the point a, at the extremity of the Major Axis, distant from the projection of the vertical diameter of the sphere by the length of a line which is the side of another Isosceles right triangle, of which the hypotenuse is $o\,a$, the radius of the Sphere. This line measures accordingly $r\sqrt{\frac{1}{2}}$.

In Fig. B, the line $o\,a$ is projected as a segment of the diameter of the sphere which is parallel to the Ground Line. It is the hypotenuse of a right triangle of which the long side is y, and the short side is z. The angle at the base is called θ. Hence $y^2 + z^2 = \frac{1}{2}\,r^2$. This triangle is similar to the larger triangle in the same figure in which the hypotenuse is r, and short side x. Since in both these triangles the hypotenuse is to the short side as $\sqrt{5}$ is to 1, the acute angle must be the angle of 26° 34′, whose Tangent equals $\frac{1}{2}$, as appears in Fig. 102 C. We have then Tan $\theta = \frac{1}{2}$, $y = 2\,z$, $z = \frac{1}{2}\,y$.

$$y^2 + z^2 = y^2 + \tfrac{1}{4}\,y^2 = \tfrac{5}{4}\,y^2 = \tfrac{1}{2}\,r^2;\; y^2 = \tfrac{4}{10}\,r^2;\; y = 2\,r\,\sqrt{\tfrac{1}{10}}. \qquad z = r\,\sqrt{\tfrac{1}{10}}.$$

$$r''^2 = r^2 - z^2 = r^2 - \tfrac{1}{10}\,r^2 = \tfrac{9}{10}\,r^2; \qquad r'' = 3\,r\,\sqrt{\tfrac{1}{10}}.$$

Hence, $y = \frac{2}{3}\,r''$, and $b = \frac{2}{3}\,r$. For the Line of Shadow cuts the radius and the half chord proportionally.

158. These results show that in the horizontal projection of the Great Circle which constitutes the Line of Shadow in a Hollow Sphere, the Major Axis of the ellipse makes with the diameter of the sphere which is perpendicular to the vertical plane of projection the angle θ, whose tangent is $\frac{1}{2}$, and with the diameter which is parallel to the vertical plane of projection the angle $90° - \theta$, whose tangent is 2. That is to say, the slope of the diameter is as 1 to 2.

159. These results furnish the following three propositions:

I. The shadow upon the surface of a Hollow Sphere cast by a Great Circle of the sphere, is itself a Great Circle of the sphere.

II. If the Great Circle that casts the shadow is parallel to the plane of projection, the projection of the circle of shadow upon that plane is an ellipse, the Major Axis of which is equal to the diameter of the sphere, and is at right angles to the projection of the rays of light. The Minor Axis lies in the direction of the projection of the rays of light. Its length is one-third of the diameter.

III. If the Great Circle that casts the shadow lies in a principal plane, perpendicular to the plane of projection, the projection of the circle of shadow upon that plane is an Ellipse, whose Major Axis is equal in length to the diameter of the Sphere, and is nearly in the direction of the light, making with the projection of the Great Circle an angle whose tangent is 2. The Minor Axis is equal to two-thirds of the Major Axis.

160. *The Method of Parallel Planes.*—The Line of Shadow in a hollow hemisphere can also be obtained by the method of parallel planes, Fig. 103 *A* and *B*.

If a hollow hemisphere is cut, as in the figure, by planes parallel to the vertical plane of projection, the lines of intersection will be projected horizontally in straight lines, as at *B*, and on the vertical projection in the circles *1*, *2*, *3*, and *4*, as at *A*. The center of the semicircular Line of Shade will cast its shadow on these planes at the points c^1, c^2, c^3, and c^4, and the shadow of the Line of Shade will fall upon these planes in the semicircles *1′*, *2′*, *3′*, and *4′*. The points in each plane where the circles are intersected by the corresponding semicircular shadows are points in the required Line of Shadow.

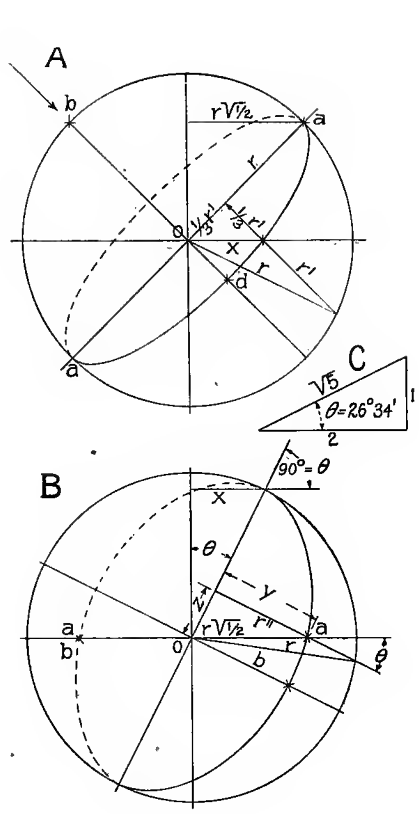


Fig. 102

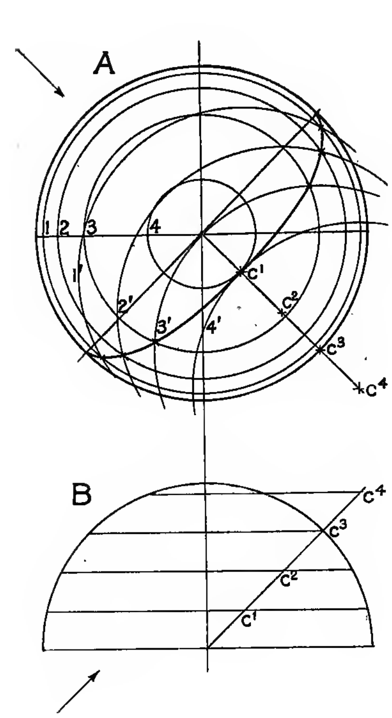


Fig. 103

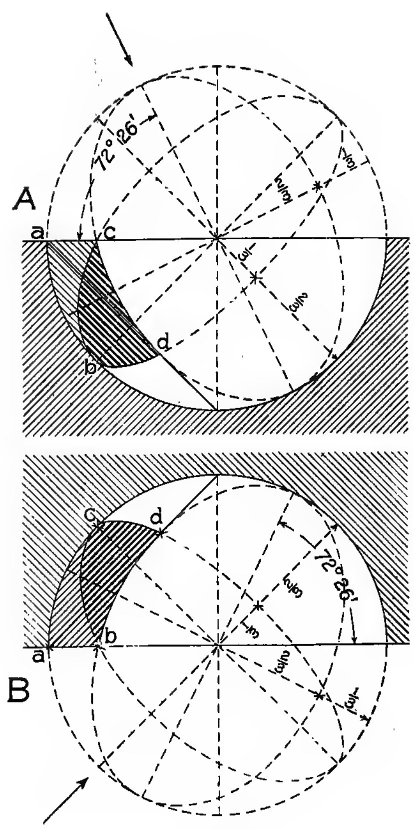


Fig. 104

161. *The Quarter Sphere.*—The interior of the surface of the Quarter Sphere receives the shadow of part of the semicircle parallel to the plane of projection and of part of that perpendicular to it (Fig. 104). The line $a\ b$ throws its shadow at $b\ d$, and the line $a\ c$ at $c\ d$, the shadow of the point a falling at d.

In both plan and elevation the Great Circle which is parallel to the plane of projection casts a shadow which appears as an ellipse with its Minor Axis in the direction of the light, and equal in length to one-third the Major Axis. The great circle which is perpendicular to the plane of projection casts a shadow which appears as an ellipse whose Minor Axis is two-thirds the Major Axis, and the Major Axis, which lies nearly in the direction of the light, makes, with the projection of the circle which casts the shadow, the angle whose tangent is 2, or 72° 26′ (= 90° − 26° 34′).

The cusp at the point d, in which the ellipses of shadow intersect, is the shadow of the point a, and lies in the 45-degree chord drawn parallel to the projection of the light, at two-thirds its length. For the elliptical shadow of the circle which is parallel to the plane of projection cuts off one-third of the chord, just as it cuts off one-third of the diameter parallel to it.

162. *Oblique Hollow Hemispheres.*—The vertical secant plane which forms the Line of Shade is sometimes inclined to the vertical plane of Projection instead of being parallel to it.

The Ellipse of the Rim or Line of Shade.—The line of intersection, or rim of the hemisphere, is of course always a Great Circle, and its vertical projection is always an ellipse whose Major Axis is the vertical diameter of the circle in which the sphere is projected. The Minor Axis is horizontal and its length depends upon the obliquity of the secant plane. These ellipses take the form shown in Fig. 105 *A*, *B*, *C*, *D*, *E*, *F*, *G*, and *H*.

The Ellipses of the Line of Shade and Shadow, and of the Line of Shadow.—This does not, of course, alter the semicircular Line of Shade upon the outside of the sphere, or the semicircular Line of Shade and Shadow upon the inside. Since these lines are, geometrically, the same Great Circle, the semiellipses in which they are projected form the two halves of a single ellipse. This Great Circle has the same position however the sphere is

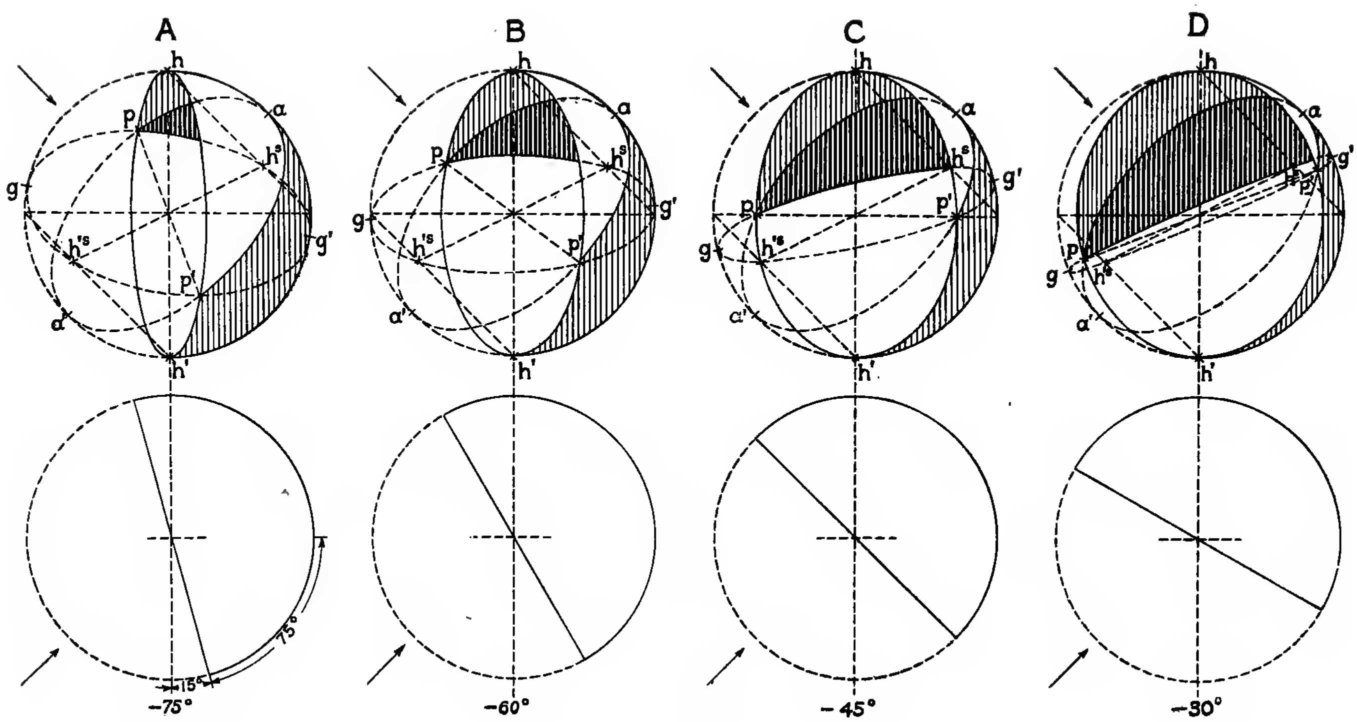


Fig. 105

divided, and its projection is always the ellipse whose Major Axis is the diameter of the circle which makes 45 degrees with the horizontal diameter, and lies across the rays of light, and whose Minor Axes, lying parallel to those rays, measures, as in Fig. 78 *A* and *B*, $\frac{2}{3}\sqrt{3}-2\sqrt{\frac{1}{3}}$. This Major Axis is lettered $a\ a'$.

The Ellipse of the Line of Shadow.—The Great Circles that constitute the different rims being all different, the shadows that they cast are all different. These shadows also are Great Circles of the sphere, as explained in Paragraph 149. These Lines of Shadow are projected as ellipses whose Major Axis, though inclined at different angles, are diameters of the same circle.

163. Two points on each curve can be got without difficulty and the symmetry of the ellipse gives two more. One of these is the point h^s, which is the shadow of the point h, at the upper extremity of the vertical diameter of the hemisphere. Since this point is common to all the vertical secant circles, or rims, its shadow is common to all the Lines of Shadow. It occurs, as was shown in Fig. 101, upon the 45-degree chord drawn through the top of the circle, parallel to the direction of the light, at two-thirds of its length from its upper end. The symmetry of the ellipse gives a second point, marked h'^s in the figures, upon the opposite side of the ellipse.

164. Two other points are furnished by the principle illustrated by Fig. E in the Introduction, in which it was pointed out that the Line of Shade that casts the Shadow; the Line of Shade and Shadow which separates the Shade from the Shadow; and the Line of Shadow, all pass through the same point. The two points in which the ellipse, which is the projection of the rim, intersects the ellipse which is the projection of the Line of Shade and Shadow are therefore points of the Line of Shadow. In each figure then the two points, lettered p and p', at which the rim cuts this ellipse are points in the elliptical Line of Shadow, the three ellipses intersecting at these points. The Line of Shadow being a Great Circle of the sphere, the ellipse in which it is projected has for its Major Axis a diameter of the circle in which the outline of the sphere is projected, and is tangent to this circle at the extremities of this diameter. If these relations are borne in mind, the points h^s and p, or any other two points, suffice to determine the ellipse of the Line of Shadow.

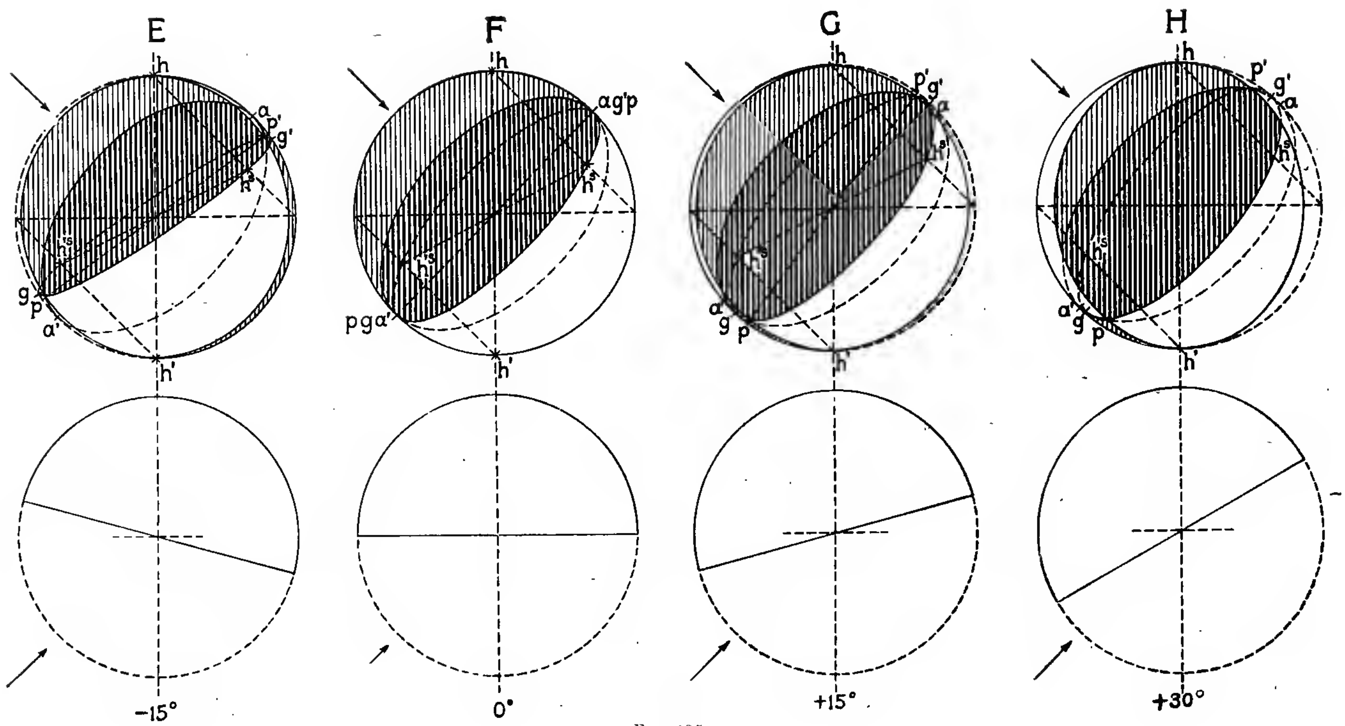


FIG. 105

165. Fig. 106 *A* shows how the axes of the ellipse may be determined, two points having been found. Let the points *1* and *1* be the two given points. These points are the projections of two points that lie in a Great Circle of the sphere, the projection of which is the required ellipse. The problem is to find its Major and Minor Axes.

A plane passed through these two points perpendicular to the plane of projection will cut the sphere in the Small Circle which is projected in the line *2 1 1 2*, cutting the plane of projection in the line *2 2*, the diameter of the Small Circle. If this Small Circle is now revolved about this diameter into the plane of projection, the points *1* and *1* will fall at *3* and *3*, and a line drawn through *3* and *3* will meet the line *2 1 1 2*, produced, at the point *4*, lying in the plane of projection. But the line *4 3 3*, in its unrevolved position, as projected at *4 1 1*, lies in the oblique plane which contains the Great Circle whose projection we wish to obtain, and since all Great Circles have their centers in the center of the sphere, the line *4 c* is the line in which this oblique plane cuts the plane of projection. The line *5 5* is then a diameter of the Great Circle and is the required Major Axis of the ellipse *5 1 1 5* in which the Great Circle is projected.

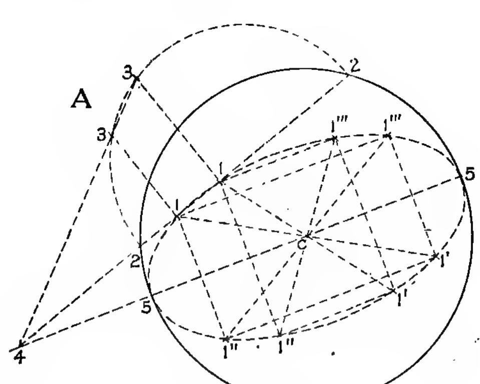


FIG. 106 *A*

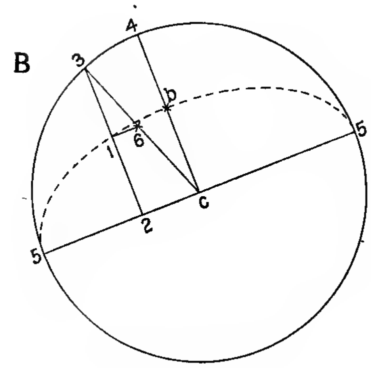


FIG. 106 *B*

The symmetry of the ellipse affords, besides the given points *1* and *1*, the six other points *1'* and *1'*, *1''* and *1''* and *1'''* and *1'''*. These eight points and the points *5* and *5*, at the extremity of the Major Axis, being determined, the required ellipse can be drawn through them with sufficient precision.

166. Fig. 106 *B* shows how the Minor Axis of this ellipse may be found, with exactness.

If the half chord *2, 1, 3* is drawn through one of the given points, perpendicular to the Major Axis, and a radius *c b 4* is drawn parallel to it, the line *cb* will be the Minor Axis, and since the semiellipse *5 b 5* divides this radius and this half chord proportionally, we have: $cb : r = 2, 1 : 2, 3$. If now we draw a second radius *c 3*, and a line *1, 6*, parallel to the Major Axis, we have the proportion: $c6 : r = 2, 1 : 2, 3$; the Minor Axis *cb* is then equal in length to the line *c 6*.

167. *The Line of Shadow Projected as a Right Line* (Fig. 107).—Fig. 105 shows that in *A*, *B*, *C*, and *D*, the semiellipse of the Line of Shadow lies above the center of the circle; in *E*, *F*, *G*, and *H* it lies below it. At *D* the secant plane makes an angle of 30 degrees with the Vertical Plane of Projection, at *E* an angle of 15 degrees. There must obviously be an angle between 30 degrees and 15 degrees at which the Line of Shadow will pass through the center, the ellipse being seen edgewise and appearing in projection as a straight line, as in Fig. 107 *A*.

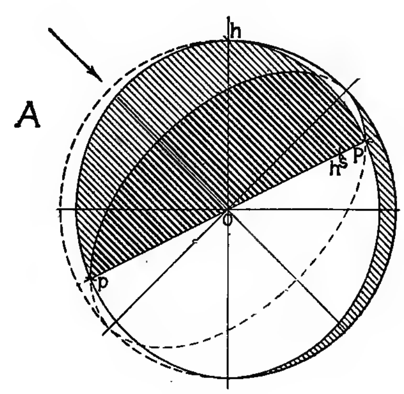


FIG. 107 *A*

Fig. 107 *A* shows the Line of Shadow thus projected as a diameter of the circle, passing through the point h^s, which is the shadow of the point h at the top of the vertical diameter of the circular rim. Since the point h^s is distant from the end of the horizontal diameter one-third of the length of the 45-degree chord passing through it, it must be distant from the horizontal diameter one-third of the radius of the circle, and from the vertical diameter two-thirds of the radius. The angle α, therefore, included between the projection of the Line of Shadow and the horizontal diameter, is the angle whose Tangent is $\frac{1}{2}$.

The angle β $(=45°-\alpha)$, included between the Line of Shadow and the 45-degree diameter, must then be the angle whose tangent is $\frac{1}{3}$. For, as the following operation shows, the sum of two angles whose tangents are, respectively, $\frac{1}{2}$ and $\frac{1}{3}$, is an angle of 45 degrees.

$$(\tan\ \alpha=\tfrac{1}{2};\ \ \tan\ \beta=\tfrac{1}{3});\ \tan(\alpha+\beta)=\frac{\tan\ \alpha+\tan\ \beta}{1-\tan\ \alpha\times\tan\ \beta}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=\frac{\frac{3}{6}+\frac{2}{6}}{1-\frac{1}{6}}=\frac{\frac{5}{6}}{\frac{5}{6}}$$

$$=1=\tan 45°.$$

Hence, $\alpha+\beta=45°$.

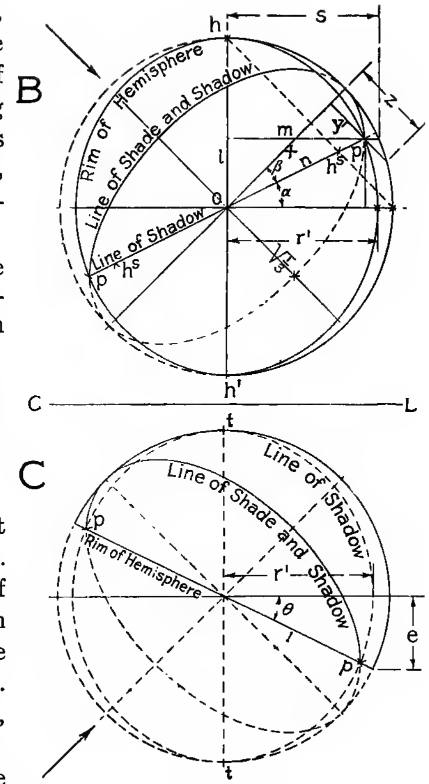


FIG. 107 *B* and *C*

168. The Vertical Projection of the Line of Shadow being thus determined, it remains to find the position of oblique circle which casts it, that is to say of the rim.

169. The point p, at which the Line of Shadow cuts the elliptical Line of Shade which is common to all the figures, is also a point in the ellipse in which the rim of the hemisphere, when found, will be projected. This projection of the rim is a vertical ellipse, and its Major Axis is the Vertical Diameter of the circle. It is required to find the length of the Minor Axis, r', as shown in Fig. 107 *B*, and the inclination of the Major Axis, shown in Fig. 107 *C* as the angle θ.

Let us call the distance of the point p from the center, n; its distance form the horizontal diameter of the circle, l; from the vertical diameter, m; from one 45-degree diameter, x; and from the other, y. The line n is the common hypotenuse of the two α and β triangles, the sides of which are l and m, and x and y. The lines s and l, and the lines x and z, are the sides of two other right-angled triangles, in both of which the hypotenuse is a radius, r. This radius is not shown in the figure.

From Fig. 107 *B* may be read the following equations, the radius of the sphere being taken as 1.

From the α and β triangles ($\tan\ \alpha=\frac{1}{2}$, $\tan\ \beta=\frac{1}{3}$): $m=2\ l.$ $x=3\ y.$ $n^2=l^2+m^2=x^2+y^2.$ $1=l^2+s^2=x^2+z^2.$

From the half chords of the ellipse, which are proportional to the half chords of the circle:

$$z:y=1:\sqrt{\tfrac{1}{3}}.\quad z=y\sqrt{3}.\quad 1=x^2+z^2=9\ y^2+3\ y^2=12\ y^2.\quad y=\sqrt{\tfrac{1}{12}}.\quad z=\sqrt{\tfrac{3}{12}}=\tfrac{1}{2}.\quad x=3\sqrt{\tfrac{1}{12}}=\tfrac{1}{2}\sqrt{3}.$$

$$s:m=1:r'.\quad r'=\frac{m}{s}.\quad n^2=x^2+y^2=\tfrac{3}{4}+\tfrac{1}{12}=\tfrac{10}{12}=\tfrac{5}{6}.\quad l^2+m^2=5\ l^2=\tfrac{5}{6}.\quad l^2=\tfrac{1}{6}.\quad l=\sqrt{\tfrac{1}{6}}.$$

$$s^2=1-l^2=\tfrac{5}{6}.\quad s=\sqrt{\tfrac{5}{6}}.\quad m=2\sqrt{\tfrac{1}{6}}=\sqrt{\tfrac{4}{6}}=\sqrt{\tfrac{2}{3}}.\quad r'=\frac{m}{s}=\sqrt{\tfrac{2}{3}}\div\sqrt{\tfrac{5}{6}}=\sqrt{\tfrac{12}{15}}=\sqrt{\tfrac{4}{5}}=2\sqrt{\tfrac{1}{5}}.$$

From the θ triangle in the horizontal projection (Fig. 107 *C*) we get the equation:

$$e^2=1-r'^2=\tfrac{1}{5}.\quad e=\sqrt{\tfrac{1}{5}}.\quad \text{Tan}\ \theta=\frac{e}{r'}=\frac{\sqrt{\frac{1}{5}}}{\sqrt{\frac{4}{5}}}=\tfrac{1}{2}.\quad \theta=26°\ 54'=\tan\ 1\tfrac{1}{2}.$$

170. The angle, between 15 degrees and 30 degrees, at which an oblique hemisphere must stand in order that the Vertical Projection of its ellipse of shadow shall be a right line, is then the angle 26° 34″, the angle whose tangent is $\frac{1}{2}$. When standing at this angle the circular ring which casts the shadow is projected as an ellipse whose Minor Axis is to the Major Axis, or diameter of the sphere, in the ratio of $\sqrt{\frac{4}{5}}$ to 1.

171. The vertical projection at Fig. 107 *A*, and the horizontal projection at Fig. 107 *C*, look like exact counterparts of one another, and in fact they are so. For the two ellipses of the Line of Shade are necessarily alike, the rays of light in the elevation making the same angle with the Ground Line as in the Plan. The two straight lines are also symmetrical, by construction, the projection of the Line of Shadow in the Elevation having the slope of $\frac{1}{2}$, just as the projection of the rim has in the Plan. If, moreover, the Great Circle of the rim that is projected in the Plan as a right line is projected in the Elevation as an ellipse whose Minor Axis is $\sqrt{\frac{4}{5}}$, the Great Circle of the Line of Shadow that is projected as a right line in the Elevation, will in the Plan appear as a similar ellipse, as shown, all the conditions being the same.

Chapter XI

RINGS AND SPINDLES

Any lines lying in a plane may be revolved about any right line also lying in the plane, as an axis, and thus give rise to a variety of forms, all of which come under the designation of Surfaces of Revolution. Besides the Cylinder, Cone, and Sphere, which form the subject of the previous chapters, the most important of these are Ovoids, Rings, and Spindles.

The Shades and Shadows of Ovoids were discussed in the Introduction.

172. *Rings*.—If a Circle, Ellipse, or Oval, is revolved about an axis, lying in its own plane, but outside its circumference, the resulting surface is a Ring, either solid or hollow. Fig. 108 illustrates the case of the Circle, showing sections of solid Rings at *A*, and of hollow Rings at *B*. The shape, or proportions, of the Ring, depends, as the figure shows, upon the ratio between *r*, the radius of the Generatrix, or generating circle, and *R*, the radius of the Directrix, or circle on which it moves.

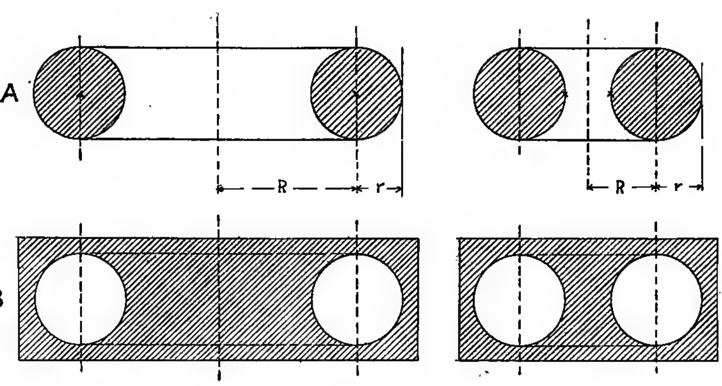


FIG. 108

173. *The Torus, Scotia, Hollow Torus, or Gorge, and Hollow Scotia*.—This affords four surfaces of revolution: the Torus, the Scotia, the Hollow Torus, or Gorge, and the Hollow Scotia (Fig. 109 *A*, *B*, *C*, *D*).

The word *Torus* properly implies a solid ring, like that shown in Fig. 108 *A*. But in Architecture, the name is regularly given to the surface of revolution which constitutes the outer half of such a ring. (Fig. 109 *A*.) The figure shows a flat circular disc with a convex edge, and consists of a short solid cylinder, surrounded by the outer half of a solid Ring.

The inner half of a hollow Ring is called a Scotia (or "shadow" surface). (Fig. 109 *B*.) The figure shows a flat disk with a concave edge, and consists of a solid Cylinder, surrounded by the inner half of a hollow Ring.

The inner half of a solid Ring is called a Hollow Torus, or Gorge (Fig. 190 *C*). The figure shows a round hole, surrounded by the inner half of a solid Ring, the sides of the hole being convex toward the center.

The outer half of a hollow Ring is called a Hollow Scotia. Like the Gorge, or Hollow Torus, the figure shows a round hole, but the sides of the hole, instead of being convex, are concave. (Fig. 109 *D*.)

The name Torus is sometimes applied, carelessly, to the Half Round, or cylindrical molding of similar section. (Fig. 109 *E*.) The name Scotia, in like manner, is sometimes applied to the Half Hollow, or cylindrical molding of similar section. (Fig. 109 *F*.)

In Fig. 109 the Shades and Shadows are put in in flat tints, the shadows being darker than the shades.

174. The Torus and Scotia are circular figures, convex in plan toward the spectator, as is the case with the Solid Cylinder, Cone, and Sphere. But the Hollow Torus, or Gorge, and the Hollow Scotia, like the Hollow Cylinder, Hollow Cone, and Hollow Sphere, cannot be seen in Elevation unless the nearer half of the surface is cut away. They then show, as here, semicircular figures, concave toward the spectator.

The Torus and the Hollow Torus, or Gorge, like the Sphere and other convex surfaces exhibit only a Line of Shade, separating the light from the Shade. The shadows they cast fall upon some other surface. But the

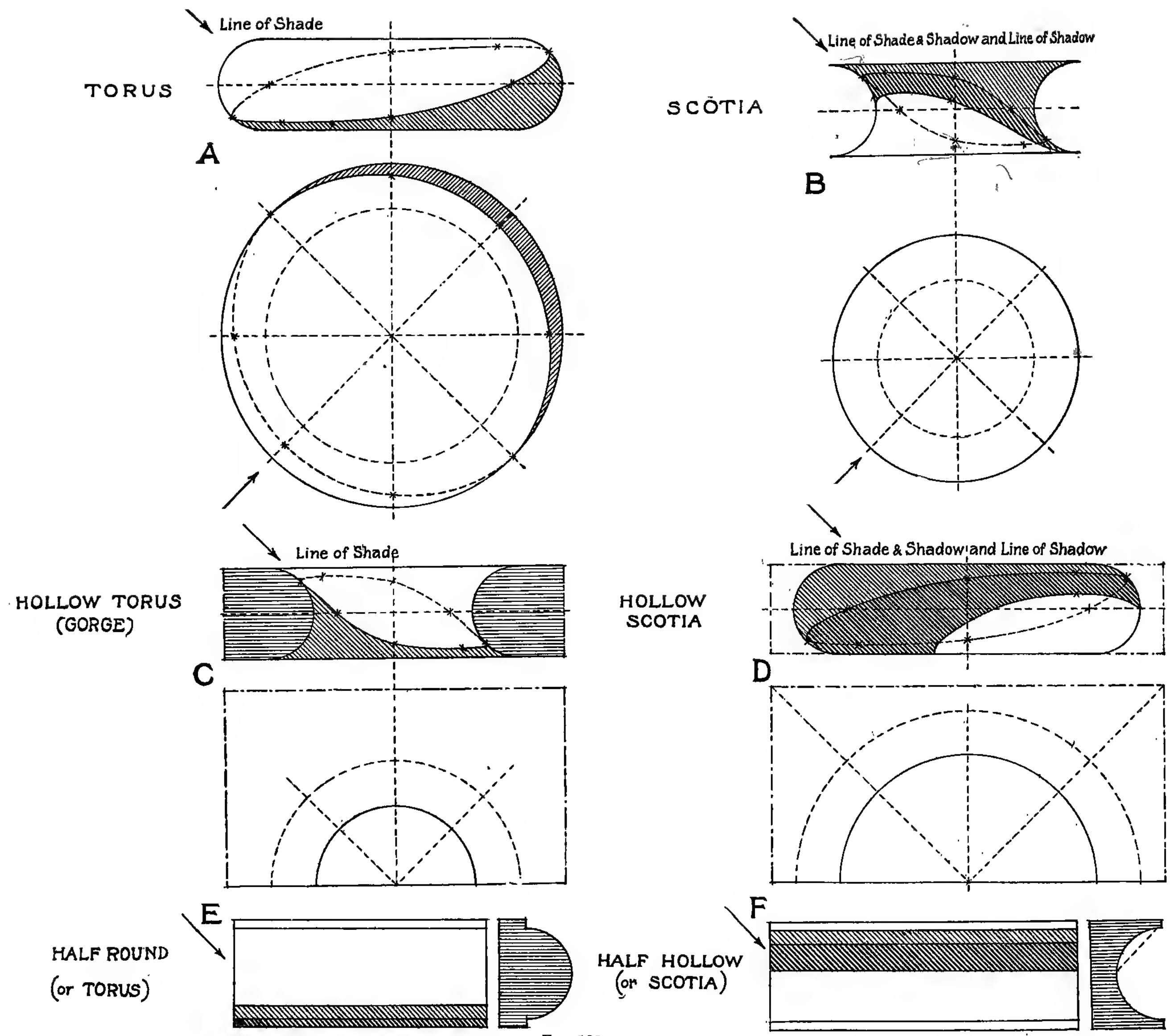


Fig. 109

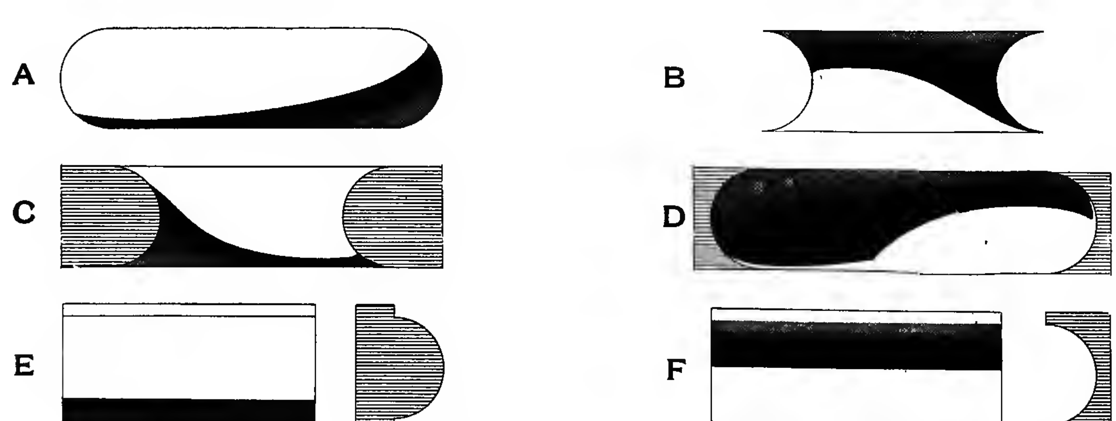


Fig. 110

Scotia and the Hollow Scotia, being concave in section, cast shadows upon themselves (just as do the Hollow Cylinder, Hollow Cone, and Hollow Sphere). Like these they exhibit both shade and shadow, separated by the Line of Shade and Shadow.

