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SLIDE-RULE NOTES

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AND
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SLIDE-RULE NOTES

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PREFACE

In 1911 the authors published a pamphlet, *Slide-Rule Notes*, now out of print, which is the basis of the present work. The material has been revised and rewritten, with additions, and a chapter on the logologarithmic scales has been added.

The work deals almost exclusively with the ordinary 25 cm. slide-rule, as designed by Colonel Mannheim. Any one familiar with this instrument will have little difficulty in learning the use of more elaborate slide-rules.

H. C. D.
C. S. J.

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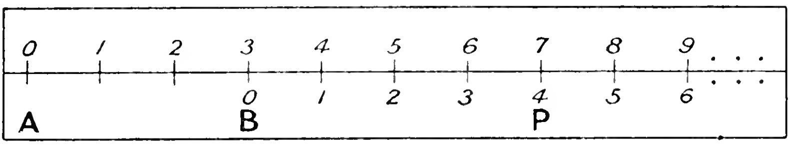
SLIDE-RULE NOTES

CHAPTER I.

ELEMENTARY EXPLANATION.

1. The following preliminary explanation may help a non-mathematical student to understand the principle of the Slide-Rule. Make a pair of paper strips and mark a scale, say of centimetres, on the edge of each. Number the marks consecutively. If now the strips are placed with their zeros say **3** cm. apart, as below,



it will be seen, for example, that they mechanically add **3** to **4**, because

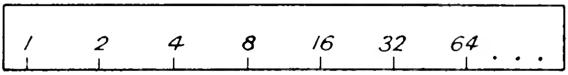
$$AB + BP = AP.$$

From the converse point of view they mechanically subtract **4** from **7**.

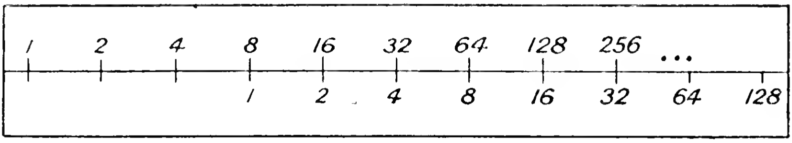
The result may be stated in a more general form, thus: for any given "setting" or relative position of the strips the **difference** between any corresponding pair of numbers, *i.e.* between any number on the upper strips and that immediately below it on the lower strip, is constant. In the above diagram this constant difference is **3**.

Let us now try to make a pair of strips for multiplying two numbers together.

We see that in the strips for addition a step of a given length to the right adds **2**; we now want instead an arrangement, for instance, in which a step of given length multiplies by 2, that is brings us to double the number at our starting point. This leads to a uniformly divided scale numbered as shewn below :



With a pair of such scales we can work out certain arithmetical problems ; in fact we have a rudimentary slide-rule. Set the strips overlapping, for instance, as below :



Notice that any number on the upper strip is 8 times that just below it, and that therefore we can, from this setting, read off the answers to certain sums in arithmetic. For example,

Multiplication.—The fact that 8 times 4 is 32 is shewn thus : 8 32 / 1 4.

Division.—The fact that 32 divided by 4 is 8 is shewn in the same way.

Proportion.—The fact that 32 is to 4 as 128 is to 16 is shewn thus : 32 128 / 4 16.

The scales may, of course, be extended to the left if desired.

Other sums may now be tried in a similar way, but with the strips of paper set to a different overlap. We observe that in the present case—For any "setting" of the strips the **Ratio** of any corresponding pair of numbers is constant.

We note that if we "invert" one of the adding scales we find the **Sum** of corresponding numbers constant, and if we "invert" one of the multiplying scales we find the **Product** of corresponding numbers constant.

NOTE. By "inverting" is meant to turn the scale end for end, so that its numerals appear upside down.

It is evident that an instrument numbered like these paper strips is not a practical one, for only a few numbers can be dealt with.

In an instrument which is to be practically useful there must be graduations for the numbers 3, 5, 6, 7, etc., and in fact for 1·1 and similar numbers, as well as for 2, 4, 8. The position for the numbers 3, 5, 7, etc., on the scales can be found roughly by a graphical construction as follows.

Draw a line AX on squared paper and set off equal intervals numbered 1, 2, 3, 4, etc. (Fig. 1). At the point 2 set up a perpendicular 1 inch long to represent the step or interval between the graduations figured 1 and 2 on the slide-rule. At the point 4 set up a perpendicular to represent the interval 1-4 on the slide-rule. This will be 2 inches long, because the distances on the slide-rule 1-2 and 2-4 are equal (Fig. 1).

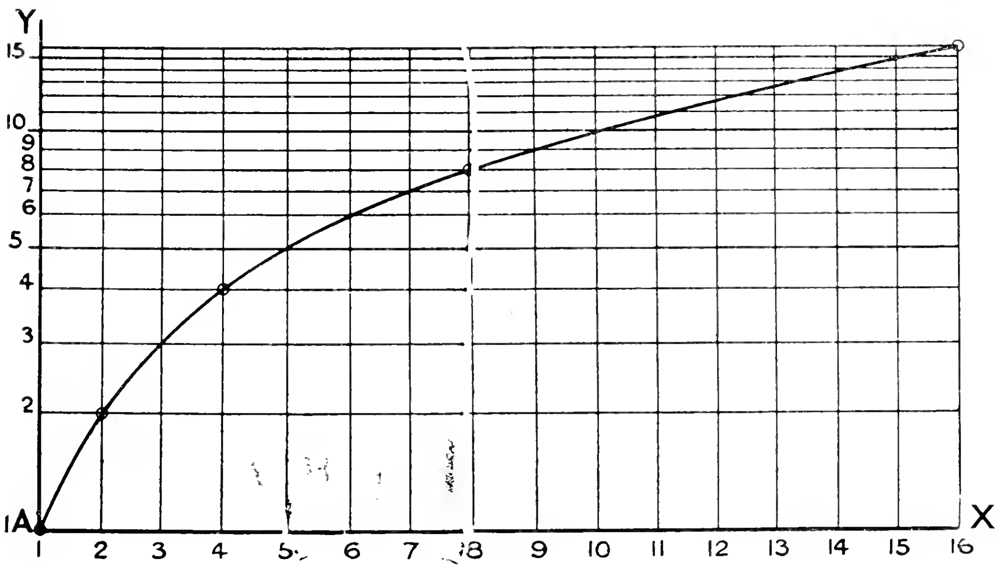


Fig. 1.

In like manner set up a perpendicular 3 inches long at the point 8 and a perpendicular 4 inches long at the point 16.

Draw a fair curve through the ends of these perpendiculars. Observe that all fair curves drawn through these points are very much alike.

Now *assume*, in order to see what happens, that the length of the perpendicular from any number on AX, say, for instance, 10, up to the curve gives the proper relative interval 1-10 on the slide-rule.

EXAMPLE. Draw parallels to AX through the points on the curve corresponding to the numbers 2, 3, 4, 5, 6, 7, 8, 9, and thus construct the scale AY (Fig. 1). Make an exact duplicate of it. Try whether by means of it you can find the product of 2 by 3, 3 by 3, or the quotient of 9 by 5.

Do the results appear to confirm the above assumption?

The interval from 1 to 10 on the slide-rule represents what is called in technical language a *logarithm* of 10, and the interval from 1 to any mark x represents the corresponding logarithm of x.

The values of the logarithms of different numbers, log 10 being taken as unity, have been calculated and printed in tables.

To use the slide-rule to the fullest advantage, a knowledge of logarithms is needful, but the non-mathematical student is recommended to skip all reference to logarithms, and merely to study the actual methods of setting and using the slide-rule given in the subsequent chapters.

For his benefit a considerable number of numerical instances have been added.

All calculations which can be made with logarithms can be made with a slide-rule. The degree of accuracy, of course, depends on the size of the slide-rule.

2. For the purpose of reference we recapitulate here the fundamental principles of logarithms.*.

Let a be any number.

Then $a = 10^{\log a}$.

This is the mathematical definition of the term logarithm. We have at once, therefore, the following equations:

(i) $a = 10^{\log a}$.

(ii) $b = 10^{\log b}$.

(iii) $ab = 10^{\log ab}$.

(iv) $\frac{a}{b} = 10^{\log \frac{a}{b}}$.

(v) $a^n = 10^{\log a^n}$.

(vi) $a^{\frac{1}{n}} = 10^{\log a^{\frac{1}{n}}}$.

(vii) $a^{\frac{p}{q}} = 10^{\log a^{\frac{p}{q}}}$.

But taking equations (i) and (ii) and multiplying, we obtain

(viii) $ab = 10^{\log a} \times 10^{\log b} = 10^{\log a + \log b}$.

* "Common logarithms" understood, here and throughout.

Dividing, we obtain

$$\text{(ix)} \quad \frac{a}{b} = 10^{\log a} \div 10^{\log b} = 10^{\log a - \log b}.$$

Further, from equation (i), taking the n^{th} power, we obtain

$$\text{(x)} \quad a^n = (10^{\log a})^n = 10^{n \log a}.$$

Taking the n^{th} root, we obtain

$$\text{(xi)} \quad a^{\frac{1}{n}} = (10^{\log a})^{\frac{1}{n}} = 10^{\frac{1}{n} \log a}.$$

Taking the $\left(\frac{p}{q}\right)^{th}$ power, we obtain

$$\text{(xii)} \quad a^{\frac{p}{q}} = (10^{\log a})^{\frac{p}{q}} = 10^{\frac{p}{q} \log a}.$$

Comparing equations

(iii) and (viii), we see that $\log ab = \log a + \log b$;

(iv) and (ix), ,, ,, $\log\left(\frac{a}{b}\right) = \log a - \log b$;

(v) and (x), ,, ,, $\log(a^n) = n \log a$.

(vi) and (xi), ,, ,, $\log\left(a^{\frac{1}{n}}\right) = \frac{1}{n} \log a$.

(vii) and (xii), ,, ,, $\log\left(a^{\frac{p}{q}}\right) = \frac{p}{q} \log a$.

These five equations shew how the operations of multiplication, division, raising to a power, finding a root, or a combination of these, can be performed with a table of logarithms.

Integral Portion of a Logarithm.

3. Since $10^0 = 1$, therefore $\log 1 = 0$,

$10^1 = 10$, ,, $\log 10 = 1$,

and $\log 10^n = n$,

$$\log(a \times 10^n) = \log a + \log 10^n$$
$$= \log a + n.$$

Hence if **n** is a positive or negative integer, log **a** and log ($a \times 10^n$) differ by an integer.

We see, therefore, that a number between **1** and **10** has a logarithm between **0** and **1**; and that for each **10** that we multiply this number by, we must *add* unity to the original logarithm to obtain the logarithm of the product; and for each **10** that we divide by, we must *subtract* unity from the original logarithm to obtain the logarithm of the quotient.

We can, therefore, write down by inspection the integral part of the logarithm of a given number. Thus:

$$\log 2 = \cdot30103 + 0 = \cdot30103,$$
$$\log 20 = \cdot30103 + 1 = 1\cdot30103,$$
$$\log 200 = \cdot30103 + 2 = 2\cdot30103,$$
$$\log \cdot2 = \cdot30103 - 1 = \bar{1}\cdot30103,$$
$$\log \cdot02 = \cdot30103 - 2 = \bar{2}\cdot30103,$$

and so on.

In $\bar{1}\cdot30103$ the – sign is placed over the integer to shew that it only affects the integral part, it being found convenient to keep the decimal part invariably positive.

The integral part of the logarithm of a number is termed its *characteristic*, and we have the following rule for finding it.

The integral part of the logarithm of a number greater than unity is positive, and one less than the number of digits in the integral part of the number.

The integral part of the logarithm of a number less than unity (and expressed as a decimal) is negative, and one more than the number of zeros which precede the first significant figure of the decimal.

For example, the characteristic of the logarithm of

137	is	2
137·45	,,	2
·00137	,,	– 3, *i.e.* $\bar{3}$.

Construction of a Logarithmic Scale, using a Table of Logarithms.

4. Take a convenient length or base, say **10** inches, to represent log **10** or unity.

Measure off a length AB to represent log 2. This will be 3·01 inches from the beginning of the scale.

Mark the point B with the number 2.

Proceed to measure off distances, AC representing log 3, AD representing log 4, etc., from the beginning of the scale, and mark the points C, D, E, etc., thus obtained, 3, 4, 5, etc., respectively.

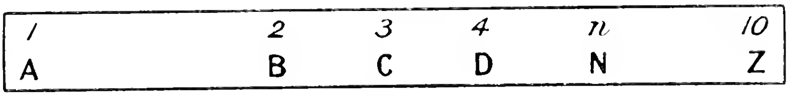


Fig. 2.

Thus, in Fig. 2, noting that the beginning of the scale is marked 1, because log 1 = 0, we have

AB	is	log 2	and	B	is marked	2,
AC	,,	log 3	,,	C	,,	3,
AD	,,	log 4	,,	D	,,	4,
AN	,,	log n	,,	N	,,	n,
AZ	,,	log 10	,,	Z	,,	10 or more often 1.

The distances log 2·1, log 2·2 and so on, might now be measured from A, and the corresponding points marked 2·1, 2·2 and so on. The scale might thus be decimally sub-divided and, if the size of the work permitted, the interval between the points 2·1 and 2·2 (for example) might be again sub-divided by means of the logarithms of 2·11, 2·12, etc., measured as before from A. We have thus constructed a logarithmic scale from 1 to 10. In like manner we could construct (as in Fig. 3) scales from 0·1 to 1, or from 10^{-n} to

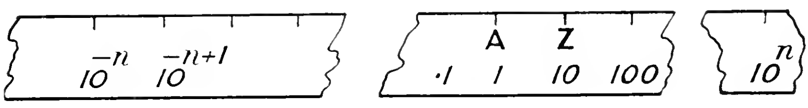


Fig. 3.

10^{-n+1}, and inasmuch as log a and log $(a \times 10^{n})$ differ only by an integer, the complete logarithmic scale would consist of an indefinite number of repetitions of the scale from 1 to 10.

EXAMPLE. Compare this scale with the scale which you made in the way shewn in Fig. 1.

Place the scales as shewn in Fig. 4, with the points marked **1** coinciding, and notice that the lines joining the points marked **2-2**, **3-3**, etc., are all parallel.

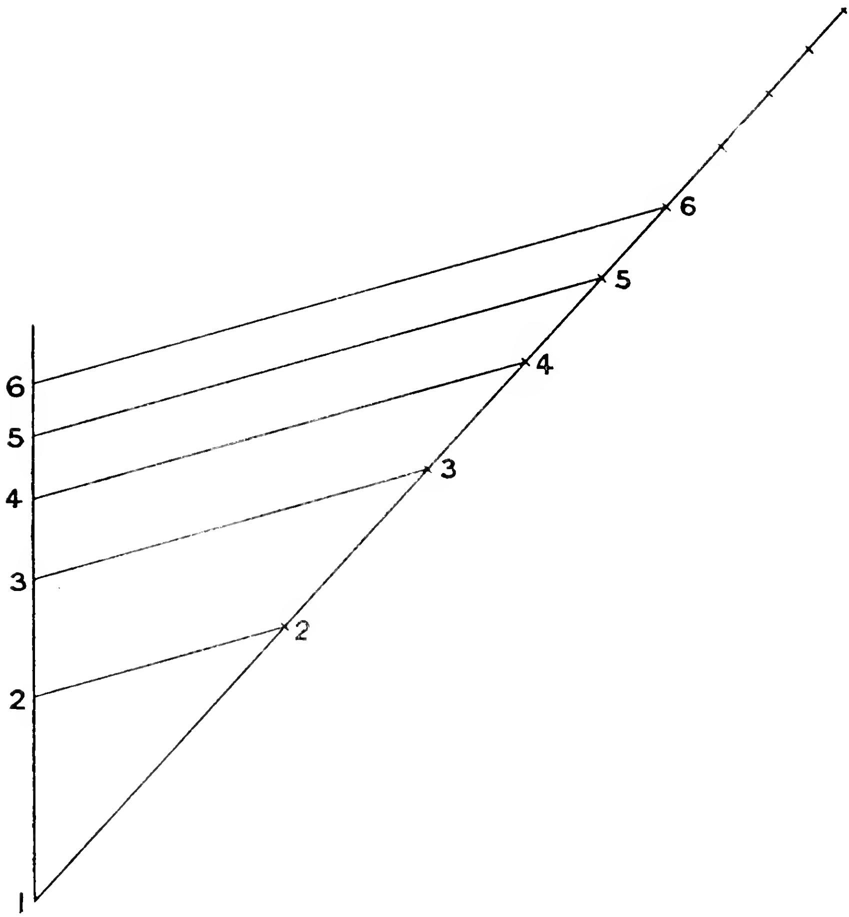


FIG. 4.