Fig. 110 shows the same surface as Fig. 109, but with the tints graded. On the Scotia and Hollow Scotia the Line of Shade and Shadow disappears, as always in concave surfaces.

175. *The Line of Shade and the Line of Shade and Shadow; the Line of Shadow.*

The Line of Shade upon a Torus or Hollow Torus (both of which have convex outlines), and the Line of Shade and Shadow upon a Scotia or Hollow Scotia (which have concave outlines), may be found by means of:

(1) Tangent Cylinders and Cones; (2) Projected Tangent Rays; (3) Revolved Tangent Rays; (4) Slicing.

The Line of Shadow cast by either a solid or a hollow Ring may be found:

(1) By Points; (2) by Tangent Cones and Cylinders; (3) by Slicing; (4) by Parallel Planes; (5) by an Auxiliary 45° Plane.

Of these seven methods Slicing is the most laborious and the least accurate; but as it gives both the Shade and the Shadow it is sometimes to be preferred.

176. These four kinds of Rings may be grouped by pairs, according to their points of resemblance, in three different ways, as follows: (Fig. 109.)

1. According as the Ring surrounds a solid disk (*A* and *B*), or an open hole (*C* and *D*).

The Torus and the Scotia (Fig. 109 *A* and *B*).—These both have in the center a short vertical cylinder, with a convex or concave edge, or profile; they thus have the form of a solid circular disc, the half which is seen in the elevation appearing as a convex semicircle.

The Hollow Torus, or Gorge, and the Hollow Scotia (Fig. 109 *C* and *D*).—Both these have in the center a circular hole, bordered by a convex or concave profile; they are represented as if seen in section, appearing in elevation as concave semicircles.

2. According as the profile is convex (*A* and *C*), or concave (*B* and *D*).

The Torus and the Hollow Torus, or Gorge (Fig. 109 *A* and *C*).—Both these have a convex edge, or profile, being the outer and inner halves of a solid ring, and exhibit a surface partly in light and partly in shade, being separated by the Line of Shade. The cast shadow falls upon some other object.

The Scotia and the Hollow Scotia (Fig. 109 *B* and *D*).—Both of these have a concave profile, each being half of a hollow ring, and exhibit a surface partly in shade, partly in shadow, and partly in light. The shade and the shadow are separated by the Line of Shade and Shadow, and the shadow and the light by the Line of Shadow.

3. According to the geometrical character of the surfaces of revolution.

The Torus and the Hollow Scotia (Fig. 109 *A* and *D*).—These two are, geometrically, the same surface, the Line of Shade which separates the light from the shade in the Torus, being the same as the Line of Shade and Shadow which separates the shade from the shadow in the Hollow Scotia.

The Scotia and the Torus or Gorge (Fig. 109 *B* and *C*).—These two surfaces also are geometrically identical, and the Line of Shade on one is, again, the same as the Line of Shade and Shadow on the other.

177. Similar Surfaces. Since the shape of the Shade and Shadow on any surface depends upon its geometrical properties, it is convenient to consider together the two rings that have geometrically the same surface, taking first the Torus and the Hollow Scotia (*A* and *D*), and then the Scotia and the Hollow Torus or Gorge, (*B* and *C*).

The Torus and the Hollow Scotia.

The Torus. *The Line of Shade Upon the Torus.*

Tangent Cylinders (Fig. 111).—The figure shows the points a and a' on the Line of Shade of a tangent horizontal cylinder, perpendicular to the Vertical Plane of Projection, the cross-section of which corresponds to that of the outline given in the Elevation; the points b and b' on that of a circular vertical cylinder; and the points c and c' on that of a horizontal cylinder, parallel to the Vertical Plane of Projection, and of the same shape as the normal cylinder. But c and c' can also be obtained from a and a', in virtue of the symmetry of the figure. A horizontal cylinder lying at 45°, and also of the same shape as the normal cylinder, gives the points d and d' (Fig. 111 *C*).

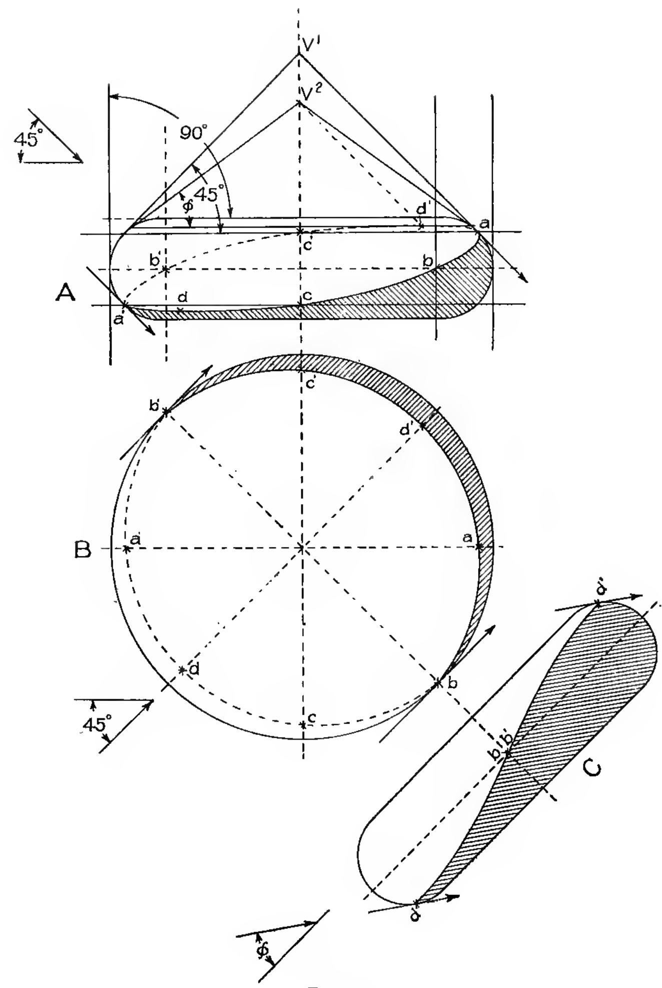


FIG. 111

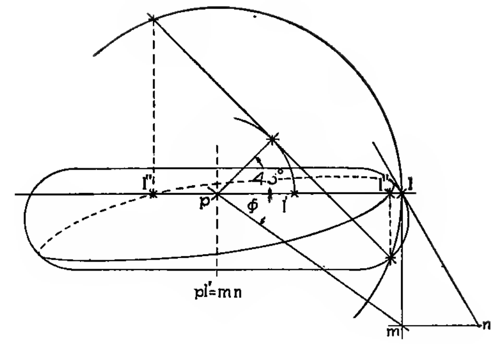


FIG. 112

178. *Tangent Cones* (Fig. 111).—Any number of additional points upon the Line of Shade can of course be determined by using a corresponding number of Cones. But, as in the case of Spheres and Ellipsoids, the six points a, a'; c, c'; and d, d', given by the two critical cones (making the angles Φ and 45° with the Horizontal Plane), and the two points b and b', given by the vertical cylinder, generally suffice.

The Line of Shade upon an Upright 45° Cone (as has been seen in Fig. 67) consists of two elements of the Cone, one of which is on the outline of the Cone farthest from the light, as at $v^1 a$ in the figure, and the other lies on the farther side of the Cone, and is projected in a vertical line, as at $v^1 c'$. This cone gives the points a and c' on the required Line of Shade.

The Line of Shade upon a Φ Cone (as was also shown in Fig. 67) consists of a single element of the Cone, lying in the direction of the rays of light, like the line $v^2 d'$ in the figure. This line gives the point d' on the required Line of Shade. The points c' and d' thus found are on the farther side of the Torus. The other points, marked a', c and d, may be obtained from a, c' and d' in virtue of the symmetry of the figure. But they may also be obtained by means of inverted cones, such as are shown in Fig. 68.

179. These Cones and Cylinders suffice to furnish eight points upon the Line of Shade, namely, a and a', b and b', c and c', d and d', all of which are easily determined "by inspection," as follows: The points a and a' are upon the outline, at the corners of the generating circles; c and c' are on the axis at the same level as a and a'; b and b' are on the corners of the "Equator," or largest Horizontal Circle, and d and d' are on the corners of the horizontal circles in which the Φ Cones are tangent to the Torus, and are on the level of the points at which the outlines of these cones are tangent to the outline of the Torus. At B is shown the horizontal projection of the Torus and its Line of Shade, and at C their projection upon an auxiliary plane, set at 45°.

180. It appears from these figures that the Line of Shade upon a Torus, though approximating in shape to an Ellipse, is not really an Ellipse, nor any plane figure. But it is symmetrical both about a horizontal plane and about a Vertical Plane set at 45° and bisecting the Torus, as appears at B and C.

181. *Projected Tangent Rays* (Fig. 111).—The same figure shows that the six points a, a'; b, b'; and d, d' can be obtained by means of the projections of rays of light drawn tangent to the outlines of the three projections of the Torus, and c and c' can be derived from a and a', as above, in virture of the symmetry of the figure.

182. *Revolved Tangent Rays* (Fig. 112).—Fig. 112 shows that any number of points on the Line of Shade can also be obtained by the Method of Revolved Tangent Rays. The Tangent Rays that slope at the angles: 90°, 45°, minus 45°, Φ, and minus Φ will indicate the eight important points, b, b'; a, a'; c, c'; d, d'.

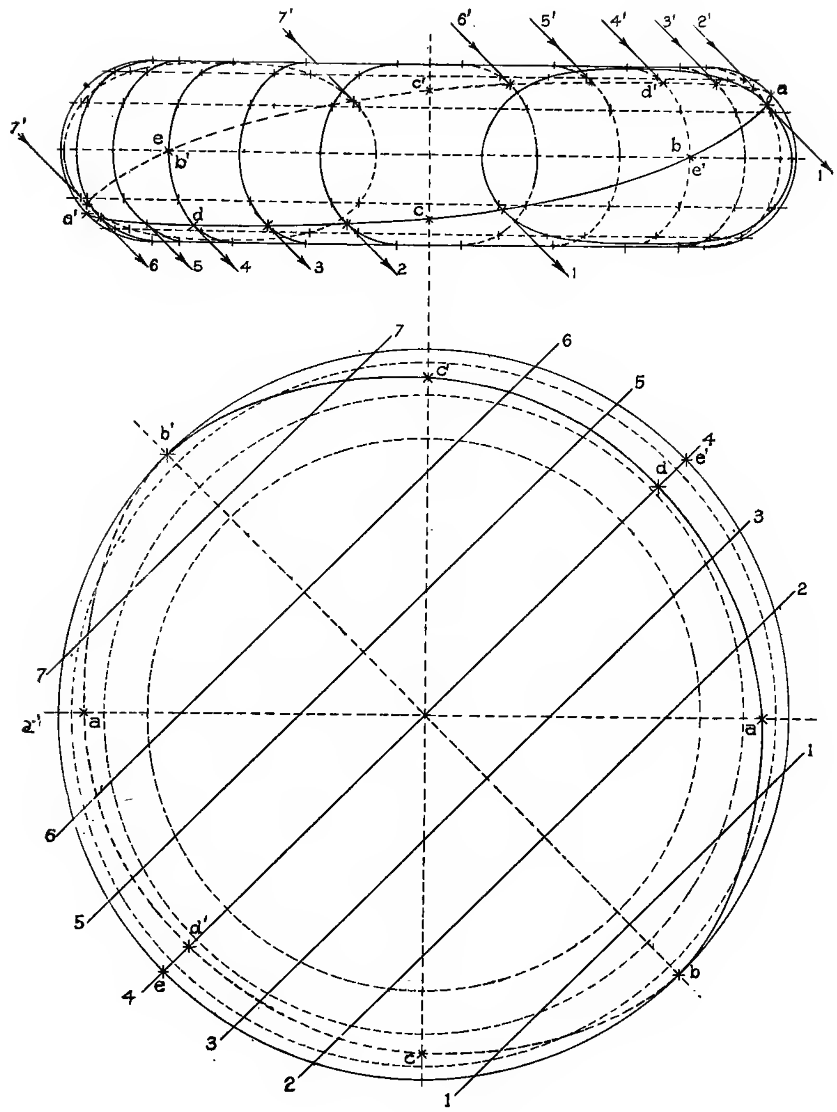


Fig. 113

183. *Slicing* (Fig. 113).—The figure shows the Torus cut by seven vertical planes which are marked in the plan: *1*, *2*, *3*, *4*, *5*, *6*, and *7*. The lines of intersection appear in the vertical projection as ovals, of which the halves that lie on the visible, or nearer, half of the Torus are drawn with a full line and the other halves with a dotted line. The first one lies wholly on the hither side and the seventh one wholly on the farther side. Upon each oval, tangent rays give two points of the required Line of Shade. The tangent rays on the hither side of the Torus are marked, *1*, *1*, *2*, *3*, *4*, *5*, and *6*, and those on the farther side *7'*, *7'*, *6'*, *5'*, *4'*, *3'*, *2'*,

The Secant Plane marked *4*, taken through the Axis of Revolution gives the points *d* and *d'* (which are the highest and lowest points of the Line of Shade), and also the vertical projection of the points *b* and *b'*, which by the symmetry of the figure, is the same as that of the points *e* and *e'*, where line *4* cuts the Equator of the Torus. But the other Secant Planes are taken at random, and the special points *a* and *a'*, *c* and *c'*, if desired, must be found by one of the methods already described.

184. *The Sphere as the Limit of the Torus.*—Fig. 114 *A* and *B* shows that as *R*, the Radius of the Directrix, grows smaller (*r*, the Radius of the Generatrix, remaining unchanged) the shape of the Torus changes until, when *R* equals zero, the Torus becomes a Sphere. The Sphere is the *limit* of the Torus. But the character of the Line of Shade does not alter, and the eight principal points upon any Torus (as also upon the Sphere) may be found as explained in Paragraph 179.

The figure shows these Toruses in elevation, first, at *A*, lighted by rays parallel to the Vertical Plane of Projection and making the angle ϕ with the Horizontal Plane; then at *B*, lighted by similar rays whose projections make angles of 45° with both planes, as in the previous figures.

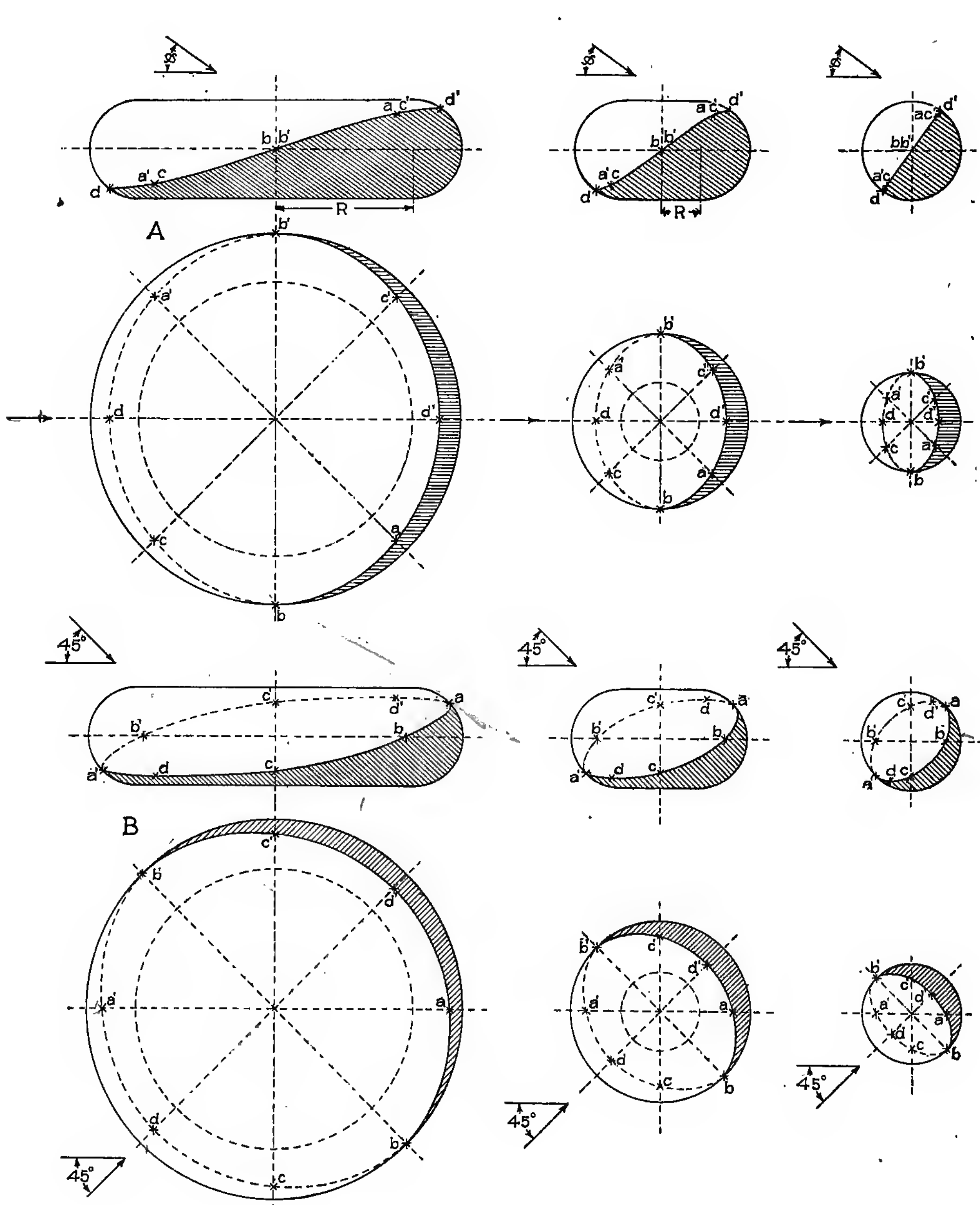


Fig. 114

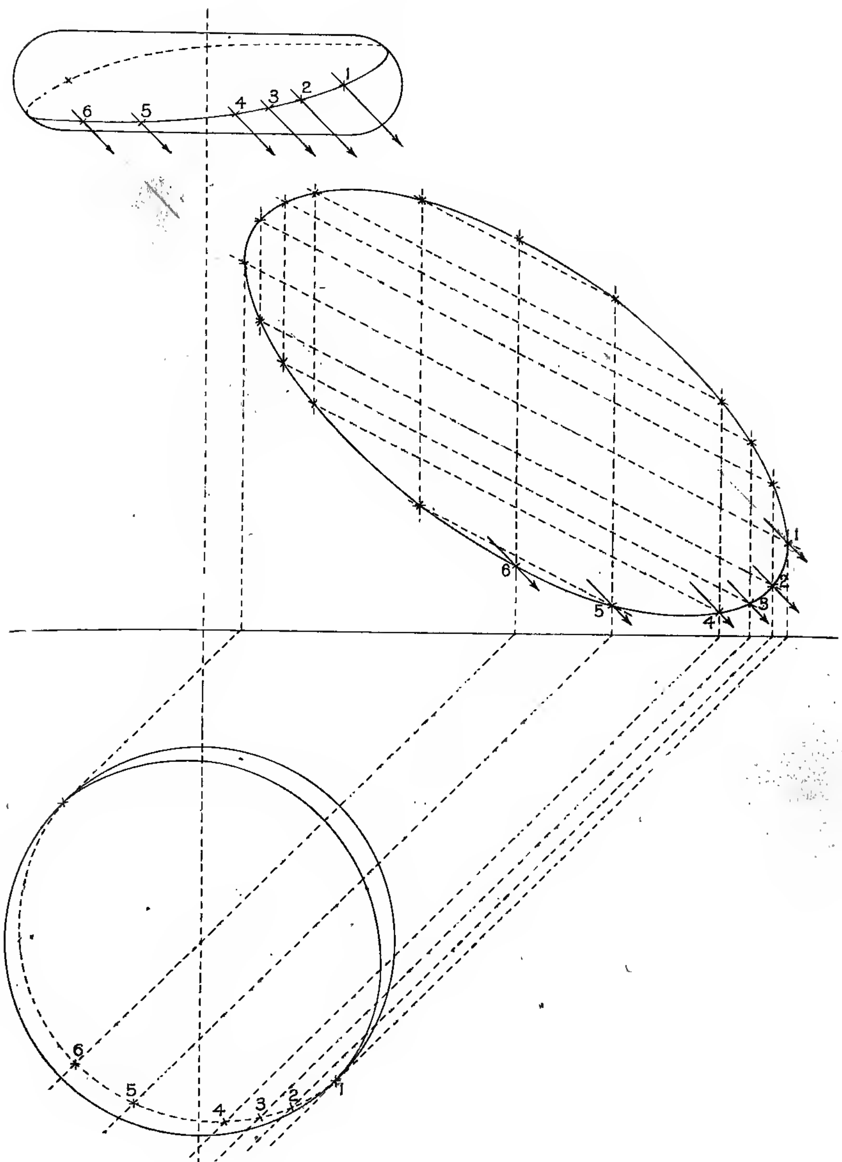


Fig. 115

185. *The Line of Shadow of the Torus Cast Upon a Plane. The Method of Points* (Fig. 115).—The Shadow cast by a Torus upon a plane surface is most easily found by the Method of Points, and the eight points in the Line of Shade given by the Vertical Tangent Cylinder and the two Critical Tangent Cones, or by Projected Tangent Rays, suffice to determine it with all necessary precision. If more points are needed, they can be taken at pleasure upon the two projections of the Line of Shade, by whatever means it has been obtained.

Fig. 115 shows the Shadow of the Torus thus cast upon a Vertical Plane of Projection. The Line of Shadow is, of course, the shadow of the Line of Shade, and it passes through the shadows of whatever points in the Line of Shade have been, by any method, determined. But, as the figure shows, if a number of points, such as *1*, *2*, *3*, and *4*, are determined on a quarter part of the curve, an equal number in each of the other three quarters result from the symmetry of the figure.

186. *Tangent Cones and Cylinders.*—The Shadow of the Torus may also be found by means of Tangent Cones and Cylinders, as was shown for the ovoid in Fig. P, 1 and 2, in the Introduction. This method has the advantage that it gives the direction, as well as the position, of the Line of Shadow at the points determined.

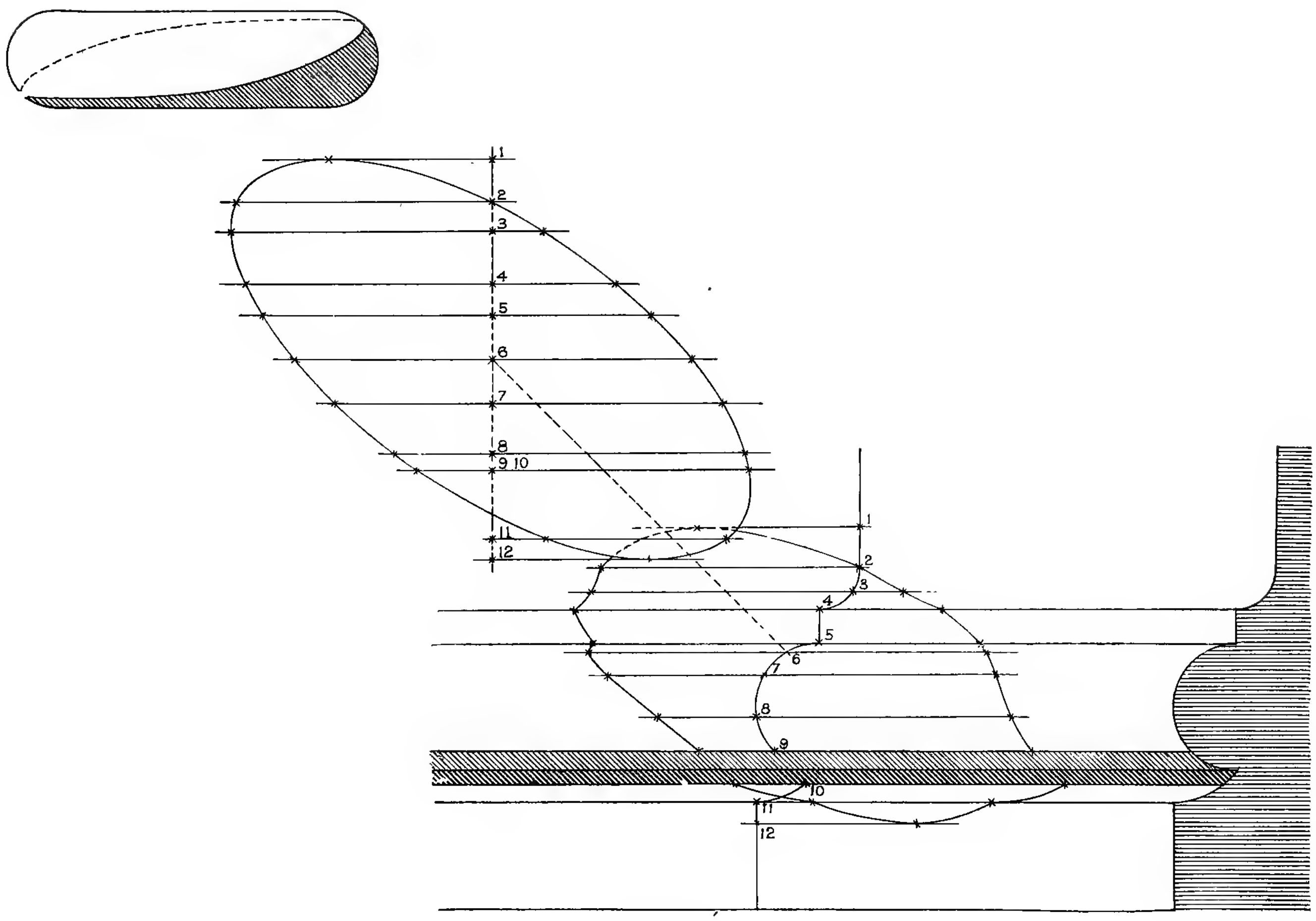


FIG. 116

187. *Slicing* (Fig. 115).—Fig. 115 also shows how, if the points on the Line of Shade have been obtained through Slicing the Torus by vertical secant planes, parallel to the light, the Shadow of the Torus upon the vertical plane of projection may be obtained by finding the shadows of these points on the vertical lines where the secant planes cut it.

188. *The Shadow of a Torus Upon an Irregular Surface.*—The shadow cast by a plane or solid figure upon an irregular surface can be obtained, in general, only by slicing both surfaces by planes parallel to the light.

But any such shadow cast upon a cylindrical surface, vertical or horizontal, may be found by the method shown in Fig. 55, Chapter VI, and again illustrated in Fig. 116. The shadow is first cast upon the plane of projection, and then transferred to the cylindrical surface just as if it were cast by a plane figure of the same shape and size as the shadow.

189. **The Hollow Scotia.**—The surface of the Hollow Scotia is geometrically the same as that of the Torus. As was shown in Fig. 109 *D*, it is concave both in the horizontal and in the vertical section.

190. *The Line of Shade and Shadow Upon the Hollow Scotia* (Fig. 117).—The Line of Shade upon the Torus and the Line of Shade and Shadow upon the Hollow Scotia are, as has been said, geometrically the same line, and the same processes that are used in Figs. 111, 112, and 113, to determine the Line of Shade upon the Torus may also be used to determine the Line of Shade and Shadow upon the Hollow Scotia. Fig. 117.

191. *The Line of Shadow Upon the Hollow Scotia.*—Fig. 117 shows also the shadow cast upon the doubly concave interior surface of a Hollow Scotia. The Line of Shadow is the shadow of the Line of Shade *1, 2, 3, 4, 5, 6, 7, 8*, which is the open circle at the top of the figure, as appears from the Plan at 117 B. The Line of Shadow is projected, in both plan and elevation, as a complete oval passing through the points 1^s, 2^s, 3^s, 4^s, 5^s, 6^s, 7^s, 8^s.

The vertical secant planes marked in the plan *2*; *1, 3*; *8, 4*; *7, 5*; and *6* (drawn through the points *2*; *1* and *3*; *8* and *4*; *7* and *5*; and *6*; respectively), cut the surface of the Hollow Scotia in curved lines, similarly marked in the elevation. Since these curves all stand edgewise to the light, the ray that passes through any of these points casts the shadow of the point upon the lower part of the same curve, and lines passed through them at 45 degrees give points on the required line of Shadow at 1^s, 2^s, 3^s, 4^s, 5^s, 6^s, 7^s, and 8^s, in both Plan and Elevation.

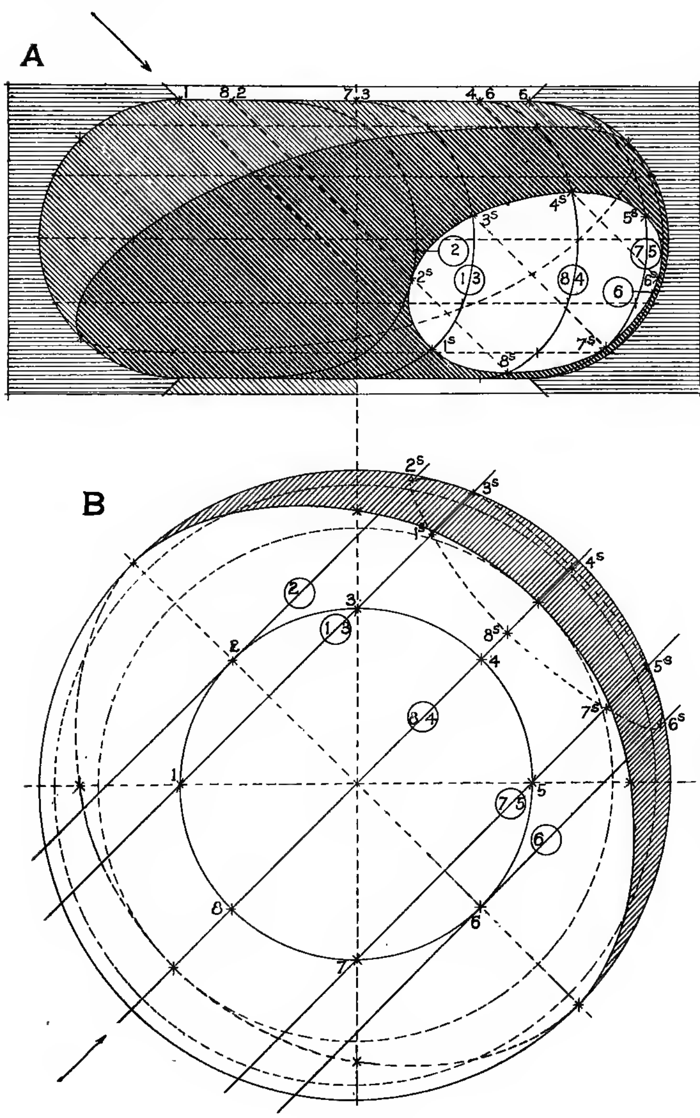


Fig. 117

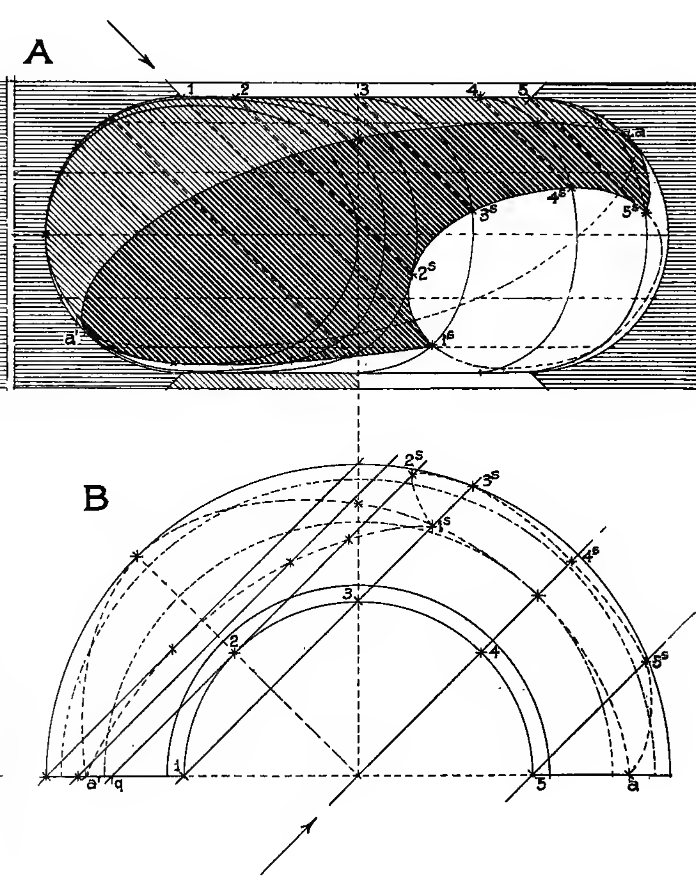


Fig. 118

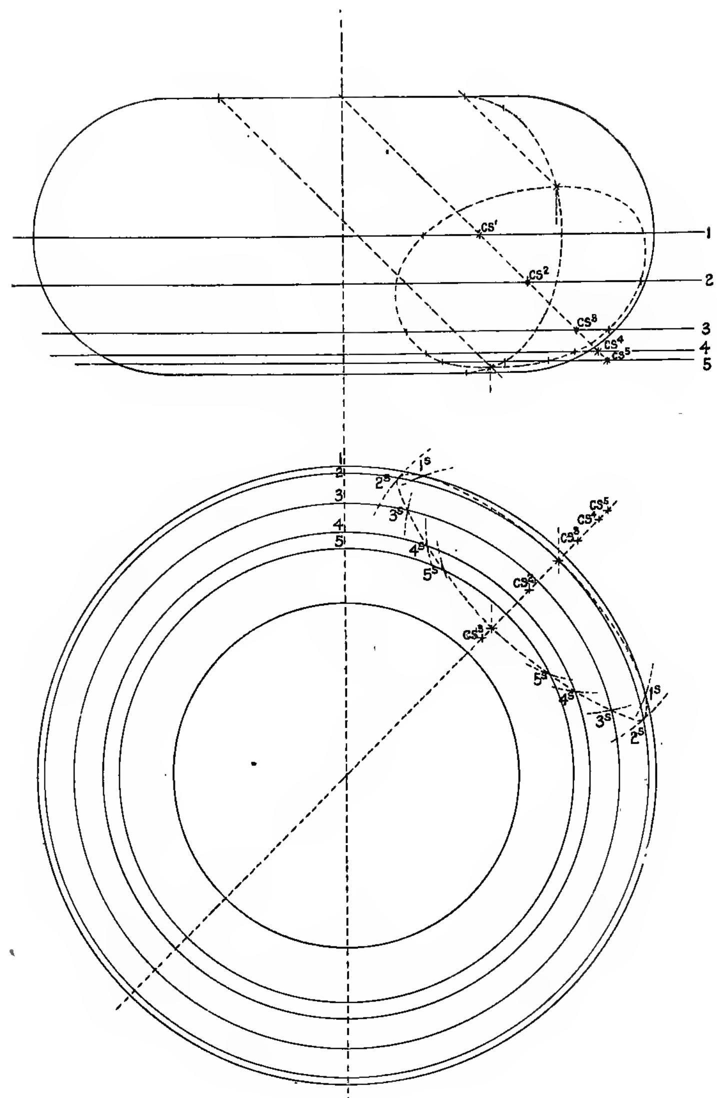


Fig. 119

192. *Slicing.*—Thus Slicing gives as many points of the Line of Shadow as may be desired, and furnishes two that are of special interest, namely the highest and lowest points of the curve, marked in the figure 4^s and 8^s. These are obviously the highest and lowest points, since both the elevation and the plan (Fig. 117 *A* and *B*) show that the ray *8*, 8^s is the longest of any, and the ray *4*, 4^s, the shortest. Since all the rays descend from the same level, 8^s must be the lowest point of the Line of Shadow and 4^s the highest.