Notice that we could thus effect, graphically, a change of base, from the scale based on powers of **2** to the scale based on powers of **10**.

Historical Note.

The graphical addition of logarithms was used very soon after their discovery. If we have a scale of logarithms we can take off, say log 2, with a pair of dividers, and add it to any other logarithm, say log 3, and thus ascertain that

$$\log 2 + \log 3 = \log 6.$$

FIG. 5.

The great advance of using a pair of logarithmic scales, which could be slid relatively to one another, seems to have been made by W. Oughtred * about 1632.

* See Cajori's *History of the Slide-Rule*, a work which contains a complete bibliography of writings on the subject down to 1909.

CHAPTER II.

DIRECT PROPORTION.

5. A slide-rule consists of two or more logarithmic scales. In its simplest form it consists of two equal logarithmic scales, one of which slides on the other. One scale may be termed the "slide" and the other the "ruler."

The reader, who is supposed now to have a slide-rule in his hand, should satisfy himself that the "top" scales (see p. 19) on the ruler and slide are a pair of equal logarithmic scales; and it will save him from numerous mistakes if he spends a little time in making out the exact values of the graduations. On most of the 10-inch or 25-centimetre slide-rules which are sold, the following statements will be found to hold good with reference to the top scales.

Between 1 and 2 the graduations correspond to numbers which increase by ·02 at each step.

Between 2 and 5 the graduations correspond to numbers which increase by ·05 at each step. Between 5 and 10 the graduations correspond to numbers which increase by ·1 at each step.

For the time being the readings on the ruler and on the slide will be taken to represent numbers from 1 to 10. It is clear, however, that to any number whatever, as far as its significant figures are concerned, there corresponds a number between 1 and 10 containing the same significant figures.

Thus the product of 13700 and ·00797 contains the same significant figures as the product 1·37 by 7·97. Hence, as far as the **significant figures** of the result are concerned, any numbers whatever can be dealt with by the methods to be explained.

Multiplication.

6. Place the slide so that 1 on the slide is opposite 2 on the ruler. We see from Fig. 6 that

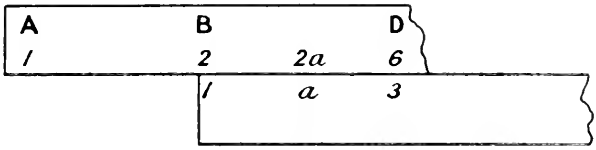


FIG. 6.

$$\begin{aligned} AD &= AB \text{ on the ruler} + BD \text{ on the slide} \\ &= \log 2 + \log 3 \text{ by the principle of the graduation} \\ &= \log 6, \end{aligned}$$

and the reading on the ruler at D over the 3 on the slide gives 6, the product of 2 by 3. More generally, if we place 1 on the slide opposite a on the ruler, we shall find ac on the ruler opposite c on the slide, unless the slide is so far out that c is beyond the end of the ruler, for in Fig. 7

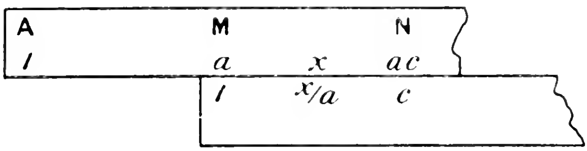


FIG. 7.

$$\begin{aligned} AN &= AM + MN \\ &= \log a + \log c \\ &= \log (ac). \end{aligned}$$

Note that this "setting" of the slide gives a table for multiplication by a, for opposite each number on the slide is a times that number on the ruler.

If the above fails to give a reading, bringing the other end of the slide opposite a on the ruler and supposing the whole length of the slide to represent log 10 or 1, we have, from Fig. 8,

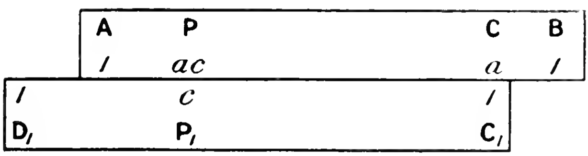


FIG. 8.

$$\begin{aligned}\mathsf{AP} &= \mathsf{AC} - \mathsf{CP}\\ &= \mathsf{AC} - \mathsf{C_1D_1} + \mathsf{D_1P_1}\\ &= \log \mathbf{a} - \log 10 + \log \mathbf{c}\\ &= \log \frac{\mathbf{ac}}{10},\end{aligned}$$

and so $\frac{\mathbf{ac}}{10}$ can be read off on the ruler opposite **c** on the slide, that is, the significant figures in **ac** can be read off.

Again, we might clearly have interchanged the functions of "ruler" and "slide," bringing **a** on the slide opposite **1** on the ruler and reading **ac** on the slide opposite **c** on the ruler.

The following equations sum the matter up:

If $$\mathbf{x} = \mathbf{ac},$$

then $$\frac{\mathbf{c}/10}{\cdot 1} = \frac{\mathbf{x}}{\mathbf{a}} = \frac{\mathbf{c}}{1}.$$

With some imagination, these equations are diagrams of the slide-rule setting; **c** is over **1**, and **x** is read over **a**.

Division.

7. If we place the mark **a** on the slide opposite **c** on the ruler, we have, from Fig. 9 (where **c** is taken greater than **a**),

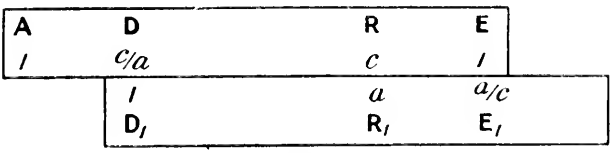


FIG. 9.

$$\begin{aligned}\mathsf{AD} &= \mathsf{AR} \text{ on the ruler} - \mathsf{RD} \text{ on the slide}\\ &= \log \mathbf{c} - \log \mathbf{a}\\ &= \log \frac{\mathbf{c}}{\mathbf{a}}.\end{aligned}$$

Thus we read $\frac{\mathbf{c}}{\mathbf{a}}$ on the ruler opposite **1** on the slide. If **c** is less than **a** the slide will be out to the left, and it is easily seen that we read the significant figures of the quotient $\frac{\mathbf{c}}{\mathbf{a}}$ on the ruler opposite the right-hand end of the slide (see

Figs. 10 and 11). To render it always possible, both in multiplication and division, to obtain a result whether the slide is out to the right or to the left, it is customary to put two segments of the infinite logarithmic scale upon the ruler, and also upon the slide.

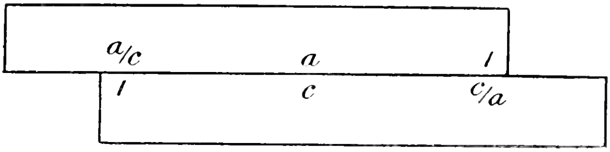


Fig. 10.

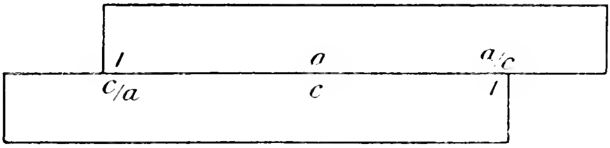


Fig. 11.

Returning to Fig. 9, where **a** on the slide is opposite **c** on the ruler, we have, if the whole length of the slide represents log **10**,

$$\begin{aligned} D_1E_1 &= AE - AD \\ &= AE - AR + D_1R_1 \\ &= \log 10 - \log c + \log a \\ &= \log \frac{10a}{c}, \end{aligned}$$

so we may read off $\frac{c}{a}$ on the ruler at D, and the significant figures of $\frac{a}{c}$ on the slide at E. In other words, the readings at D and E_1 in Fig. 9 may be regarded as reciprocals, so that if we require the reciprocal of a single number **n** we may pull out the slide until an end of the slide is opposite **n** on the ruler, and read $\frac{10}{n}$ (slide right) or $\frac{1}{n}$ (slide left) on the slide opposite the left or right end of the ruler, as the case may be, in either case reading the significant figures of $\frac{1}{n}$. Again, if we "invert" the slide, that is, turn it end for end, we see that if **1** on the slide is opposite **a** on the ruler, then opposite **x** on the slide is $\frac{a}{x}$ on the ruler, unless the slide is so far out that **x** on the slide is beyond the end of the ruler, for in Fig. 12

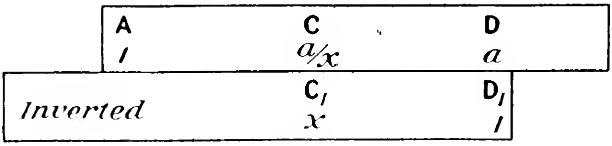


FIG. 12.

$$\begin{aligned} AC &= AD - C_1D_1 \\ &= \log a - \log x \\ &= \log \frac{a}{x}. \end{aligned}$$

It is clear that this mode of using the instrument enables us to read off the quotients of a given number **a** by a series of divisors. If this method fails with the slide to the right it will succeed with the slide to the left, as in the cases already explained.

The remark may be repeated once for all that the functions of "slide" and "ruler" may be interchanged.

EXAMPLE. In working a long division sum by ordinary arithmetic, the slide-rule may be used to give the quotient by one or two figures at a time. Any uncertainty as to the next figure in the quotient is obviated.

Thus, to divide 3849517 by 4831, the slide-rule gives 79 as the first two figures of the quotient.

```
        796·8
4831 ) 3849517
       33817
       -----
        467817
        43479
        -----
         33027
         28986
         -----
          40410
          38648
          -----
          ....
```

Having performed the corresponding steps in the division, 33027 remains for division. The slide-rule gives 68 as the next two figures of the quotient, and so on.

Proportion.

8. The operations of Multiplication and Division have been explained, but it is preferable to regard Proportion as the standard method of which multiplication and division are particular cases.

Let **b** on the slide be brought under **a** on the ruler, and suppose that in this position underneath any number **x** on the ruler we find **y** on the slide.

The distance between **x** and **a** represents $\log \mathbf{x} - \log \mathbf{a}$, say, equals $\mathbf{k}(\log \mathbf{x} - \log \mathbf{a})$. The distance between **y** and **b** represents $\log \mathbf{y} - \log \mathbf{b}$, say, equals $\mathbf{k}(\log \mathbf{y} - \log \mathbf{b})$.

But these distances are equal. Therefore (Fig. 13)

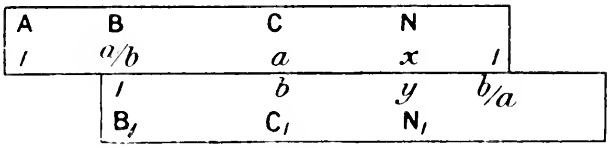


FIG. 13.

$$\log \mathbf{x} - \log \mathbf{a} = \log \mathbf{y} - \log \mathbf{b},$$

or

$$\frac{\mathbf{x}}{\mathbf{y}} = \frac{\mathbf{a}}{\mathbf{b}}.$$

Making **b** equal to unity, we have the case of multiplication

$$\mathbf{x} = \mathbf{ay},$$

and making **y** unity, we have the case of division

$$\mathbf{x} = \frac{\mathbf{a}}{\mathbf{b}}.$$

It is important to notice that proportion is worked with one setting of the instrument, and that if we have to deal with such a case as

$$\frac{x}{\cdot 0037} = \frac{913}{23000},$$

we find the significant figures of the answer by working with the significant figures of the data.

The position of the decimal point in the answer is most easily found by common sense or rough approximate calculation.

The habit of seeing about what the answer will be (whether **170** or **1700** yards, one ton, ten tons, or one tenth of a ton) should be cultivated.

We have obtained the equation $\frac{x}{y}=\frac{a}{b}$ by expressing the obvious fact that the distance between any two points on the ruler is equal to the distance between the corresponding points on the slide. This method is typical of all slide-rule calculations.

The form of the equation suggests a very important **mnemonic**, which is particularly useful when a series of results are required from one setting of the instrument.

The equation is

$$\frac{x}{y} = \frac{a}{b},$$

where we will suppose that **a** and **b** are constants, **x** a variable of which a series of values are known; and **y** a variable of which a series of values are sought.

The point to be particularly noticed is that the two variables are brought under one another.

With a little aid from the imagination, the equation may be supposed to be a picture of the actual readings on the slide-rule, and, therefore, if the horizontal lines indicating the fractions are regarded as continuous, then two of the four quantities involved are of course above the dotted line and two below it, and the following statement holds good:

> The quantities above the dotted line being on the ruler, the quantities below the dotted line will be on the slide.

Thus, the setting for $x=ab$ is indicated by the equation $\frac{x}{a}=\frac{b}{1}$, and the setting for $x=\frac{c}{d}$ is indicated by the equation

$$\frac{x}{1}=\frac{c}{d}.$$

An example will render the method clear.

A map is drawn to a scale of $\frac{1}{12500}$. Set the slide-rule to read yards for any given number of inches measured on the map.

In this case **12500** inches on the ground is represented by **1** inch on the map. Hence, if **y** yards, *i.e.* **36y** inches on the ground is represented by **x** inches on the map, we have

$$\frac{36y}{x} = \frac{12500}{1} \quad \text{or} \quad \frac{y}{x} = \frac{12500}{36},$$

which indicates the method of solution, namely, "Bring 36 on the slide under 125 on the ruler; and over **x** on the slide read **y** on the ruler."

Some results are tabulated below:

x	y
2·9 inches,	1010 yards.
4·9 ,,	1700 ,,
8·7 ,,	3020 ,,

EXAMPLES.

1. Given 1 kilogram = 2·204 pounds, set the instrument to read pounds corresponding to kilograms, and grains corresponding to milligrams.

2. A Russian verst is ·663 of a mile. Set the slide-rule to read the number of miles in any given number of versts.

3. What rate of interest is yielded by a four per cent. stock at £77 per £100 stock?

4. In a certain examination there are 1300 candidates, of whom 645 offer Latin, 121 Greek, 1018 French, and 70 German. What are the percentage proportions offering each language?

5. Set the rule to give velocities in feet per second, being given velocities in miles per hour.

6. Obtain the percentage composition of MnO_2 and of $KClO_3$.

7. Given 1 metre = 1·0936 yards,
shew that 1 yard = ·915 metre.

Set the rule to read

(*a*) miles corresponding to kilometres,
(*b*) inches corresponding to centimetres.

Verify the rough approximations

5 miles = 8 kilometres,
4 inches = 10 centimetres,

8. A map is drawn to a scale of $\frac{1}{25000}$. Set the slide-rule so as to give miles on the ground corresponding to centimetres on the map.

9. Set the slide-rule to read off circumferences of circles from their diameters.

10. Set the rule to read weight of various lengths of iron rod weighing 13·4 kilos per 100 metres.

11. Set the rule to read price in pence per yard or per pound corresponding to prices in francs per metre or per kilogram.

CHAPTER III.

SQUARES AND SQUARE ROOTS.

9. The instrument specially referred to in this chapter is made by J. Davis & Co., Derby, and consists of three parts: the Ruler, the Slide, and the Cursor. It is about **10** inches long. As it is of the ordinary type, most of what follows will apply to any ordinary slide-rule.

The equations to the various scales on the instrument will be found in Chapter X.

The following definitions are collected here for purposes of reference :

The **Back** of the **Ruler** is that surface upon which there are no scales.

The **Front** of the **Ruler** is parallel to the back.

The **Front** of the **Slide** is that surface of the slide on which there are two scales.

The **Back** of the **Slide** is that surface of the slide on which there are three scales.

We shall use the terms **Left** and **Right** on the supposition that the instrument is held in such a manner that the scales on the ruler read from left to right.

On the front of the ruler, and also on the front of the slide, will be found a scale on which is engraved the following numerals :

1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1.

We shall speak of these scales as the **Top Scales** of the **Slide** and **Ruler.**

On the front of the ruler, and also on the front of the slide, will be found a scale on which is engraved the following large numerals :

1 2 3 4 5 6 7 8 9 1.

We shall speak of these scales as the **Bottom Scales** of the **Slide** and **Ruler.**

On the back of the slide there are three scales. One is usually marked with an S, and will be termed the **S Scale.** Another is usually marked with a T, and will be termed the **T Scale.** The third lies between the S and T scales, and will be termed the **Central Scale.** The description and particulars of these scales will be found in Chapters IV. and V.

In whatever position the slide may be, all these scales will preserve their names.

Particular attention is invited to the preceding remark, especially as regards the top and bottom scales of the slide.

When the slide is in the ruler, if the top and bottom scales of the slide are in full view and read from left to right, we shall consider the **slide** as in its **normal position** and not draw special attention to the matter.

Starting with the slide in its normal position, if we take it out and replace it in the ruler, so that

(1) The top and bottom scales are still in view, but turned end for end, so that the numerals on them are upside down as compared with those on the ruler, the slide is said to be **Inverted** ;

(2) The S and T scales are in view, with the numerals on them the same way up as those on the ruler, the slide is said to be **Upset** ;

(3) The S and T scales are in view, but turned end for end, so that the numerals on them are upside down as compared with those on the ruler, the slide is said to be **Upset and Inverted.**

A **Scale** is said to be **read backwards** when it is read in the direction opposite to that in which the numerals on it run.

The top scale of the ruler and the top scale of the slide may each be considered to contain the logarithmic scale from **1** to **10** duplicated, or to contain any two segments (not always necessarily consecutive) of the infinite logarithmic scale of Fig. 1*a*. The bottom scale of the ruler and the bottom scale of the slide may each be considered to contain the logarithmic scale from **1** to **10**, or to contain any one segment of the infinite logarithmic scale, the distance taken to represent log **100** or **2** on the top scales representing log **10** or **1** on the bottom scales.

The operations of multiplication, division, and finding proportionals can be performed by means either of the top or of the bottom scales and in a variety of ways, and for these purposes the top and bottom scales may be considered as independent.

It may amuse the reader to find the product of, say, **21** by **31** in as many ways as he can.

It is possible to read this product in **31** ways with **20** different settings, though these are not all equally convenient or accurate.

The bottom scales are to be preferred to the top scales if absolute, not relative, accuracy is required in a series of readings or calculations.

That is to say, we can read off a single answer more accurately from the bottom scales, but if a series of results are to be read from the bottom scales we shall very probably find ourselves obliged to reset the slide, and thereby to introduce an error which will affect some of the readings and not others, and will therefore diminish the *relative* accuracy of the series.

Squares and Square Roots.

10. It is clear from what has been said above that if a point P is marked to *read* $\mathbf{x}$ on the top scale (see Fig. 14), then the distance AP represents $\log \mathbf{x}$, and the same *distance* on the bottom scale *represents* $\frac{1}{2} \log \mathbf{x}$ or $\log \sqrt{\mathbf{x}}$; the point P_2 on the bottom scale is therefore marked to read $\sqrt{\mathbf{x}}$.

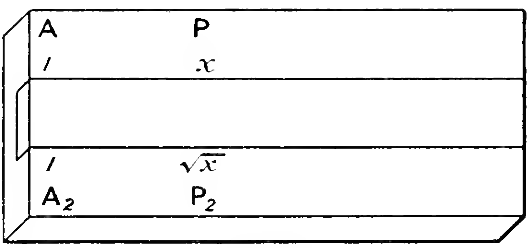


Fig. 14.

Hence we have below each number on the top scale its square root on the bottom scale; and above each number on the bottom scale its square on the top scale.

This statement is equally true of the two scales on the ruler and of the two scales on the slide. Roots and squares are usually read by means of the *Cursor*, or hair line, which slides along the ruler and is always perpendicular to its edge. The cursor is generally made use of when the scales on which corresponding readings are to be taken are not in contact.

In finding square roots care is requisite not to confuse such numbers as 20 and 2, 17 and 1·7, or ·7 and ·07.

If we look on the bottom scale as running from 1 to 10 it follows, from the mode of construction of the instrument, that the top scale must be regarded as running from 1 to 100. Thus the first or left-hand half of the top scale reads from 1 to 10, and the second or right-hand half reads from 10 to 100. Consequently 17 is on the right half of the top scale of the ruler, and 1·7 is on the left, and the square roots of these quantities, namely 4·123 and 1·304, are found at the corresponding points on the bottom scale of the ruler. If the square root of a number not lying between 1 and 100 is required, we may express it in terms of a number between 1 and 100.

Thus, if we require $\sqrt{\cdot 0017}$, we write it $\frac{1}{100}\sqrt{17}$, and if we require $\sqrt{1700000}$ we write it $1000\sqrt{1 \cdot 7}$.

Or, we may proceed thus:

We have looked upon the top scale of the ruler as representing numbers from 1 to 100, and the bottom scale of the ruler as representing numbers from 1 to 10.

But, inasmuch as multiplication of numbers by any integral power of 10 does not alter the decimal portions of their logarithms, we see that (n being an integer*)

* In other words, if a logarithmic scale were constructed of indefinite extent it would consist, as already stated, of repetitions of the scale from 1 to 10, the successive portions, from left to right, running

$$\ldots 10^{-n} \text{ to } 10^{-n+1}; \quad \ldots \cdot 010 \text{ to } \cdot 1; \quad \cdot 1 \text{ to } 1; \quad 1 \text{ to } 10; \quad \ldots 10^{n} \text{ to } 10^{n+1} \ldots.$$

Only one or two portions of this indefinitely long scale are constructed. (See Fig. 3.)

We see incidentally that the corresponding scales of the ruler and of the slide need not represent the same portion of the infinite logarithmic scale, but that, in every case, if the corresponding readings are:

$$\text{Ruler } 10^{n} \ldots\ldots\ldots x \ldots\ldots\ldots y \ldots\ldots\ldots 10^{n+1},$$
$$\text{Slide } 10^{m} \ldots\ldots\ldots x_1 \ldots\ldots\ldots y_1 \ldots\ldots\ldots 10^{m+1},$$

we have
$$\log y - \log x = \log y_1 - \log x_1,$$
$$\log x - \log 10^{n} = \log x_1 - \log 10^{m}.$$

By these considerations it is clearly possible (but not desirable in practice) to place the decimal point, thus giving precise values to the numbers used, at every step in the work.

if the top scale is looked on as representing numbers from	then the bottom scale must be looked on as representing numbers from
1 to **100**	**1** to **10**
100 to **10,000**	**10** to **100**
10000 to **1,000,000**	**100** to **1000**
·01 to **1**	**·1** to **1**
·0001 to **·01**	**·01** to **·1**
10^{2n} to 10^{2n+2}	10^{n} to 10^{n+1}

and the following procedure will hold good.

Find by inspection, as already explained, the *characteristic* of the logarithm of the number whose square root is required.

(i) If the characteristic is **even**, take the number in the **left half** of the top scale; if it is **odd**, take the number in the **right half**.

(ii) If the characteristic is even, its half is the characteristic of the logarithm of the square root. If the characteristic is odd, its half, increased numerically by $\frac{1}{2}$, is the characteristic of the logarithm of the square root.

Thus, for $\sqrt{\cdot 0017}$, characteristic is $\bar{3}$. Therefore the characteristic for the square root is $-\frac{3}{2}-\frac{1}{2}$ or $\bar{2}$, and the square root has *one* zero before the significant figures. So that the required square root is **·04124**.

In practice mental arithmetic will show by inspection the first figure of the root.

Another method of finding square roots is to "*invert*" the slide so that the top scale of the slide is in contact with the bottom scale of the ruler. Now, bringing the end of the inverted slide opposite the given number on the top scale of the ruler, find (using the cursor) where the readings on this scale and on the top scale of the slide are the same. Either reading is the required square root. Care is required, as in the other method, to distinguish between $\sqrt{n}$ and $\sqrt{10n}$. The same method might be carried out using the bottom scales.

Ex. 1. Set the rule to read (*a*) area of circles when diameter is given, (*b*) radius of circle when area is given.

Ex. 2. The time of oscillation of a simple pendulum is given by the formula

$$t = \pi\sqrt{\frac{l}{g}},$$

where l = length of the pendulum in feet,

g = acceleration due to gravity = 32·2 f./s^2.

Find the times of oscillation of pendulums 2, 20, and 50 feet long.

Ex. 3. Find the length of the side of a square whose area is 370 sq. yards.

Ex. 4. Plot on squared paper the curves whose equations are

$$y = x^2 \quad \text{and} \quad y = x^{\frac{1}{2}},$$

taking 10 inches to represent unity in each case, and taking x between 0 and 1.

Ex. 5. The size of a map is 15″ × 12″.

What will be the sides of a new map of the same extent of country, but of $\frac{3}{5}$ths of the area of the old map?

Ex. 6. Obtain a series of fractions which are approximately equal to $\sqrt{2}$.

[$\frac{17}{12}$, $\frac{24}{17}$, $\frac{41}{29}$ are some answers.]

A similar problem occasionally arises in deciding on the number of teeth in a pair of cog wheels which are required to approximate to a given velocity ratio, *e.g.* in the case of an orrery.

Combined Use of Slide and Cursor.

11. If the value of y is required from the equation

$$y = \frac{abcd}{pqr},$$

we may proceed, using the top scale, as in the following example:

Given that 1 cubic foot of water contains 6·25 gallons, that the cross section of a pipe of uniform bore is 1·7 square inches, and that the pipe delivers 35 gallons of water per minute; find the velocity of flow in feet per second.

We have at once, if this velocity is denoted by **v**,

$$\frac{1 \cdot 7}{144} \times \mathbf{v} \times 60 = \text{cubic feet delivered per minute} = \frac{35}{6 \cdot 25},$$

whence
$$\mathbf{v} = \frac{35 \times 144 \times 1 \times 1}{1 \cdot 7 \times 60 \times 6 \cdot 25}.$$

The steps in the work are shewn in Figs. 15, 16, and 17.

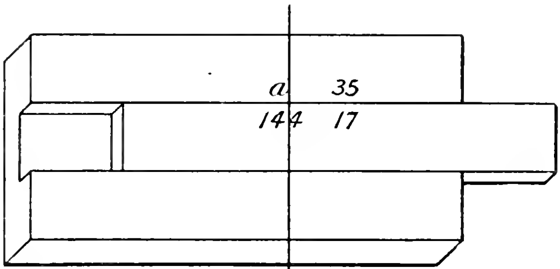


Fig. 15.

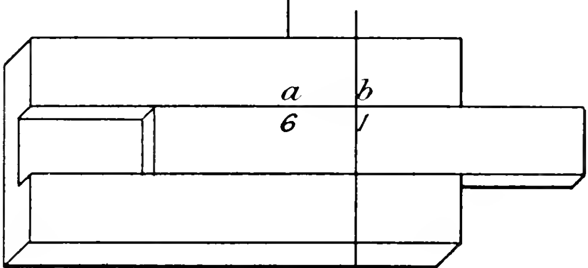


Fig. 16.

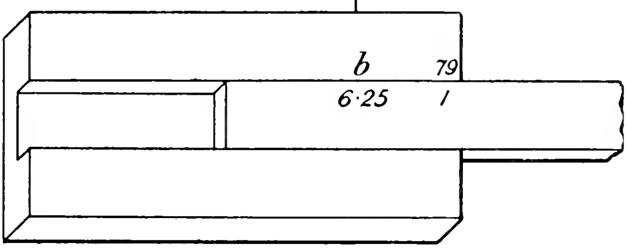


Fig. 17.

(1) Underneath **35** on the top scale of the ruler set **17** on the top scale of the slide (Fig. 15).

(2) Without moving the slide, set the cursor to **144** on the top scale of the slide.

We *could* now read the value of $\frac{35 \times 144}{17}$ on the top scale of the ruler, but as we do not require this value we do not trouble to read it, but proceed to divide it by **6**. Thus,

(3) Without moving the cursor set the slide (Fig. 16) until **6** on its top scale is underneath the cursor.

(4) Without moving the slide, set the cursor to **1** on the top scale of the slide.

(5) Without moving the cursor, set the slide (Fig. 17) until **625** on its top scale is underneath the cursor.

(6) Read the result **79** on the top scale of the ruler opposite **1** on the top scale of the slide.

(7) Place the decimal point by rough cancelling or inspection.

Thus the answer is **7·9** feet per second.

An equation of this form is of very frequent occurrence, and the following points are worthy of note:

(1) The factors are taken in the order **a, p, b, q, c, r, d,** ... .

(2) It will be found convenient to make the number of factors in the numerator exceed the number in the denominator.

Thus, to find **y**, where

$$\mathbf{y} = \frac{\mathbf{ab}}{\mathbf{pqrs}},$$

we should write the equation

$$\mathbf{y} = \frac{\mathbf{a} \times \mathbf{b} \times 1 \times 1 \times 1}{\mathbf{p} \times \mathbf{q} \times \mathbf{r} \times \mathbf{s}}.$$

(3) The position of the decimal point should be left till the end. In ninety-nine cases out of a hundred it is seen by inspection. When that fails the roughest arithmetical cancelling will fix it.

(4) The cursor should be used near the middle of the ruler, and the slide should not be pulled much more than half way out.

(5) The first factor of the numerator, and the final answer, are read on the ruler. All the other factors are read on the slide. The slide is moved when setting factors belonging to the denominator. The cursor is moved when setting factors belonging to the numerator.

In many cases, of which the following example is a type, the top and bottom scales are both employed.

If W = weight of a projectile in pounds,
$\mathbf{d}$ = diameter of its base in inches,
$\mathbf{v}$ = its initial velocity in feet per second,
l = length of bore in feet,

then $\mathbf{p}$ the pressure (average) on the base of the projectile as it passes down the bore is given in tons per square inch by the equation

$$\mathbf{p} = \frac{W\mathbf{v}^2}{2g \times 2240 \times \frac{\pi \mathbf{d}^2}{4} \times l}, \qquad \mathbf{p} = \frac{W\mathbf{v}^2 \times 1 \times 1 \times 1 \times 1}{16 \times 2240 \times \pi \mathbf{d}^2 \times l}.$$

The steps are:

(1) Underneath W on the top scale of the ruler set **16** on the top scale of the slide.

(2) Without moving the slide, set the cursor to $\mathbf{v}$ on the bottom scale of the slide. Under the cursor we could read $\frac{W\mathbf{v}^2}{16}$ on the top scale of the ruler.

(3) Divide by **2240**, multiply by **1**, divide by π, multiply by **1** as before, in the manner already explained.

Taking the factor $\mathbf{d}^2$ next,

(4) Without moving the cursor, set the slide until $\mathbf{d}$ on its bottom scale is underneath the cursor.

(5) Without moving the slide, set the cursor to l on the top scale of the slide.

The process is really more clear when represented symbolically. Thus,

(1) and (2) $\frac{W}{16} = \frac{\mathbf{x}}{\mathbf{v}^2}$. We let the slide give the value of $\mathbf{v}^2$, taking $\mathbf{v}$ on the bottom scale.

(2) $\frac{\mathbf{x}}{2240} = \frac{\mathbf{y}}{1}$.

(3) $\frac{\mathbf{y}}{\pi} = \frac{\mathbf{z}}{1}$.

(4) $\frac{\mathbf{z}}{\mathbf{d}^2} = \frac{\mathbf{u}}{1}$.

(5) $\frac{\mathbf{u}}{l} = \frac{\mathbf{p}}{1}$.