These two points being determined in both plan and elevation, the Line of Shadow can be drawn through them without serious error, if it is borne in mind that in the Plan the curve is symmetrical about the line 8^s, 4^s, and is tangent at its extremities to the rays drawn through the points *2* and *6*. In the Elevation it is tangent to the rays drawn through the points *1* and *5*.

193. Fig. 118 shows the shape of the shadow cast upon the surface of the Hollow Scotia when, as in Fig. 109 *D* the nearer half of the ring is cut away and the figure seen as a concave semicircle. The Line of Shade then consists, in part, of the horizontal semicircle *1*, *2*, *3*, *4*, *5*, and in part of the curves *5a* and *1a′*, on the outline of the figure. The upper segment of the Line of Shadow starts accordingly at the point *a*, where it is tangent to the outline, and terminates at the point 5^s, where there is a cusp. The lower segment, in like manner, lies between the point *a′*, and the point 1^s.

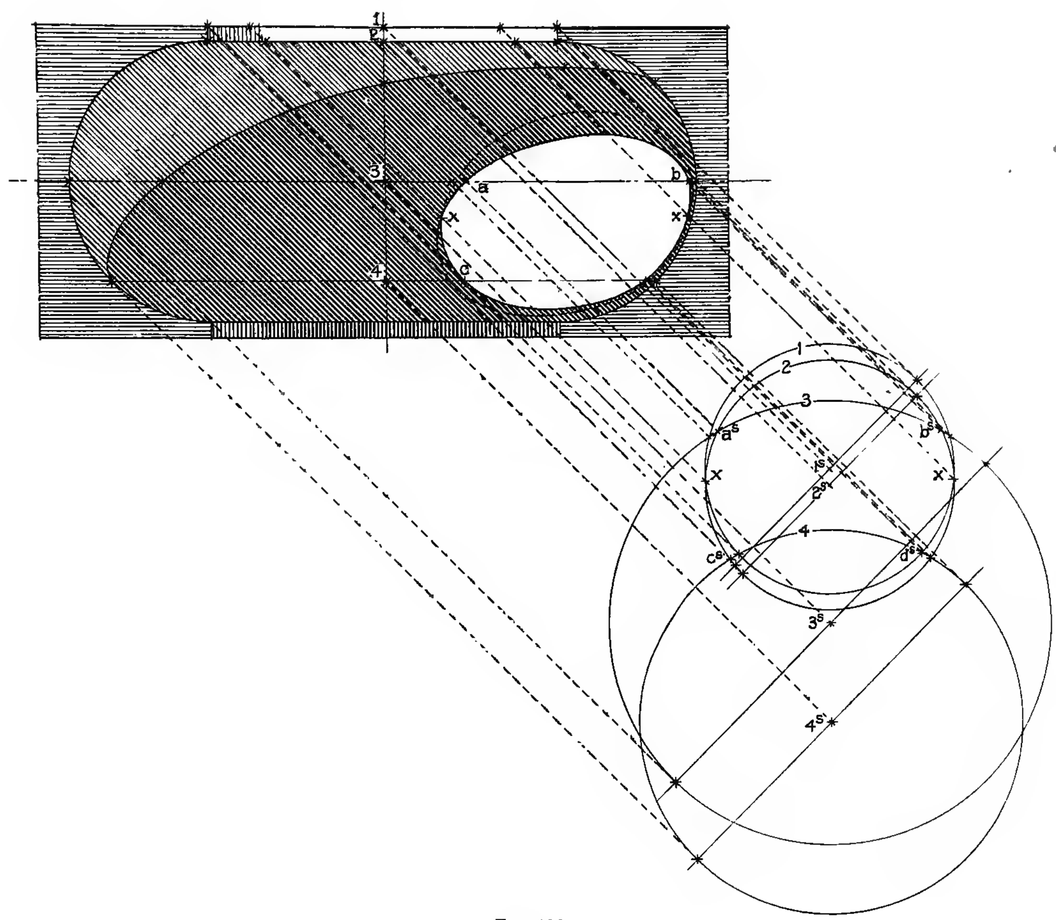


FIG. 120

In both Fig. 117 and Fig. 118 the shadow is represented as being cast by the horizontal circle that constitutes the upper edge of the Hollow Scotia itself. But in actual practice the Hollow Scotia is generally surmounted by a short vertical cylinder, as in Fig. 120, and the actual shadow is composed partly of the shadow of the half of the upper edge of the cylinder nearest the light, and partly of the Shadow of the half of the upper edge of the Hollow Scotia farthest from the light, as shown in the figure, the two Lines of Shadow crossing at the points x and x.

The Method of Slicing has the advantage not only that it gives both the Line of Shade and Shadow and the Line of Shadow, but that it gives them both in the Vertical projection, where they are most wanted.

194. *Parallel Planes.*—The Method of Parallel Planes (Fig. 119) gives only the Line of Shadow, and gives that only in the horizontal projection, or Plan, from which it has to be transferred, point by point, to the Elevation.

195. *The Auxiliary 45° Plane.*—Fig. 120 shows how the Line of Shadow upon the Hollow Scotia may be obtained by means of a 45° Auxiliary Plane, which is always available when the line that casts the shadow is a horizontal circle.

Neither the Method of Parallel Planes nor the Method of the Auxiliary 45° Plane gives the highest and lowest points of the curve, which must be obtained as in Fig. 117.

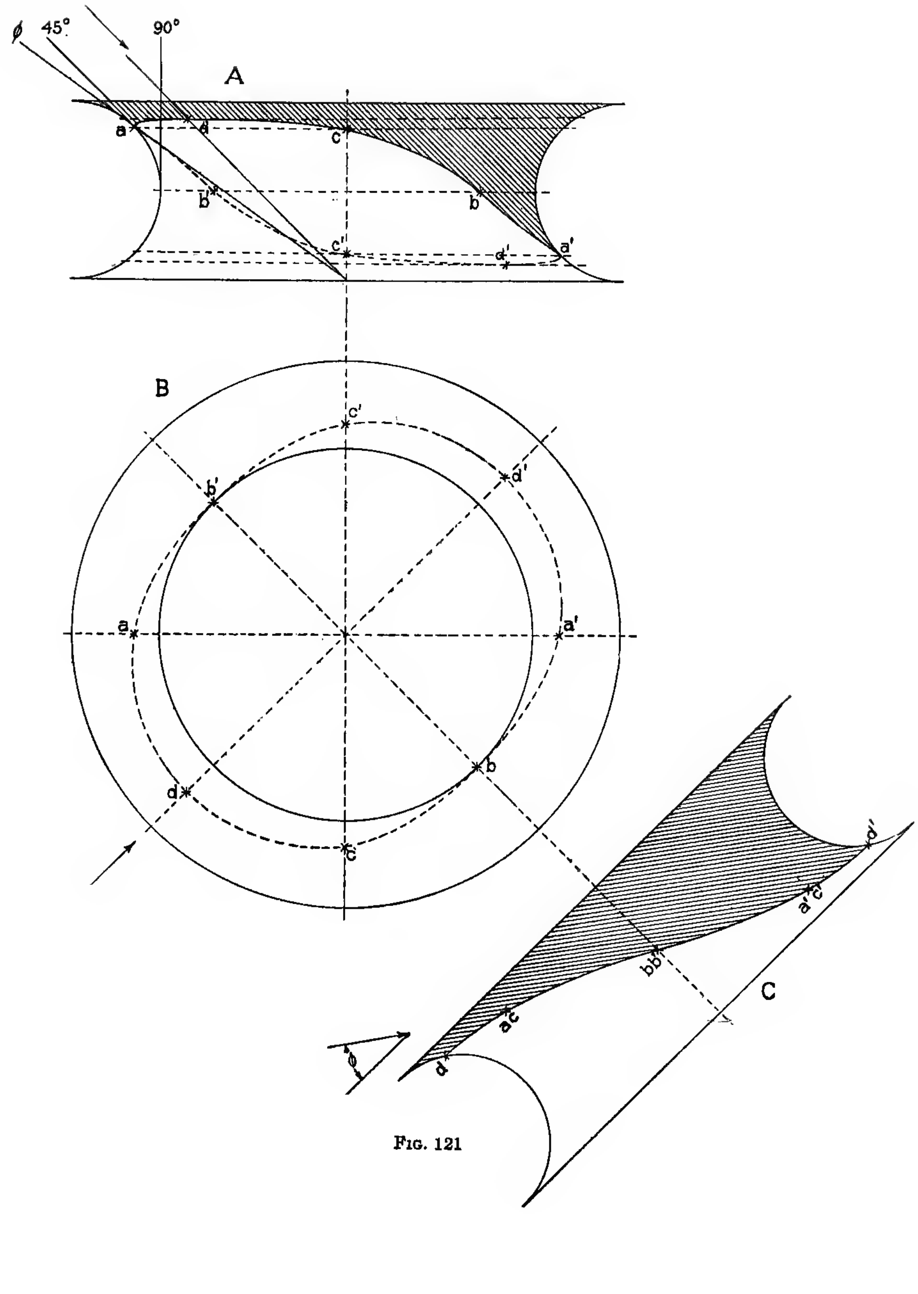


FIG. 121

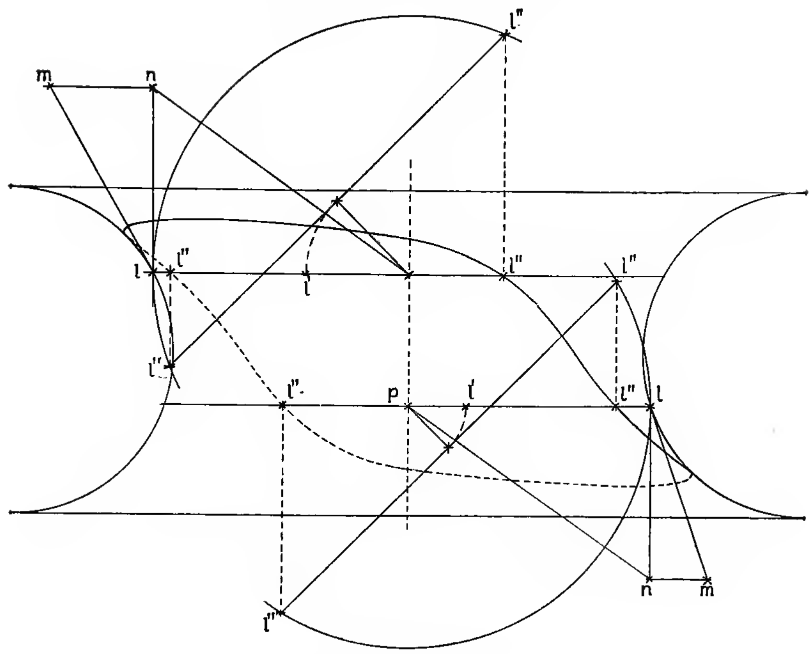


FIG. 122

196. *The Scotia and the Hollow Torus, or Gorge.*

The Line of Shade and Shadow upon the Scotia is the same as the Line of Shade upon the Hollow Torus, the surfaces of these two rings being, geometrically, the same.

THE SCOTIA. *The Line of Shade and Shadow upon the Scotia.*

Tangent Cylinders, Tangent Cones, and Projected Tangent Rays.—Fig. 121 shows eight points on the Line of Shade and Shadow of the Scotia, corresponding to the eight principal points upon the Line of Shade of the Torus, shown in Fig. 110. These are easily determined by means of Tangent Cylinders, Tangent Cones, or Projected Tangent Rays, or, as that of the Torus was determined in Fig. 111, by simple "inspection," as follows:

The points a and a' are on the corners of the generating semicircles that form a part of the outline of the Scotia; b and b' are on the corners of the Equator; c and c' are on the axis of revolution at the same levels as a and a'; d and d', the highest and lowest points of the curve, are at the corners of horizontal circles drawn through the points of the semicircles which form the outline where they slope at the angle 45°. There is a point of contrary flexure between a and b, one between a' and b', and others at b and b'.

197. *Revolved Tangent Rays.*—Fig. 122 shows how any desired number of points on the Line of Shade and Shadow of the Scotia, such as l'', can be obtained by the Method of Revolved Tangent Rays.

198. *Slicing.*—Fig. 123 (a) shows the Scotia cut by five vertical planes drawn parallel to the direction of the light, as was done in the case of the Torus, in Fig. 113. The lines of intersection are projected in the plan as right lines and in the elevation as curves, each of which has two branches, one of which, for each of the planes marked *2* and *3*, is on the farther side of the Scotia and is not shown. The second branch of plane No. *1*, also on the farther side of the Scotia, is shown by a dotted line. In the Secant Plane No. *4*, almost the whole of the second branch is visible. In Plane No. *5*, which is drawn tangent to the cylindrical core of the Scotia, the two branches meet at the point b.

The curves of intersection cut by Plane No. *1* lie at two of the corners of the Scotia, and the other two corner curves, cut by a plane at right angles to No. *1*, have the same projections as these two.

As in the previous case of the Line of Shade upon the Torus, the Line of Shade and Shadow upon the Scotia is drawn through the points of these curves at which the rays of light are tangent to them, that is to say, the points at which the projections of the curves have a slope of 45°. It crosses line No. *5* also at the point b. This is the point where the Line of Shade and Shadow crosses both the *Equator* of the Scotia and the line upon the corner of the Scotia. Its position is independent of the slope of the rays of light. Fig. 123 (b) shows the same for a Scotia of different proportions.

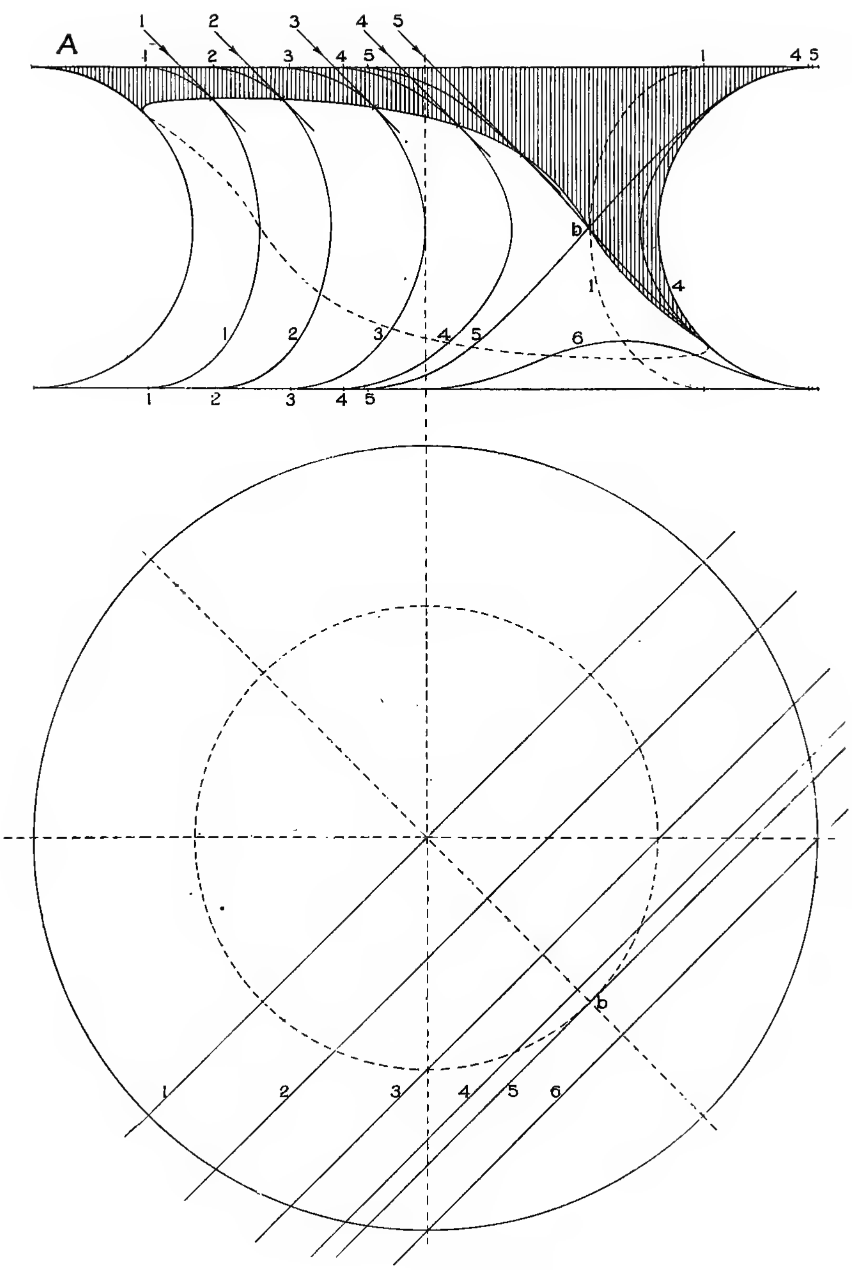


FIG. 123 (a)

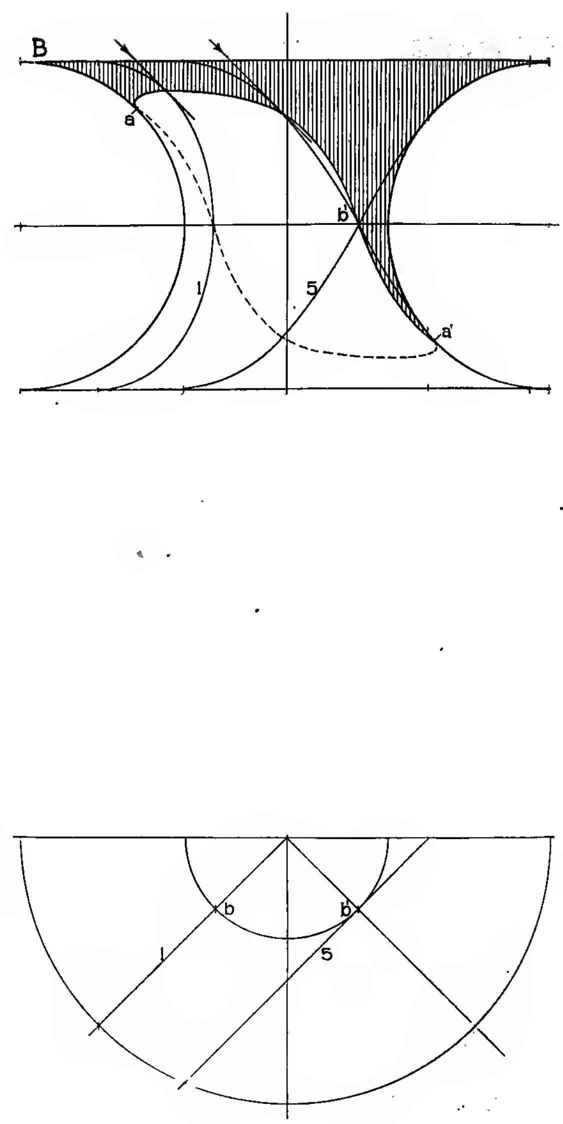


FIG. 123 (b)

199. *The Line of Shadow Upon the Scotia. Slicing.*—Fig. 124 shows the elevation and plan of a Scotia cut by Vertical Planes set at 45° and parallel to the direction of the light, as in Fig. 123, with the Convex semi-circle of the upper edge of the Scotia casting its shadow upon the surface below, a surface which is convex in horizontal section and concave in vertical section.

At *A* is the Vertical projection of the Scotia, showing curves of intersection similar to those of Fig. 123. Since these curves all stand edgewise to the light, the rays of light that pass through the points *1*, *2*, *3*, *4*, *5*, and *6* upon the convex Line of Shade which forms the upper edge of the Scotia cast the shadows of these points upon the lower parts of these same curves, and lines drawn through them at 45° gives points of the required Line of Shadow at 1^s, 2^s, 3^s, 4^s, 5^s, and 6^s.

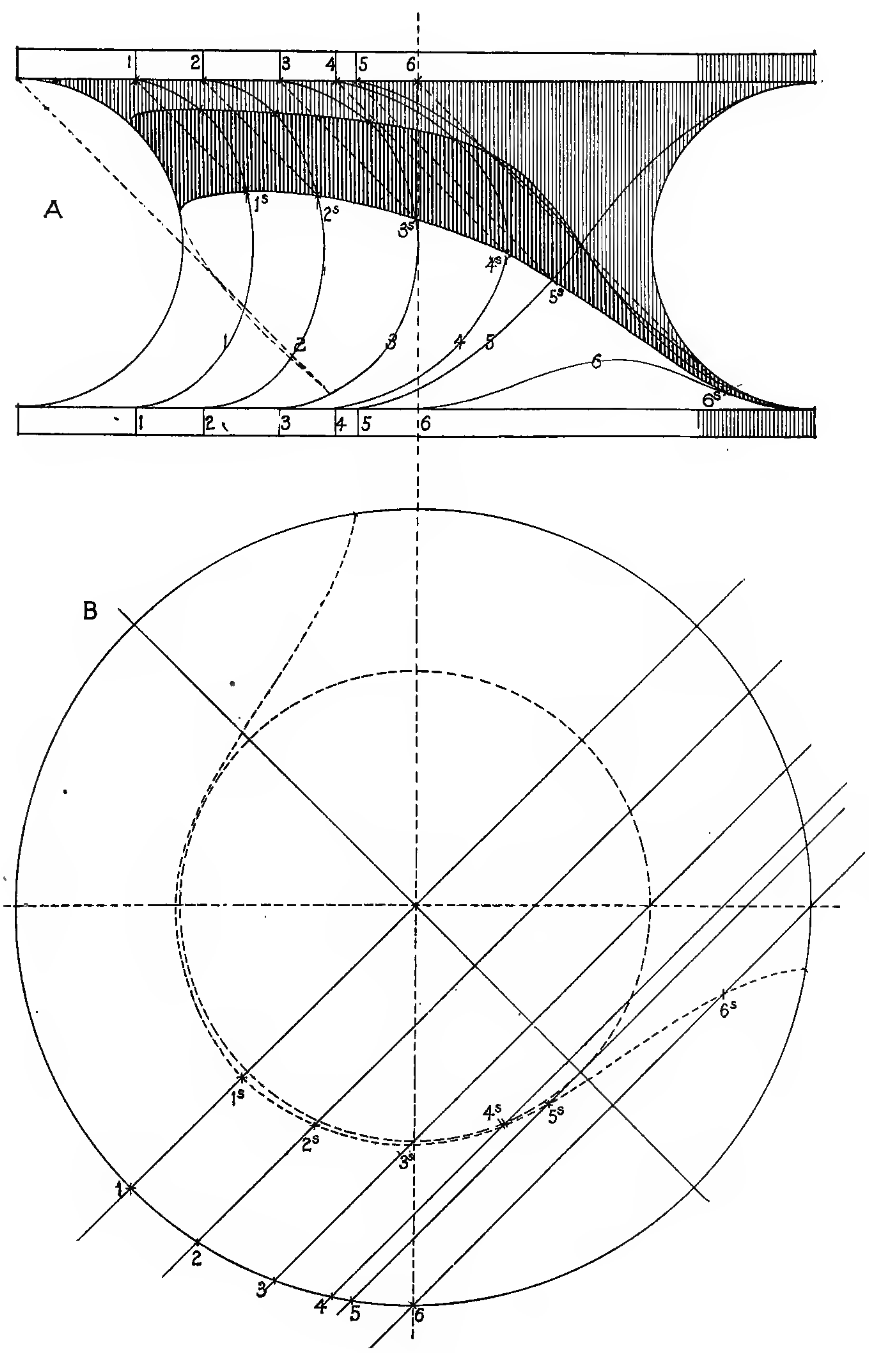


Fig. 124

200. Here, as in the Hollow Cylinder, Hollow Cone, and Hollow Sphere, the Line of Shade and Shadow separates the upper part of the concave surface, which is turned away from the light, from the lower part, which is turned toward the light, and is consequently either in light or in the shadow cast by the upper part. The Line of Shadow separates the portion of the lower part which is in the shadow from the portion which is in the light. The shade thus extends from the Line of Shade (which is the upper edge of the Scotia) to the Line of Shade and Shadow; the shadow extends from the Line of Shade and Shadow to the Line of Shadow.

Any number of points of the Line of Shade and Shadow and the corresponding points of the Line of Shadow, can thus easily be obtained. But it generally suffices to obtain the highest points of these curves, passing only a single 45° Vertical Plane through the axis of revolution, like Plane No. *1* in Fig. 124. The resulting line of intersection, on the nearer side of the Scotia, lies in the corner of the Scotia nearest to the light and the highest points of both lines lie in it. The points at which these lines are tangent to the outline of the Scotia are at the same level as those where they cross the axis.

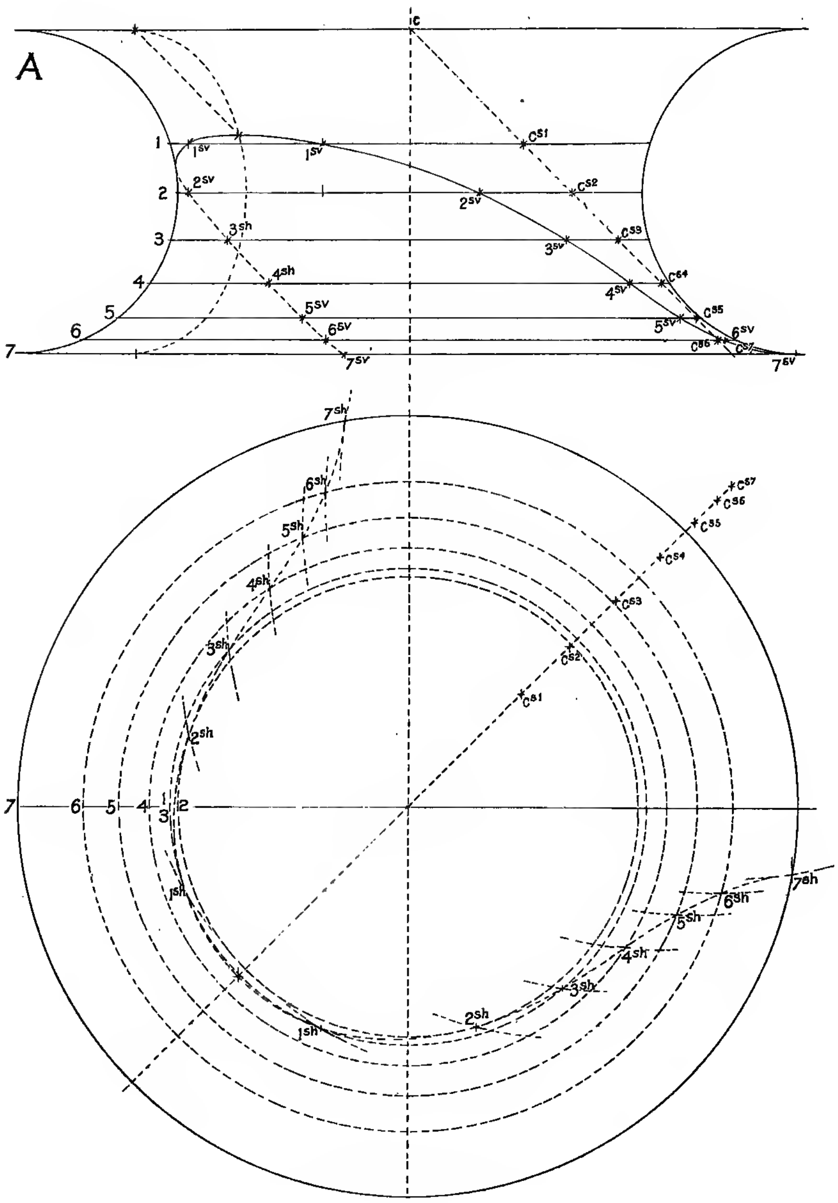


Fig. 125

The Line of Shadow can then be drawn through this point with a fair degree of accuracy, without obtaining any other points, if it is borne in mind (as is to be seen in figure 124) that the highest point of the Line of Shade and Shadow is also upon this corner curve, that the two lines are farthest apart at these points, and that though they grow gradually together, as they descend, they do not cross, since a shadow cannot fall within a shade. Moreover the Line of Shadow, like the Line of Shade and Shadow, crosses the outline nearest the light at a point which is on a level with the point where it crosses the axis of revolution, and crosses the outline farthest from the light near its lowest point. At both points the Line of Shadow and the Outline, lying in the concave surface, are tangent to planes which are tangent to the surface, and as at the outline each plane is seen edgewise, and is projected as a right line, the projections of the two lines are tangent to the outline. It is plain indeed that when a line lying in a Surface of Revolution reaches the outline, its projection makes an angle with the projection of the outline, as in Figs. 111 *C* and 121 *C*, if at that point the line is normal on the Plane of Projection. Otherwise, its projection is tangent to the projection of the outline, as in Figs. 111 *A* and 121 *A*.

Fig. 110 *B* shows a Scotia, with the Shade and Shadow forming a continuous gradation, lighted by diffused reflected light coming in a direction opposite to that of the Sun's rays. The shading is darkest where the reflected light is least strong, namely, along the lower edge of the Shadow and on the corner of the Scotia nearest the Sun, and the Line of Shade and Shadow entirely disappears.

201. *Parallel Planes.*—Fig. 125 shows how the Line of Shadow is obtained by the Method of Parallel Planes. While the Method of Slicing gives both the Line of Shade and Shadow and the Line of Shadow, and gives both of them in the Elevation, where they are usually most wanted, the Method of Parallel Planes gives only the Line of Shadow, and gives it in the Plan, from which it has to be transferred to the Elevation.

On the other hand the Method of Parallel Planes is more accurate than Slicing, even when the Line of Shadow is wanted only in the Elevation.

202. *The Auxiliary Plane at 45°.*—Fig. 126 shows how the Line of Shadow upon the Scotia may be determined by means of an Auxiliary 45° Plane. The Scotia is cut, by a series of horizontal planes, in a series of horizontal circles. The required Line of Shadow is the shadow cast upon them by the lower edge of the fillet at the top, which is also a horizontal circle. The shadows cast by all these circles upon the Auxiliary Plane are projected in true circles. (See Paragraph 58, Fig. 47, Chapter V.) The points *2′*, *2′*, *3′*, *4′*, *5′*. *6′*, and *7′*, in which the circular shadow of the edge of the fillet cuts the other circular shadows, are the shadows of points *2″*, *2″*, *3″*, *4″*, *5″*, *6″*, and *7″* in the required Line of Shadow. These points and the Line of Shadow itself are found in the elevation by tracing back the rays of light from the points of intersection until they cut the corresponding horizontal lines in the elevation.

203. *The Hour-Glass Scotia* (Fig. 127).—When the radius of the Directrix of a Torus equals the radius of the Generatrix, the semicircles that define the Generatrix are tangent to one another, and the cylindrical core of the Scotia is at its center diminished to a point. This form may be called the Hour-Glass Scotia. It is the limit of the Scotia just as the Sphere is the limit of the Torus.