We do not actually read **x**, **y**, **z**, and **u**, but employ the cursor to record their values as long as they are needed.

In the problem just worked out, we found squares on the top scales by setting **v** and **d** on the bottom scales. If the converse case arises, and we have to deal with $\sqrt{x}$ by working the problem on the bottom scales and setting **x** on a top scale, care is necessary to select the right square root and not $\sqrt{10x}$. It may be useful to deal with the square root factor last, as thought may be saved by a knowledge of the approximate answer.

General advice.—The reader is advised:

(1) To use the instrument constantly.

(2) To remember the very important mnemonic given in Sec. 8 (p. 16).

(3) If in doubt to work an easy case, *e.g.* 2×3; square of 2; square root of 4 or ·04; cube of 2; cube root of 8 or of ·125; sine or tangent of 30°.

(4) Never to run the slide further out than can be helped. When there is a choice, to take the readings close together. To leave the decimal point alone as long as possible.

(5) Never to allow any portion of the cursor to be off the rule. Thus, if $\sqrt{94}$ is required, use the end of the slide as a cursor.

EXAMPLES.

1. Find the weight of a cubic foot of copper, steel, and cast iron, given their specific gravities to be 8·8, 8, and 7·6 respectively.

2. Find the weight of 100 yards of No. 5 B.W.G. copper wire, sectional area ·0353 of a square inch. Find the diameter of the above wire.

3. Set the instrument so as to read off the weights of 100 yards of copper wire of diameter $\frac{1}{16}$, $\frac{1}{8}$, $\frac{3}{16}$ in. respectively.

4. The energy of a projectile is

$$E = \frac{wv^2}{2g \times 2240} \text{ foot-tons,}$$

where w = weight in pounds,
v = velocity in feet per second,
g = acceleration due to gravity = 32·2 f./s.s.

Find E in the following cases:

v	w	Diameter in inches.
2050	215 grains	·303
1900	1250 pounds	13·5
2100	1250 ,,	13·5
1800	100 ,,	6
1500	45 ,,	4·7.

5. Captain Orde Browne's rough rule for penetration **x** of wrought-iron armour is

$$x = \cdot001\, vd \text{ inches},$$

where v = striking velocity in f.s.,
d = diameter of shot in inches.

Compute **x** for the cases given in Q. 4.

6. The mean pressure **p** in tons per square inch on the base of a projectile during its passage down the bore of the gun is

$$p = \frac{wv^2}{2g \times 2240 \times \frac{\pi d^2}{4} \times l} \text{ tons} = \frac{4E}{\pi d^2 l},$$

where l = length of bore in feet

$$= \frac{nd}{12} \text{ suppose.}$$

Compute **p** for the last two cases of Question 4, taking **n** to be 40.

7. The greatest bending moment on a beam resting on supports at its extremities and uniformly loaded is

$$M = \frac{Wl}{8} \text{ pounds inches},$$

where W = total load in pounds,
l = length between support in inches.

Calculate M when

l =	10	20	24 feet
W =	970	1320	3700.

8. The moment of resistance in a beam of rectangular section is

$$M_r = \frac{1}{6}Sbd^2 \text{ pounds inches,}$$

where b = breadth of beam in inches,
d = depth " "
S = the greatest intensity of tension in the fibres of the beam in pounds per square inch.

Calculate M_r when

b = 5, 10·5, or 13 inches,
d = 5, 10·5, or 13 inches,
S = 3000 lb. per square inch.

9. If the greatest admissible value of S in a timber beam is 3000, find W, the greatest admissible uniformly distributed load on a beam of length l supported at the ends, by equating the bending moment to the moment of resistance in Q. 8.

10. By means of Q. 7 and 8 verify the approximate rule:

Safe dead load at centre of a rectangular timber beam resting on supports at its ends is $\frac{bd^2}{l}$ hundredweights, l being the length in feet, b and d the breadth and depth in inches.

The maximum bending moment due to a load at the centre is double that due to the same load uniformly distributed.

11. The rough rule for the breaking load on a wire rope is:

Breaking strain in tons,

S = square of circumference in inches.

Find S for ropes of $\frac{1}{4}''$, $\frac{1}{2}''$, $\frac{5}{8}''$, and $1\frac{1}{4}''$ diameters.

12. If a beam of rectangular section is supported at its ends and loaded with a weight of W pounds, the deflection D at the centre is given by

$$D = \frac{1}{4}\frac{Wl^3}{Ebd^3} \text{ (load W at centre)}$$

$$= \frac{5}{32}\frac{Wl^3}{Ebd^3} \text{ (load W uniformly distributed),}$$

l, b, d being length, breadth, and depth in inches, and E the modulus of elasticity.

Find D for an oak beam 20 feet long, taking

$$E = 1700000, \qquad W = 1500 \text{ pounds},$$
$$b = 8,\ 12,\ 15, \qquad d = 10,\ 12{\cdot}5,\ 17.$$

13. From the formula $P = \frac{WRN^2}{g} \times \frac{4\pi^2}{3600}$ find the centrifugal pull P lb. of a weight W rotating, at a distance R feet from an axle, at N revolutions per minute.

Take $W = 50$ lb., $R = \frac{1}{6}$ foot, $N = 137$.

14. The tractive force exerted by a locomotive engine being assumed to be

$$P = \frac{d^2pl}{D} \text{ pounds},$$

where d = diameter of cylinders in inches,

l = stroke ,, ,,

p = pressure of steam in pounds per square inch,

D = diameter of driving wheel in inches.

Compute P when $d = 10\frac{1}{2}$, $l = 18$, $p = 140$, $D = 36$.

15. The indicated horse-power (I.H.P.) of an engine being given by

$$\text{I.H.P.} = \frac{2\frac{\pi d^2}{4} \times p \times l \times n}{33000 \times 12},$$

where d = diameter of the cylinder in inches,

l = stroke ,, ,,

p = mean pressure in pounds per square inch,

n = number of revolutions per minute.

Compute I.H.P. where

	(1)	(2)	(3)
d =	12	15	17
l =	16	18	24
p =	100	150	150
n =	70	90	40

16. The piston speed **v** of a locomotive in feet per second is given by

$$\mathbf{v} = \frac{2l\mathrm{V} \times 1760 \times 3}{3600\pi \mathrm{D}} = \cdot 934 \frac{\mathrm{V}l}{\mathrm{D}},$$

where D = diameter of driving wheels in feet or inches,

l = stroke of cylinders in feet or inches,

V = train speed in miles per hour.

Take D = 6 ft. 2 in.,

l = 22 inches,

V = 55.

17. When the train runs round a curve of radius R feet with a velocity of **v** feet per second, then, in order that there may be little lateral pressure on the rails, the outer rail must be elevated about **x** feet above the inner, where

$$\mathbf{x} = \frac{\mathbf{v}^2}{32 \cdot 19\mathrm{R}} \times l,$$

l being the gauge in feet = 4 ft. $8\frac{1}{2}$ in. on British narrow-gauge railways.

Notice that **x** varies at different speeds, so that in practice the rail is set for an intermediate speed.

Find **x** in inches when

V = 30 miles an hour,

R = (*a*) 600 yards, (*b*) 900 yards, (*c*) 937 yards,

l = (1) 4 ft. $8\frac{1}{2}$ in., (2) 1 metre.

18. Compute the buoyancy of a cask by Major Collins' formula, viz.:

Buoyancy in pounds = $5\mathbf{c}^2 l - \mathbf{w}$,

where **c** = mean circumference in feet,

l = length measured along a stave in feet,

w = weight of cask in pounds.

Take **c** = 7 ft. 3 in.,

l = 4 ft. 9 in.,

w = 70 pounds.

19. An electrical current liberates x cubic centimetres of mixed gases (H and O) per second by decomposition of water.

The current C in amperes is given by

$$C = \frac{x}{\cdot 1733},$$

the barometer being at 760 mm. and temperature 0° Cent. If the barometer stands at h mm., and the temperature is t° Cent., the formula becomes

$$C = \frac{x}{\cdot 1733 \times \frac{760}{h} \times \frac{(273+t)}{273}}.$$

Take temperature 7°,
pressure 783 mm.,
gas liberated = 37 c.c. in 4 minutes.

20. A troopship costs £400 a day for provisions, wages, etc., and at 8 knots the daily consumption of coal is 50 tons, costing 10s. per ton, and the daily coal consumption varies as the cube of the speed.

Draw a graph shewing the relation between the cost of a voyage of given length and the speed for speeds ranging from 8 to 16 knots.

CHAPTER IV.

CUBES AND CUBE ROOTS. INVERSION OF THE SLIDE. QUADRATIC AND CUBIC EQUATIONS.

Cubes and Cube Roots.

12. To cube and extract cube root we must use the slide as well as the ruler.

When "1" on the bottom scale of the slide is opposite **x** on the bottom scale of the ruler, then, as already shown, $\mathbf{x}^2$ on the top scale of the ruler is opposite "1" on the top scale of the slide, and by the principle of multiplication (Fig. 7, p. 11), opposite **x** on the top scale of the slide we find $\mathbf{x}^3$, say **n**, on the top scale of the ruler [and $\mathbf{x}^{\frac{3}{2}}$ at the corresponding point of the bottom scale of the ruler, provided care is taken to read the correct square root on the latter scale as already explained].

If the reader will try to cube **5** in this manner he will find that, if the slide is out to the right, the **5** on the top scale

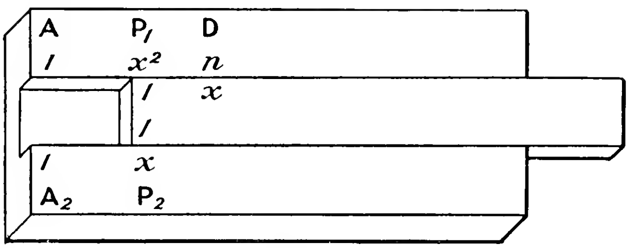


Fig. 18.

of the slide is beyond the right end of the ruler, and a reading fails. But pulling the slide to the left until the right-hand **1** on the bottom scale of the slide is opposite **5** on the bottom scale of the ruler, we read above the **5** on the top scale of the slide **125** on the top scale of the ruler.

The operation of finding cube root *may* be performed by trial by reversing the above process. Thus, if the slide is moved in and out until the reading, say, **x** on the top scale of the slide opposite, say, **n** on the top scale of the ruler is the same as the reading on the bottom scale of the ruler opposite the end of the slide, we have in Fig. 18

Reading at $P_2 = \mathbf{x}$,

and therefore Reading at $P_1 = \mathbf{x}^2$;

and consequently, the distance PQ being the same whether measured along the ruler or along the slide, we have

$$\log \mathbf{n} - \log \mathbf{x}^2 = \log \mathbf{x} - \log 1,$$

whence $$3 \log \mathbf{x} = \log \mathbf{n},$$

or $$\mathbf{x} = \mathbf{n}^{\frac{1}{3}}.$$

Again, in Fig. 19, we have, equating the two expressions for the distance PD,

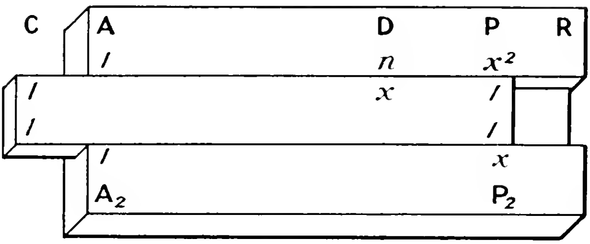


FIG. 19.

$$\log \mathbf{x}^2 - \log \mathbf{n} = \log 100 - \log \mathbf{x},$$

or $$3 \log \mathbf{x} = \log 100\mathbf{n},$$

so that $$\mathbf{x} = (100\mathbf{n})^{\frac{1}{3}},$$

assuming in both cases that the top scale runs from 1 to 100 and the bottom scale from 1 to 10.

As this adjustment of the slide by trial is rather difficult, better practical methods will be given subsequently (see pp. 39, 40).

Inversion of the Slide.

13. If the slide is "*inverted*" and run home, we see in Fig. 20 that

$$AP = \log \mathbf{x},$$

$$B_1P_1 = \log \mathbf{z},$$

and $\log x + \log z = \log 10^n$, since $AP + B_1P_1$ makes up the whole length of the ruler.

Inverting the slide is convenient when several reciprocals are wanted, but when the reciprocal of a single number only is required, it may be less trouble to proceed as in Section 7, p. 13.

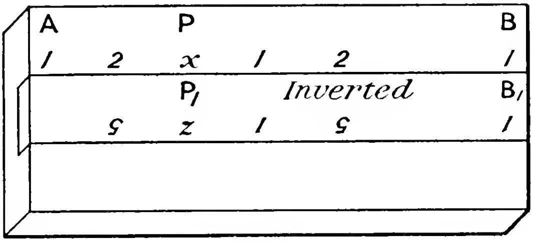


Fig. 20.

If we require the value of z from the equation $z = \frac{ac}{x}$ for a series of values of x, we may proceed as follows:

Inverting the slide, set a on, say, the top scale of the slide, immediately below c on the top scale of the ruler. We can now read * z or $\frac{ac}{x}$ on the top scale of the ruler above x on the inverted top scale of the slide, for in Fig. 21 we have, equating the two expressions for the distance MP,

$$\log z - \log c = \log a - \log x,$$

whence $$z = \frac{ac}{x},$$

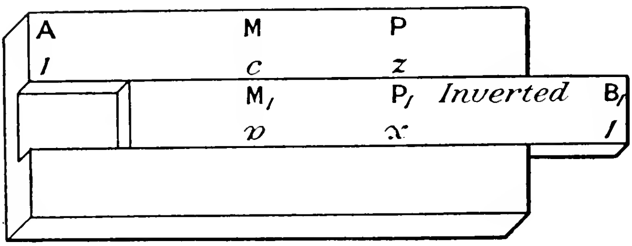


Fig. 21.

so that the value of z is read on the top scale of the ruler at P. We might have interchanged c and a without making any

* For particular values of ac or x a setting may be found for which x on the slide is beyond the end of the ruler. But such a setting need never occur if we remember that we only require the significant figures of the answer from the significant figures of the data.

difference, and we might, of course, have employed the bottom scales.

We have only to suppose either **x** or **z** to equal unity to see how multiplication can be performed with the slide inverted.

We thus obtain Fig. 22, the verification of which may be left as an exercise for the student.

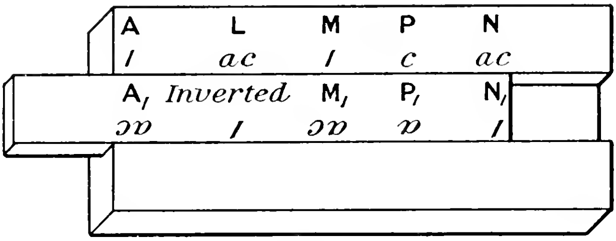


Fig. 22.

Alternative Discussion of Inversion.—If we have four numbers **p**, **q**, **r**, **s** connected by the equation

$$\frac{p}{q}=\frac{r}{s},$$

the equation gives also a diagram of the slide-rule setting, namely, "Bring **q** under **p** and read **r** over **s**."

If **p** and **q** are constants and **r** and **s** variables, the setting is both correct and convenient, for we can read off the **series** of values of **r** and **s** without resetting the slide.

But if **p** and **s** are constants and **q** and **r** variables, the above setting, though still, of course, correct, is no longer convenient; for each fresh value of **q** will have to be set under **p** before **r** can be read off.

To avoid this, we take out the slide and replace it, so that the figures appear upside down relatively to the rule, obtaining

$$\frac{p \qquad\qquad r}{s \qquad\qquad q}.$$

The same relation still connects the four readings, for the distance **p** to **r** is still equal to the distance **q** to **s**; but the constant **s** is now under the constant **p**, and the set of values of **r** corresponding to given values of **q** can be read off without moving the slide.

If we write the original equation in the form

$$\frac{p}{1/s} = \frac{r}{1/q},$$

we may again consider this equation as a diagram of the slide-rule setting.

Ex. 1. A gun is situated 121 feet above sea level. Set the slide-rule to read angles of depression **m** in minutes for a series of long ranges in thousands of yards.

Here, if y = angle of depression in minutes and the range $= x$ yards $= 3x$ feet,

$$\frac{121}{3x} = \sin y \text{ or } \tan y = \text{circular measure of } y \text{ nearly}$$

$$= \frac{\pi y}{180 \times 60} = \frac{y}{3440}.$$

We write the equation

$$\frac{121}{1/1146} = \frac{1146}{1/121} = \frac{x}{1/m}$$

$$= \frac{m}{1/x}.$$

To guard against mistakes as to the decimal points, note that when $m = 121$, $x = 1146$.

A few results are tabulated below:

x		y
7,000 yards	- - -	59·3 minutes.
8,000 ,,	- - -	52 ,,
9,000 ,,	- - -	46·1 ,,
10,000 ,,	- - -	41·5 ,,

Ex. 2. Plot the hyperbola whose equation is $xy = 12$.

Ex. 3. A merchant has the monopoly of the sale of articles, each of which costs him three shillings in all. Assuming that the number sold varies inversely as the square of the selling price, plot graphs shewing the relation between selling price, number sold, and total profit if there is an annual sale of 20,000 when the price is four shillings.

[For the justification of the assumption, see Jevons, *Theory of Political Economy*, p. 158.]

Cube Root by Inversion of the Slide.

14. Invert the slide and pull it out until the "1" at the centre of the inverted top scale of the slide is below the number on the top scale of the ruler whose cube root is required.

Then, using the cursor, find the points at which the reading on the bottom scale of the ruler is the same as that immediately above it on the inverted top scale of the slide. We might alternatively find where the reading on the top scale of the ruler is the same as that immediately below it on the inverted bottom scale of the slide.

It will be found that for every setting of the instrument two such coincidences of reading can be found by either alternative.

Care must be taken to select the proper reading, that is, to find the cube root of **n**, the number sought, and not of **10n** or of **100n**.

Figs. 23 and 24 illustrate this.

We will suppose that the final results are to be read on the lower scale of the ruler.

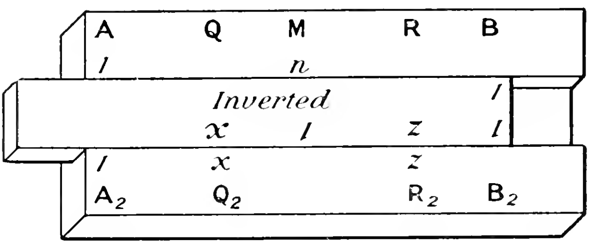


FIG. 23.

Supposing the top scale of the ruler to read from **1** to **100**, and **n** to be a number between **1** and **10**, then in Fig. 23, at the coincidence Q, which, going from left to right, is reached first, we have

$$\text{Reading at } Q_2 = \mathbf{x},$$

and therefore

$$\text{Reading at } Q \ = \mathbf{x}^2\,;$$

and consequently, equating the two expressions for the distance QM, we have

$$\log \mathbf{n} - \log \mathbf{x}^2 = \log \mathbf{10x} - \log \mathbf{10},$$

whence

$$\mathbf{x} = \mathbf{n}^{\frac{1}{3}},$$

while, if the second coincidence occurs at R, we have, in like manner,

$$\log z^2 - \log n = \log 10 - \log z,$$

so that

$$z = (10n)^{\frac{1}{3}}.$$

That is, if we set the central "1" on the inverted top scale of the slide to a number on the first half of the top scale of the ruler, we shall find two points Q_2, R_2 on the bottom scale of the ruler at which the readings coincide with those immediately above them on the inverted top scale of the slide.

The coincidence to the left gives the cube root of **n**, and the coincidence to the right gives the cube root of **10n**.

Now, let $m = 10n$ be a number between 10 and 100. Then in Fig. 24 we have, at the coincidence R_2 on the left, a reading **z**, and

$$\log m - \log z^2 = \log 10z - \log 10,$$

whence

$$z = (m)^{\frac{1}{3}} = (10n)^{\frac{1}{3}};$$

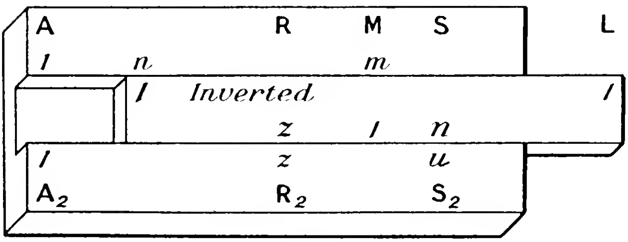


FIG. 24.

while at S, where the second coincidence occurs, we have a reading **u**, and

$$\log u^2 - \log m = \log 10 - \log u,$$

so that

$$u = (10m)^{\frac{1}{3}} = (100n)^{\frac{1}{3}}.$$

Thus, if we follow the plan described of

(*a*) reading the final results on the bottom scale of the ruler,

(*b*) setting the central "1" on the inverted top scale of the slide to the number on the top scale of the ruler whose cube root is required, the following statements hold good:

I. A number, the characteristic for which is a multiple of 3 (such as 3, 0, or $\bar{3}$), may be taken on the left-hand half of the top scale of the ruler, and its cube root will be given by the coincidence on the left, namely at Q_2 in Fig. 23.

II. A number, the characteristic of which is one more than a multiple of 3 (such as 1 or $\bar{2}$), may be taken on either half of the top scale of the ruler, and its cube root will be found at the right-hand coincidence R_2 in Fig. 23, or the left-hand coincidence R_2 in Fig. 24.

III. A number, the characteristic for which is *two* more than a multiple of 3 (such as 2 or $\bar{1}$), may be taken on the right-hand half of the top scale of the ruler, and its cube root will be found at S_2, the right-hand coincidence in Fig. 24.

It will be observed that if the cube root of **n** is read on the bottom scale of the ruler, the reading above it on the top scale of the ruler gives the value of $n^{\frac{2}{3}}$.

The reader will find on trial that the cube root of any number *can* be found from either half of the top scale of the ruler, but that adherence to the above directions will avoid resettings due to the slide having been pulled too far out or in the wrong direction. In practice the first figure of the required cube root can always be found by inspection, and it is therefore at once seen whereabouts on the bottom scale of the ruler to look for the answer, and consequently which way to run the slide.

Ex. 1. Plot on squared paper the curves

$$y = x^3,$$

$$y = x^{\frac{1}{3}},$$

(*a*) for values of **x** between 0 and 1,

(*b*) for values of **x** ranging from 1 to 10.

Ex. 2. If a shell of diameter 3 inches weighs 14 pounds, what will be the weight of a geometrically similar shell of the same material 3·3 inches in diameter?

Ex. 3. The powder charge of a common mine in earth is given by the formula

$$c = \frac{l^3}{10} \text{ pounds},$$

where l = line of least resistance in feet.

Find **c** when $l = 12, 15, 18$ feet.

Ex. 4. A spherical leaden bullet which will just enter the bore of a "20" bore gun weighs $\frac{1}{20}$th of a pound. Taking the S.G. of lead as 11·4, find the diameter of a 4, 12, 16, 20, and 28 bore gun.

Solution of Quadratic Equations* (with Real Roots).

15. The principle upon which the solution depends will be readily understood by considering a simple example.

Suppose we have to solve the equation

$$x^2 + 4x = 7,$$

or, as we may write it, $x(x+4) = 7$.

We see at once (Fig. 25), using the bottom scales, that if 1 on the slide is over the correct value of x on the ruler, then the

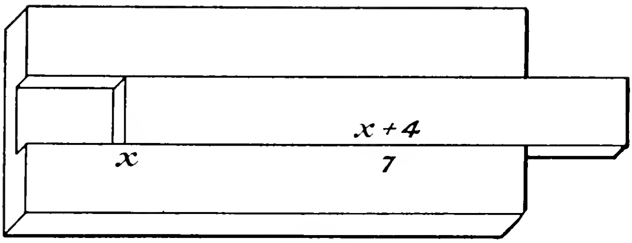


Fig. 25.

value of $x+4$ will be found on the slide over the 7 on the ruler; consequently the problem is reduced to that of getting the slide into this position. This is effected by trial by observing that the values of x and of $x+4$ differ by 4. Therefore, we must move the slide until the readings on the ruler below the end of the slide and on the slide over 7 on the ruler differ by 4. We thus readily find 1·315 as one value of x.

Algebraic Theory of the Method.

16 Taking $x^2 + px + q = 0$ as the standard quadratic equation, the ordinary process of solution gives for the roots

$$\text{(i)}\quad x = \frac{-p + \sqrt{p^2 - 4q}}{2} \quad\text{or}\quad \frac{-p - \sqrt{p^2 - 4q}}{2};$$

* A considerable number of slide-rule solutions which have been suggested for the quadratic and cubic equations are referred to in the *Encyclopädie der Mathematischen Wissenschaften*, vol. i. p. 1057.

calling these α and β, we have at once, on adding,

$$\alpha + \beta = -\mathbf{p},$$

and on multiplying together,

$$\alpha\beta = \mathbf{q},$$

or, taking $[\alpha]$ $[\beta]$ $[\mathbf{p}]$ $[\mathbf{q}]$ as the positive numerical values of α, β, $\mathbf{p}$, $\mathbf{q}$, we have

$$[\alpha]\,[\beta] = [\mathbf{q}].$$

$$\text{(i)}\quad [\alpha] + [\beta] = [\mathbf{p}] \text{ if } \mathbf{q} \text{ is positive,}$$

$$\text{(ii)}\quad [\alpha] \sim [\beta] = [\mathbf{p}] \text{ if } \mathbf{q} \text{ is negative.}$$

Now, using the bottom scales, bring the slide by trial into such a position that the reading on the ruler under the end of the slide and the reading on the slide over **q** on the ruler,

amount to **p** in case (i),

differ by **p** in case (ii).

These readings give the two roots. Their signs must be obtained by noticing that in case (i) both roots have the same sign, the opposite to that of **p**, and in case (ii) the roots have opposite signs, the numerically smaller root having the same sign as **p**.

From the fact that $\alpha\beta = \mathbf{q}$ we see that the second root of the equation $\mathbf{x}^2 + 4\mathbf{x} = 7$ is the value of the other factor on the slide rule, namely 5·315, and that its sign is negative.

An alternative process, handy when familiarity with the use of the inverted slide has been obtained, is to invert the slide, and, using the bottom scales, bring the end of the slide opposite **q** on the ruler, and, using the cursor, find where the readings under the cursor on ruler and slide,

amount to **p** in case (i),

differ by **p** in case (ii).

In both methods, care is requisite to solve the equation given, and not, for instance, the equation

$$\mathbf{x}^2 + \mathbf{px} + 10\mathbf{q} = 0.$$

It will be found that using the top scales sometimes renders a mistake of this kind less likely. (See Fig. 26.)

The slide may be out either to the right or to the left, and the position of the decimal point is most easily obtained by a knowledge of the theory of quadratics, which enables the approximate value of the roots to be seen by inspection.

The value of **q** may always be supposed to lie between **1** and **100**, for the roots of the quadratic in question may be multiplied or divided by **10** until this is effected.

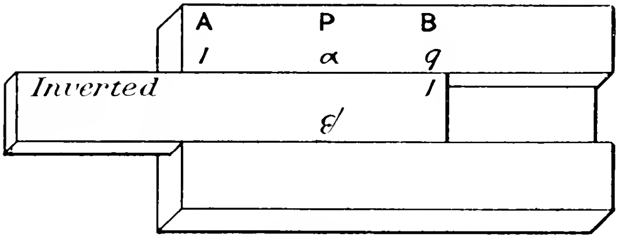


FIG. 26.

Thus, the roots of

$$x^2 - 19x + 3100 - 0$$

are respectively ten times the roots of

$$x^2 - 1{\cdot}9x + 31 = 0.$$

As an example, Fig. 27 shows the solution of the equation

$$x^2 - 8x + 13 = 0,$$

using the bottom scales. The roots are **5·73** and **2·27**.

It is generally quite easy to see the approximate value of a root. Thus $x^2 - 7x - 24$ is negative when $x = 9$, and positive when $x = 10$. Therefore the equation $x^2 - 7x - 24 = 0$ has a root between **9** and **10**. As an alternative, the formula (i) might be used to give a rough value of a root.

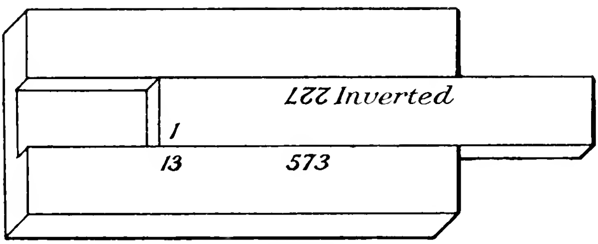


FIG. 27.

EXAMPLES.

$$x^2 - 5x + 6 = 0, \quad [2 \text{ and } 3]$$

$$x^2 - 11x + 30 = 0, \quad [5 \text{ and } 6]$$

$$x^2 - 155x + 750 = 0,$$

$$x^2 + 65x + 750 = 0,$$

$$x^2 - 65x + 300 = 0,$$
$$x^2 - 35x + 300 = 0,$$
$$x^2 - 5x + 2 = 0,$$
$$x^2 - 5x - 2 = 0,$$
$$x^2 + 500x + 90 = 0,$$
$$x^2 - 50x + 2 = 0,$$
$$x^2 - 50x + 20 = 0,$$
$$x^2 - 5x - 200 = 0.$$

Solution of a Cubic Equation.

17. The principle upon which the solution depends is analogous to that adopted in the case of the quadratic.

Thus, taking as an example the equation

$$x^3 + 2x - 5 = 0,$$

which we may write $x(x^2 + 2) = 5$,

we see that if the slide is in such a position that its end is over the correct value of **x** on the bottom scale of the ruler, then over **5** on the bottom scale of the ruler must be the value of the other factor $x^2 + 2$ on the bottom scale of the slide.

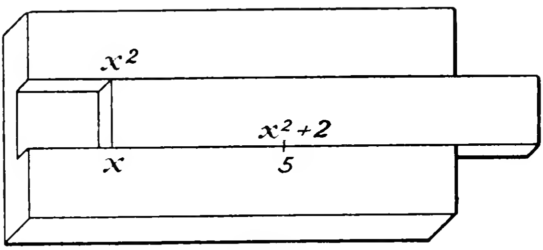


FIG. 28

The reading on the top scale of the ruler over **x** on the bottom scale is, of course, x^2. Hence we effect the solution by getting the slide, by trial, into a position such that the reading on the bottom scale of the slide over **5** on the bottom scale of the ruler is **2** more than the reading on the top scale of the ruler over the end of the slide. The root proves to be **1·345**.

EXAMPLE. Give an analogous method of dealing with the equation

$$x^3 + x^2 = 12, 13, \text{ or } 14.$$

Algebraic Theory of the Method.

18. Taking $x^3 - 3bx^2 + 3cx + d = 0$

as the general form of a cubic, if we replace x by $z - b$ it is easily seen that the coefficient of z^2 disappears, and, as x is of course known as soon as z is known, we may take $z^3 + pz + q = 0$ as a form to which any cubic equation may be reduced. This equation may have one real root or three.

Let α be the single real root, or the numerically largest of the three real roots, as the case may be. Then if β and γ are the other roots, $z - \alpha$, $z - \beta$, and $z - \gamma$ are factors of $z^3 + pz + q$; and multiplying these factors together and comparing the result with $z^3 + qz + q$, we see that

$$\alpha + \beta + \gamma = 0, \quad \text{.........................(1)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = p, \quad \text{....................(2)}$$

$$\alpha\beta\gamma = -q. \quad \text{.............................(3)}$$

Hence, putting $\beta\gamma = \lambda$, we have

$$\alpha\lambda = -q,$$

$$\lambda - \alpha^2 = p.$$

We notice that λ is essentially a positive quantity; for, if β and γ are real, they have like signs, and if β and γ are imaginary, their product is positive.