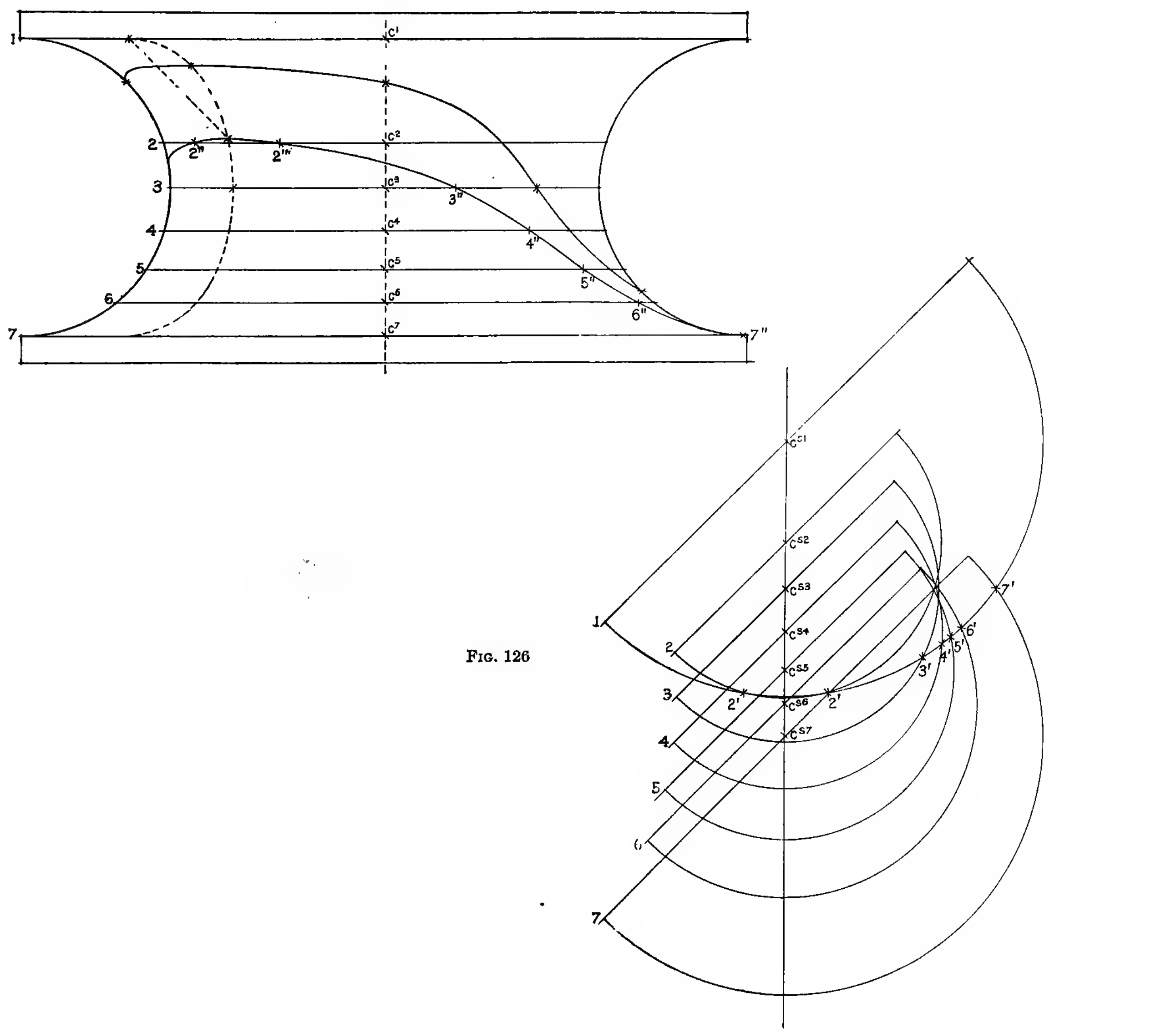


Fig. 126

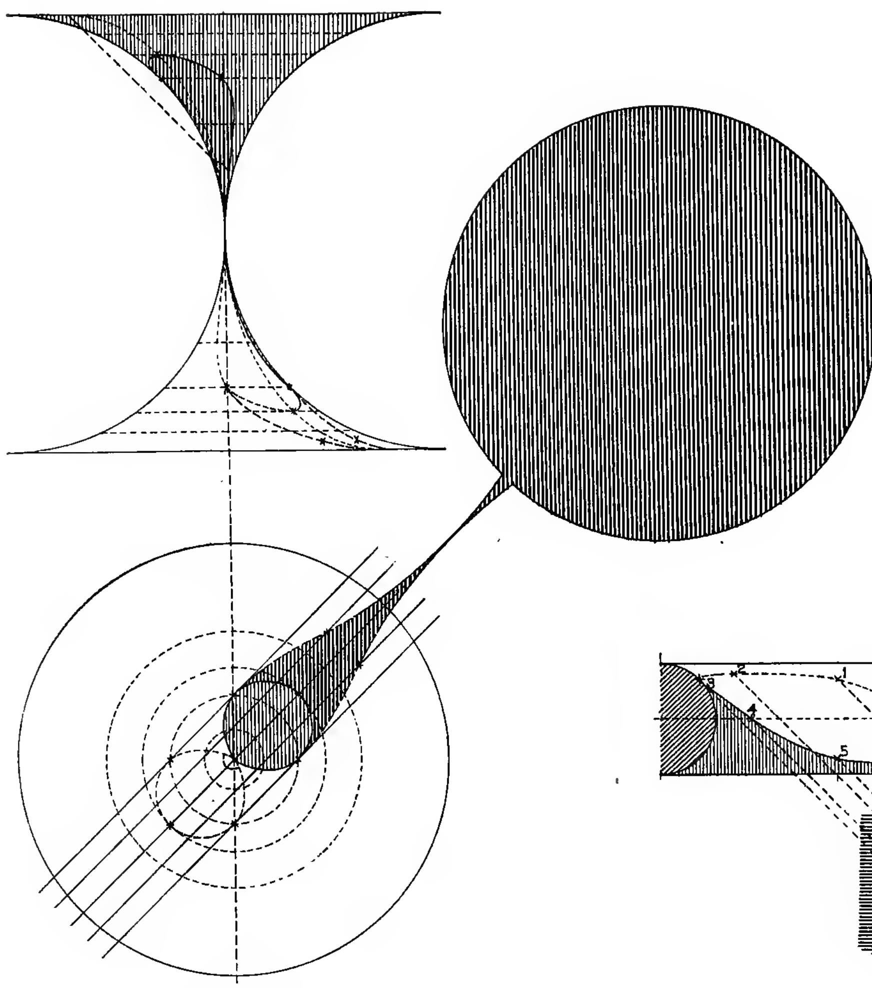


Fig. 127

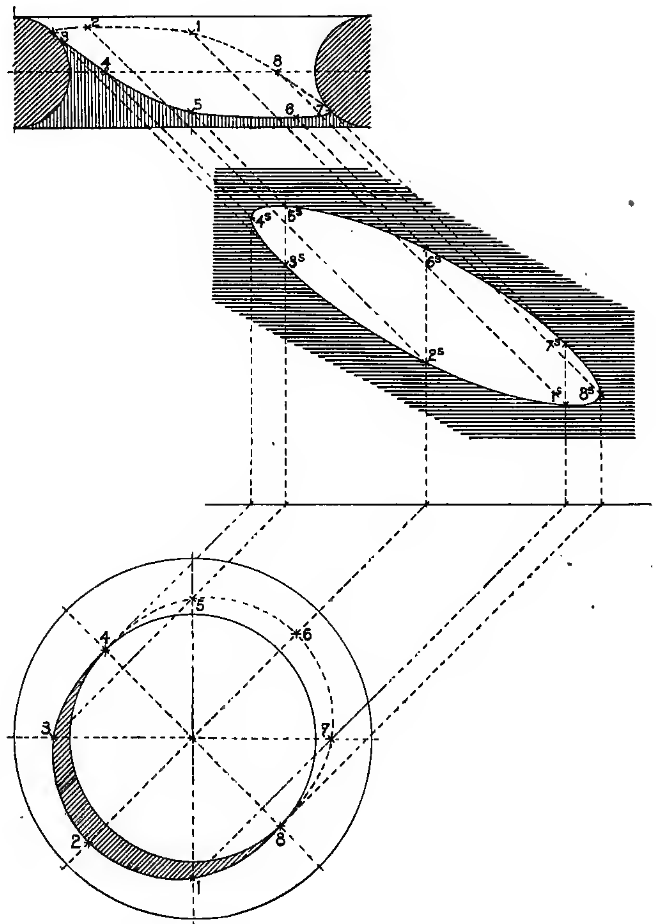


Fig. 128

204. *The Hollow Torus, or Gorge* (Fig. 128).—*The Line of Shade Upon the Gorge.*—The Line of Shade of the Hollow Torus, or Gorge (Fig. 129), is, as has been said, the same as the Line of Shade and Shadow upon the Scotia (Fig. 122), the geometrical form of the two surfaces being identical.

205. *The Line of Shadow of the Gorge.*—Fig. 128 shows also how the Shadow of the Hollow Torus can be obtained from the Line of Shade by means of Points, just as was done for the Torus in Fig. 115.

206. *Upright Rings.*—Fig. 129 shows the aspect presented by the four kinds of Ring, when the axis of revolution is parallel to the Ground Line, the Shade and the Shadow being shown as in Fig. 109, with flat tints. Fig. 130 shows the same with graded tints, as in Fig. 110.

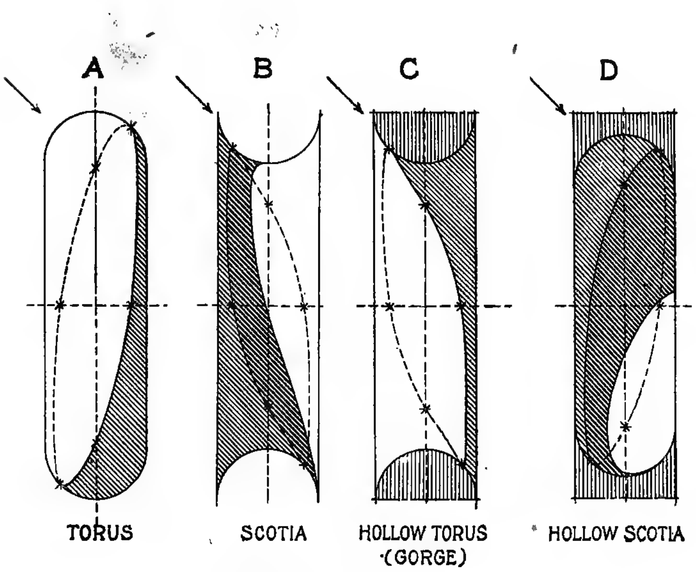


Fig. 129

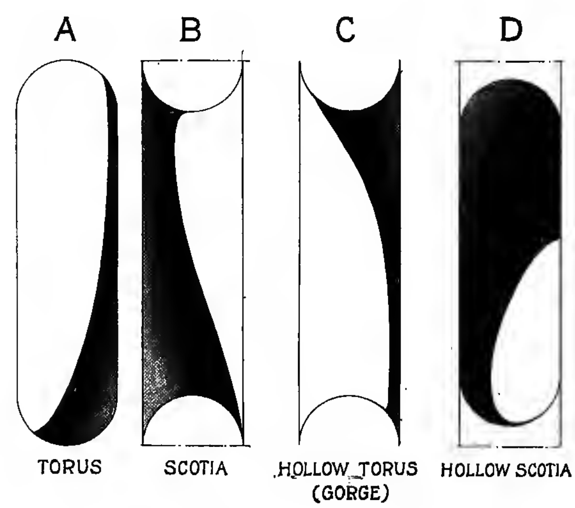


Fig. 130

207. *Spindles* (Figs. 131, 132).—When R, the radius of the Directrix of a Torus is smaller than r, the radius of the Generatrix (the axis of revolution being inside the circumference of the generating circle, so that the Generatrix is only an arc of a circle), the resulting surface of revolution is called a *Spindle*, from its resemblance in shape to the bobbins used in spinning. In these figures the radius of the Generatrix is longer than that of the Directrix, the difference being greatest at A, and diminishing until at E the two radii are of the same length. R then equals r' and the result is a Sphere. The Sphere is accordingly the *limit* of the Spindle, as well as of the Torus.

In these figures the light is shown as making the angle ϕ with the horizontal plane. In Fig. 131 it is parallel to the vertical plane of projection; in Fig. 132, it makes with it the angle 45°.

208. It is to be noticed that although the five Spindles regularly increase in diameter from A to E (the angle at the vertex decreasing from 60° to 0°), the Line of Shade as shown in the plans changes shape irregularly. The Cusp, which is rather blunt at A, grows sharper in B and C, and then more blunt until it disappears altogether at E. This, however, is not really as unreasonable as it seems. For the five figures constitute not a single sequence, but two. In the figures of the first series, A, B, and C, the Line of Shade starts at the vertex of the Spindle, the difference in the three figures being that the angle the light makes with the surface of revolution grows less and less, until at C it is parallel with the surface. Under these circumstances the cusps grow sharper and sharper. But in the second series, C, D, and E, the highest point of the Line of Shade has begun to recede from the vertex of the spindle. The rays of light are in each case parallel to the surface of revolution at the highest, or initial point, of the Line of Shade, which constantly gets farther and farther from the Vertex. Under these circumstances the cusp grows more and more obtuse until it disappears altogether.

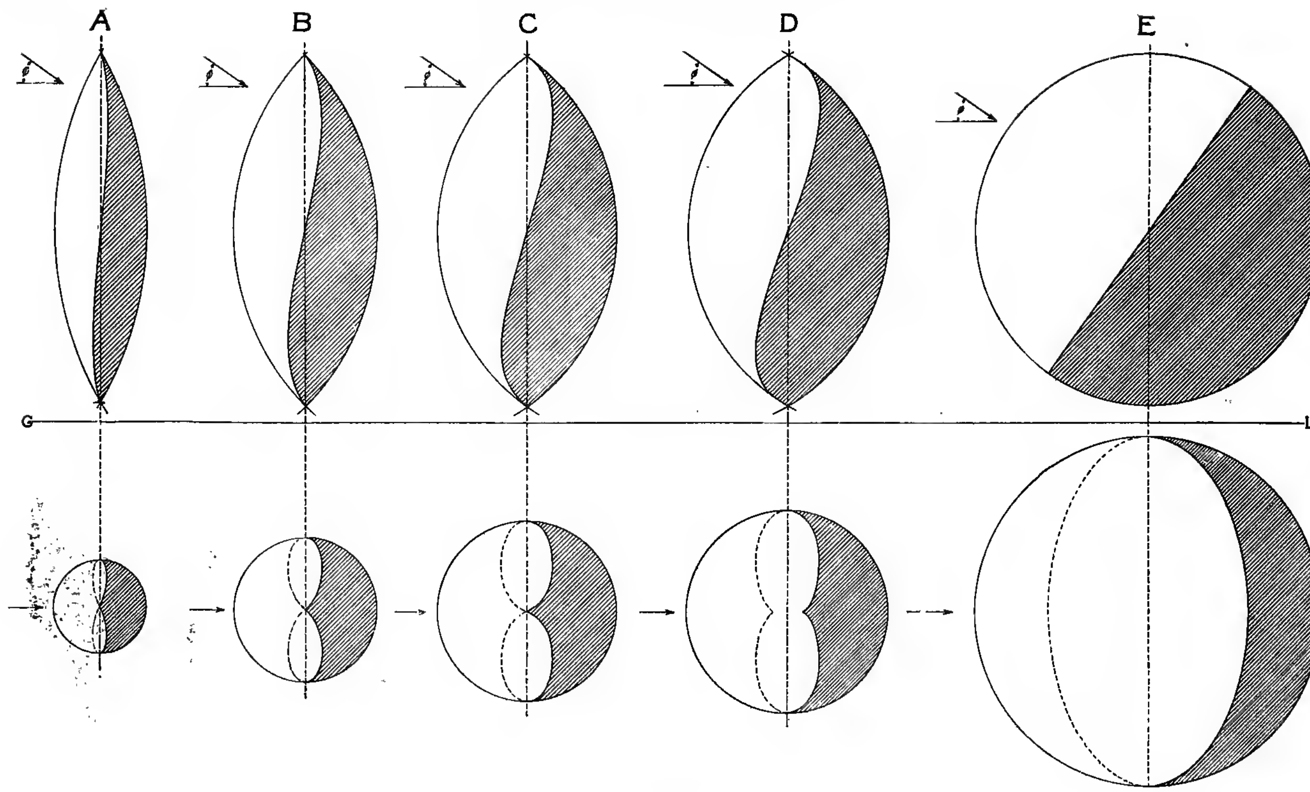


FIG. 131

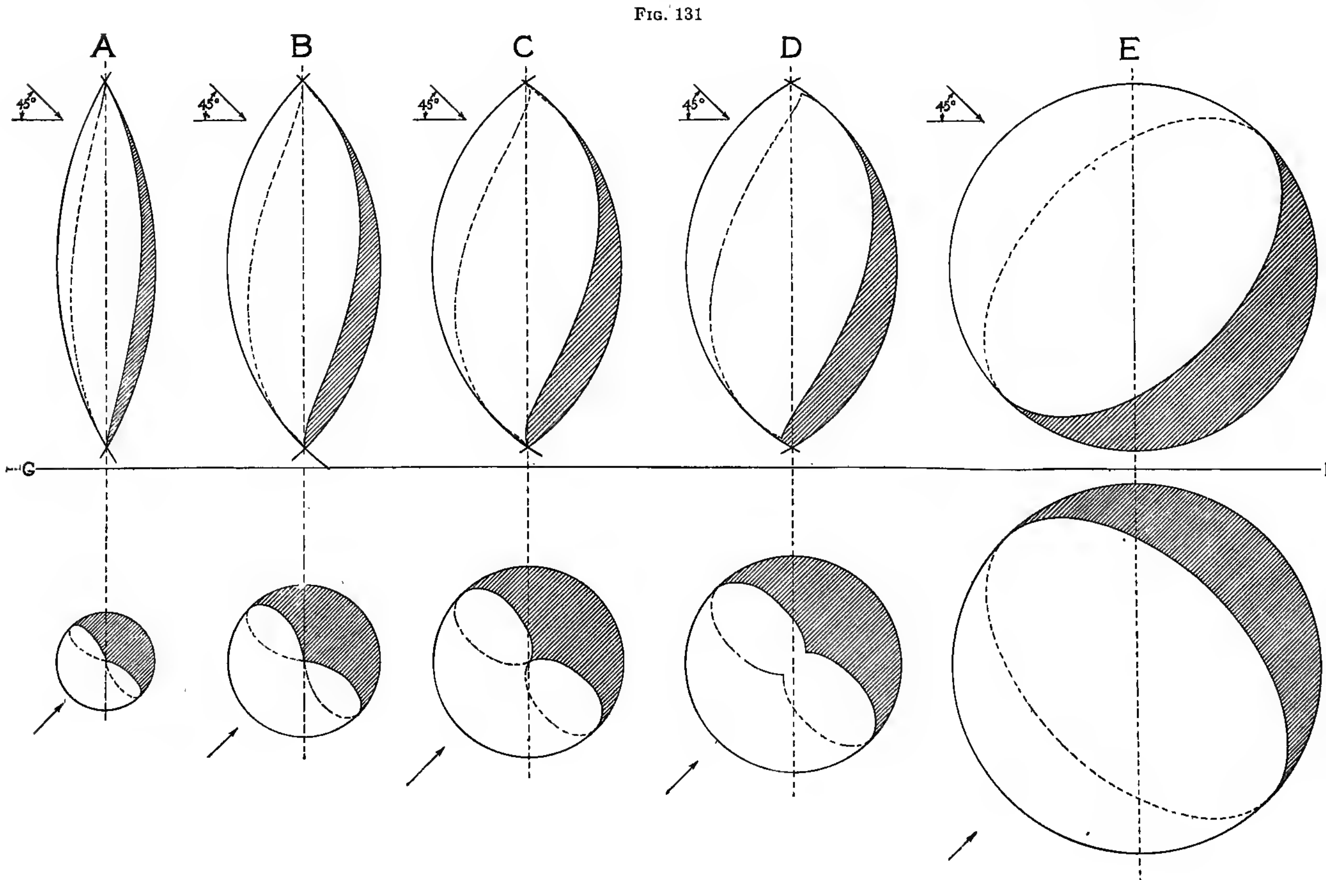


FIG. 132

209. Fig. 133 shows a series of concave surfaces of revolution which resemble Spindles, in that the Generatrix is an arc of only a few degrees, the curve being so flat that no shadow is cast, and they exhibit only the Line of Shade. These figures, which are shaped much like the Capstan of a ship, obviously consist of the central segment of a Scotia.

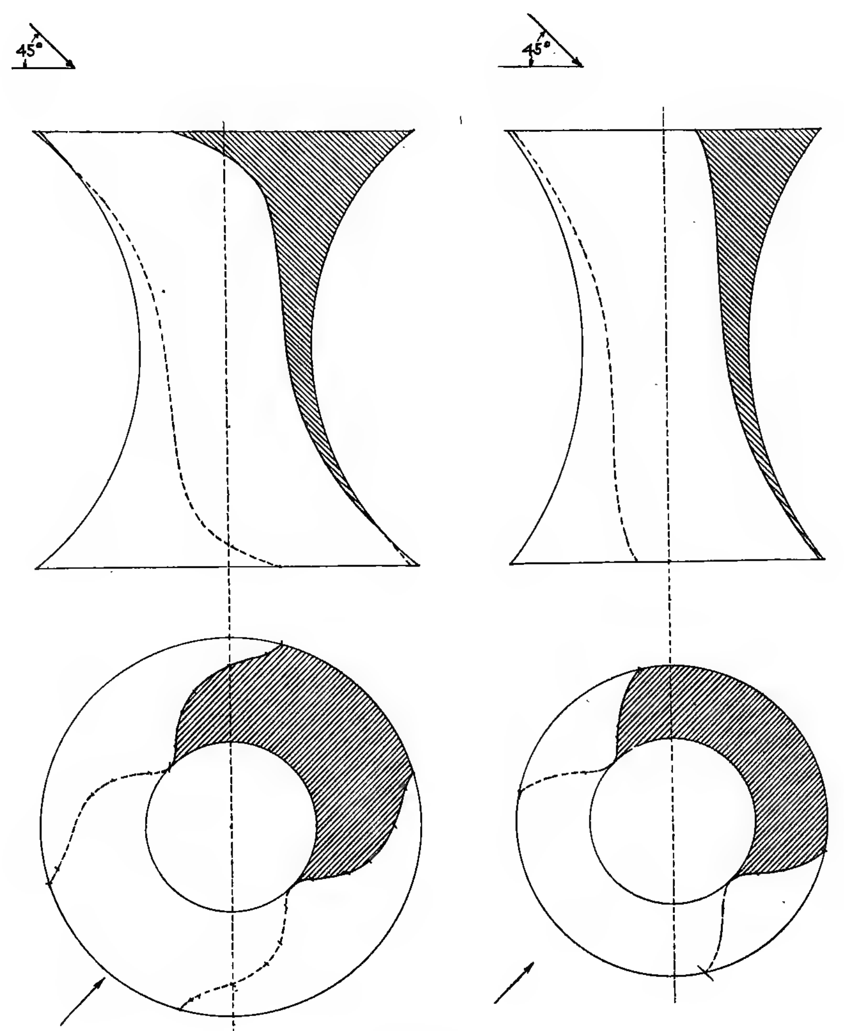


FIG. 133

Chapter XII

COMPOSITE FIGURES OF REVOLUTION

210. The Surfaces of Revolution already considered, namely the Cylinder, Cone, and Sphere, the Ovoids and Ellipsoids, and the varieties of Torus, Scotia and Spindle, have for their Generatrices single geometrical lines, straight or curved.

If two such lines, straight or curved, are combined, either tangentially or at an angle, surfaces of revolution result which present some interesting features. Forms of this character frequently occur in Architecture, both in the larger masses of which buildings are composed, and in their decorative details.

211. The Lines of Shade, the Lines of Shade and Shadow, and the Lines of Shadow, in these figures may be found in any of the ways described in the preceding chapters. If the scale of the drawings is not too large, the five points in the Line of Shade given for surfaces of Revolution by the Tangent Cylinder and the two Critical Tangent Cones will generally suffice, especially if it is borne in mind that when the axis is vertical both the shades and the shadows lie symmetrically on either side of a vertical plane taken through the axis of revolution parallel to the ray of light, so that when they are projected upon the vertical plane of projection, the highest and lowest points come on the corners of the circle of revolution, and the points on the axis of the figure are in, the elevation, on a level with those on the margin nearest the light.

212. *Tangent Surfaces.*—In Figs. 134, 135, 136, although the different portions of the surfaces of revolution which are generated by the different lines are tangent to one another, the Lines of Shade meet at an angle.

Fig. 134 *A*, *B*, and *C*, shows the Sphere tangent to the Ovoid, Cylinder, and Cone; at *D* the Half Torus and Half Scotia are tangent to the Cylinder; at *E* the Half Torus and Half Scotia are variously combined with each other.

213. In Fig. 135 *A* and *B* the Quarter Torus and the Quarter Gorge are tangent at their extremities, and in *C* and *D* a similar combination is made of the Quarter Scotia and Quarter Hollow Scotia.

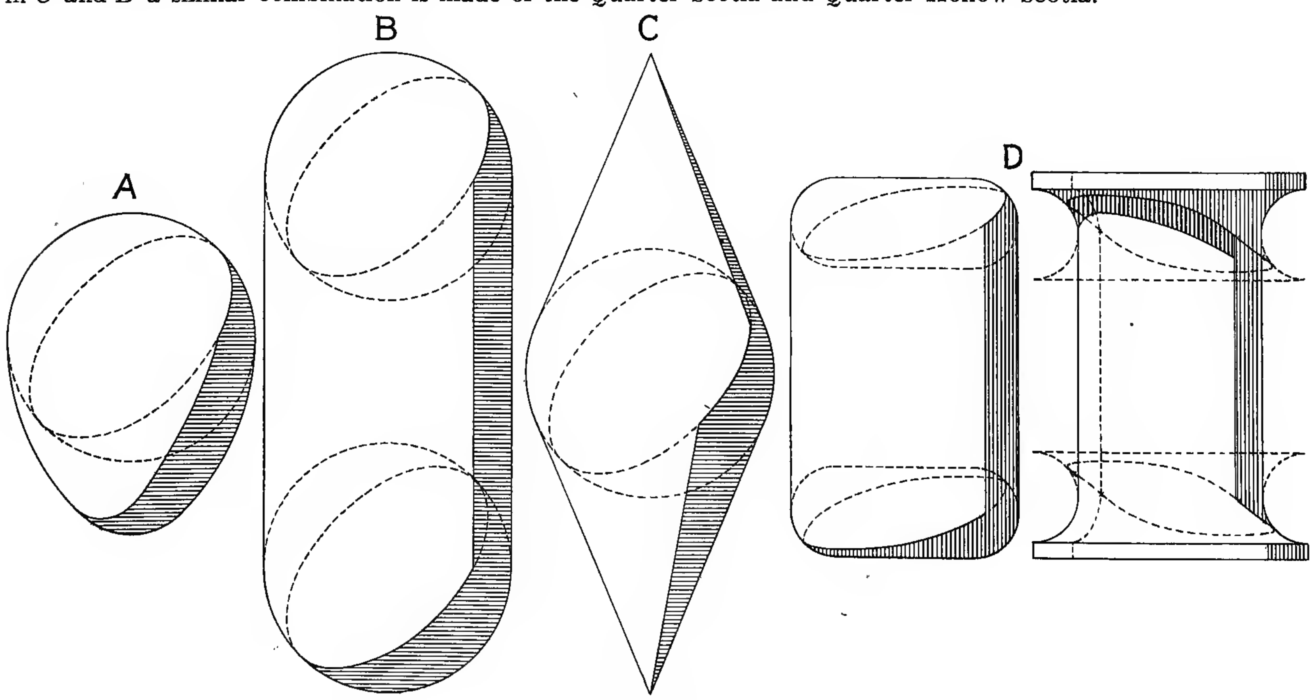


Fig. 134

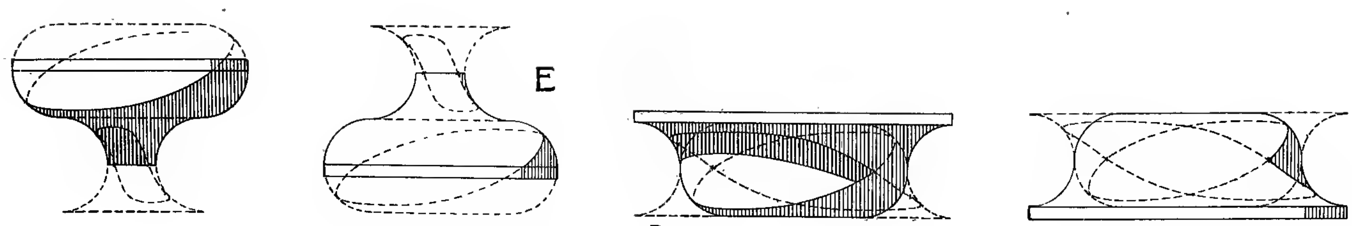


Fig. 134

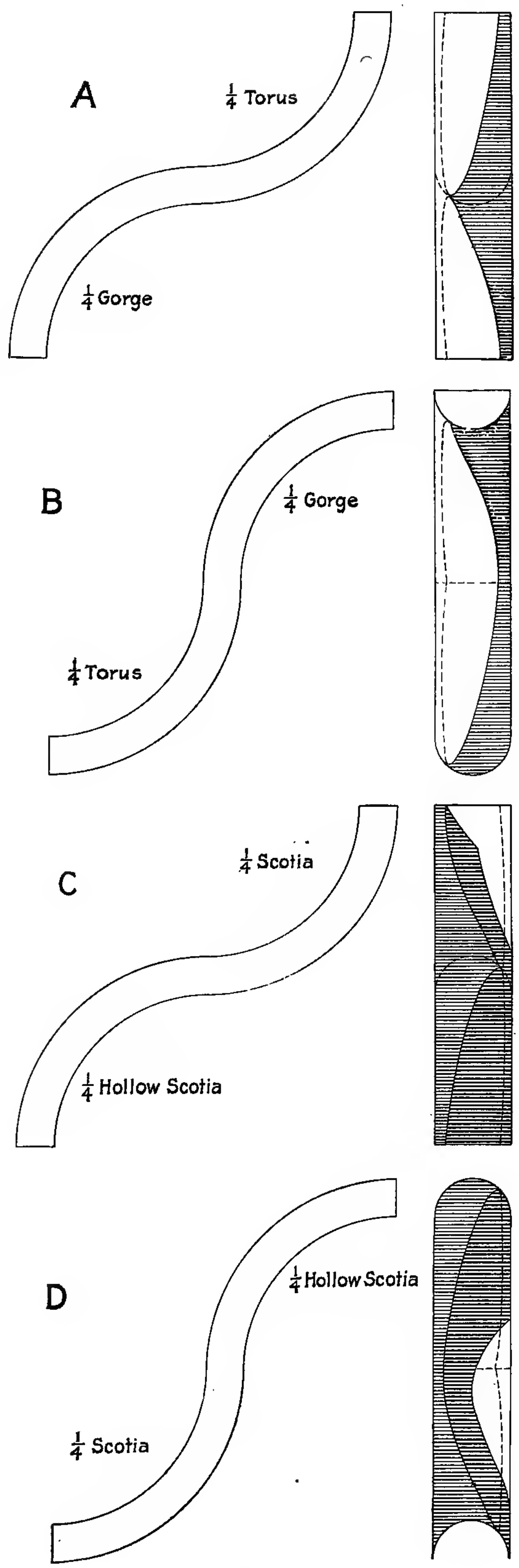


Fig. 135

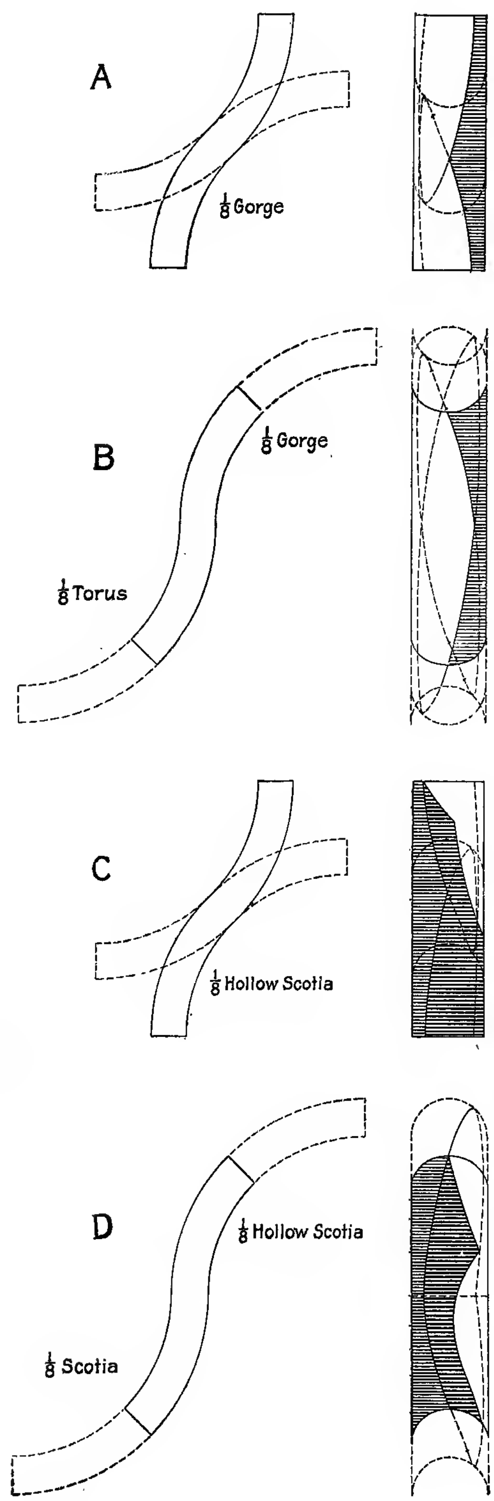


Fig. 136

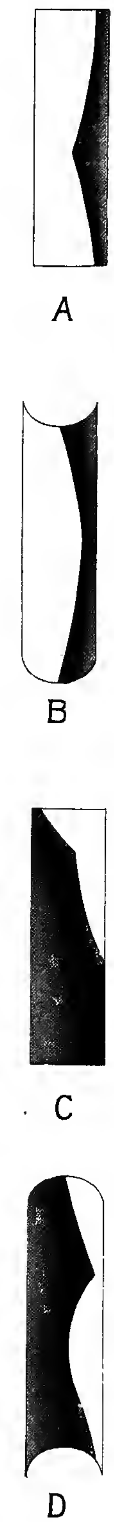


Fig. 137

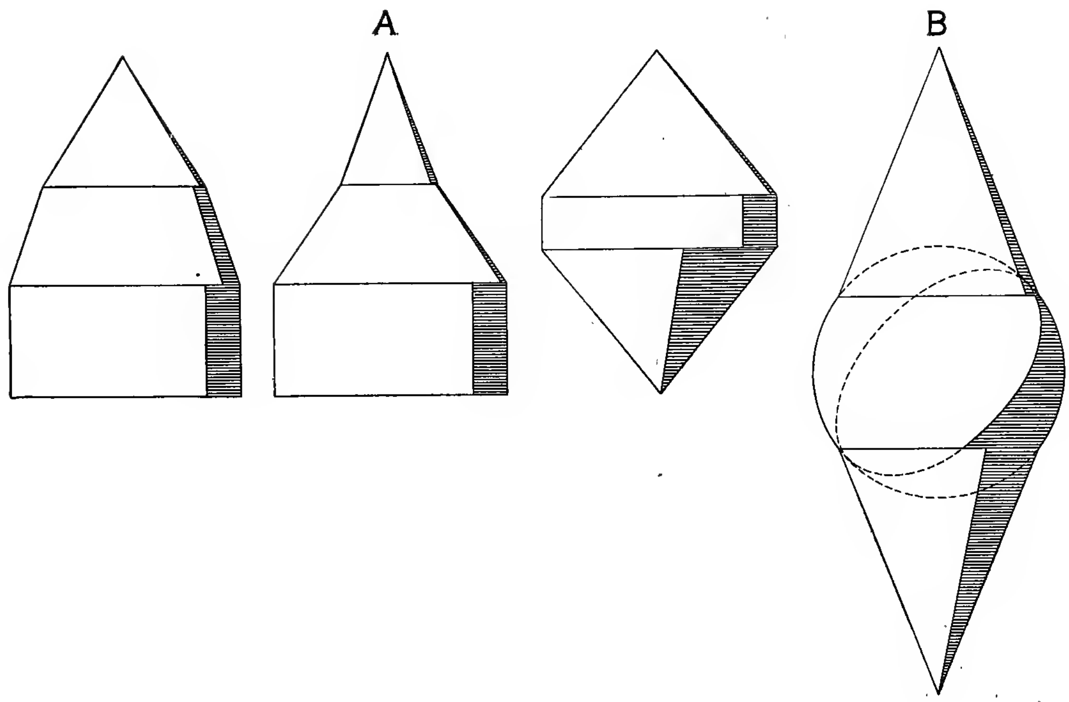


FIG. 138

214. In Fig. 136 *A*, *B*, *C*, and *D*, these combinations are repeated with arcs of 45 degrees. Fig. 137 shows the same, with the Shades and Shadows graded.

215. *Surfaces Meeting at an Angle.*—In Fig. 138 *A* and *B*, the different portions of the surfaces of revolution meet at an angle, as is seen on the outline, and their Lines of Shade do not meet, but are connected by the Lines of Shade upon the solid angles that lie between the surfaces. At *A* Cones of different slopes are superposed, and the case of the Cone and the Cylinder, previously considered in Chapter VIII, is again illustrated. At *B* segments of the Sphere are met by segments of Cones, Sphere, or Cylinders. In this figure the upper cone would cast a small shadow on the spherical surface, and that surface would cast a small Shadow on the lower cone.

216. *Niches.*—The most important of the compound surfaces of revolution is the Niche, upright or inverted. This name is given to a combination of the Hollow Cylinder and the Hollow Sphere, or, rather, of the Hollow Half Cylinder and the Hollow Quarter Sphere.

217. *The Upright Niche* (Fig. 139 *A*).—In the Upright Niche the elliptical Line of Shade and Shadow makes, at the point marked x in the figure, an obtuse angle with the Vertical Line of Shade and Shadow upon the cylinder.

The Line of Shadow in an Upright Niche consists of three parts, or segments.

In the first segment, which extends from a to b^s, a portion of the circular edge of the Niche a b casts its shadow upon the spherical surface. This is part of the Ellipse of Shadow shown in Fig. 101, Chapter X. The Major Axis lies at 45 degrees and is as long as the diameter of the Sphere, and the Minor Axis is one-third as long. The curve is concave toward the light.

In the second segment the arc b c throws its shadow on the cylindrical surface at b^s c^s. The arc b^s c^s is part of the curve shown in full in Figs. 53 and 65 *F*, Chapter VII. These figures show the shadow thrown upon a Hollow Vertical Half Cylinder by a vertical circle which has the same radius as the Cylinder, and which lies in the plane which divides the Cylinder in half.

The arc $b^s c^s$, as projected, is neither elliptical nor circular, but it very nearly coincides with the arc of a circle, whose radius is the diagonal of the radius of the given circle. In Fig. 140 this arc is shown by a dotted line.

The line $b^s c^s$ is tangent both to the curve $a\ b^s$ above it, and to the right line $c^s d^s$ below it. It is convex toward the light, and the Line of Shadow thus experiences a contrary flexure at the point b^s.

In the third section the edge of the Cylinder at $c\ d$ throws its shadow on the cylindrical surface in a vertical line, at $c^s d^s$.

218. *The Inverted Niche* (Fig. 139 *B*).—The Shadow of the Inverted Niche is in like manner composed of three parts.

In the first segment the edge of the Cylinder at $f\ b$ throws its shadow on the axis of the cylinder at $f^s b^s$.