We notice also the sign of α is the opposite to that of q, and that α^2 is greater or less than λ, according as p is negative or positive. To solve the equation, ascertain first a rough approximation to the value of α.

It is generally quite easy to find the consecutive integers between which the numerically greatest root lies; for, as already stated, the sign of α is opposite to that of q.

Thus $x^3 - 17x + 12$ is negative when $x = -5$, and positive when $x = -4$; consequently it changes sign, by passing through the value zero, for a value of x between -4 and -5; or, in other words, the equation

$$x^3 - 17x + 12 = 0$$

has a root between -4 and -5, and this is the numerically greatest root; for, since the sum of the roots is zero, and the numerically greatest root certainly negative, there can be no other negative root.

To solve the equation, adjust the slide by trial until the readings on the top scale of the ruler over the end of the slide, and on the bottom scale of the slide over **q** on the bottom scale of the ruler differ by **p**.

An alternative method, which avoids the necessity of continually shifting the slide during the process of adjustment, is to invert the slide and bring its end opposite **q** on the bottom scale of the ruler (Figs. 29 and 30). Find where the readings

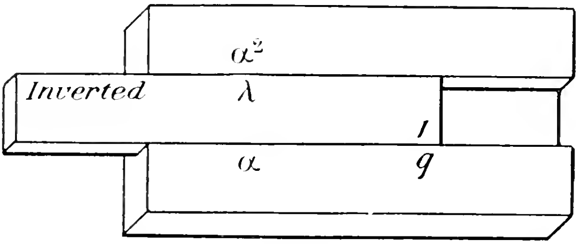


Fig. 29.

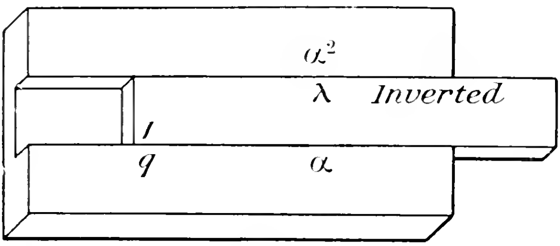


Fig. 30.

on the top scale of the ruler and on the bottom scale of the slide *differ* by **p**, and then by means of the cursor read α at the corresponding point of the bottom scale of the ruler, using the approximate value as a check. The other two roots, if real, may be found without altering the setting by noticing that $\alpha\beta$ is essentially negative, and thus we have from equation (2) above, putting $\alpha\beta = -\mu$,

$$\gamma^2 + \mu = -\mathbf{p}.$$

Hence, if we find where the *sum* of the readings on the top scale of the ruler and on the bottom scale of the slide *amounts* to **p**, we read by means of the cursor β and γ at the corresponding points on the bottom scale of the ruler.

The solution of the equation

$$x^3 - 7x + 7 = 0$$

by this method is exhibited in Fig. 31.

When the reading on the top scale of the ruler is 9·3, the reading on the bottom scale of the slide is 2·3.

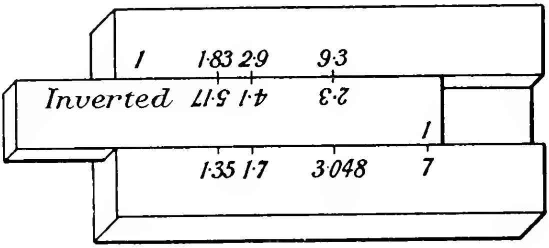


Fig. 31.

These differ by 7; consequently the numerical value of the largest root is 3·048, and, since **q** is positive, the root is negative, namely, − 3·048.

Where the *sum* of the readings amounts to 7, we read 1·7 and 1·35, or more accurately, 1·692 and 1·356, as the corresponding roots.

EXAMPLES.

1. Solve the equations:

$x^3 - 19x - 30 = 0$, [Roots 5 − 2 − 3]

$x^3 - 28x - 48 = 0$,

$x^3 - 2x - 5 = 0$, [Root 2·094]

$x^3 - 91x - 330 = 0$, [Roots 11 − 5 − 6]

$x^3 - 2x - 500 = 0$,

$x^3 - 20x - 5 = 0$,

$x^3 - 200x - 5 = 0$,

$x^3 - 20x - 50 = 0$,

$x^3 - 7x + 2 = 0$,

$x^3 + 7x + 2 = 0$,

$x^3 + 7x + 20 = 0$,

$x^3 - 3x^2 + 7 = 0$.

2. A loan of £100 is to be repaid by three equal instalments of £36, at intervals of 6, 12, and 18 months. Find the true rate of compound interest charged.

3. The volume of a sphere is $\frac{4}{3}\pi \mathbf{r}^3$, and the volume of a spherical cap of thickness **h**, cut from a sphere of radius **r**, is

$$\frac{\pi \mathbf{h}^2}{3}(3\mathbf{r} - \mathbf{h}).$$

Find the thickness of a cap whose volume is $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$ of that of the sphere.

4. A body of specific gravity **s** which floats in water has **s** of its volume below the water line. What is the depth of immersion of a sphere of specific gravity ·3, ·5, ·7, and radius 2 feet?

CHAPTER V.

THE TRIGONOMETRICAL SCALES.

19. When a quantity **y** is given in terms of a quantity **x** by an equation, **y** is said, in mathematical language, to be a function of **x**.

Thus $$y = \sin x, \qquad y = \cos x,$$ $$y = \tan x, \qquad y = \cot x,$$

as well as $$y = x^n,$$ $$y = e^x,$$ $$y = \log x,$$

are all instances in which **y** is a function of **x**.

Let us use the symbol **f(x)** to denote any of these functions of **x**.

We may construct a logarithmic scale for the function **f(x)** in the following manner.

Let $f(a) = 1$. Then, taking a starting point and marking it **a**, because $$\log f(a) = \log 1 = 0,$$

set off a distance $\log f(x)$, and mark the extremity **x**.

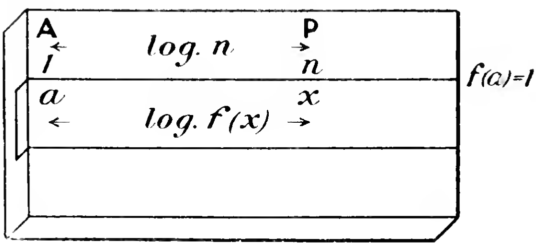


FIG. 32.

If this scale is used in conjunction with a common logarithmic scale, as in Fig. 32, we see at once, equating the expressions for the distance AP, that $\log \mathbf{f}(\mathbf{x}) = \log \mathbf{n}$, and therefore that $\mathbf{f}(\mathbf{x}) = \mathbf{n}$ and $\mathbf{f}(\mathbf{a}) = 1$. That is, we can read off the numerical value of $\mathbf{f}(\mathbf{x})$ for any given value of $\mathbf{x}$ within the limits of the scale.

Again, in Fig. 33, shewing the same scales in *any* relative position, we have, equating the two expressions for the distance PQ on either scale,

$$\log \mathbf{f}(\mathbf{y}) - \log \mathbf{f}(\mathbf{x}) = \log \mathbf{n} - \log \mathbf{m},$$

whence
$$\frac{\mathbf{m}}{\mathbf{f}(\mathbf{x})} = \frac{\mathbf{n}}{\mathbf{f}(\mathbf{y})}.$$

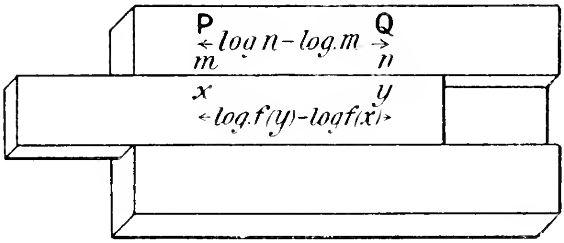


Fig. 33.

We note once more that, by a little stretch of the imagination, the equation in this form may be thought of as shewing how the two scales are set in order to solve it.

The trigonometrical functions of $\mathbf{x}$ dealt with by means of the ordinary slide-rule are $\sin \mathbf{x}$ (and hence $\cos \mathbf{x}$, $\operatorname{cosec} \mathbf{x}$, and $\sec \mathbf{x}$), $\tan \mathbf{x}$ (and hence $\cot \mathbf{x}$), and these we proceed to consider.

The Sine Scale.

20. On the back of the slide are three scales. Dealing first with that which is marked S, and which we shall term the S scale; if the slide is "**upset**" and run home, then immediately below each graduation on the top scale of the ruler, regarded as running from ·01 to 1 (see § 10), is marked on the S scale the angle which has that number for its **sine**, so that in Fig. 34

$$\begin{aligned} AP &= \log \mathbf{x} - \log(\cdot 01) \\ &= \log \sin \alpha - \log \sin 34', \end{aligned}$$

whence
$$\mathbf{x} = \sin \alpha.$$

The graduations on the S scale run from 34′ to 90° (sin 34′ = ·01). The gradations by which the graduations proceed vary slightly in different types of instrument. The student must familiarise himself with the instrument he intends to use (see p. 10).

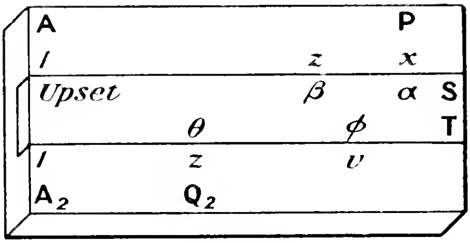


FIG. 34.

The reader will notice that if the numbers on the top scale of the ruler are supposed to run from 10^{2n} to 10^{2n+2}, we no longer have $\log x = \log \sin \alpha$, but we do have (if x and z on the ruler are opposite α and β on the slide)

$$\log x - \log z = \log \sin \alpha - \log \sin \beta,$$

whence
$$\frac{x}{\sin \alpha} = \frac{z}{\sin \beta},$$

and as two particular cases (n and m being integers)

$$\log 10^{2n+2} - \log x = \log \sin 90^\circ - \log \sin \alpha,$$
$$\log x - \log 10^{2m} = \log \sin \alpha - \log \sin 34',$$

whence $\frac{x}{10^{2n+2}} = \sin \alpha$ from either equation.

Small Angles.

21. We find from trigonometrical tables

$$\tan 6^\circ = \cdot 10510$$
$$\sin 6^\circ = \cdot 10453$$
$$\text{difference} = \cdot 00067$$

or about one part in 1500.

Hence the *tangents* of angles less than 6° can be read off on the S scale as well as the sines, for the sine and tangent of any angle less than 6° are equal as far as the instrument will shew.

The reader will note that beyond 70° the graduations on the S scale are very close together. For a reason explained subsequently (§ 38), this is of little importance for many calculations. In any special case, we may often avoid the use of this part of the scale by a knowledge of elementary trigonometry; thus, if we require sin 86°, we may say (see Fig. 35)

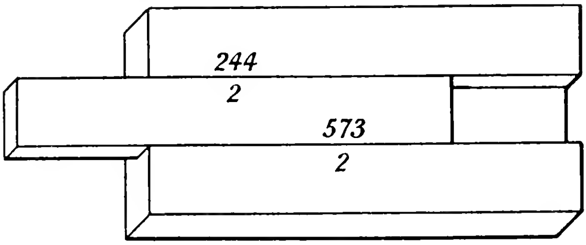


FIG. 35.

$$\sin 86^\circ = \cos 4^\circ = 1 - 2\sin^2 2^\circ = 1 - 2\left(\frac{2\pi}{180}\right)^2$$

$$= 1 - 2 \times \left(\frac{2}{57 \cdot 3}\right)^2 \text{ nearly} = 1 - \cdot 00244$$

$$= \cdot 99756, \text{ which is correct to five places.}$$

The Tangent * Scale.

$$f(x) = \tan x.$$

22. The scale marked T (termed the T scale) is constructed in an analogous way; that is, the slide being upset and run home, above each graduation on the bottom scale of the ruler, regarded as running from $\frac{1}{10}$ to 1 (§ 10), is marked on the T scale the angle which has that number for its **tangent**, so that in Fig. 34

$$A_2Q_2 = \log z - \log(\cdot 1)$$
$$= \log \tan \theta - \log \tan(5^\circ\ 43'),$$

whence $z = \tan \theta.$

The graduations on the T scale run from 5° 43′ to 45°.

If the numbers on the bottom scale of the ruler are regarded as running from 10^n to 10^{n+1}, we have, as in § 10 note, p. 22,

* In some makes of slide-rule, the T scale is made for use in conjunction with the top scales. It then runs from 45° (tan 45° = 1) to about 35′ (tan 35′ = ·01). In these instruments the settings for the T scale will correspond exactly to the settings for the S scale.

if $\mathbf{v}$ and $\mathbf{z}$ on the ruler are opposite ϕ and θ on the T scale of the slide,

$$\log \mathbf{v} - \log \mathbf{z} = \log \tan \phi - \log \tan \theta,$$

whence

$$\frac{\tan \phi}{\mathbf{v}} = \frac{\tan \theta}{\mathbf{z}},$$

and as particular cases ($\mathbf{n}$ being an integer)

$$\log 10^{n+1} - \log \mathbf{z} = \log \tan 45^\circ - \log \tan \theta,$$

$$\log \mathbf{z} - \log 10^{n} = \log \tan \theta - \log \tan 5^\circ\ 43',$$

and $\dfrac{\mathbf{z}}{10^{n+1}} = \tan \theta$ from either equation.

Ex. 1. Plot the curve of sines, taking **1** inch horizontal to represent **50°** and **10** inches vertical to represent unity.

Ex. 2. Plot graphs representing the curves

$$\mathbf{y} = \sin \mathbf{x}, \quad \mathbf{y} = \mathbf{x}, \quad \mathbf{y} = \tan \mathbf{x},$$

$\mathbf{x}$ denoting the circular measure of an angle. Take **1** inch horizontal to represent **10°** and **10** inches vertical as unity. Note that the graph of $\mathbf{y} = \mathbf{x}$, using these scales, will *not* be inclined at **45°** to the horizontal.

Ex. It may be shewn by the aid of the calculus that

(i) $$\frac{\pi}{4} = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x \ldots \quad (0 < x < \pi).$$

(ii) $$\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x = \frac{\pi}{2} \quad \left(0 < x < \frac{\pi}{2}\right)$$
$$= 0 \quad \left(\frac{\pi}{2} < x < \pi\right).$$

(iii) $$\sin x - \frac{1}{9} \sin 3x + \frac{1}{25} \sin 5x \ldots = x \quad \left(0 < x < \frac{\pi}{2}\right)$$
$$= \pi - x \quad \left(\frac{\pi}{2} < x < \pi\right).$$

Plot the first four terms of each of these series, shewing how the compound curve obtained in each case indicates an approach towards the limiting form.

HINT. Begin by plotting the first term. Obtain the result of the first and second terms by plotting the ordinates of the second on to the first, and so on. (Byerley, *Fourier Series.*)

The Indices at the Back of the Ruler.

23. Instead of "upsetting" the slide, we may sometimes proceed as follows:

At the back of the ruler there is a small mark exactly corresponding to the right-hand end (B in Fig. 36) of the scales of the ruler. Consequently, if the slide is drawn out

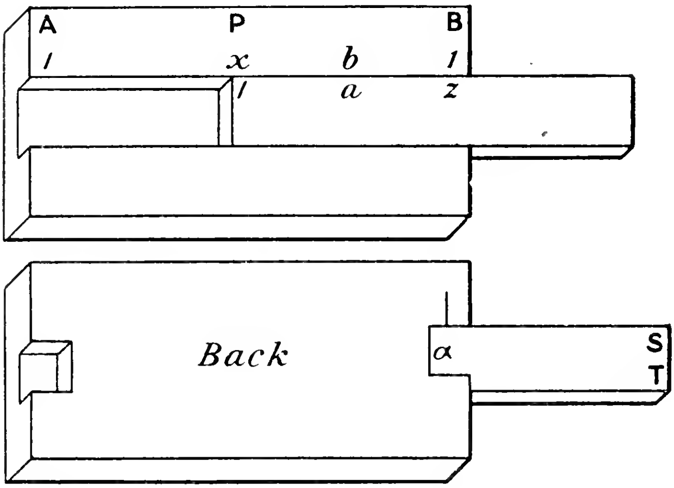


FIG. 36.

to the right until α on the S scale is opposite this mark, then in Fig. 36 the slide projects as much on the right as it is drawn in on the left. Hence,

$$AP = \log \sin 90^\circ - \log \sin \alpha = \log \operatorname{cosec} \alpha, \quad \text{and } AP = \log \mathbf{x},$$

so that the graduation on the top scale of the ruler opposite "1" on the top scale of the slide reads **x** where $\mathbf{x} = \operatorname{cosec} \alpha$ (the graduations on the top scale of the ruler being supposed to run from 1 to 100). Further, the point on the top scale of the slide opposite the extreme right-hand "1" on the ruler reads z where $z = \sin \alpha$ (the graduations on the top scale of the slide being supposed to run from ·01 to 1).