In the second segment the edge of the Cylinder $b\ c$ throws its shadow across the spherical surface at $b^s c^s$. This is the intersection of the Sphere by a plane. The line is part of a small Circle of the Sphere, the projection of which on the vertical plane is an ellipse whose Major Axis is equal to the square root of 2, which, as appears in the plan, is the diameter of the Small Circle, and whose Minor Axis is 1, and equal to the radius of the sphere, this radius being taken as unity. The line $b^s c^s$ is an arc of this ellipse.

In the third segment the arc $c\ d$ throws its shadow on the concave spherical surface at $c^s d$. This is a portion of the oblique ellipse of shadow shown in Fig. 101, Chapter X, whose Minor Axis is one-third of the diameter of the Cylinder.

219. Fig. 140 shows how the Shadows in a Niche are obtained by the Method of Parallel Planes.

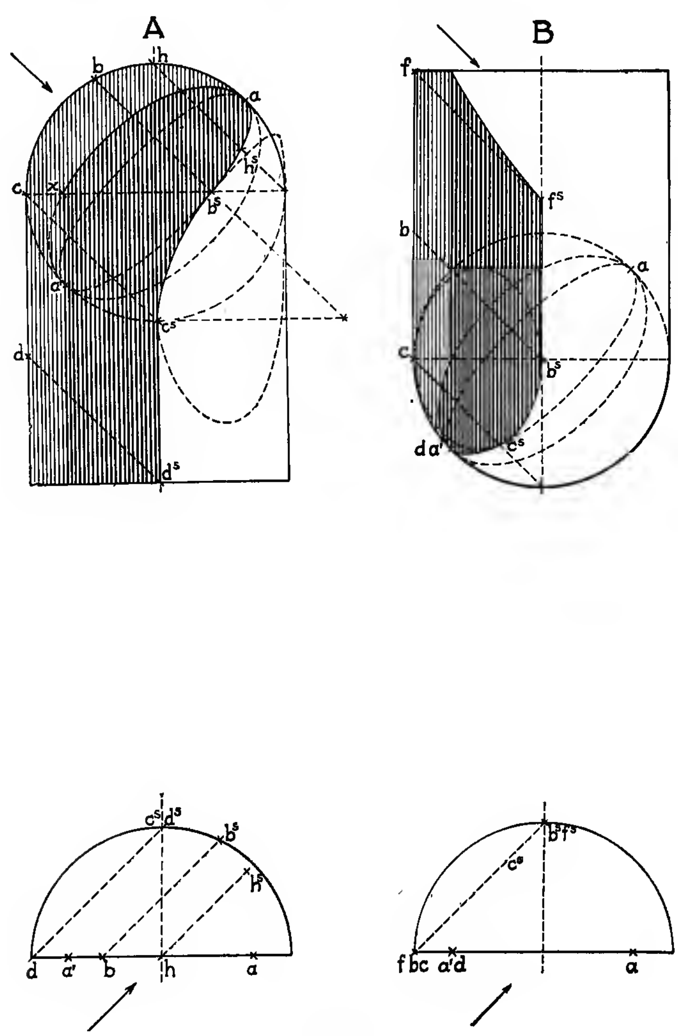


Fig. 139

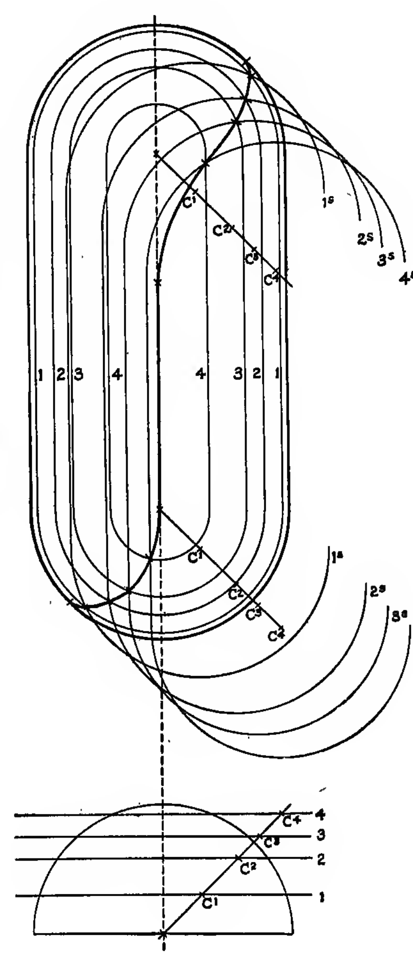


Fig. 140

The surface of the Niche is here cut by four planes, parallel to the vertical plane of projection, numbered in both projections 1, 2, 3, and 4. The centers of the two semicircular portions of the rim cast their shadows on these planes at the points c^1, c^2, c^3, c^4, and the shadows of the semicircles themselves fall upon the secant planes in the eight semicircles marked 1^s 2^s 3^s 4^s. The eight points in which these eight semicircles of shadow cut the four lines in which the secant planes intersect the surface of the Niche, are points in the required Lines of Shadow.

220. *Oblique Niches.*—Fig. 141 *A, B, C, D, E, F, G,* and *H* shows the shadows cast in Vertical Niches, the faces of which stand at different angles with the vertical plane of projection. These shadows are composed of the shadows cast by their rims upon the corresponding hemispheres (as shown in Fig. 105 Chapter X), and of the shadows cast upon the corresponding semicylinders by circles lying in the secant plane (as shown in Fig. 65, Chapter VII).

221. In all these figures the upper portion shows Upright Oblique Niches. The degree of obliquity is shown in the plans which occupy the middle of the figure. Since, as in Fig. 105, the Quarter Spheres which form the head of the Niche are in all the figures portions of the same Sphere, the Line of Shade and Shadow is in all the figures the same, as is also the point in the Line of Shadow marked h^s, which is the shadow of the highest point of the circle that casts the shadow, being the point at the top of the Niche, at the extremity of the axis of the Cylinder. As this point is the same in all the figures and the spherical surface is the same, the point of shadow is always h^s, as has been said in paragraph 163.

The Line of Shadow on the Quarter Sphere, at the top of the Niche, is in every case, as in Fig. 140, made up of two parts or segments. The first segment is the shadow of the arc $p\ h\ c$, which is a part of the circular rim of the Niche. This shadow $p\ h^s\ c^s$ is cast upon the hollow spherical surface and is an arc of the elliptical Line of Shadow $p\ h^s\ c^s\ p$. The second segment is the shadow of the line $c\ d$, cast upon the cylindrical surface at $c^s\ d^s$. This line is part of the line shown in Fig. 14. In *A, B, C, D,* and *E* a portion of this Line of Shadow is hidden by the right-hand edge of the rim of the Niche.

In all these figures (except in *C*) the Line of Shadow $h^s\ c^s$, where it crosses the horizontal diameter of the head of the Niche, is tangent to the Line of Shadow $c^s\ d^s$ cast by the Line $c\ d$ upon the inner surface of the Hollow Cylinder.

222. The lower portions of these figures show Inverted Oblique Niches.

The Line of Shadow is in each case, as in Fig. 140 *B*, made up of the segments of two ellipses.

The first or upper segment is the shadow of the vertical edge of the Cylinder at $d\ c$ cast upon the hollow spherical surface at $d^s\ c^s$. This line is an arc of the vertical ellipse whose Minor Axis is the line $c\ d^s$ and whose Major Axis is a vertical line equal in length to the 45-degree chord $d\ c^s$, shown in the plan. This ellipse is the vertical projection of a Small Circle of the Sphere, whose diameter is of that length.

The second segment of the Line of Shadow, $c^s\ p'$, is an arc of the elliptical shadow $p\ h^s\ d^s\ c^s\ p'$. It is the shadow of portion of the circular rim which lies between c and p', and it is tangent to the arc $d^s\ c^s$ at c^s.

223. In Fig. C the light, falling at 45 degrees, casts no shadow at all upon the Inverted Niche, which directly faces it, as has been said above. In Fig. F the shadows are the same as in Fig. 137, since the Niche is turned neither to the right nor to the left.

In all the figures the Line of Shade and Shadow, dividing the shade from the cast shadow, is made up partly of the elliptical Lines of Shade and Shadow upon the spherical surfaces, and partly of the vertical rectilineal line upon the cylindrical surface.

224. All these niches occur in the Round Altar which, in Example C V, constitutes the last of the Architectural Illustrations at the end of this volume. They are similar to the small niches which terminate the channels of Ionic and Corinthian Columns.

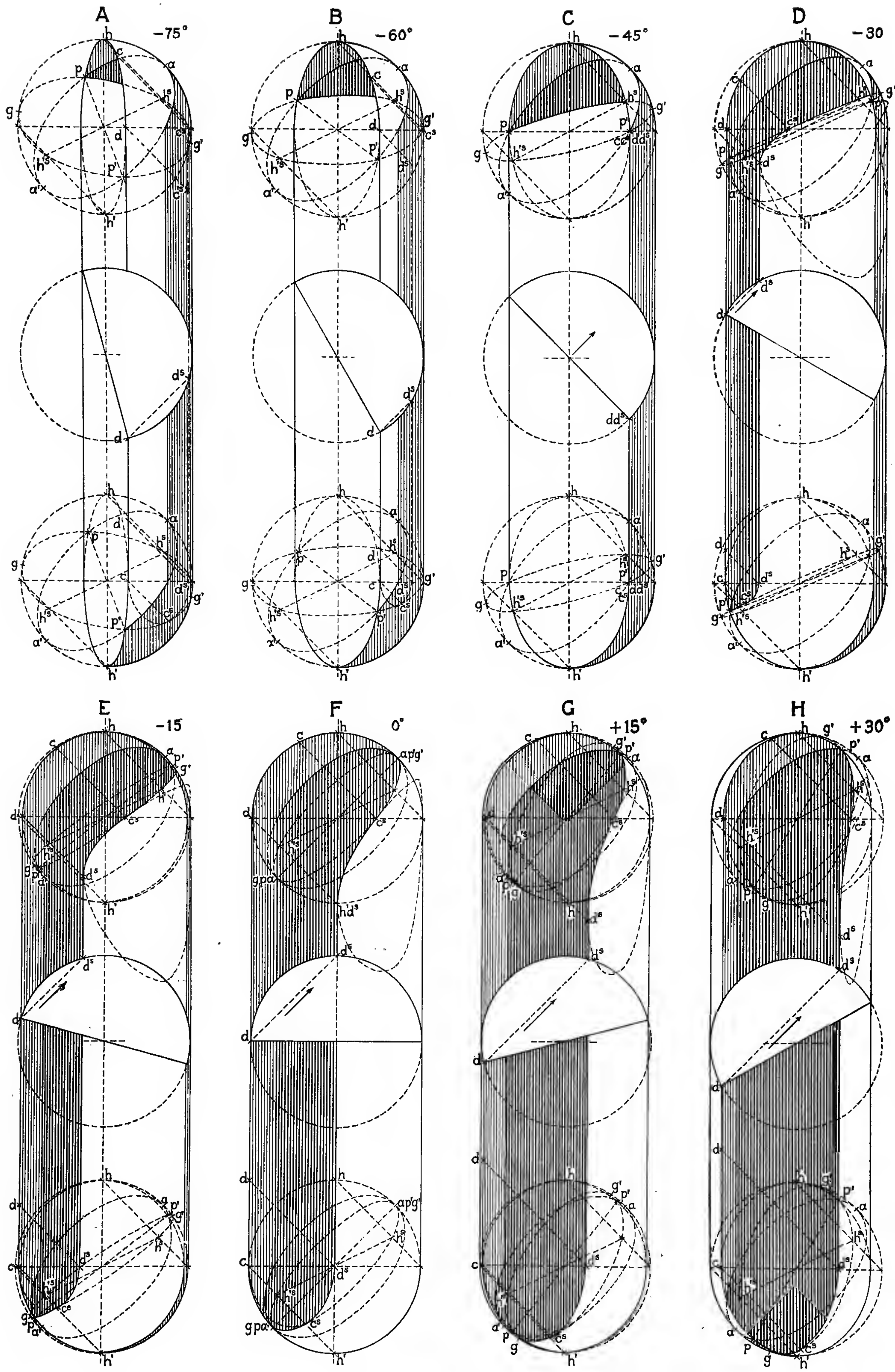


Fig. 141

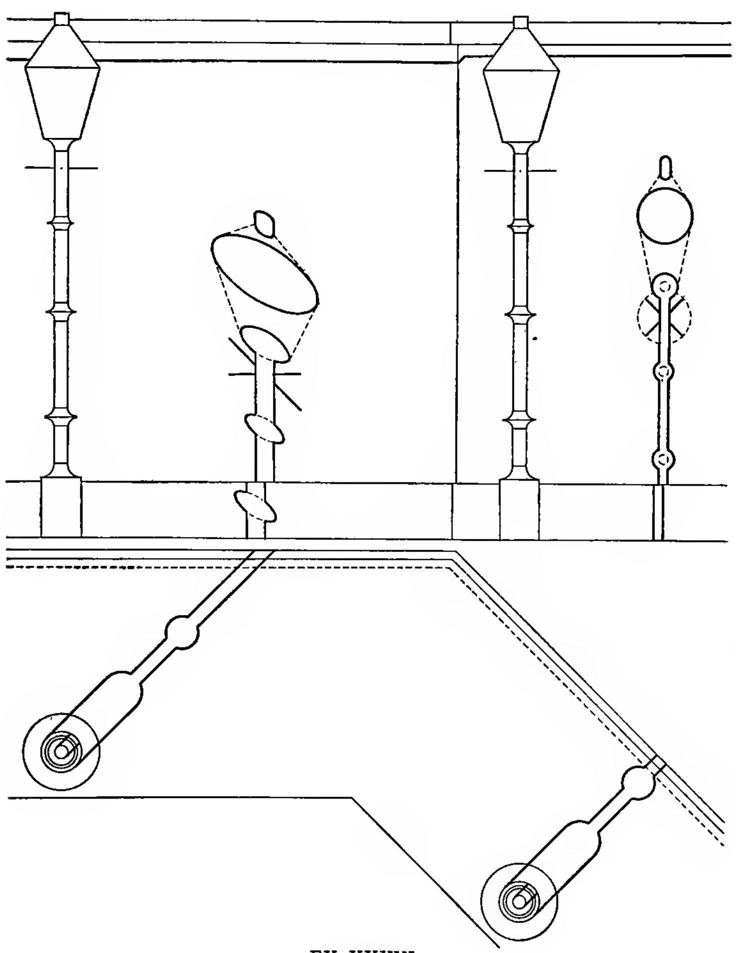
APPLICATIONS TO ARCHITECTURAL DETAILS

CHAPTER V. CIRCLES AND ELLIPSES

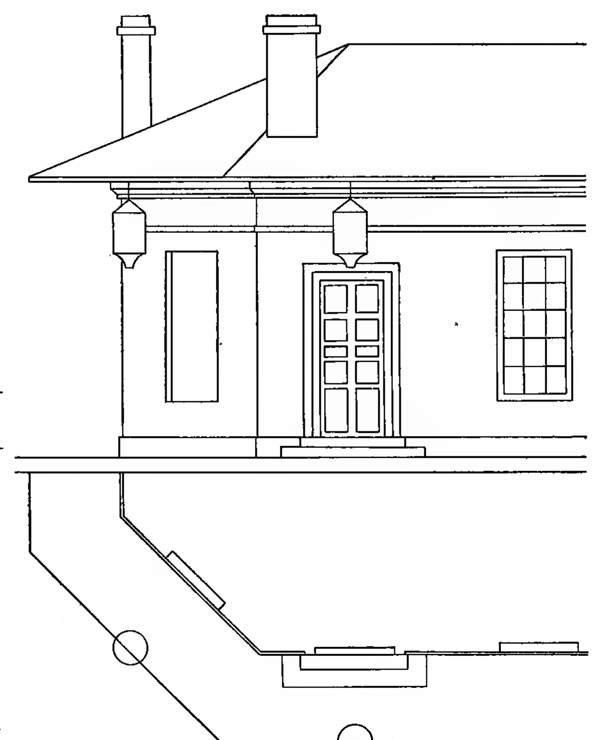
Circles. Example XXXVI. Two Lamp Posts. Plan and Elevation.

These Lamp Posts are composed chiefly of horizontal circles. The shadow of the first one is thrown upon the portion of the wall which is parallel to the Vertical Plane of Projection and is composed mainly of ellipses of the shape already made familiar.

The shadow of the second Lamp Post, cast upon the portion of the wall which turns at an angle of 45 degrees, is composed mainly of circles, as has been explained in Paragraph 58, Fig. 47, Chapter V.



EX. XXXVI

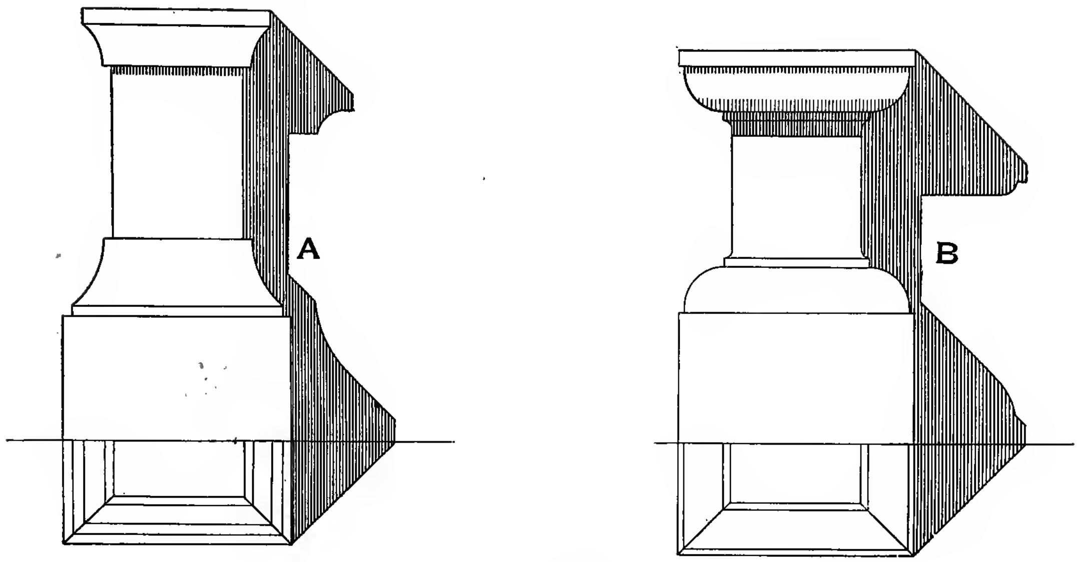


EX. XXXVII

Example XXXVII. A Shed With Two Circular Lanterns. Plan and Elevation.

This example, like the previous one, illustrates the shadow cast by horizontal circles upon vertical planes, one of which is parallel to the Plane of Projection and one stands at 45 degrees with it.

These shadows are left for the Student to draw.



EX. XXXVIII

Example XXXVIII A. A Post with Concave Mouldings.
Example XXXVIII B. A Post with Convex Mouldings.

These shadows are bounded by arcs of ellipses, such as are cut in figure 48, *D*, *E*, *F*, and *G*, page 85.

CHAPTER VI. CYLINDERS

Example XXXIX. A Cylindrical Shelf Supported By a Cylindrical Pillar. Plan and Elevation.

The line of Shade upon the shelf consists of the half of the lower edge which is nearest the light, from *6* to *0*, passing through (the points *0, 1, 2, 3, 4, 5, 6*) and the elements of the cylinder which connect these two semicircles. One of these elements occupies the front right-hand corner of the shelf at *0*, the other the back left-hand corner at *6*. The line of Shade on the pillar occupies the corresponding corners of the pillar.

The shadow cast by the shelf upon the pillar may be ascertained, as is done in the figure, by taking, in both projections, a number of points in the line of shade upon the shelf, and finding their shadows on the surface of the pillar. These will be points in the required line of shadow. Five such points are sufficient, if taken as here, so that the shadows numbered 1^s, 2^s, and 5^s, shall fall on the elements at the corners of the pillar, the one numbered 2^s on the front element, and the one numbered 4^s, on the outline. It is obvious that 1^s and 5^s will be at the same level and so will 2^s and 4^s, and that 3^s will be the highest point of the curve: Also that the projection of the curve will be horizontal at 3^s and vertical at 4^s.

Example XXXIX shows the shadow cast upon a vertical plane set at 45 degrees with the Vertical Plane of Projection, as explained in Fig. 47, page 84.

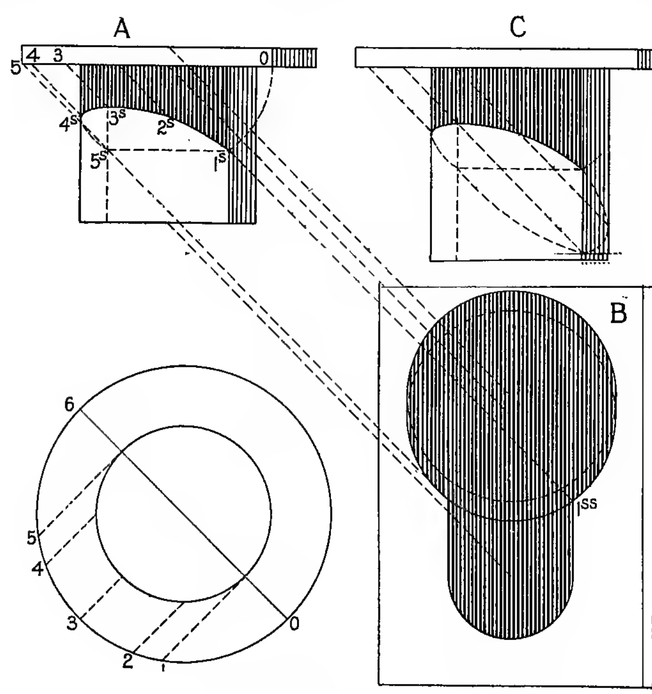
The shadows of the horizontal circles are projected as circles whose radius is half the diagonal of the given horizontal circle.

Fig. C shows that the line 5^s, 4^s, 3^s, 2^s, 1^s, is not an arc of an ellipse, but half of a broken curve which is the line in which the large elliptical cylinder of shadow cuts the small vertical circular cylinder.

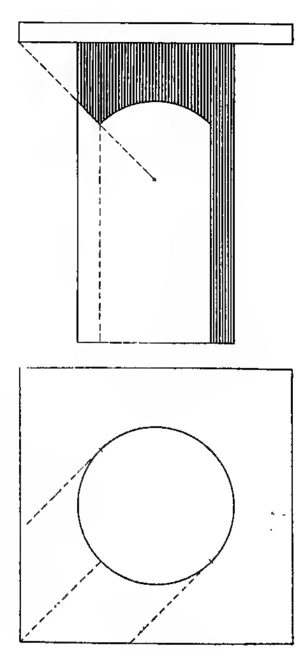
The shadow cast by such a pillar and shelf upon a vertical plane parallel to the vertical plane of projection is composed of the shadow of the shelf and of the pillar, overlapping, just as do the shadows upon the 45-degree plane at *B*. These shadows are left for the Student to draw out.

Example XL. A Square Shelf Supported By a Cylindrical Pillar. Elevation.

The corner of the square shelf which is nearest to the light casts its shadow upon the corner of the cylinder; the left-hand edge toward the light, being a normal line, casts a shadow whose projection is a line at 45 degrees, and the front edge casts a shadow which is a true section of the cylindrical pillar and occupies an arc of 90 degrees, extending from one corner of the pillar to the other. (See Fig. 52 *A*, page 88.)



EX. XXXIX



EX. XL

Example XLI. A Circular Shelf Upon a Square Pillar. Plan and Elevation.

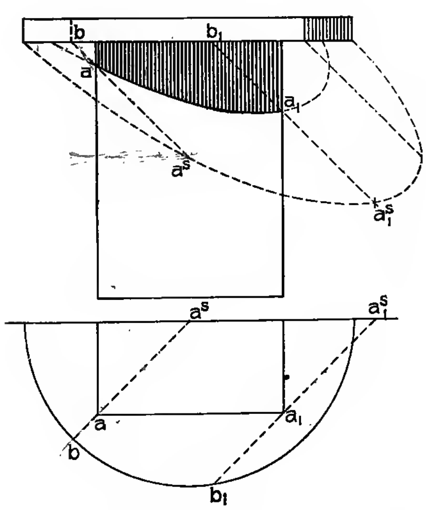
45 degree lines drawn in the plan through a and a_1, taken at the corners of the pillar, show that the shadows of these points will fall first at a and a_1, and then at the points a^s and a_1^s, a and a_1 being Points of Flight. The elliptical arc $a\ a_1$ in the elevation will obviously be similar to the arc a^s and a_1^s. The semiellipse shown by the dotted line through a^s and a_1^s represents the shadow of the semicircular shelf cast upon a vertical Plane of Projection taken through the middle of the shelf. This shadow is drawn as explained in Fig. 45, Paragraph 56, in Chapter V, page 83.

Example XLII. An Octagonal Shelf Upon a Circular Pillar. Plan and Elevation. Shade and Shadow.

The left-hand edge of the Octagonal shelf, perpendicular to the Vertical plane of projection, casts upon the Cylindrical pillar a shadow which is projected as a line of 45, like the corresponding shadow in Example XL. The only visible portion of this shadow lies between the points 1^s and 2^s.

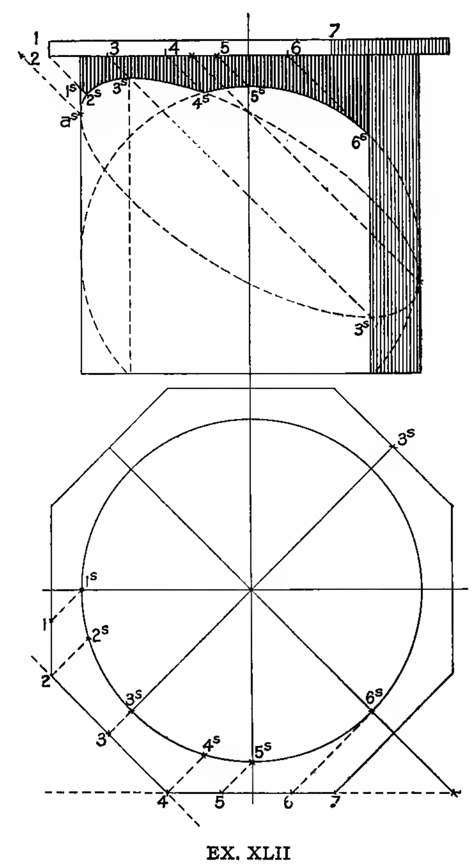
The edge of the Octagonal shelf which lies between the points *2* and *4* makes an angle of 45 degrees with the Vertical Plans and is part of an imaginary line the shadow of which upon the cylindrical pillar would be an ellipse lying in the surface of the cylinder. This would be projected in the ellipse shown in the figure as passing through the points a^s, 2^s, 3^s, 4^s, and 3^s. The elliptical arc 2^s–4^s is the shadow of the line *2*–*4*.

The front edge of the Octagonal shelf, between the points marked *4* and *7*, is part of an imaginary line, parallel to the Ground Line, the shadow of which would be a true section of the circular pillar, like the shadow shown in Example XL. The segment of this line comprised between the points marked *4* and *6*, the shadow of which would fall upon the pillar, has for its shadow the circular arc $4^s\ 5^s\ 6^s$.



EX. XLI

Thus the shadow of a horizontal circle falling upon a 45° plane is projected in a vertical circle, just as is the shadow of a horizontal line falling upon a vertical cylinder, as shown in Figs. 51 and 52.



EX. XLII

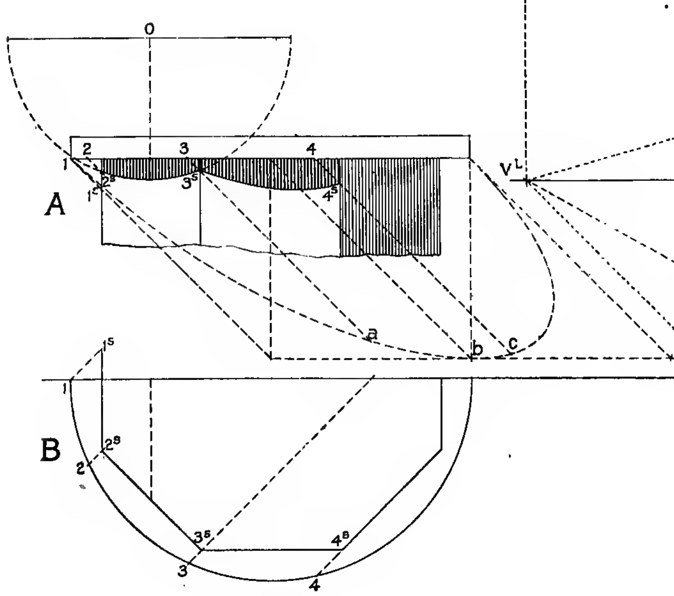
Example XLIII. A Semicircular Shelf Upon a Semioctagonal Pillar. Plan and Elevation.

The shadow cast by a semi-circular shelf upon a plane parallel to the vertical plane of projection would be, as explained in Paragraph 56, Fig. 45, a semiellipse enclosed in a parallelogram consisting of a square, one side of which is equal to the radius of the semicircle, and two half squares. Such a semiellipse is Shown in Fig. XLIII A, passing through the points *a*, *b*, and *c*.

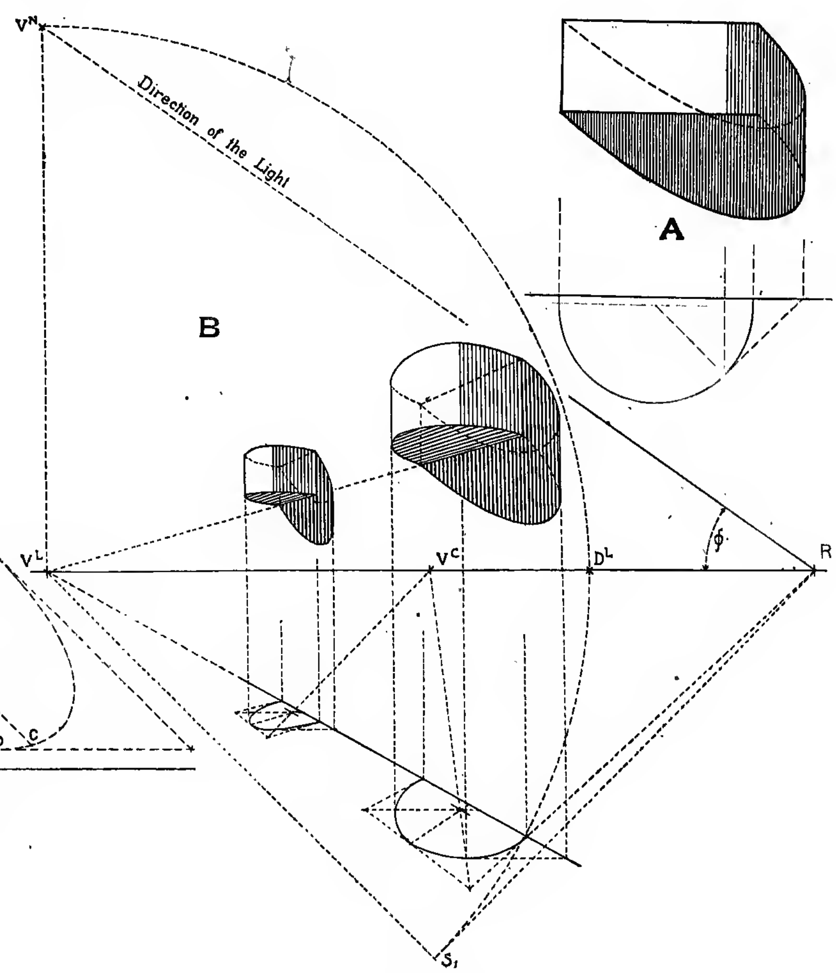
Fig. XLIII B shows that the portion of the semicircle that throws its shadow upon the front face of the octagon lies between the points *3* and *4*, and Fig. XLIII A shows that the shadow of this segment which falls upon the plane passed through the center of the semicircle is the elliptical arc, having its lowest point at *b*. The shadow cast by this segment upon the front face of the octagonal pillar, which is parallel to the plane of projection, is obviously similar to the arc, as in the figure.

The shadow cast upon the oblique side of the octagonal pillar, between the points 2^s and 3^s, is part of the shadow cast upon a vertical plane standing at 45 degrees with the vertical plane of projection. But the projection of such a shadow upon the vertical plane of projection is an arc of a circle whose radius is half the diagonal of the radius of the horizontal circle which casts the shadow. This is explained in Fig. 47, page 84, and again illustrated in Example XXXIX, page 156. The points 2^s and 3^s being given, and the length of the radius shown, the center *O* is easily found.

Example XLIV A. A Semicylindrical Gutta. Plan and Elevation.
Example XLIV B. (See Fig. 45.) Perspective Plan and Perspective.

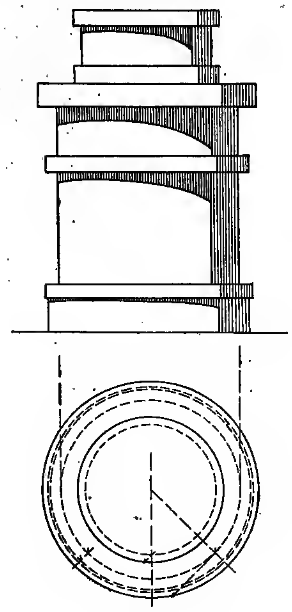


EX. XLIII

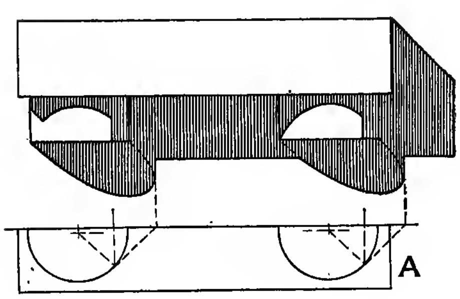


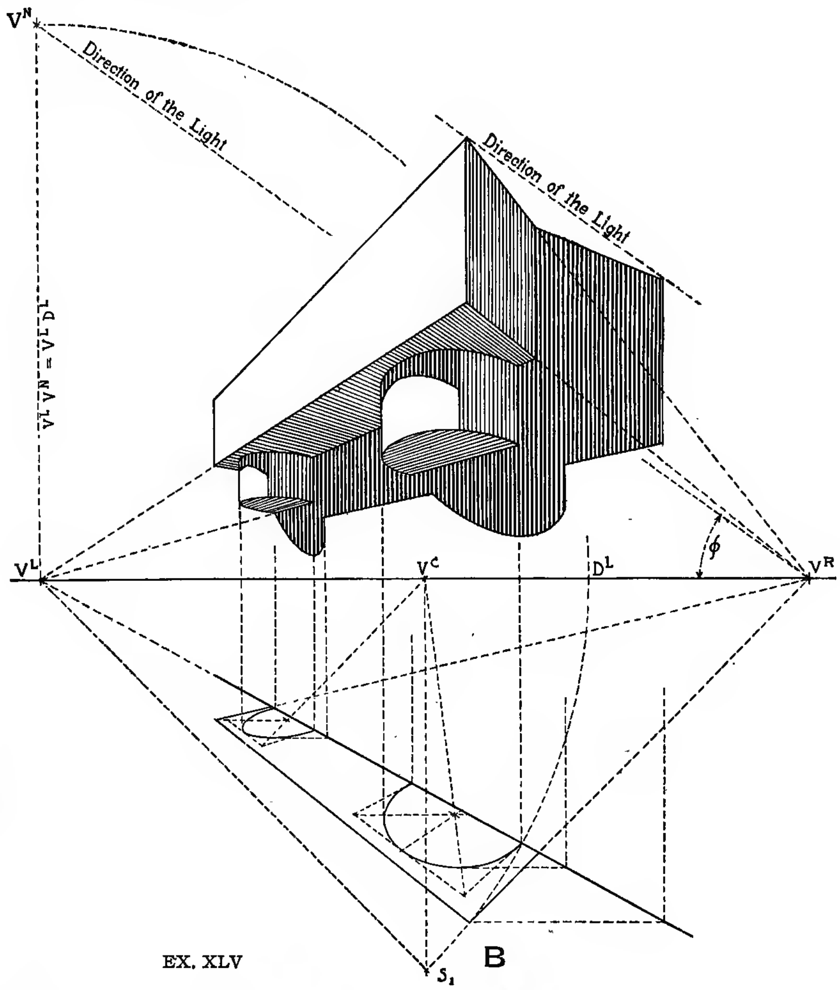
EX. XLIV

Example XLV A. A Greek Doric Regula and Guttæ. Plan and Elevation.
Example XLV B. The Same. Perspective Plan and Perspective.
In the Greek Doric order the Guttæ, though sometimes conoidal, with concave sides, are often cylindrical, as here shown.
Example XLVI. The Same, With the Lower Part of the Triglyph.