Tangents may be read in a similar manner, using the small mark at the back of the left-hand end of the ruler (Fig. 37). If the slide is drawn out on the left until α on the T scale is opposite this mark, we read **x** where $\mathbf{x} = \tan \alpha$ on the bottom scale of the slide opposite "**1**" on the bottom scale of the ruler (the graduations of the bottom scale of the slide running from **·1** to **1**), and we read **z** where $\mathbf{z} = \cot \alpha$ on the bottom scale of the ruler opposite "**1**" on the bottom scale of the slide (the graduations on the bottom scale of the ruler running from **1** to **10**).

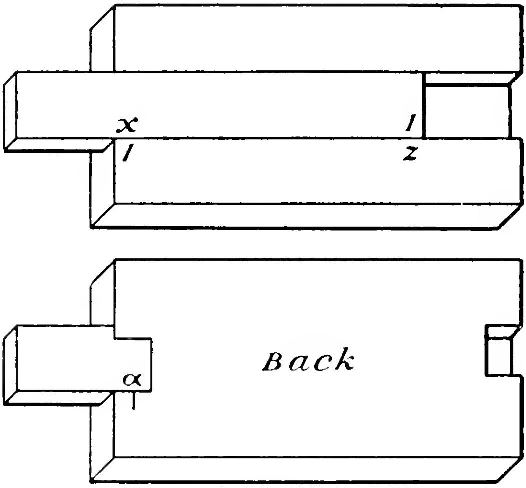


FIG. 37.

If the slide is inverted we can use the T scale in conjunction with the right-hand mark at the back of the ruler. Pulling out the inverted slide to the right until α on the T scale is opposite the right-hand mark at the back of the ruler, we read tan **a** on the bottom scale of the ruler opposite the end of the slide and $\tan^2\alpha$ (or on adding unity to the reading $\tan^2\alpha + 1$, which is $\sec^2\alpha$) on the top scale of the ruler above the same end of the slide.

Some Important Applications.

24. To find θ from the equation

$$\sin \theta = \frac{\mathbf{a}}{\mathbf{c}}.$$

First Method (see Figs. 38 and 39).—"Upset" the slide and pull it out until an end of the S scale is opposite **c** on the top scale of the ruler. Then, the slide being out to the *left*, we have in Fig. 38

$$\log\sin 90^\circ - \log\sin\theta = \log \mathsf{c} - \log \mathsf{a},$$

or, in mnemonic form,

$$\frac{\mathsf{a}}{\sin\theta} = \frac{\mathsf{c}}{\sin 90^\circ},$$

whence opposite a on the top scale of the ruler is θ° on the S scale.

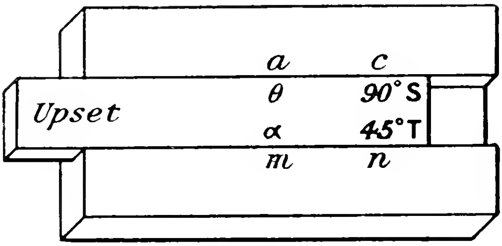


FIG. 38.

If the slide is out to the right we have in Fig. 39 no reading c on the top scale of the ruler opposite the right-

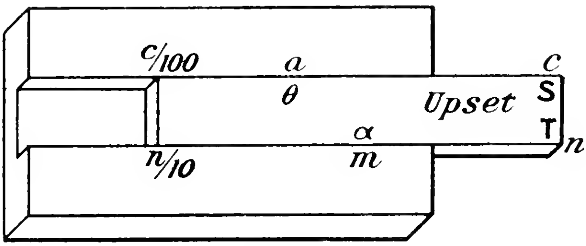


FIG. 39.

hand end of the slide, but we have a reading $\frac{\mathsf{c}}{100}$ opposite the left-hand end, and hence

$$\log\sin\theta - \log\sin 34' = \log \mathsf{a} - \log\frac{\mathsf{c}}{100},$$

or

$$\frac{\mathsf{a}}{\sin\theta} = \frac{\mathsf{c}/100}{\sin 34'},$$

$$\sin\theta = \frac{\mathsf{a}}{\mathsf{c}}.$$

Notice that the trigonometrical equations are

$$\frac{c/100}{\sin 34'} = \frac{a}{\sin \theta} = \frac{c}{\sin 90^\circ},$$

and one pair will always give a reading.

It should be observed that this method enables us, for a given value of **c**, to read off a series of corresponding values of **a** and θ.

In other words, this method gives a convenient solution of the equation

$$x = c \sin \theta \quad \text{or} \quad \sin \theta = \frac{x}{c},$$

where **c** is a constant and **x** and θ are variables.

In practice, if we remember that

$$\begin{aligned} \sin 90^\circ &= 1, \\ \sin 30^\circ &= \cdot 5, \\ \sin 5^\circ\ 45' &= \cdot 1, \\ \sin 34' &= \cdot 01, \end{aligned}$$

as suggested by Major Von Donop, R.A., there will generally be no difficulty in avoiding mistakes.

Ex. 1. Find θ from

$$\sin \theta = \frac{2 \cdot 7}{131} \qquad (\theta = 1^\circ\ 11'),$$

$$\sin \theta = \frac{27}{131} \qquad (\theta = 11^\circ\ 53'),$$

$$\sin \theta = \frac{13 \cdot 1}{270} \qquad (\theta = 2^\circ\ 47'),$$

$$\sin \theta = \frac{131}{270} \qquad (\theta = 29^\circ\ 1').$$

Ex. 2. Set the slide-rule to read "group differences" for a "group" G situated at a distance of 72·3 yards from a range-finding instrument at R.

We have, if the zero line of the rangefinder is at right angles to RG, and if the angle read on its bottom plate is **y**,

$$\text{"Group difference"} = x = 72 \cdot 3 \sin y;$$

$$\therefore \frac{x}{\sin y} = \frac{72 \cdot 3}{\sin 90^\circ},$$

which gives

$$\frac{\mathbf{x} \text{ and } 723 \text{ on the top scale of the ruler}}{\mathbf{y} \text{ and } 90^\circ \text{ on the S scale of the slide upset}}.$$

Some results are tabulated below :

x		y
0	yards,	0,
$12\frac{1}{2}$	,,	$9^\circ\ 57'$,
$37\frac{1}{2}$	,,	$31^\circ\ 16$,
$62\frac{1}{2}$	,,	$59^\circ\ 45'$.

In deciding whether the Group Differences should be added or subtracted, the slide-rule affords no assistance. Trigonometry will always solve the difficulty, but the best plan is to see whether the rangefinder is in front or behind.

Ex. 3. In the sine galvanometer a current C_1 of known amount produces a deflection **a**. Find the current **x** which will produce a deflection **y**.

The formula is $$\frac{x}{C_1} = \frac{\sin y}{\sin a}.$$

That is, $$\frac{x}{\sin y} = \frac{C_1}{\sin a},$$

and we have

$$\frac{\mathbf{x} \text{ and } C_1 \text{ on the top scale of the ruler}}{\mathbf{y} \text{ and } \mathbf{a} \text{ on the S scale of the slide upset}}.$$

Ex. 4. To solve a plane triangle, given **a**, **B**, **C**,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Find A from the equation $A + B + C = 180^\circ$. Bring A on the S scale under **a** on the top scale of the ruler. Over B on the S scale read **b** on the top scale of the ruler, and over C on the S scale read **c** on the top scale of the ruler.

The proof may be left as an exercise for the reader. If one of the angles is obtuse, we must use its supplement, *i.e.* its defect from 180°, which is most easily done by reading the S scale backwards from 90°. Thus, 110° or $90^\circ + 20^\circ$ may be read by reading 20° backwards from 90°, thus making up 110°.

25. **Second Method** of dealing with the equation

$$\sin\theta = \frac{a}{c}.$$

Without "upsetting" the slide, pull it out to the right until a on the slide is opposite c on the ruler; then opposite the right-hand mark on the back of the ruler read θ on the S scale as already explained (§ 23). (Cf. Fig. 36.) This method may fail to give a reading if θ is less than $5°\ 44'$, but is sometimes convenient when only one angle has to be dealt with. Notice that it gives also the solution of

$$\operatorname{cosec}\theta = \frac{c}{a}.$$

Ex. Find θ from the equations:

$$\sin\theta = \tfrac{1}{2}.$$
$$\sin\theta = \tfrac{11}{17}.$$
$$\sin\theta = \tfrac{17}{100}.$$

26. **Third Method** (see Figs. 40 and 41).—"Upset" and "Invert" the slide and pull it out until an end of the S scale is opposite a on the top scale of the ruler. Then, if the slide is out to the *right*, we have in Fig. 40

$$\log\sin 90° - \log\sin\theta = \log c - \log a,$$

or

$$\sin\theta = \frac{a}{c},$$

whence opposite c on the top scale of the ruler is θ on the S scale.

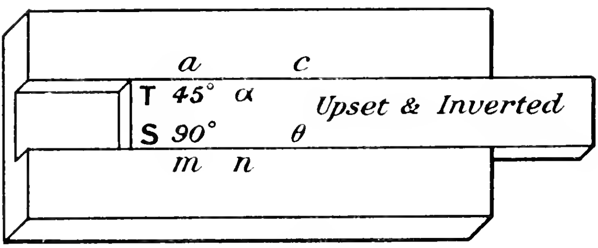


FIG. 40.

If the slide is out to the *left*, we have in Fig. 41 no reading a on the top scale of the ruler opposite the left-hand end of the slide, but we have a reading **100a** opposite the right-hand end, and hence

$$\log \sin \theta - \log \sin 34' = \log \mathbf{100a} - \log \mathbf{c},$$

whence
$$\sin \theta = \frac{\mathbf{a}}{\mathbf{c}}.$$

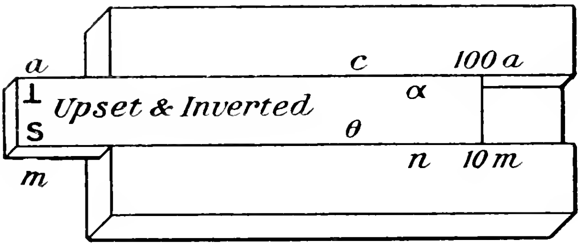


FIG. 41.

Notice that this setting enables us for a given value of **a** to read off a series of corresponding values of **c** and θ. The method is therefore convenient for solving the equations

$$\mathbf{x} = \frac{\mathbf{a}}{\sin \theta}, \quad \text{or} \quad \sin \theta = \frac{\mathbf{a}}{\mathbf{x}}, \quad \text{or} \quad \operatorname{cosec} \theta = \frac{\mathbf{x}}{\mathbf{a}}$$

for a series of values of **x** or θ. An example will be found at p. 102.

Another way of understanding the use of the inverted slide is to take the original setting by the first method,

$$\frac{\mathbf{a} \quad \mathbf{c} \quad \text{on top scale}}{\theta \quad 90^\circ \quad \text{on S scale}},$$

and notice that it is inconvenient if the constants are **a** and **90°**.

But, inverting the slide, we do not alter the relation between the four readings, and now we have

$$\frac{\mathbf{a}}{1/\sin 90^\circ} = \frac{\mathbf{c}}{1/\sin \theta},$$

and, the constants **a** and **90°** being together, no *resetting* is necessary in order to obtain a series of corresponding values of **c** and θ.

The trigonometrical equations may be written

$$\frac{\mathbf{c}}{1/\sin \theta} = \frac{\mathbf{a}}{1/\sin 90^\circ} = \frac{\mathbf{100a}}{1/\sin 34'},$$

and one of this pair will always give a reading.

Solution of Plane Triangles.

27. (i) Given the base a and the base angles B, C. This has been already dealt with (Ex. 4, page 59).

(ii) To solve a triangle, given two sides a, b and the included angle C.

First Method.—Take the exterior angle C_1, say.

In practical applications this is as readily measured as the angle C, and so no subtraction sum is required.

Get the slide by trial into the position indicated by the following diagram:

a b and c on the top scale of the ruler

A and B making C_1 on the S scale of the upset slide.

The method is facilitated by having two cursors, setting one at a and the other at b and moving the slide until the sum of the readings on the S scale, under the cursors, make up C_1. It may be necessary to read the S scale backwards, if the angle B happens to be obtuse.

If the angle A is less than 34′, we may solve the triangle whose sides are 10a and b and included angle C.

This will give us, very approximately, the value of 10A, whence A is known and then B deduced.

An alternative method consists in breaking the triangle up into two right-angled triangles.

Taking a numerical example, let (Fig. 42)

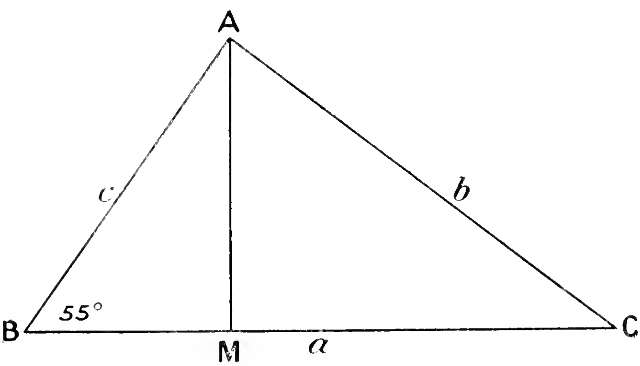


Fig. 42.

$$BC = a = 2150 \text{ yards},$$

$$AB = c = 375 \text{ yards},$$

$$ABC = B = 55°.$$

Letting fall AM perpendicular to BC, we first find AM and BM.

Upsetting the slide and bringing 90° on the S scale opposite 375 on the top scale of the ruler, we read off opposite 55° on the S scale,

$$AM = 375 \sin 55^\circ = 307,$$

and opposite 35° or $90^\circ - 55^\circ$,

$$BM = 375 \cos 55^\circ = 375 \sin 35^\circ = 215\,;$$

consequently, in the triangle AMC, we have

$$AM = 307 \text{ yards},$$

$$CM = 2150 - 215 = 1935 \text{ yards}\,;$$

and consequently $$\tan C = \frac{307}{1935}.$$

So that bringing the end of the T scale above 1935 of the bottom scale of the ruler, we find above 307 on the bottom scale of the ruler $C = 9^\circ\ 1'$ on the T scale.

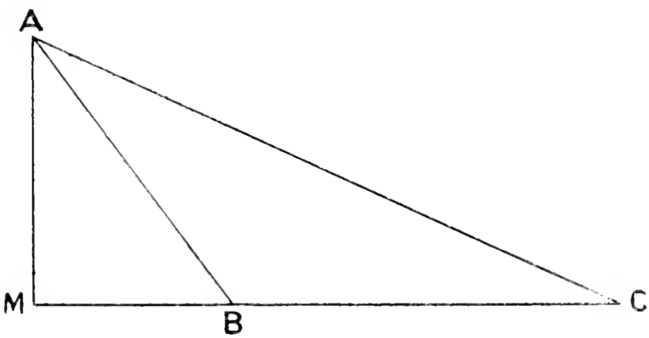


FIG. 43.