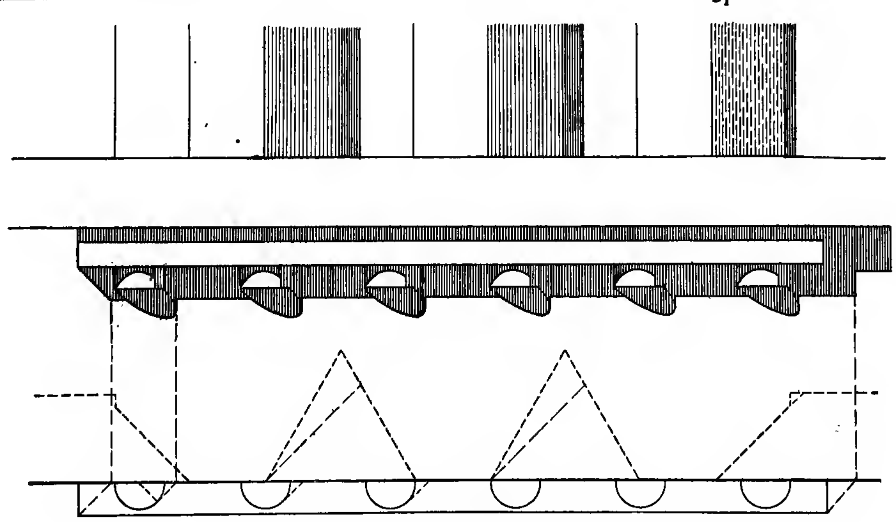


EX. XLVII





EX. XLV



EX. XLVI

Example XLVII. A Cylindrical Tomb. Plan and Elevation.

Care must be taken that the highest parts of the shadows shall come upon the corners of the cylinders that are nearest the light.

Example XLVIII. A Greek Doric Freze.

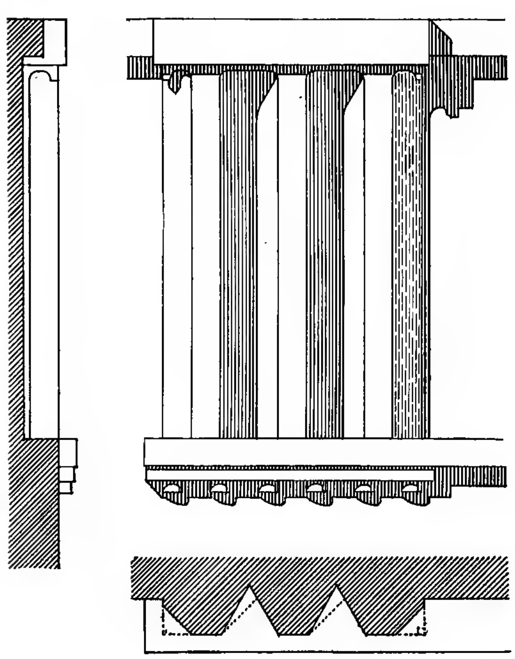
In the Greek Orders there is much variety of treatment, no two examples being exactly alike. In general, however, the Doric Architrave has only one Band or Fascia, instead of the two bands of Roman Order; the guttæ are cylindrical and short, being one-third as high as the Tænia, two-thirds, the two together being as high as the Tænia. The face of the Triglyph is flush with the Architrave below it, and the Metope is set back, instead of having the Metope on the same plane as the Architrave and the Triglyph set forward as in the Roman Doric Order; the Triglyph itself is thicker, the channels being cut at an angle of 60 degrees, so that they have the section of an equilateral triangle; the chamfers at the edges having a slope of 45 degrees. The Upper Tænia is wider over the Triglyph than over the Metope, as shown in the Example. It follows that one side of the channel is in shade and the other receives a cast shadow. The chamfer, or half channel, that is farthest from the light is a surface of light and shade, being parallel to the light, and it casts a narrow shadow upon the Metope.

Example XLIX. A Doric Pilaster Capital and Base; Elevation.

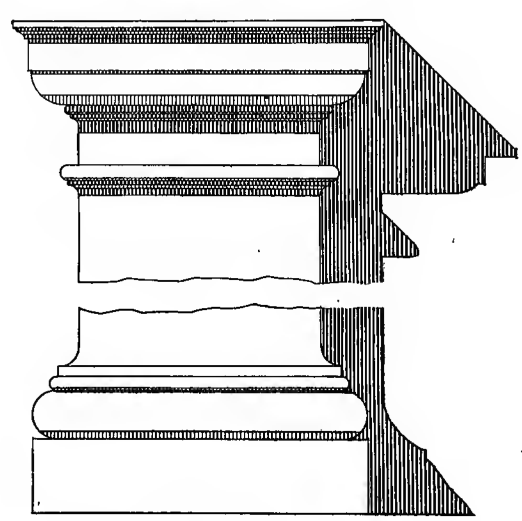
Example L. A Square Baluster, Elevation.

The *belly*, or widest part of a Baluster, generally measures, in width, about a third of the height, and comes about one-third of the way up.

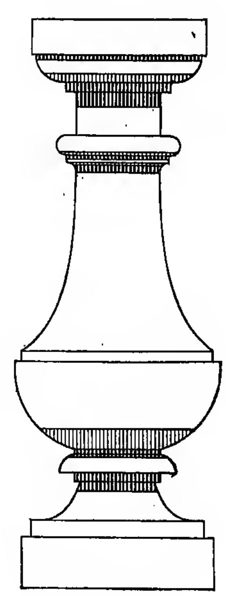
Square balusters vary in shape, just as Round balusters do.



EX. XLVIII



EX. XLIX



EX. L

CHAPTER VII. HOLLOW CYLINDERS

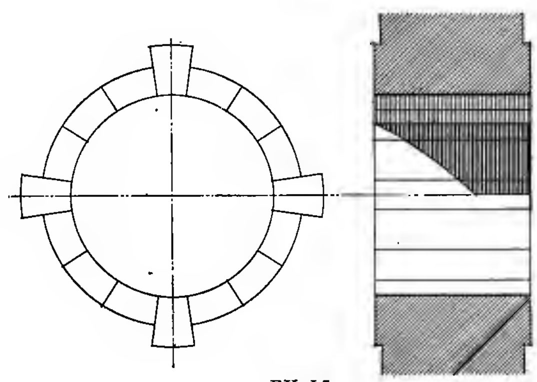
Example LI. A Round Window, Oeil-de-Boeuf, or Hole in the Wall; Elevation and Section.

Example LII A. A Fountain, Discharging the Overflow From a Reservoir, Plan, Elevation, and Section.

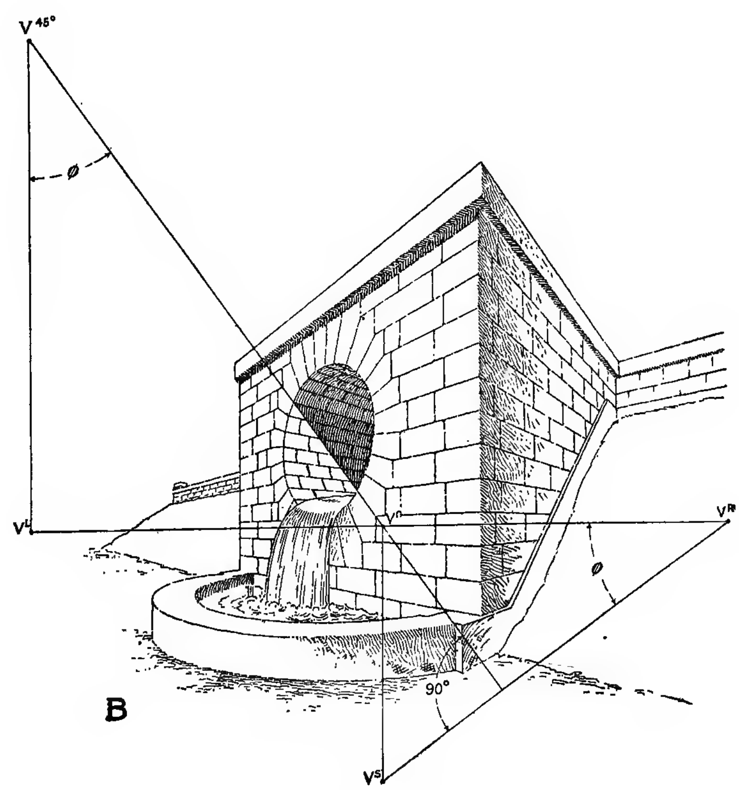
This Example is left for the Student to draw, in conformity with the following Example:

Example LII B. The Same; Perspective.

The circular tunnel is so situated that the Line of Shadow appears in the Perspective as a right line. (See Fig. 62, page 95.)



EX. LI



EX. LII

Example LIII A. An Arch, Elevation, and Section. With the Shadow of the Whole Arch Indicated.

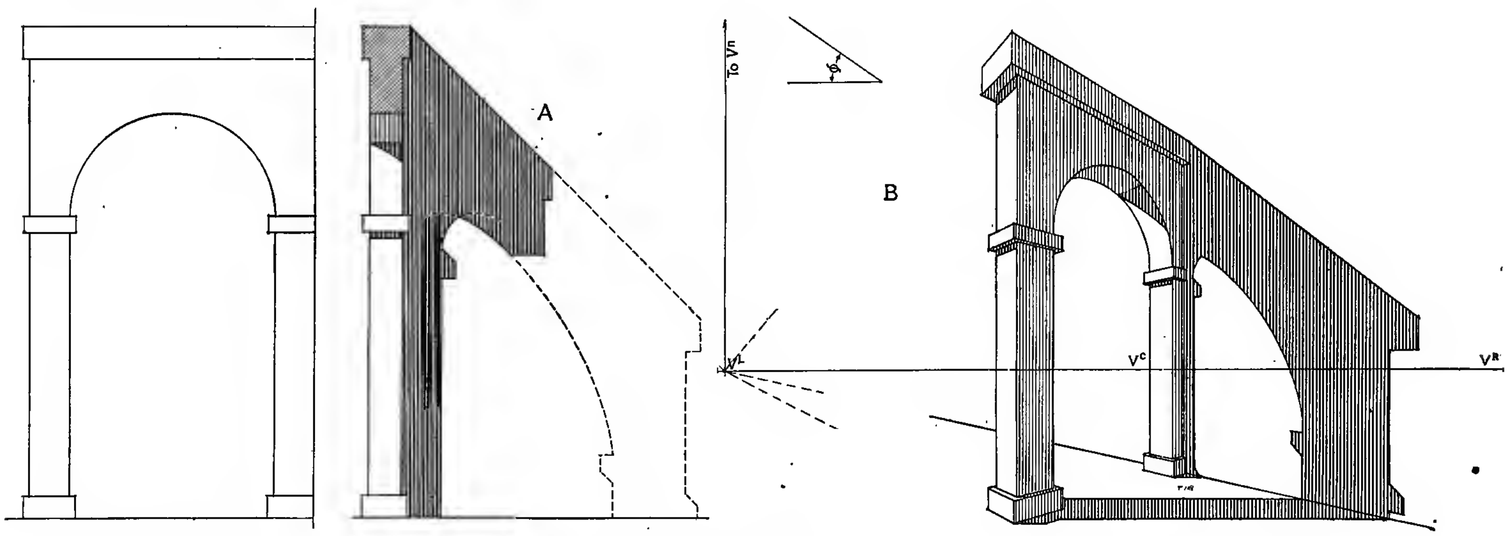
Example LIII B. The Same. Perspective.

The line $V^L V^N$ equals in length the diagonal of the line $V^l V^r$. (See Part 1, Appendix.)

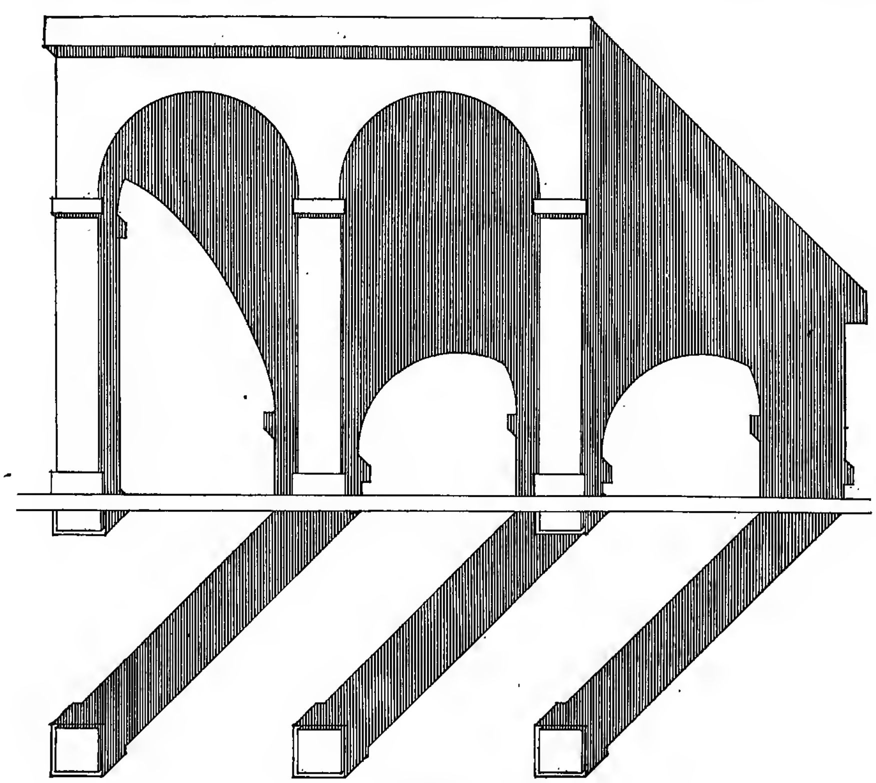
Example LIV. An Arcade. Elevation.

In the case of the two arches that are parallel to the wall, the outer edge of the intrados casts upon the wall a semicircular shadow similar to itself, which intersects the similar shadow of the inner edge of the intrados.

In the same way the shadows of the outer and inner edges of the arch which stands at right angles to the wall are bounded by intersecting semiellipses.

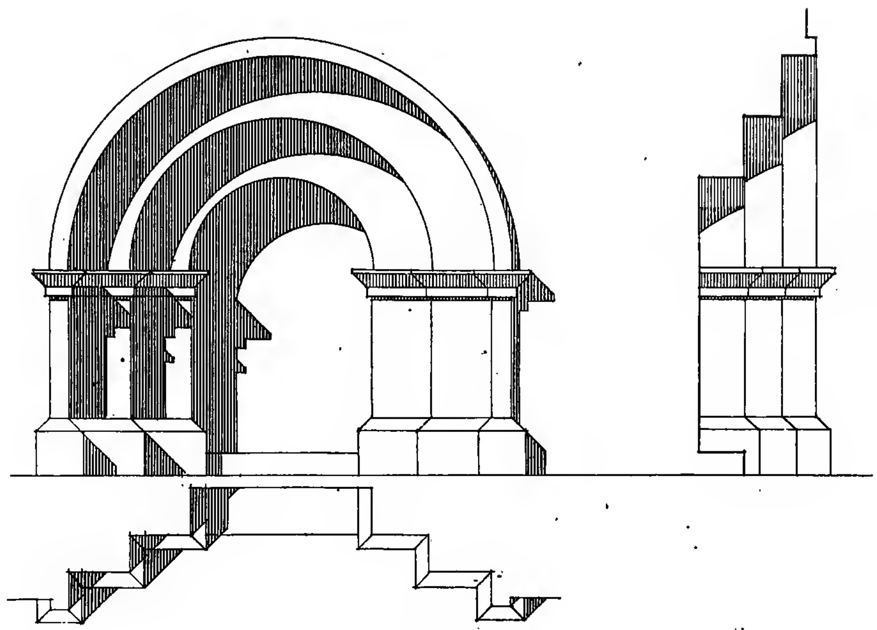


EX. LIII

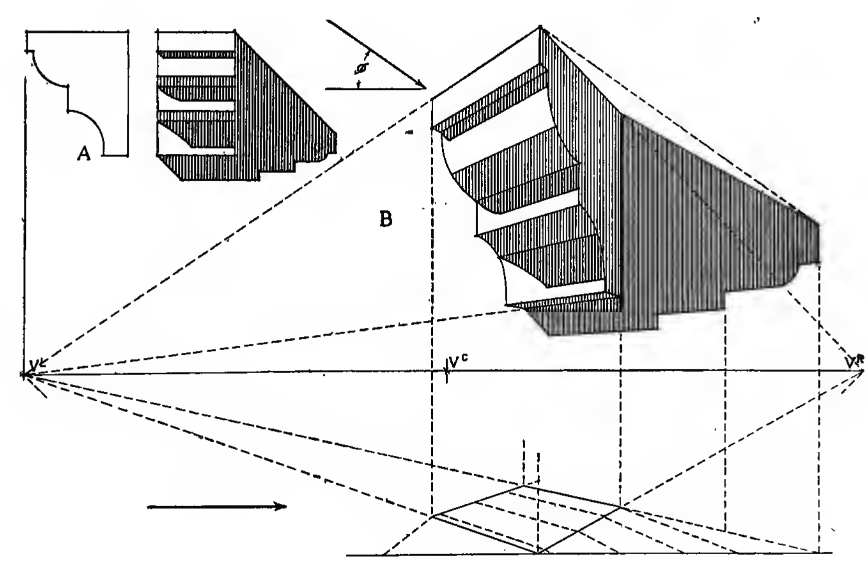


EX. LIV

Examples LV. Recessed Arches.
Example LVI A. A Bracket With Ovolo and Cavetto. Two Elevations.
Example LVI B. The Same. Perspective Plan and Perspective.

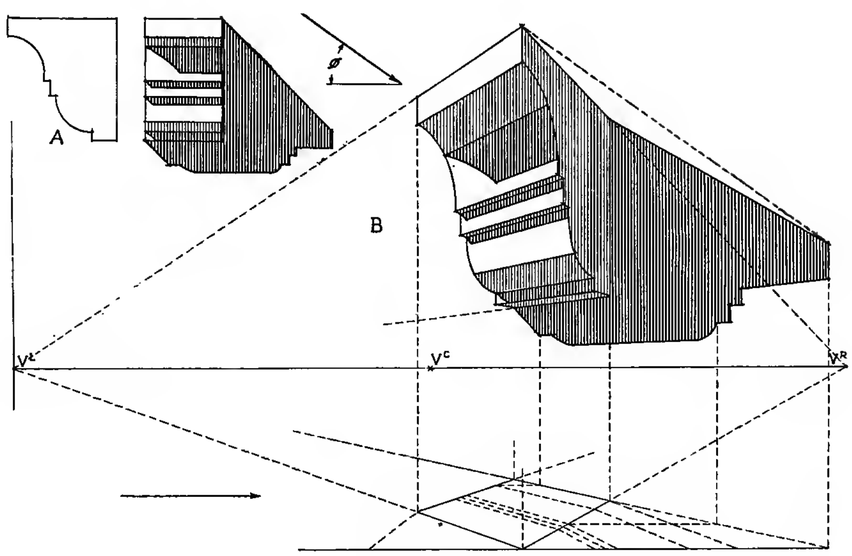


EX. LV

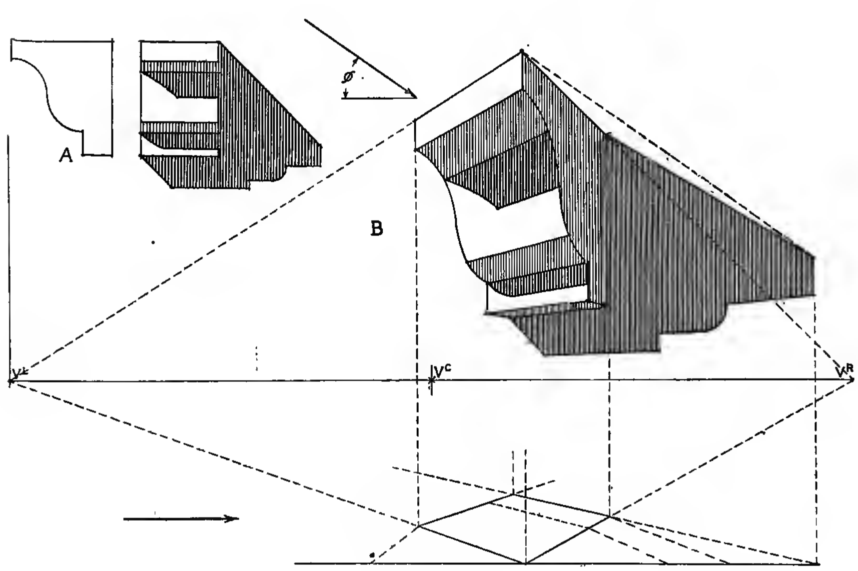


EX. LVI

Example LVII A. A Bracket With Cavetto and Ovolo. Two Elevations.
Example LVII B. The Same. Perspective Plan and Perspective.
Example LVIII A. A Bracket With Cyma Recta. Two Elevations.
Example LVIII B. The Same. Perspective Plan and Perspective.

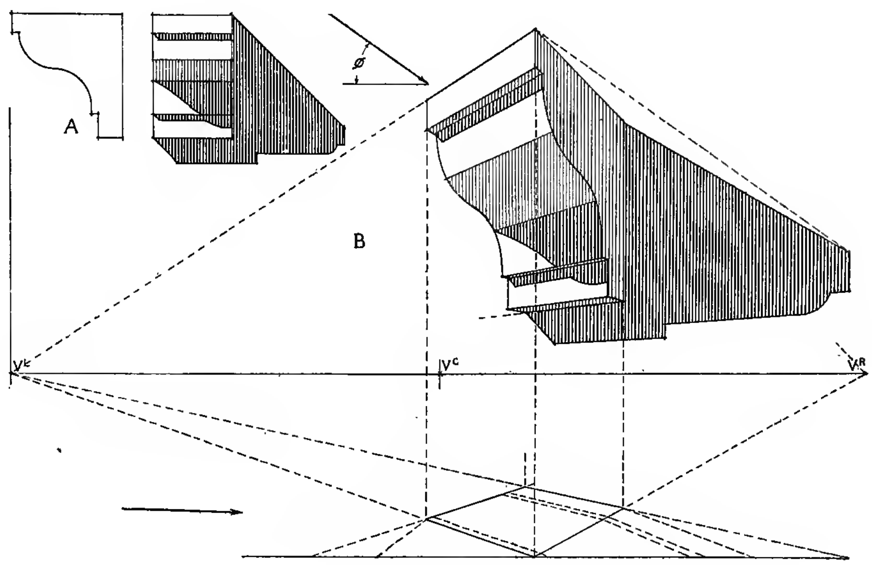


EX. LVII

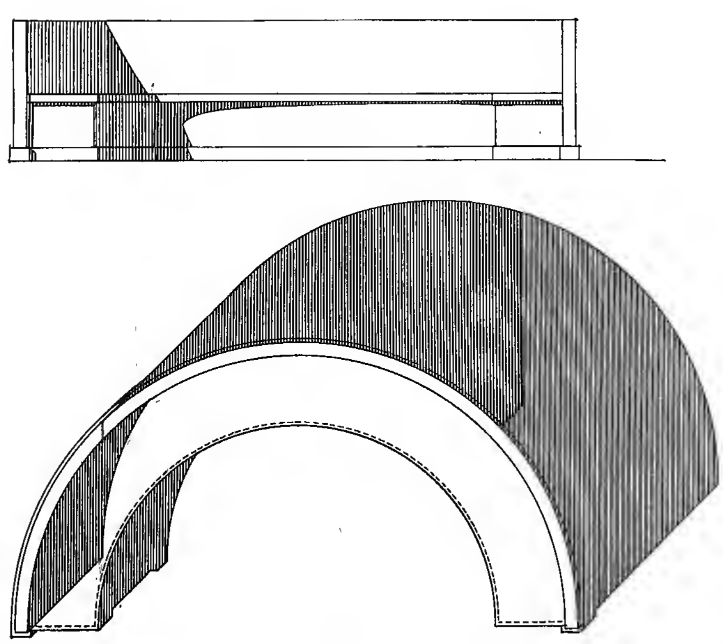


EX. LVIII

Example LIX A. A Bracket With Cyma Reversa. Two Elevations.
Example LIX B. The Same. Perspective Plan and Perspective.
Example LX. An Exedra. Plan and Elevation.



EX. LIX



EX. LX

Example LXI A, B, and C. Pediments.

Pediments.—The discussion of Pediments, among the Architectural Illustrations of Chapter IV, considered only the relations between the slope of the Pediments and the slope of the Cymatium, or Gutter, which forms the upper member of an Entablature. In the figures illustrating this discussion it was convenient to reduce the Cymatium to a straight line—a condition seldom met in actual practice.

The cylindrical surfaces, convex or concave, now to be considered, differ from those discussed in Chapter VI and Chapter VII, in that the axes of the cylinders are not Principal Lines, parallel to two planes of projection; but, though parallel to the vertical plane of projection, are inclined to the horizontal plane.

Example LXI, A B C, shows how these conditions modify both the Shades and the Shadows of the convex Molding (or Ovolo), of the concave Molding (or Cavetto), and of the Cyma Recta, which is a curve of double curvature. The case of the Cyma Reversa, which is of less frequent recurrence as a crowning member, is not illustrated. It is left for the Students to work out, if they care to do so.

The discussion and illustration of Curved Pediments, which involve the consideration of Conical surfaces, is taken up among the Architectural Illustrations of Chapter VIII.

In all these Drawings the Shades and Shadows are shown in flat tints, the Shadows being darker than the Shades. The effects of the Reflected Light upon the curved surfaces would change the flat tints to graded tints, and the Shades and Shadows would coalesce.

In this Example, as in Examples A and B, the miter lines are really elliptical, though projected as arcs of circles, and so much of the right-hand ones as are exposed to the light cast shadows which are arcs of such ellipses as are shown in Fig. 48, page 85.

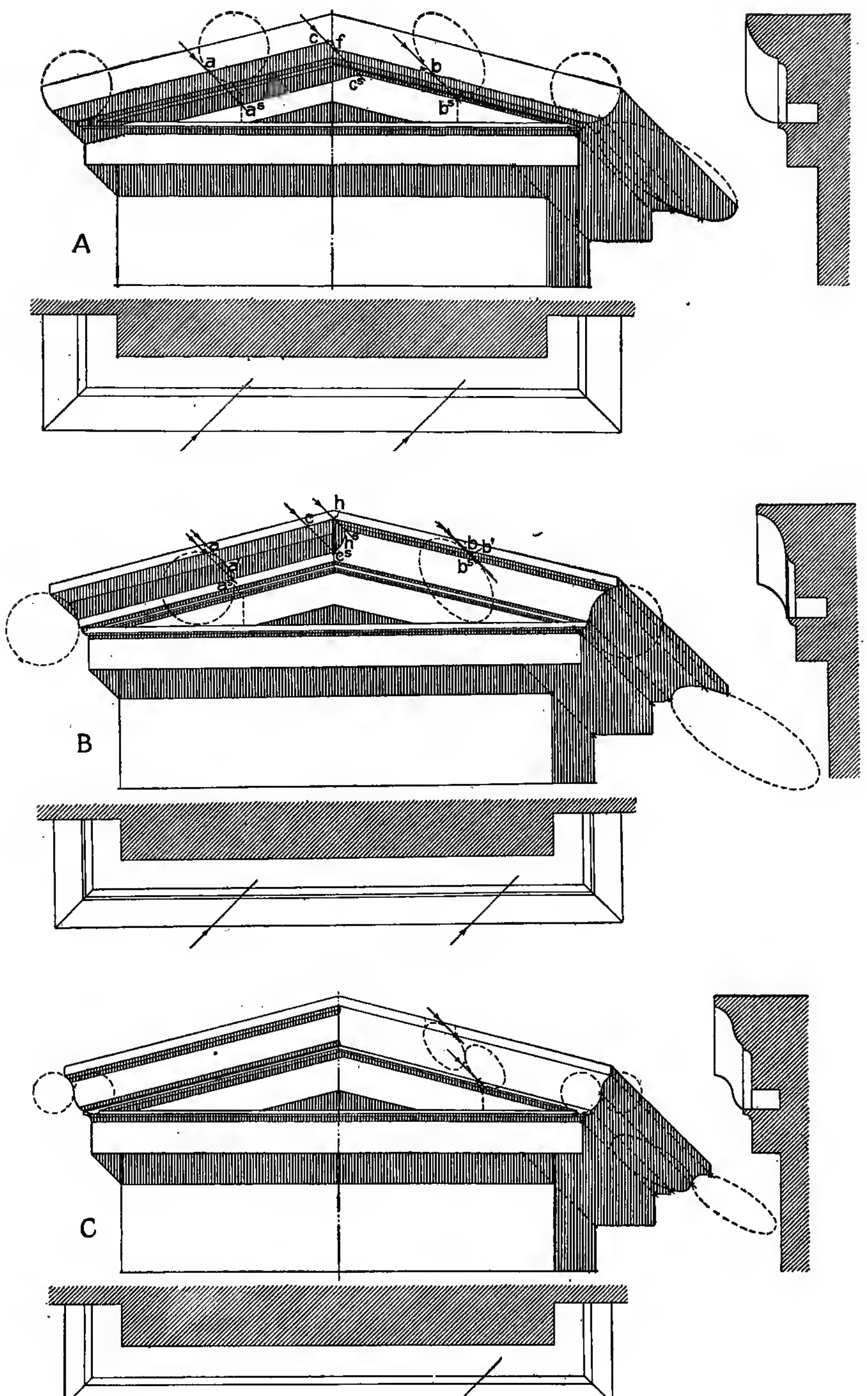
In *A*, the cymatium is convex, the molding being an Ovolo; at *B*, it is concave, the molding being Cavetto; and in C, it unites the two, the molding being a Cyma Recta.

The Lines of Shade upon the convex cymatium shown in Example LXI A, the lines of Shade and Shadow upon the concave cymatium shown in Example LXI B, and the corresponding lines in Example LXI C, are found by cutting across the cylindrical surface by vertical planes parallel to the direction of the light, that is to say, making 45 degrees with the Vertical Plane of Projection. The elliptical line of intersection upon the First Rake is similar to the miter line, and like it is projected as the arc of a circle (this line of intersection is accordingly not needed, and in C it is omitted); that upon the Second Rake is projected as the arc of an ellipse. Rays of light drawn tangent to these curves give the position of the Lines of Shade in A, at a and b, and of the Lines of Shade and Shadow in B at a' and b'.

In Example LXI A, the Lines of Shade passing through a and b cast shadows parallel to themselves passing through a^s and b^s. These Lines of Shadow meet at the point c^s, which is the shadow of the point c upon the Line of Shade of the First Rake, and also of the point f, which is a Point of Flight; c and f are found by tracing back a 45-degree ray from c^s. The short segment of the Line of Shade above c, on the First Rake, throws a small elliptical shadow upon the Second Rake. This shadow terminates at f, as appears in the figure. In Example LXI B, the concave arc does not measure quite 90 degrees.

The Line of Shade which casts the shadow is the lower edge of the fillet which crowns the concave cymatium. In the First Rake the shadow of the point a falls near the lower edge of the cymatium itself at a^s, and in the Second Rake the shadow of b is about a quarter of the way down at b^s. The point c is the last point that throws such a shadow, and the rest of the Line of Shade, from c to h, casts a shadow upon the Second Rake, from c^s to h^s, as shown, which, as well as its vertical projection, is an arc of an ellipse.

In Example LXI C, the circular arcs that compose the cymatium measure only 60 degrees, not 90 degrees, and the Lines of Shadow are modified accordingly. There is no shade upon the Second Rake, and only the two small shadows which are cast by the Lines of Shade of the First Rake.



EX. LXI

CHAPTER VIII. CONES

Example LXII. A Conical Roof and Dormer. Plan and Elevation.—The Lines of Shadow cast upon the conical roof by the straight lines of the Dormer are of course Conic Sections, two of them being ellipses, of which one is seen edgewise and is projected as a right line. The Shadow cast upon the roof by the vertical corner of the Dormer is an arc of a hypebrola, as is also its projection upon the Vertical Plane of Projection.

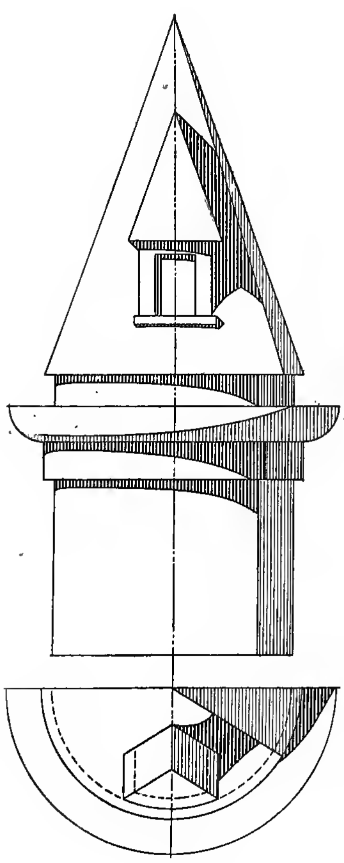
The two Lines of Shadow cast upon the cylindrical tower below by two horizontal circles are not ellipses, but are like those shown in Example XXXIX C. That cast by the Line of Shade upon the Ovolo, though resembling an Ellipse is, also, not a Conic Section.

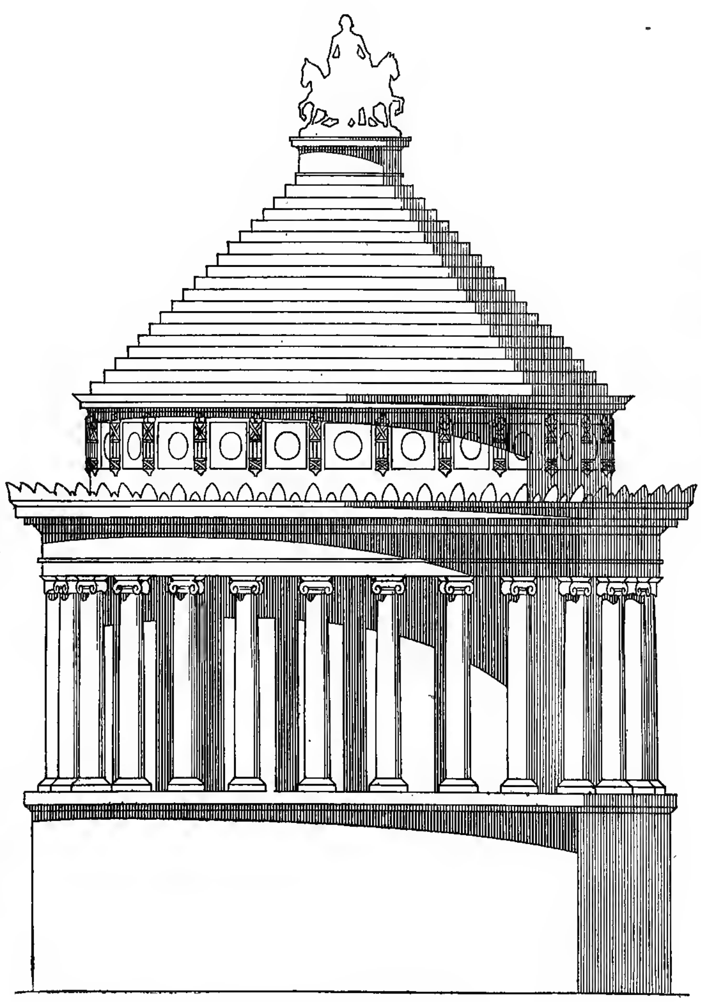
The Roman Doric order often has Guttæ shaped like the frusta of cones, instead of the pyramidal forms shown in Example XII.

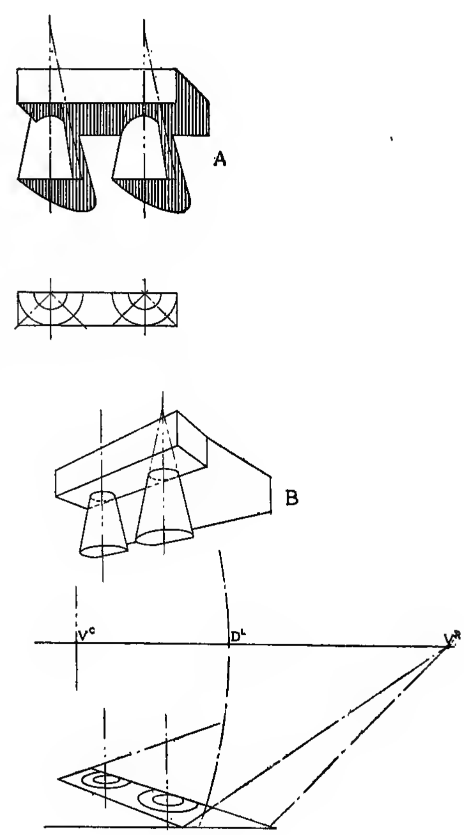
Example LXIII A. Conical Guttæ. Plan and Elevation.

B. The Same. Perspective.

Example LXIV. General Grant's Tomb. Elevation.—This exemplifies the Paradox of the Cone, as set forth in Fig. 75, Paragraph 106.


Ex. LXII


Ex. LXIV



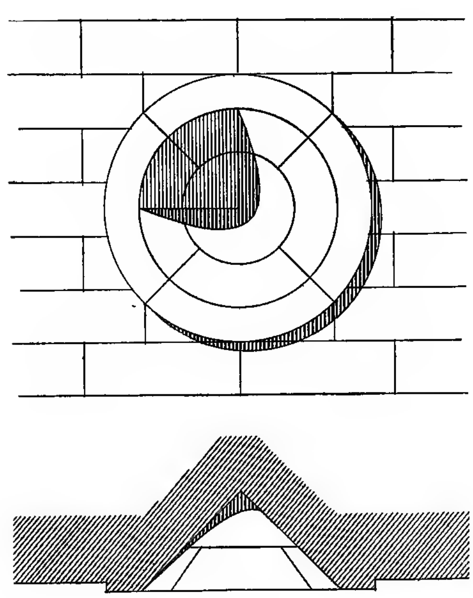
Ex. LXIII

Hollow Cones play even a less conspicuous part in Architecture than do Solid Cones, neither the larger features nor the details of ornamentation often assuming this form. The only noticeable cases are the Conical Niche, the Conical Pendentive, and Curved Pediments.

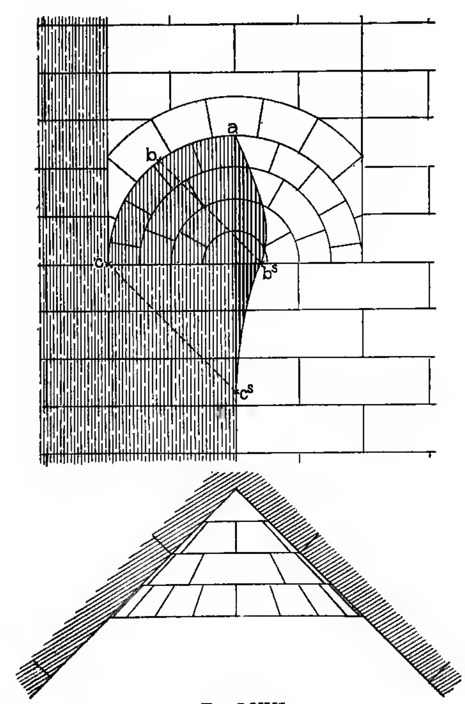
Example LXV. A Conical Niche. Plan and Elevation.

Example LXVI. A Conical Pendentive. Plan and Elevation.—A Pendentive, effecting the transition from square to octagonal walls, sometimes has the form of the upper half of a Conical Niche.

The Figure shows such a Niche in Elevation. Below is the rectangular corner, the walls of which, as appears from the plan, stand at 45 degrees. The Shadow of the Arc a b falls upon the conical surface in the elliptical line a b^s, as in the previous Example, and in Fig. 73 C. The Shadow of the arc b c falls upon the right-hand wall in the line b^s, c^s, which is an arc of the elliptical line of Shade shown in Fig. 47 B, Paragraph 56. The left-hand wall, being parallel to the light, is a Surface of Light and Shade, and neither receives nor casts any Shadow.



Ex. LXV



Ex. LXVI

Curved Pediments.—Curved Pediments have generally an arc of 90 degrees. The Cymatium, or Gutter, rises at an angle of 45 degrees and projects in front just as much as along the eaves. The miter line at the intersection of the two lies in a vertical 45-degree plane.

The profile of the horizontal Cymatium along the wall determines the precise shape of this miter line, and this determines the profile of the pediment molding, or *vice versa.*

If, as in the examples discussed in Chapter IV, we substitute an inclined line for the curved profile of the Cymatium, the Cymatium of the Curved Pediment will be a segment of a hollow cone with a horizontal axis, like the upper quarter of the niche shown in Example *LXV*. A section through the top of the pediment will give the true inclination of the elements of the cone. The three cases which it is worth while to consider are illustrated in Example LXVII A and B.

Example LXVII A. The 45-Degree Cymatium.—If the Cymatium along the wall slopes at 45 degrees, as is usually the case, it is a plane of Light and Shade. It is seen in the Elevation as the hypotenuse of an inverted right-angled triangle, the vertical and horizontal sides of which are equal. If this dimension is taken as a unit of measure, the length of this hypotenuse is $\sqrt{2}$. The miter line at the corner, that is to say that the line in which this cymatium is intersected by a vertical plane cutting it at 45 degrees, is the hypotenuse of a second right-angled triangle lying in this diagonal plane, as is shown alongside, the height of which is 1 and the base, $\sqrt{2}$. The length of the miter line is $\sqrt{3}$. It makes the angle ϕ with the horizontal plane. This miter line at the bottom of the pediment and the hypotenuse of the cross-section at the top of the pediment are elements of the hollow cone of which the curved cymatium is a segment. They make the angle ϕ with the base of the cone.

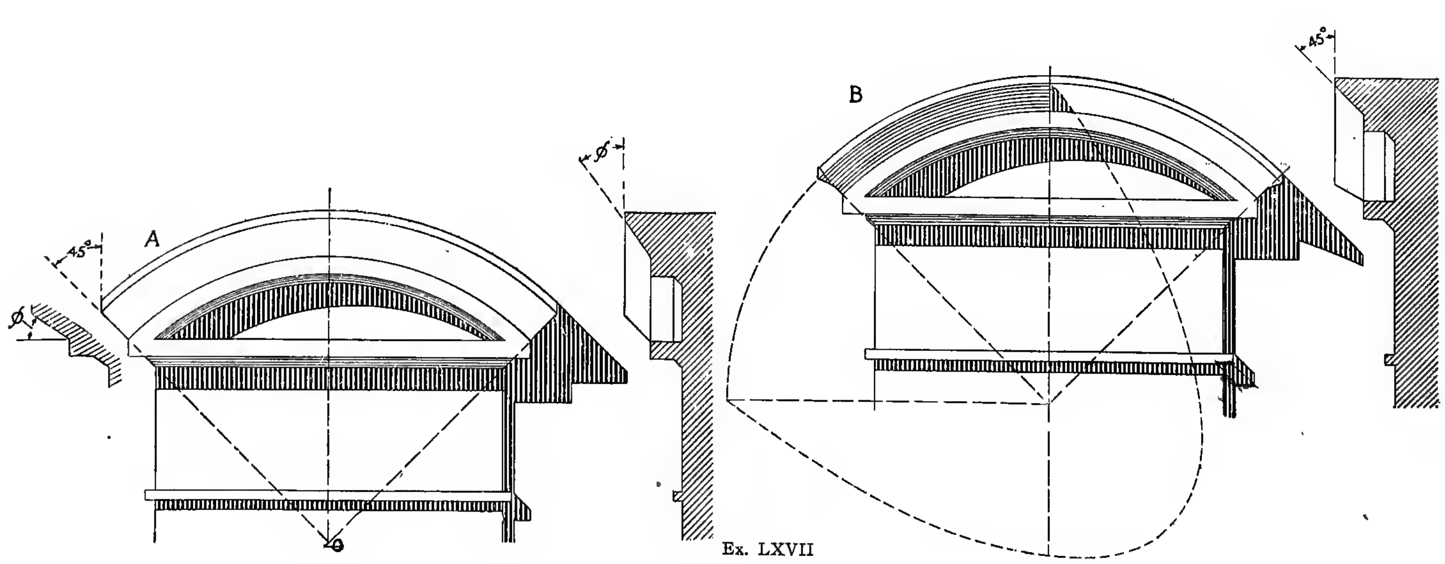
The element which coincides with the miter line lies in the direction of the light and is, as in Fig. 73 D, neither in the light nor in the shade. All the rest of the interior surface of the cone is in light, as in the figure.

Example LXVII B. If the horizontal cymatium along the wall shows in profile, as seen in elevation, a less slope than 45 degrees, then this cymatium is in shade, being turned away from the light, and the cymatium of the curved pediment, which is part of the interior surface of a hollow cone, is more or less in shade and shadow, according to the slope.

If this angle is exactly 45 degrees, as is the cone in Fig. 73 C, then just half the Cymatium is in shade; the shadow upon the other half is like that shown in Example LXVII B.

If the horizontal cymatium, along the wall, shows in profile a line steeper than 45 degrees, then that cymatium is wholly in light, and the conical cymatium also.

In the Greek-provinces, and in the modern so-called *Neo-Grec* Style, the cymatium along the wall is often so steep as to be vertical. The miter line is then also a Vertical Line, and the whole curved cymatium lies in a Vertical Plane.



Ex. LXVII

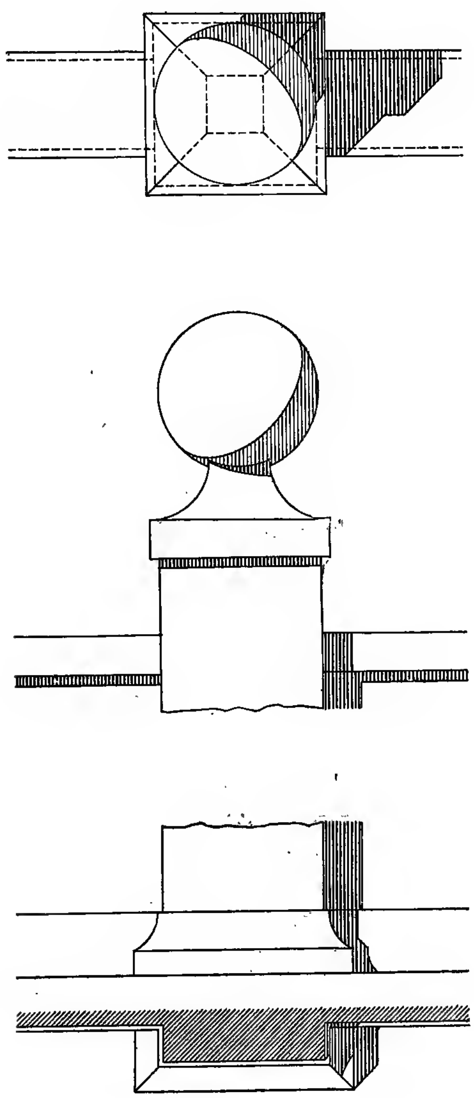
CHAPTER IX. SPHERES AND ELLIPSOIDS.

Example LXVIII. A Post and Ball. Plan and Elevation.

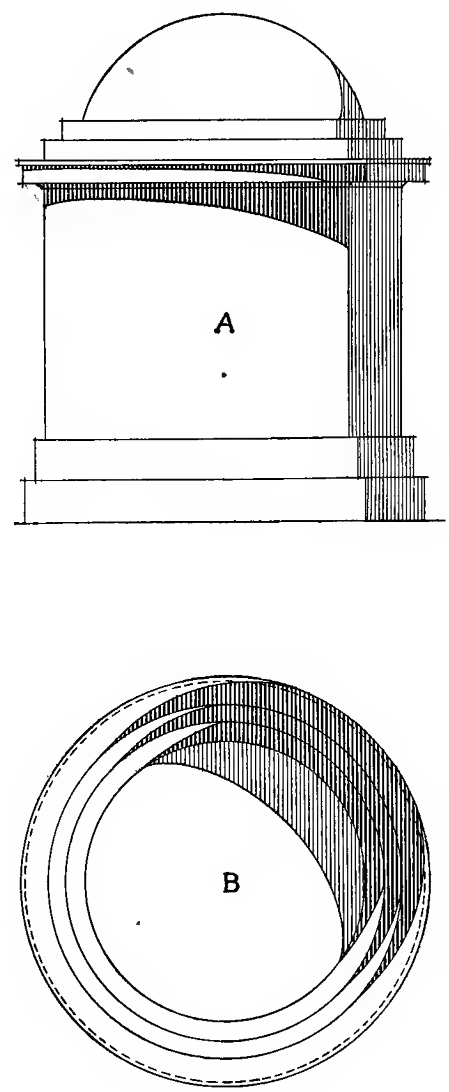
Example LXIX A. A Half Dome. Plan and Elevation.

Example LXIX B. The Same. Perspective Plan and Perspective.—This is left to be drawn by the Student.

The center of the Sphere must come just above the Center of the Picture at V^c, or the ellipses representing the horizontal circles will have inclined axes.

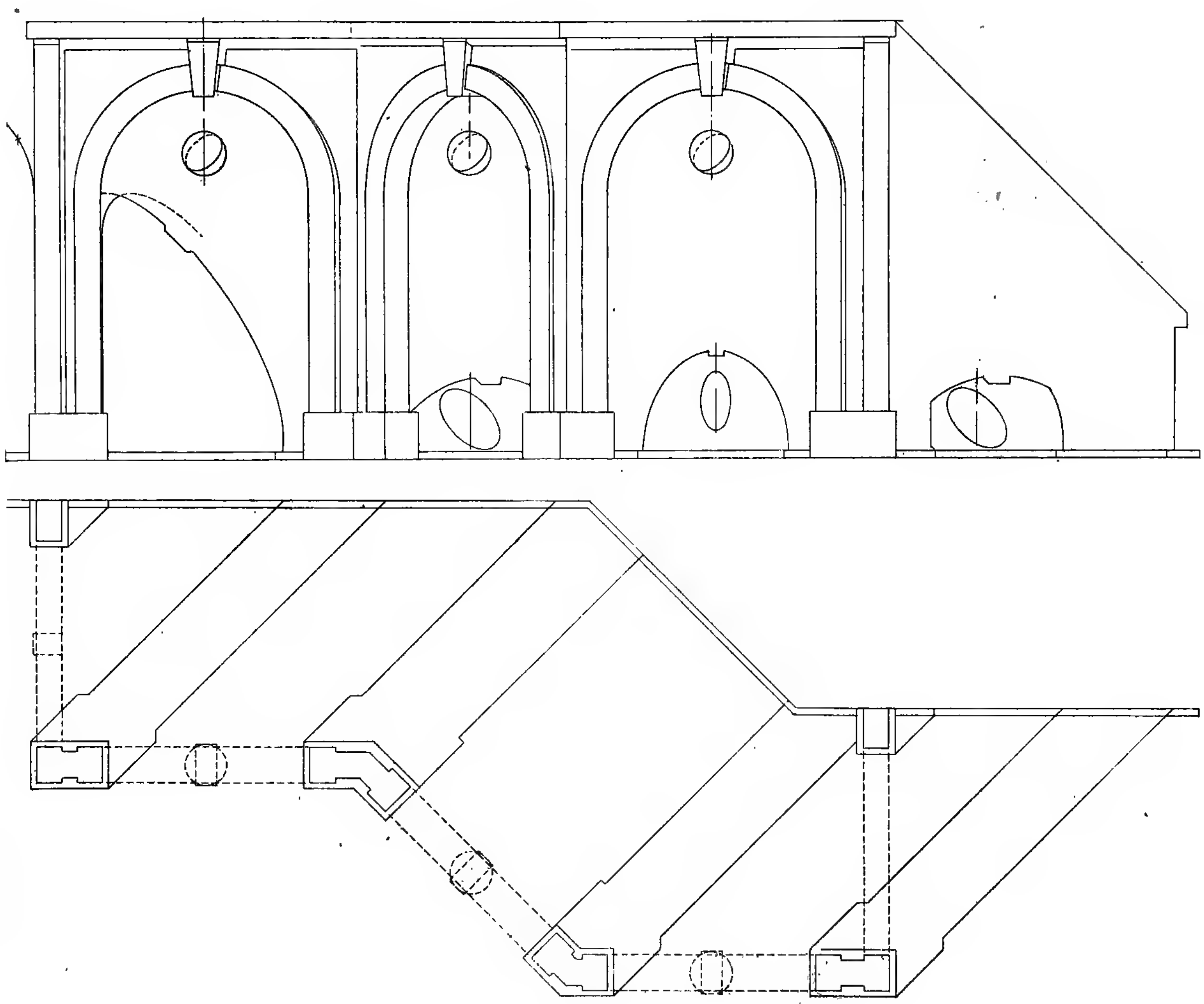


Ex. LXVIII

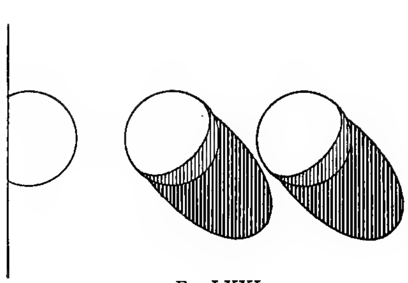


Ex. LXIX

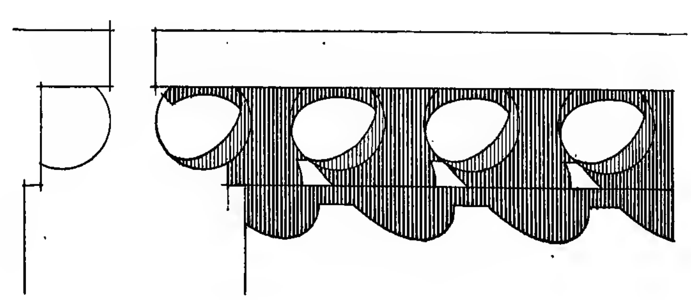
Example LXX. An Arcade and Spherical Lanterns. Plan and Elevation.
Example LXXI. Beads.
Example LXXII. Beads and Fillet.



Ex. LXX



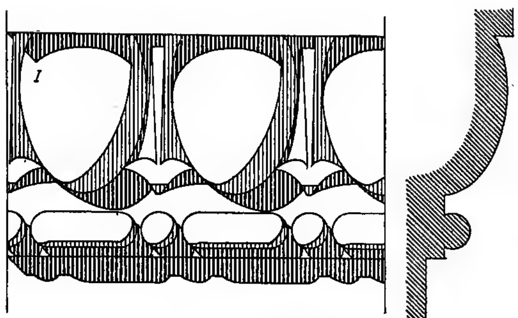
Ex. LXXI



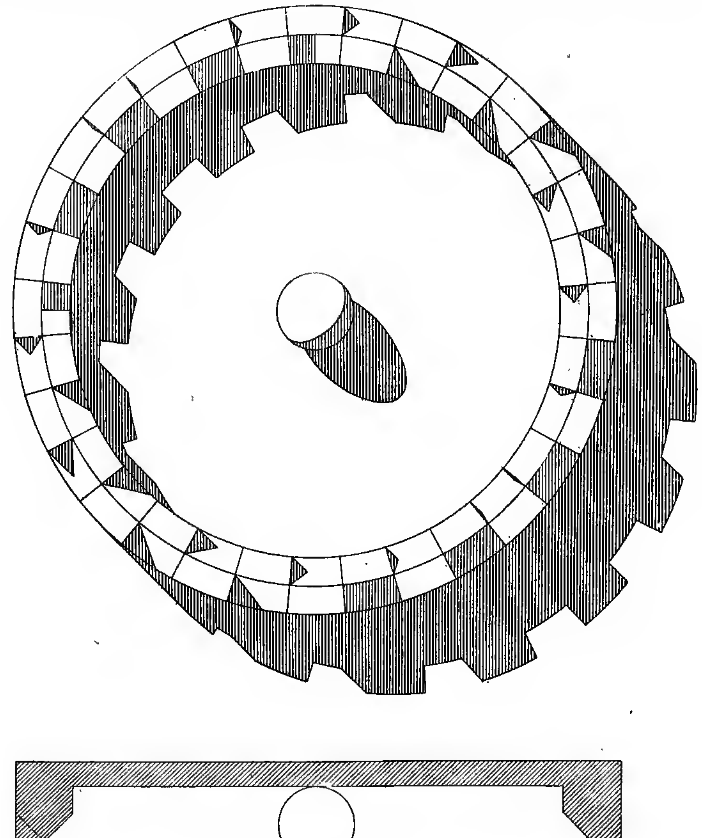
Ex. LXXII

Example LXXIII. Eggs and Darts. Elevation and Section.

Example LXXIV.—Some of the buildings in Venice are decorated with circular discs of marble, enclosed in a ring of what Mr. Ruskin calls *Venetian Dentils*, and carrying at their centers a small sphere of polished marble.



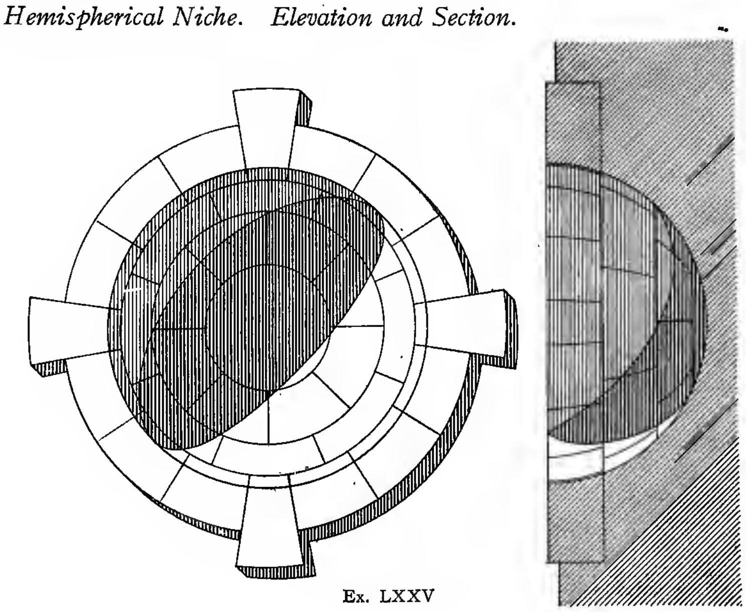
Ex. LXXIII



Ex. LXXIV

CHAPTER X. HOLLOW SPHERES. EXAMPLE LXXV

Example LXXV. A Hemispherical Niche. Elevation and Section.



Ex. LXXV

CHAPTER XI. RINGS

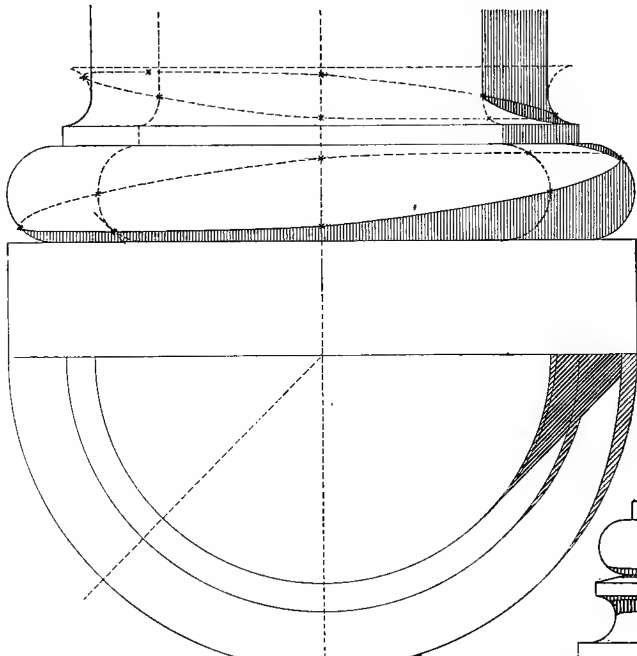
Example LXXVI. A Tuscan Base. Elevation and Plan.—The Tuscan Base consists of a square plinth surmounted by a Torus, above which is a broad Fillet, called the *Cincture*, which, however, is properly to be considered as the lowest member of the shaft. The height of the Torus and Cincture taken together is the same as that of the Plinth, and measures $\frac{1}{4}$ D, that is to say, one-quarter of the diameter of the Shaft taken just above the Cincture. The height of the Tuscan Base is accordingly $\frac{1}{2}$ D, including the Cincture. All Bases are generally $\frac{1}{2}$ D high.

Example LXXVII. A Greek Attic Base. Elevation.—Greek Bases have no plinths.

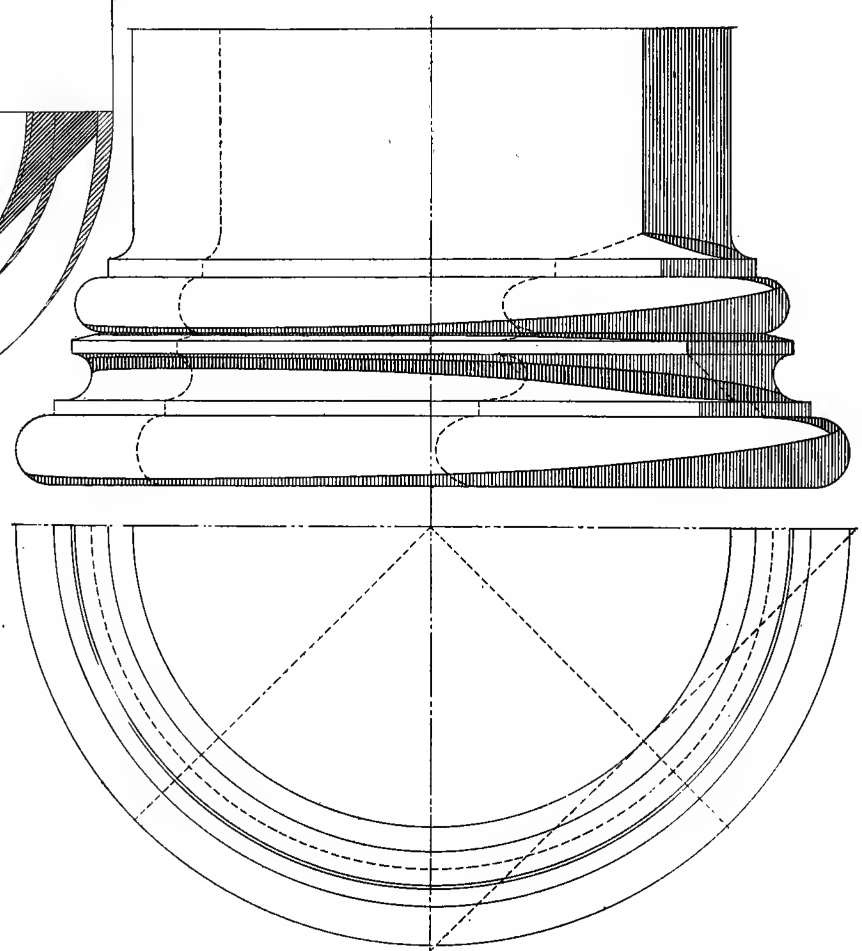
The Greek Attic Base consists of two Toruses, of which the lower one is the largest. Between them is an elliptical Scotia and two Fillets, the upper one of which projects as much as does the Torus above it. These Toruses sometimes have an elliptical section.

The shaft terminates in a large congé and a rather large Cincture.

The height of the Base is $\frac{1}{2}$ D exclusive of the Cincture.



Ex. LXXVI



Ex. LXXVII.

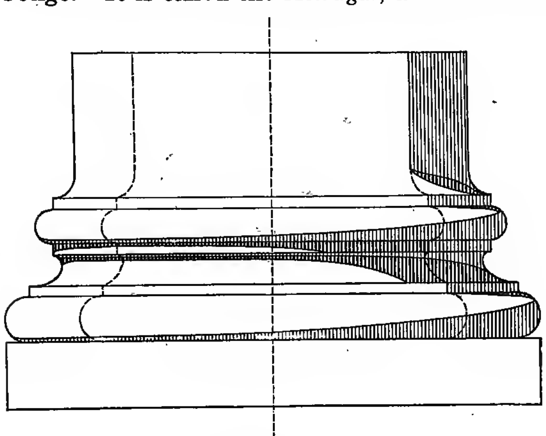
Example LXXVIII. A Roman Attic Base. Elevation.—The Roman Attic Base is $\frac{1}{2}$ D in height. It also does not include the Cincture. But as it has a Plinth $\frac{1}{6}$ D in height. The Torus and Scotia are smaller than in the Greek Attic Base, which otherwise it much resembles, but the upper Fillet does not project beyond the center of the Torus above it.

Example LXXIX. A Gothic Base. Elevation.—A simple Gothic Base of this form often occurs in France consisting, like the Tuscan Base, of only a Torus and a Plinth. There is no Cincture, or Congé. Their place is often supplied by a Cyma Reversa, or by a small Cavetto; these, however, are attached to the base and the joint is above them, the Shaft terminating without a molding. In the Greek and Roman Bases the joint comes below the Cincture.

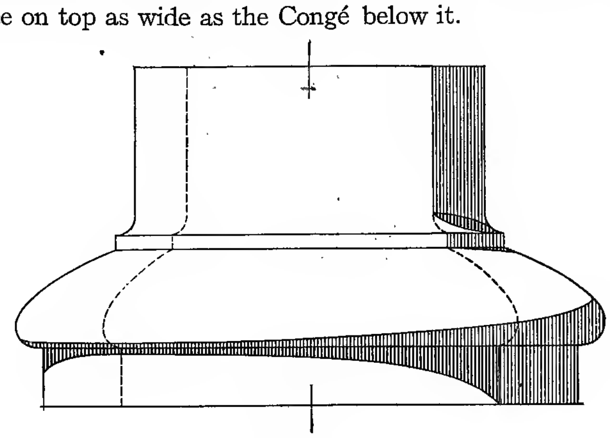
Example LXXX. A Greek Ionic Base. Elevation.—In the Greek Ionic Bases the principal feature is the Scotia, and the upper Torus is larger than the lower one, which, as here, is sometimes omitted altogether. This example is taken from one of the Choragic Columns, with triangular capitals, on the south side of the Acropolis at Athens.

Example LXXXI. Vignola's Tuscan Capital. Elevation.—The Tuscan Capital, according to Vignola, consists of a square Abacus crowned by a Fillet and Congé, below which is an Echinus of the shape of a half Torus, or Ovolo, the outline of which is a circular arc of 90 degrees; below this is a short cylinder as large in diameter as the upper part of the Shaft. This is called the Necking. It terminates in a Congé and Fillet, which support the Echinus. The Abacus, the Echinus and Fillet, and the Necking and Congé form three equal divisions, each $\frac{1}{6}$ D in height. The height of the capital is thus $\frac{1}{2}$ D, the same as that of the Base.

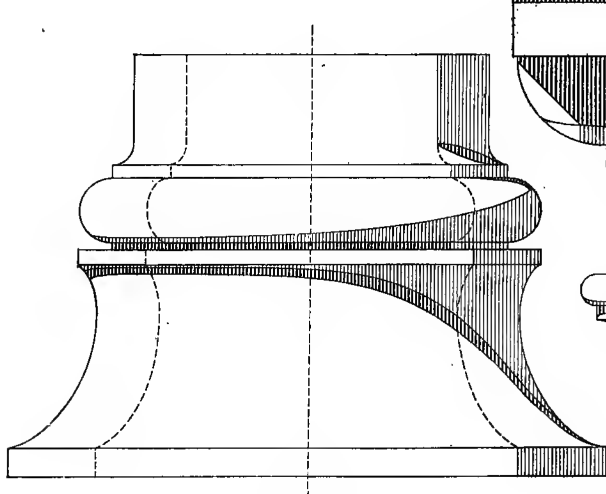
The Shaft below the Necking terminates in a Bead or small Torus which is supported by a third Fillet and a large Congé. It is called the *Astragal*, and has a flat surface on top as wide as the Congé below it.



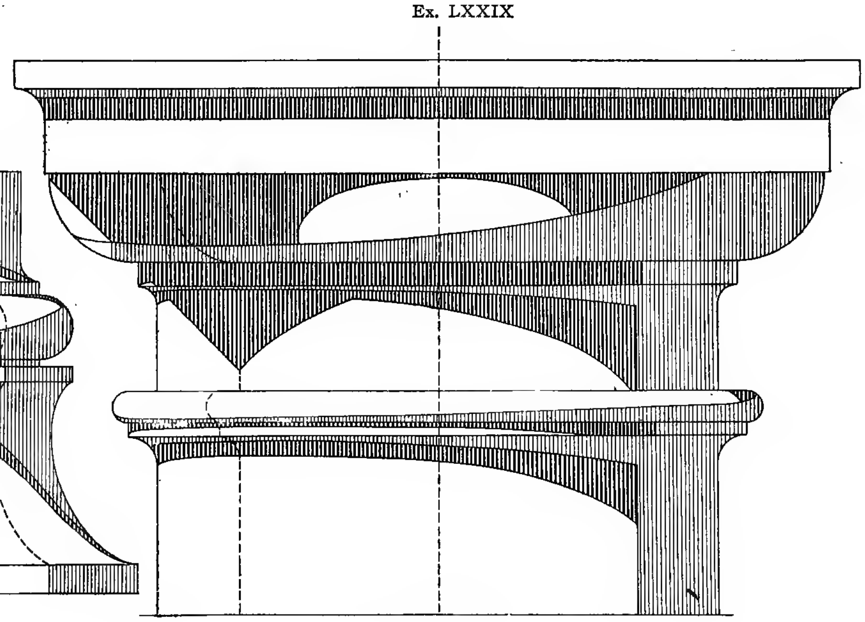
Ex. LXXVIII



Ex. LXXIX



Ex. LXXX



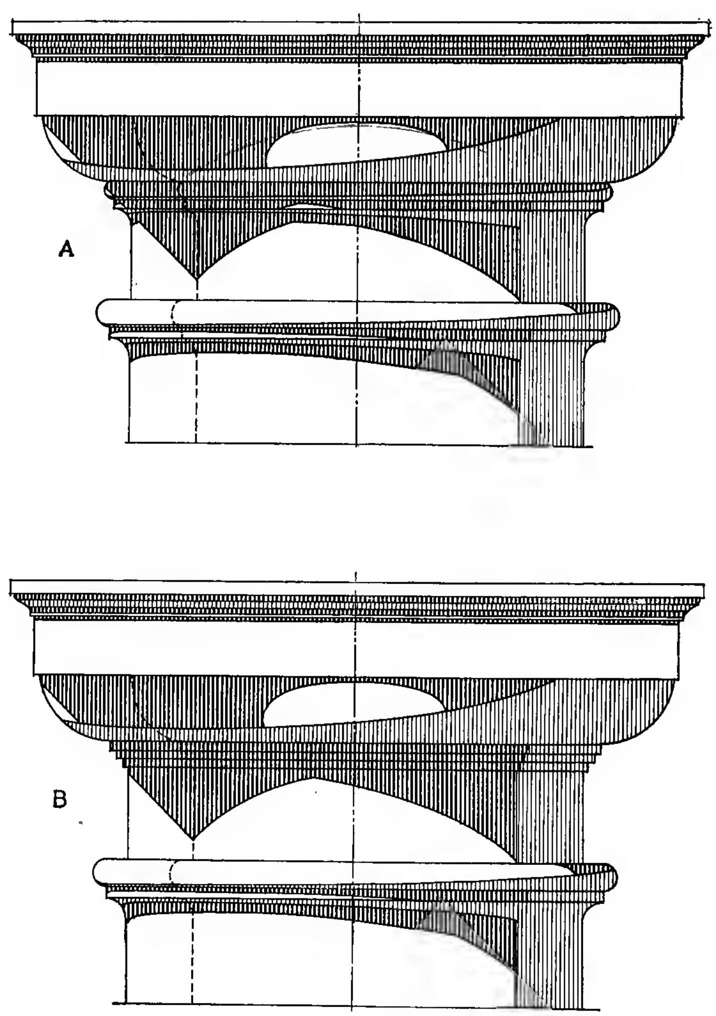
Ex. LXXXI

The Fillet and Congé may thus be considered characteristic features of the Tuscan Order. They occur for a fourth time at the bottom of the Shaft, as in all Classical Orders, and as appears in the previous examples.

Example LXXXII A and B. Vignola's Roman Doric Capital. Elevation.—The Roman Doric Capital, like the Tuscan, is $\frac{1}{2}$ D in height, and like it consists of three equal parts, each measuring $\frac{1}{6}$ D. But the Fillet which surmounts the Abacus is narrow and is supported by a small Cyma Reversa in place of the Tuscan Congé. The $\frac{1}{6}$ D which constitutes the middle part is itself divided into thirds. The upper two-thirds are occupied by an Echinus which is like the Tuscan Echinus, but smaller. The lower third is again divided into thirds, and the Echinus is supported either, as at *A*, by a smaller Torus, Fillet, and Congé (like the Astragal, though smaller), or, as at *B*, by three equal Fillets.

When the Shaft is fluted it has 20 shallow channels which meet on an edge, or *arris*, and terminate below the Astragal in small elliptical niches, These are not shown in this figure.

In the Roman Orders the face of the Architrave comes, in the Elevation, just over the upper diameter of the Shaft, though in perspective it overhangs it.

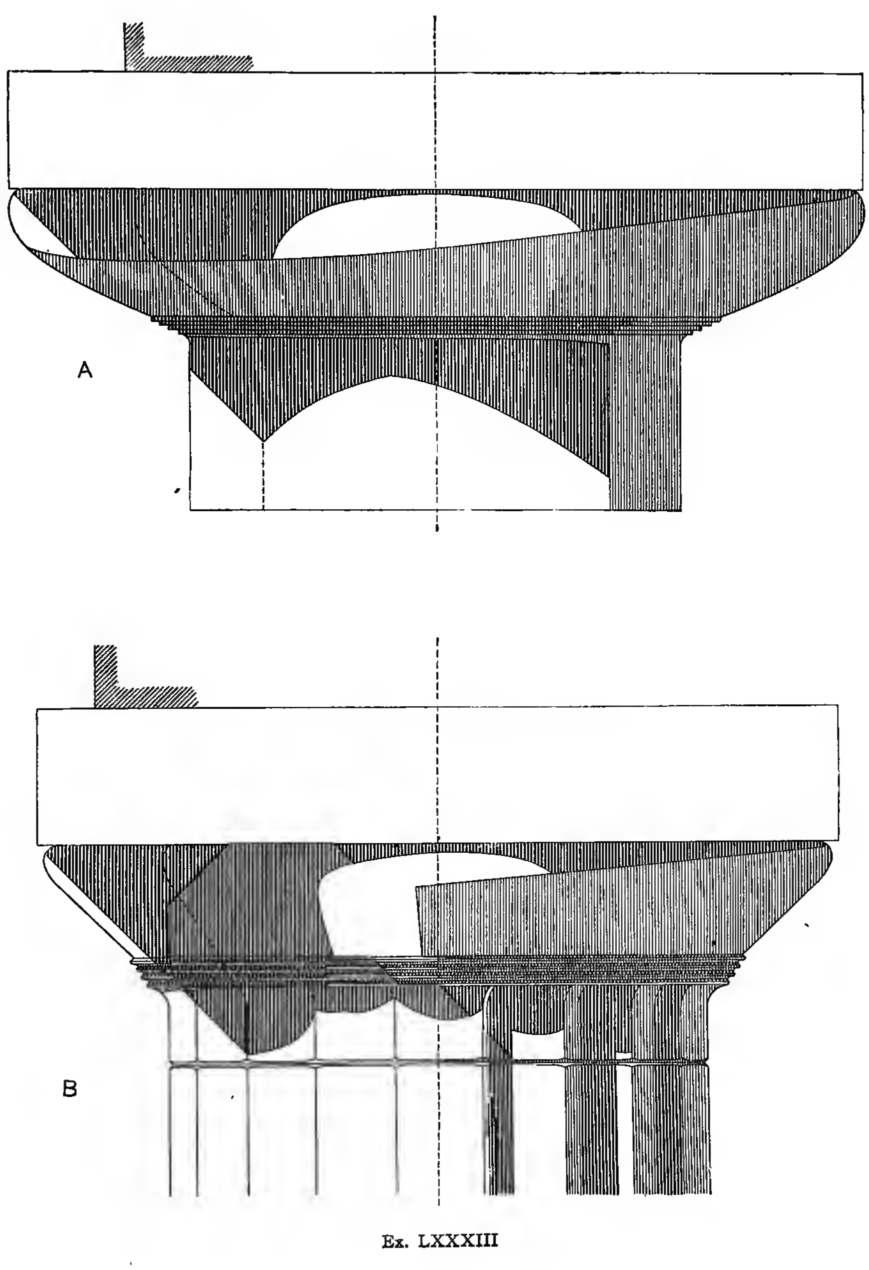


Ex. LXXXII

Example LXXXIII A and B. Two Greek Doric Capitals. Elevation.—The Greek Doric Capital has a plain Abacus, and the Echinus, which is supported by two or more angular Fillets, is elliptical or hyperbolic.

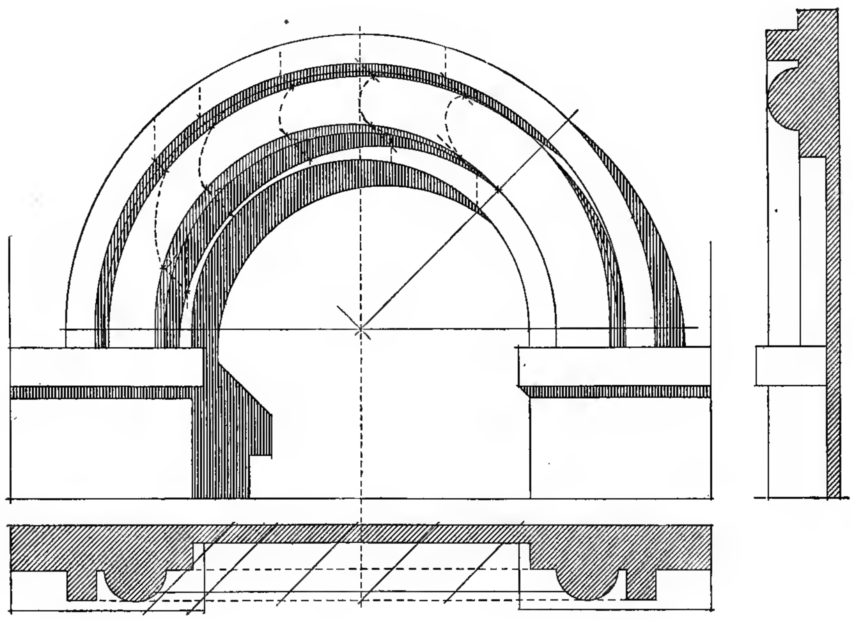
Below the Necking, instead of an Astragal, is a groove called a *sinkage*, or *Apophyge*, and the channels of the shaft are carried past it through the Necking. These are not shown in Fig. A. As in the Roman order, the channels are separated by an Arris.

In the Greek Doric Order the Architrave has only one Fascia or Band. When the Echinus has an elliptical outline and projects a good deal, the face of the Architrave comes in elevation over the upper diameter of the shaft, as in the Roman order; but when the Echinus is steep, with a hyperbolic outline, it overhangs, even in the elevation They all overhang in perspective.

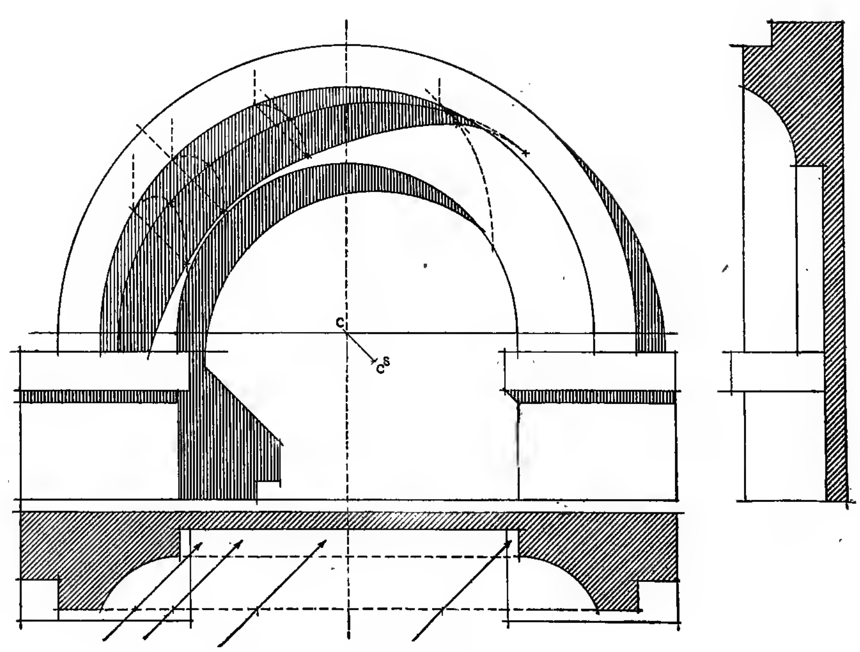


Ex. LXXXIII

Example LXXXIV. An Archivolt With Convex Section. Elevation.
Example LXXXV. An Archivolt With Concave Section. Elevation.



Ex. LXXXIV



Ex. LXXXV

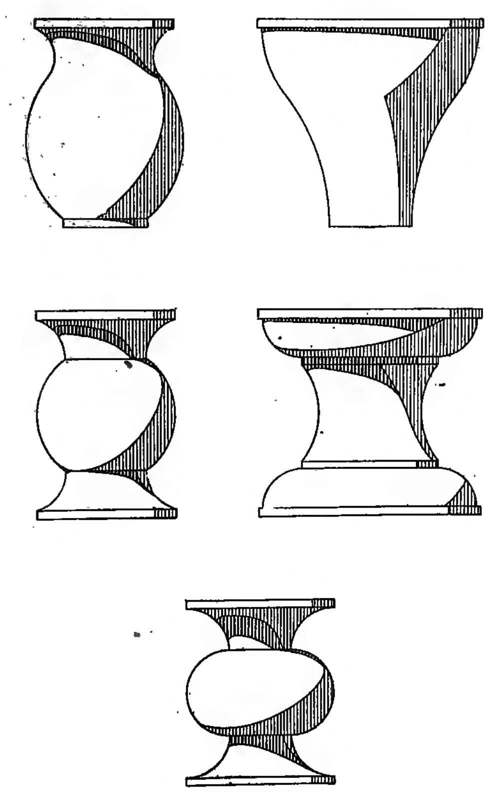
CHAPTER XII. COMPOSITE FIGURES

Example LXXXVI. Vases. Elevations.—The Student may profitably substitute for the forms here given others of his own devising, finding the Line of Shade upon each. It is an interesting exercise to use all five of the principal kinds of outline (the straight line, the convex line, the concave line, the Cyma Recta, and Cyma Reversa), making three vases of each kind, one narrow and high, one broad and flat, and one about as high as it is wide.

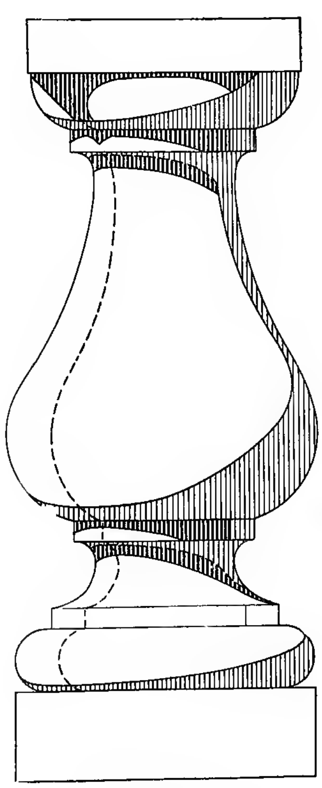
Example LXXXVII. Finials, in Stone or in Wood.—These also it is left for the Student to design.

Example LXXXVIII. A Baluster. Elevation.—A Baluster is a kind of stunted column with a simple Tuscan or Doric Cap and a Tuscan or Attic Base. For the straight line of the shaft is substituted a Cyma Reversa, called the *Sleeve*, or sometimes a Beak Molding, which often has a Fillet between the concave and convex portions, as in Examples L and XC—B. Balusters are used in a Parapet or low walls, which have a base of their own beneath them, and above them a simple Rail, resembling a Cornice, or small entablature. The Base and Rail are each about one-sixth the height of the Baluster. The widest part of the Baluster occurs at about one-third of its height, and this is their breadth at that point.

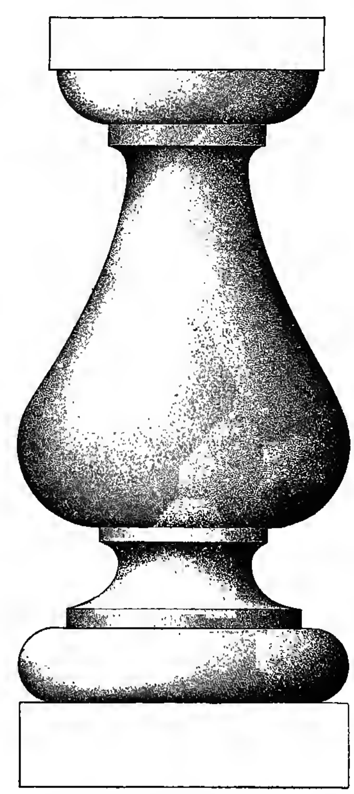
Example LXXXIX. The Same, in Diffused Light.



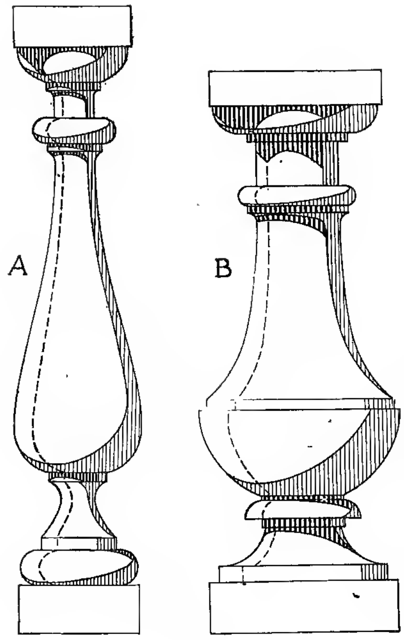
Ex. LXXXVII



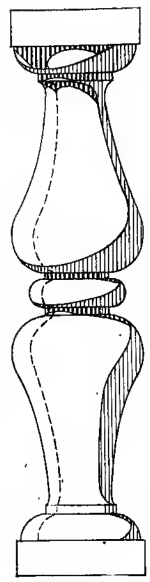
Ex. LXXXVIII



Ex. LXXXIX

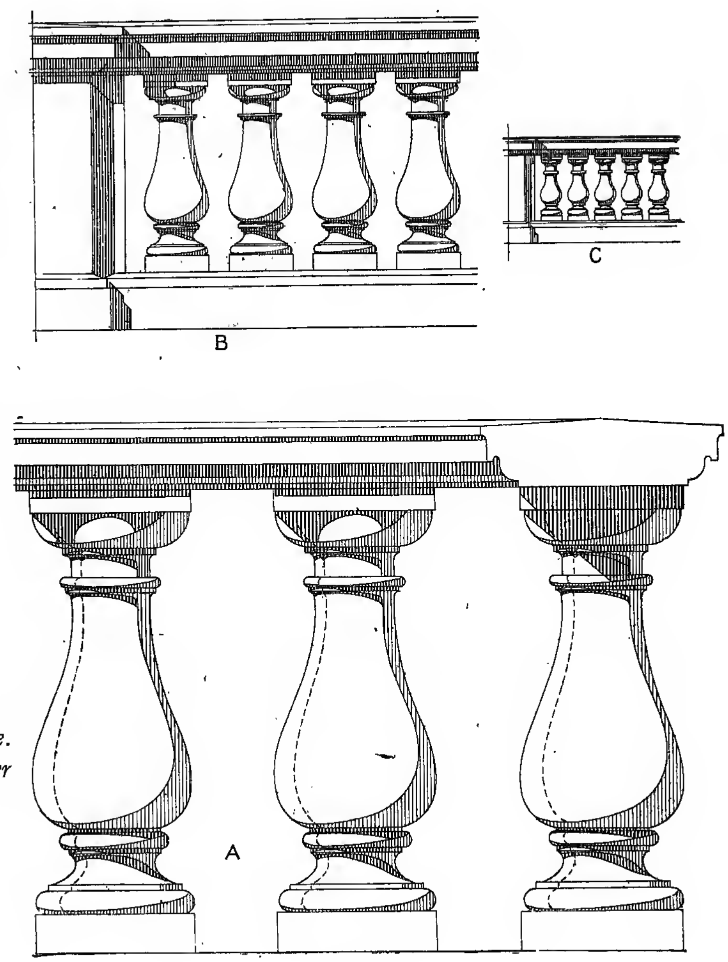


Ex. XC



Ex. XCI

Example XC and B. Two Other Balusters.
Example XCI. A Double Baluster.



Example XCII A. A Balustrade. Elevations.
B. The Same, on a Smaller Scale.
C. The Same, on a Still Smaller Scale.

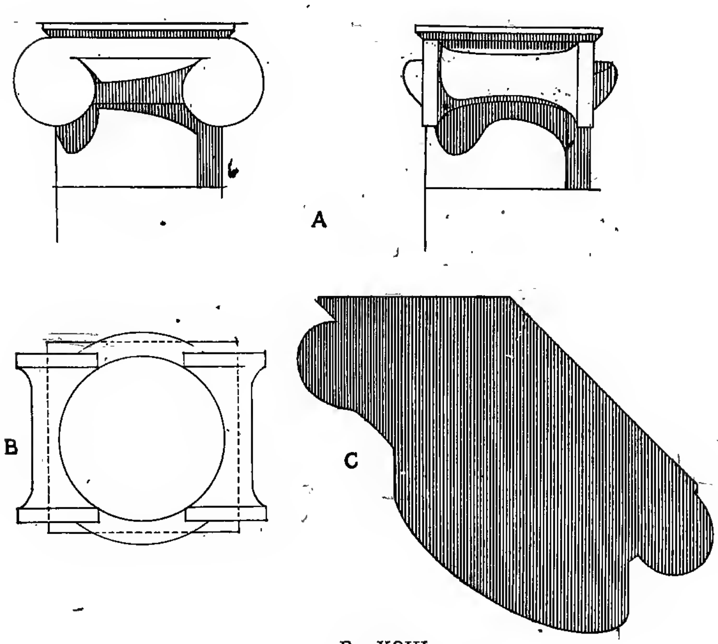
Ex. XCII

Balusters often occupy the whole space between one Post or Pedestal and the next, forming a Balustrade. If the distance is so great that the Cap has to be made of separate lengths of stone, a Block called an Uncut Baluster is placed under the joint. Not more than two Balusters should occur without such an interruption. The Cap is generally one-fourth the height of the Balusters and so is the Base, of which the Scotia is generally the principal member.

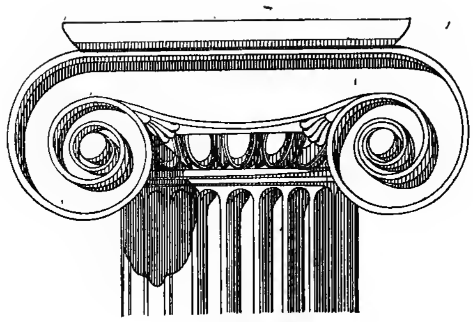
Fig. B shows that the points at which the lines of Shade and the lines of Shadow cut the outlines which are nearest the light are at the same level as the corresponding points at which they cut the axis; and that the points at which they cut the corner line which is nearest the light are the highest point of the Lines of Shade, and the lowest points of the Lines of Shadow. Moreover, the points at which the cylindrical surfaces of the shaft and the upper and lower fillets, and the middle lines or "equators" of the two Toruses, pass from light to shade, and the middle fillet passes from shadow to shade, all fall upon the "conner" line which is farthest from the light. At these six points these six surfaces are vertical.

If these relations are borne in mind, the shades and shadows of capitals, Bases, Vases, Finials, and Balusters can be drawn with, in all essential particulars, a close approach to accuracy.

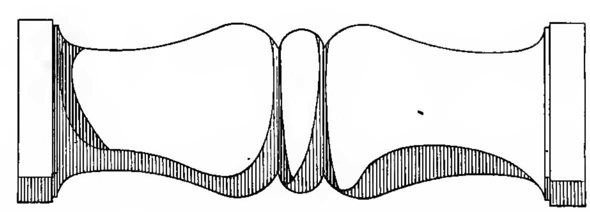
Example XCIII. A. B. C. An Ionic Capital, in Block, with Shadow.



Ex. XCIII



Ex. XCIV



Ex. XCV

Example XCIV The Same, Detailed. *B.*

Example XCV. The Baluster of an Ionic Capital. Elevation.—The scrolls at the side of an Ionic Capital much resembles a Double Baluster. But they vary a good deal, and sometimes have leaves carved upon them.

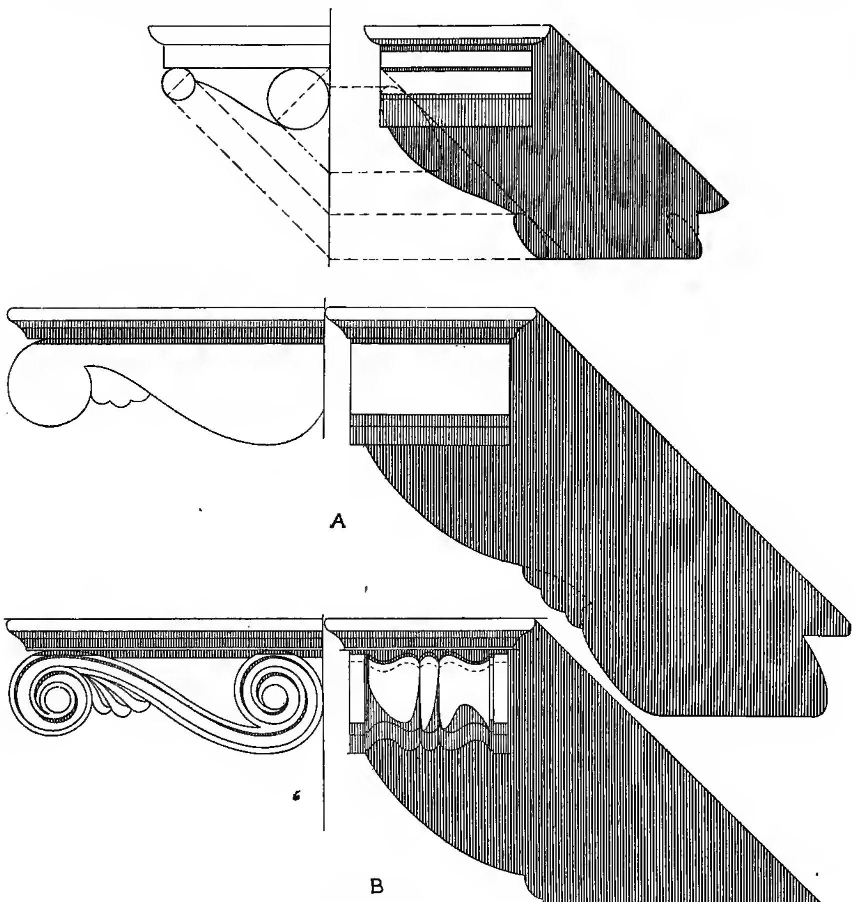
Example XCVI. A. A Modillion in Block. A Modillion is a bracket, longer than it is high, which consists of a large Baluster next the wall, and a smaller one at the outer end, with moldings between them which have outlines of a Cyma Recta. Modillions carry a small Abacus which has the profile of a Cyma Reversa, and crowns the interval of wall between them.

B. The Same, Detailed. Two Elevations.

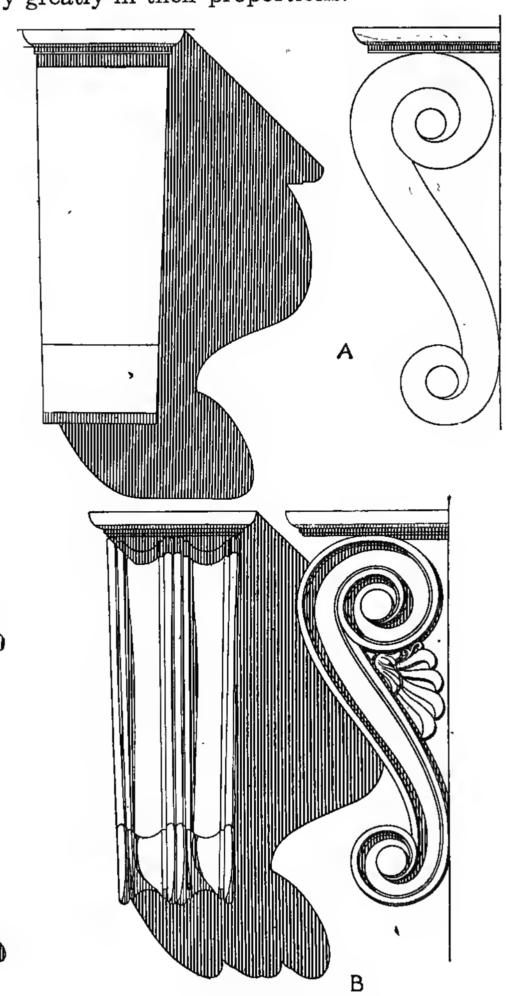
Example XCVII. A Console, in Block. Two Elevations.—A Console is an upright Modillion. It has the outline of a Cyma Reversa; the small Baluster coming below, the larger one above. It has an Abacus like that of the Modillion.

Example XCVIII A and *B. Another, in Block and Detailed.*

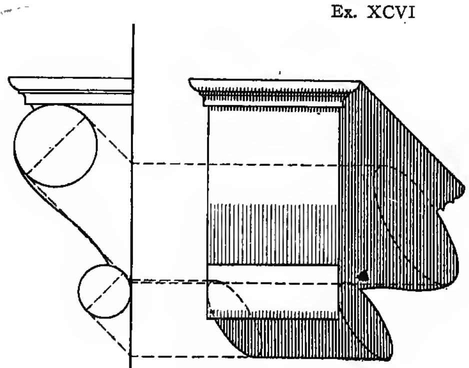
Example XCIX. Another Console. Both Consoles and Modillions vary greatly in their proportions.

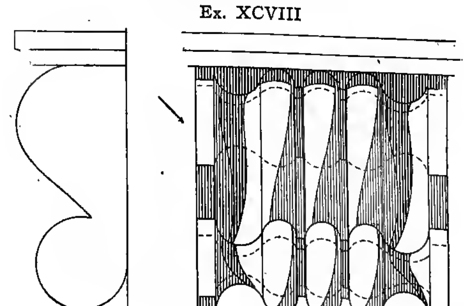


Ex. XCVI

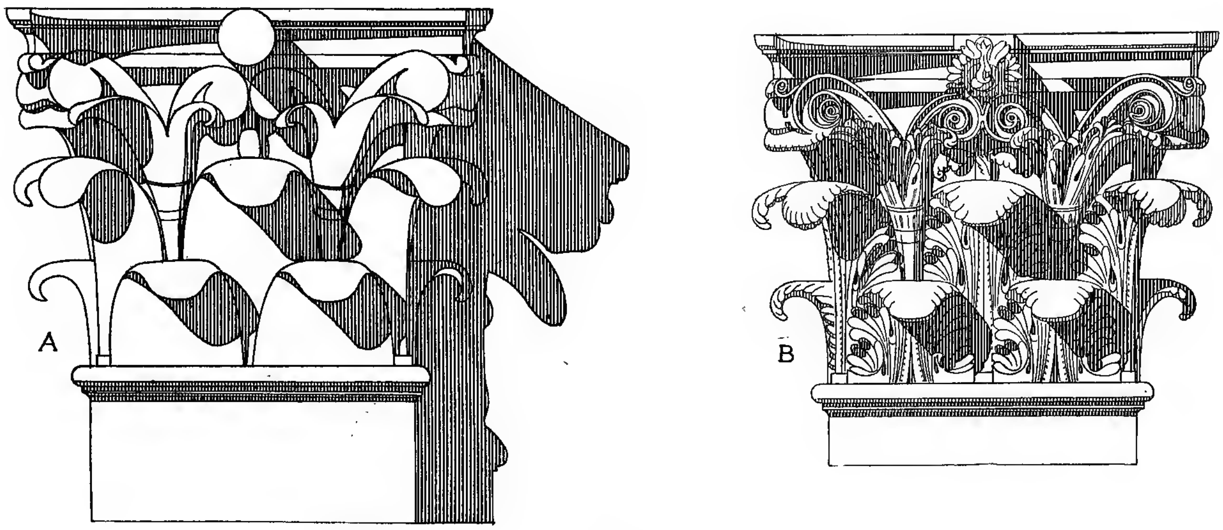


Ex. XCVIII

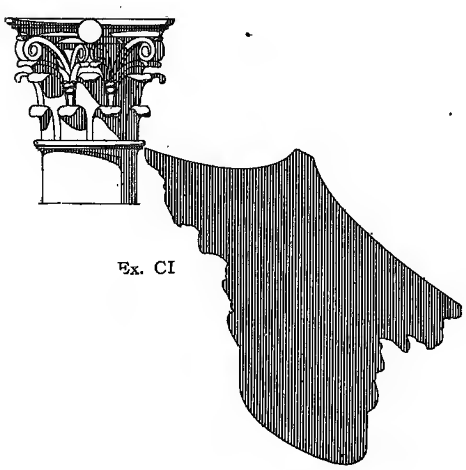

Ex. XCVII


Ex. XCIX

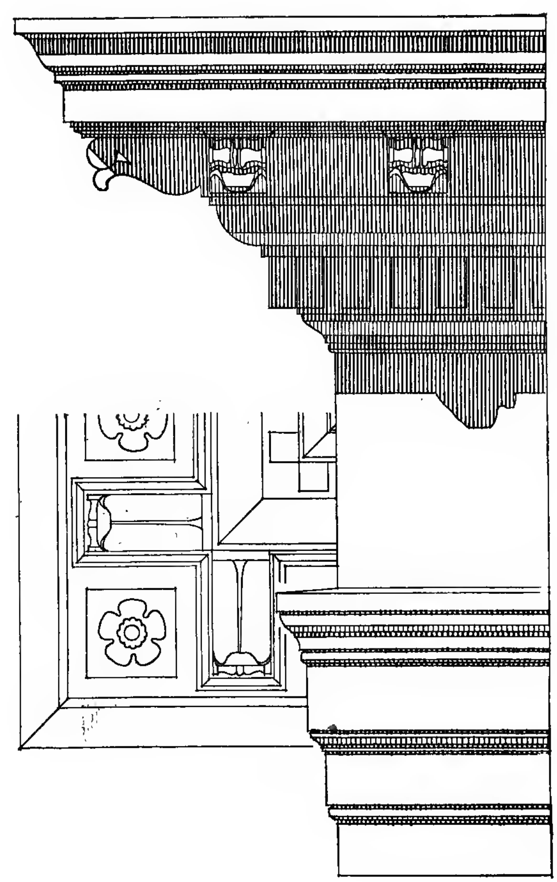
Example C A. *A Corinthian Pilaster Capital, in Block.*
B. *The Same, Detailed.*
Example CI. *A Corinthian Capital, in Block. Elevation, With Shadow.*
Example CII. *A Corinthian Entablature. Elevation.*



Ex. C



Ex. CI

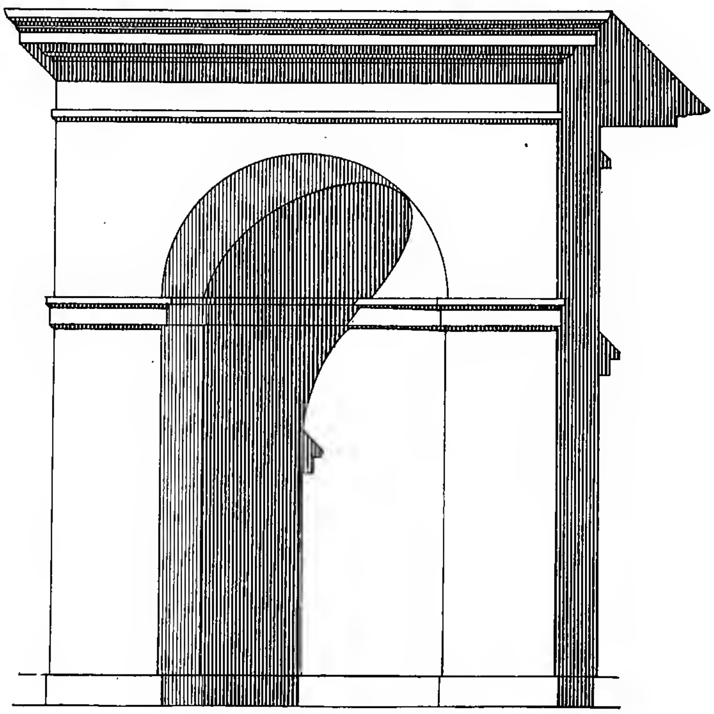


Ex. CII

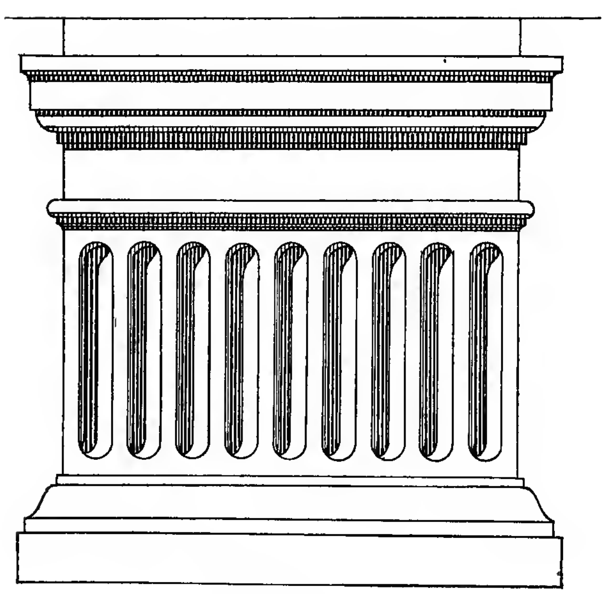
Example CIII. A Niche and Entablature.

Example CIV. An Oblong Altar, With Niches. Elevation.

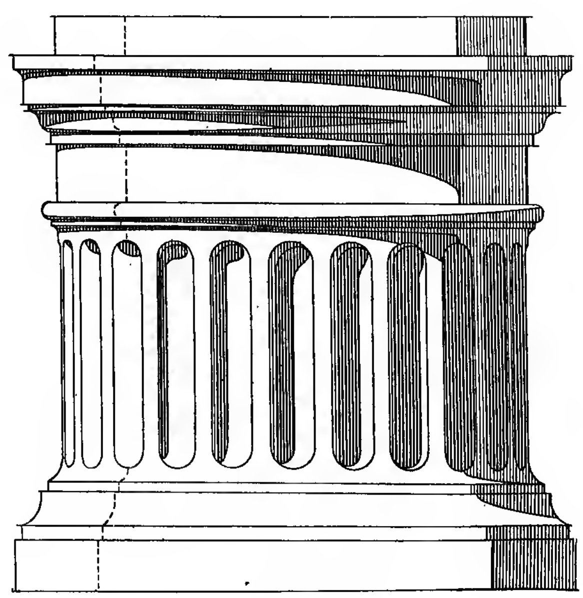
Example CV. A Round Altar, With Niches. Elevation.—There are in this example twenty-four niches separated by plain Fillets, as in Ionic and Corinthian Columns.



Ex. CIII



Ex. CIV



Ex. CV