The exterior angle at A is then

$$55^\circ + 9^\circ\ 1' = 64^\circ\ 1',$$

and, since $AC \cos 9^\circ\ 1' = 1935$, to find AC we set 81° on the S scale, immediately below 1935 on the top scale of the ruler, and above 90° on the S scale we read $AC = 1965$ yards.

If the given angle at B is obtuse, it is at once clear on drawing a figure (Fig. 43) that we work with $180^\circ - B$, and that BM must be *added* to BC to give CM.

Ex. 1. If $a = 257$ yards,

$b = 2350$ yards,

$C_1 = 4° \ 11', 52°, 111°, 170°$, find c.

Ex. 2. Shew that if C_1 is under $20°$ or over $160°$, then the third side $c = b + a \cos C_1$ nearly.

Ex. 3. Shew how to solve a plane triangle in which a, A, and b are given.

Consider $a = 240$,

$b = 355 \quad A = 37°$.

Cosines.

28. To find $\cos\theta$ we utilise the fact that

$$\cos\theta = \sin \overline{90° - \theta},$$

and find the sine of $90° - \theta$ by one of the preceding methods.

In practice, instead of performing the operation of subtracting θ from $90°$ by mental arithmetic or on paper, we let the slide-rule do the work for us by counting the degrees backwards from $90°$ on the S scale,

counting $0°$ at $90°$,

$20°$ at the point marked 70,

$30°$,, ,, 60,

and so on.

Ex. 1. Plot the curve defined by the equations

$$x = 10 \cos\theta,$$

$$y = 5 \sin\theta + 5 \sin 2\theta,$$

for values of θ between $0°$ and $180°$.

Ex. 2. Plot the curve

$$2 \cos y + \cos x = 0.$$

Solution of Spherical Triangles.—A section of a sphere by a plane passing through the centre of the sphere is a great circle.

A figure bounded by three arcs of great circles, each arc being less than a semicircle, is a spherical triangle.

The angles between the planes of the great circle are the angles of the spherical triangle. The angles which the sides of the triangle subtend at the centre of the sphere are termed, by an ellipsis, the sides of the triangle.

If A, B, C are the angles and a, b, c the sides of a spherical triangle, then

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \quad \text{............(1)}$$

with similar formulae for cos b and cos c, while

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \quad \text{.....................(2)}$$

Ex. 3. Suppose

$$b = 41^\circ\ 7',$$
$$c = 74^\circ\ 39',$$
$$A = 84^\circ\ 15'.$$

Required a. The formula is

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

We evaluate the terms cos b cos c and sin b sin c cos A separately.

The steps in the work are:

(1) Upset the slide and run it home.

(2) Count 41° 7′ back from the right-hand end of the S scale. Bring the cursor to the point reached. We could now read the value of cos 41° 7′, viz. ·753 on the top scale. But we do not require this.

(3) Bring the end of the S scale under the cursor and count 74° 39′ from the right-hand end of the S scale. Bring the cursor to the point reached Read the value of cos 41° 7′ cos 74° 39′, viz. ·1985, under the cursor on the top scale.

(4) Now, in like manner work out

$$\sin 41^\circ\ 7' \sin 74^\circ\ 39' \cos 84^\circ\ 15'.$$

This proves to be ·0632.

(5) Hence $\cos a = \cdot 1985 + \cdot 0632 = \cdot 2617.$

(6) Bring the cursor to ·2617 on the top scale, run the slide home, and count backwards from the right-hand end of the S scale to the point under the cursor. The result is a. We find

$$a = 74^\circ\ 50';\ \text{more accurately, } 74^\circ\ 46'.$$

Notice that if one of the given parts of the triangle is obtuse, then one of the terms in the formula will become negative.

The slide-rule will not disclose the sign of the terms.

Ex. 4.
$$b = 101^\circ\ 17',$$
$$c = 73^\circ\ 40',$$
$$A = 17^\circ.$$

Find a.

We know that
$$\sin 101^\circ\ 17' = \sin 78^\circ\ 43',$$
$$\cos 101^\circ\ 17' = -\cos 78^\circ\ 43'.$$

We find
$$\cos 78^\circ\ 43' \cos 73^\circ\ 40' = \cdot 0551,$$
$$\sin 78^\circ\ 43' \sin 73^\circ\ 40' \cos 17^\circ = \cdot 895,$$

whence
$$\cos a = -\cdot 0551 + \cdot 899$$
$$= \cdot 844, \quad \text{whence} \quad a = 32^\circ\ 25'.$$

Notice also that, if a is a small angle, this formula is not well suited to give an accurate result, as the small error in the numerical value of the cosine will involve a large error in the angle.

The fundamental formula may be modified in various ways, *e.g.*

$$1 - \cos a = 1 - \cos b \cos c - \sin b \sin c \cos A,$$

i.e.
$$2 \sin^2 \frac{a}{2} = 1 - \cos b \cos c - \sin b \sin c + \sin b \sin c\,(1 - \cos A)$$
$$= 1 - \cos (b - c) + 2 \sin b \sin c \sin^2 \frac{A}{2},$$

or
$$\sin^2 \frac{a}{2} = \sin^2 \frac{b - c}{2} + \sin b \sin c \sin^2 \frac{A}{2},$$

which, with a little more labour, will give a very accurate result when a is a small angle.

Ex. 5.
$$b = 42^\circ\ 45',$$
$$c = 47^\circ\ 15',$$
$$A = 11^\circ\ 12'.$$

The above formula gives

$$\sin^2 \frac{a}{2} = \sin^2 (2^\circ 15') + \sin 42^\circ 45' \sin 47^\circ 15' \sin^2 5^\circ 36'$$

$$= \cdot 00154 + \cdot 00473$$

$$= \cdot 00627, \quad \text{whence} \quad \sin \frac{a}{2} = \cdot 079,$$

$$\frac{a}{2} = 4^\circ 31',$$

$$a = 9^\circ 2'.$$

Ex. 6. Make a table of values of ϕ and ϕ' from the refraction formula

$$\sin \phi = 1 \cdot 5 \sin \phi'.$$

Ex. 7. Make a table of values of ϕ for various crank-angles θ from the equation

$$n \sin \phi = \sin \theta \quad (n = 2,\ 3,\ 10).$$

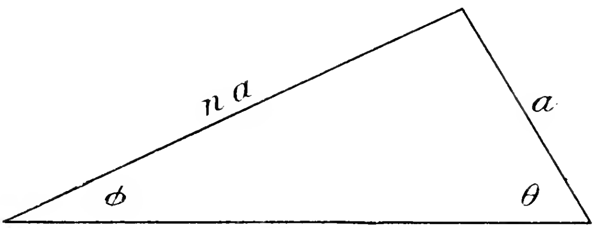


FIG. 44.

Ex. 8. Use Ex. 2 to make a diagram of turning moment from the formula

$$M = \frac{Pa \sin \overline{\theta + \phi}}{\cos \phi} \qquad (Pa = 100).$$

Complete the columns in order. (See p. 81.)

θ	ϕ	$\overline{\theta + \phi}$	M

Summary.

29. Denoting the angle whose sine is ·01 by α, the equation

$$\mathbf{x} = \mathbf{r} \sin \theta$$

may be written in slide-rule form thus,

$$\frac{r/100}{\sin \alpha} \overleftarrow{=} \frac{x}{\sin \theta} \overrightarrow{=} \frac{r}{\sin 90}$$

(convenient if **r** is constant), and *either* the first pair *or* the last pair will certainly give a reading.

The equations may also be written

$$\frac{100x}{1/\sin \alpha} = \frac{r}{1/\sin \theta} = \frac{x}{1/\sin 90}$$

(convenient if **x** is constant), and *either* the first pair *or* the last pair will certainly give a reading.

Tangents.

30. To find α from the equation

$$\tan \alpha = \frac{m}{n} \quad \left(\frac{m}{n} \text{ being a proper fraction}\right).$$

We notice that if **n** is less than **m** the angle α must be greater than **45°**.

First Method (see Figs. 38 and 39).—Upset the slide and pull it out until an end of the T scale is immediately above **n** on the bottom scale of the ruler.

Then, if the slide is out to the *left*, we have, in Fig. 38,

$$\log \tan 45° - \log \tan \alpha = \log n - \log m \quad \text{or} \quad \frac{\tan \alpha}{m} = \frac{\tan 45°}{n};$$

$$\therefore \tan \alpha = \frac{m}{n};$$

and if the slide is out to the *right* we have the same equation; but as **n** is off the scale, we must use instead the equation

$$\frac{\tan 5° 43'}{n/10} = \frac{\tan \alpha}{m}.$$

Thus, in Fig. 39,

$$\log \tan \alpha - \log \tan 5° 43' = \log m - \log \frac{n}{10},$$

$$\tan \alpha = \frac{m}{n}.$$

We notice that for a given value of $\mathbf{n}$ this setting enables us to read off a series of corresponding values of $\mathbf{m}$ and α; or, in other words, to solve the equations

$$\mathbf{x} = \mathbf{n} \tan \alpha \quad \text{or} \quad \tan \alpha = \frac{\mathbf{x}}{\mathbf{n}} \quad \text{or} \quad \mathbf{x} \cot \alpha = \mathbf{n}$$

for a series of values of α or $\mathbf{x}$.

Second Method.—The mode of using the left-hand mark on the back of the ruler in conjunction with the T scale has been already explained. This method may fail to give a reading.

Third Method (see Figs. 40 and 41).—"Upset" and "invert" the slide and pull it out until an end of the slide is opposite $\mathbf{m}$ on the bottom scale of the ruler.

Then, if the slide is out to the *right*, we have, in Fig. 40,

$$\log \tan \mathbf{45}^\circ - \log \tan \alpha = \log \mathbf{n} - \log \mathbf{m},$$

whence

$$\frac{\mathbf{1}/\tan \mathbf{45}^\circ}{\mathbf{m}} = \frac{\mathbf{1}/\tan \alpha}{\mathbf{n}},$$

$$\tan \alpha = \frac{\mathbf{m}}{\mathbf{n}},$$

whence above $\mathbf{n}$ on the bottom scale of the ruler is α on the T scale.

If the slide is out to the *left* (Fig. 41) we have the same equation; but as it fails to give a reading, because $\mathbf{m}$ is now off the scale, we must use instead the equation

$$\log \tan \alpha - \log \tan \mathbf{5}^\circ\ \mathbf{43}' = \log \mathbf{10m} - \log \mathbf{n},$$

whence

$$\tan \alpha = \frac{\mathbf{m}}{\mathbf{n}}.$$

This method enables us for a given value of $\mathbf{m}$ to read off a series of corresponding values of $\mathbf{n}$ and α, or, in other words, to solve the equation

$$\mathbf{x} = \frac{\mathbf{m}}{\tan \alpha} = \mathbf{m} \cot \alpha \quad \text{or} \quad \tan \alpha = \frac{\mathbf{m}}{\mathbf{x}}$$

for a series of values of α or $\mathbf{x}$.

Another way of stating the matter.—The trigonometrical equation $\tan\alpha = \frac{m}{n}$ may be written in either of the following slide-rule forms:

$$\frac{\tan 5^\circ 43'}{n/10} = \frac{\tan\alpha}{m} = \frac{\tan 45^\circ}{n} \quad \text{(convenient if n constant)}$$

or $$\frac{1/\tan 5^\circ 43'}{10m} = \frac{1/\tan\alpha}{n} = \frac{1/\tan 45^\circ}{m} \quad \text{(convenient if m constant).}$$

To find the Cotangent of an Angle less than 45°.

31. If the slide is inverted and upset and run home, we have, in Fig. 45, considering the bottom scale of the ruler as reading from 1 to 10,

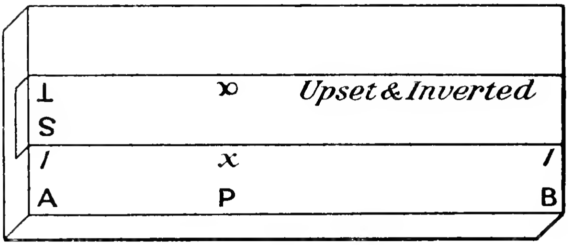


FIG. 45.

$$\log x - \log 1 = \log\tan 45^\circ - \log\tan\alpha,$$

or $$\log x = \log\cot\alpha,$$

whence $$x = \cot\alpha,$$

so that opposite α on the T scale we read its cotangent on the bottom scale of the ruler, if we consider the latter as reading from 1 to 10; in other words, if $\theta = 90 - \alpha$ we read $x = \tan\theta$ on the bottom scale of the ruler, regarded as reading from 1 to 10.

Alternative Method of finding Cotangents.—The equation

$$\cot\theta = \frac{m}{n} \quad \begin{pmatrix} m < n, \\ \text{so } \theta > 45^\circ \end{pmatrix}$$

may be written $$\tan(90 - \theta) = \frac{m}{n},$$

so $$\frac{\tan(90-\theta)}{m} = \frac{\tan 45^\circ}{n}.$$

$90 - \theta$ may be found by counting backwards from 45°.

If $$\cot\theta = \frac{m}{n} \qquad (m > n),$$

the equation may be written

$$\frac{\tan\theta}{n} = \frac{\tan 45^\circ}{m}.$$

To find the tangent of an angle greater than 45°.

If $\theta = 90^\circ - \alpha$ and θ is greater than 45°, α is less than 45°, and since

$$\tan\theta = \cot\alpha,$$

we have the value of $\tan\theta$ by finding the value of $\cot\alpha$ by the preceding method.

Instead of subtracting α from 90° mentally, or on paper, we may use the slide-rule to perform the operation mechanically by counting the degrees on from 45°. Thus, at the point marked

40	we count	50°,
30	,,	60°,
20	,,	70°,

and so on.

In order to avoid mis-readings, it is convenient to recollect that

$$\tan 5^\circ 43' = \cdot 1 = \cot 84^\circ 17',$$
$$\tan 26^\circ 36' = \cdot 5 = \cot 63^\circ 24',$$
$$\tan 45^\circ = 1 = \cot 45^\circ,$$
$$\tan 63^\circ 24' = 2 = \cot 26^\circ 36',$$
$$\tan 84^\circ 17' = 10 = \cot 5^\circ 43'.$$

EXAMPLES.

1. To plot a given angle AB from a given point O in a given line OA. The ordinary six-inch protractor is not always sufficiently accurate when it is desired to plot an angle. A good method is to take out the sides of a right-angled triangle containing the given angle from a traverse table, and to construct the triangle with the given sides, using a uniformly divided plain scale. Using the slide-rule instead of a traverse table, we may proceed thus:

Take a distance OA as long as practicable, say 11·5 inches. Suppose the angle that is to be plotted to be, say, 17° 30′.

Find the value of $x = 11{\cdot}5 \tan 17^\circ 30'$, namely 3·63, using one of the preceding methods.

For instance, upset the slide and bring the left-hand end of the T scale above 115 on the bottom scale of the ruler, and beneath $17^\circ 30'$ on the T scale we read 3·63 on the bottom scale of the ruler. At the point A erect a perpendicular AB 3·63 inches long and join BO. AOB is the required angle. As a check measure BO. It should be

$$\frac{OA}{\cos(17^\circ 30')}.$$

This can be worked on the slide-rule, as already described, by setting $90^\circ - 17^\circ 30'$ on S scale upset under 115 on the top scale of the ruler and reading 12·05 on the top scale of the ruler above the 90° mark on the S scale.

2. Use the slide rule to complete the following columns from a Traverse, drawing a rough sketch first.

Distance in feet.	Bearing 0–360° clockwise from North (magnetic).	Northing.	Southing.	Easting.	Westing.
527	53° 7′				
1420	123° 14′				
673	194° 7′				
947	107° 4′				
514	37° 11′				
1120	74° 19′				

3. In a map of the world on the stereographic projection, the centre of projection being at the S. pole, the parallels of latitude are represented by concentric circles, the radius of the circle corresponding to N. latitude λ being

$$a \tan \frac{90^\circ - \lambda}{2}, \qquad a \text{ being a constant.}$$

Draw the series of circles corresponding to latitudes 5°, 10°, 15°,...85° N.

4. In a map of the north polar regions on an equal-area projection, the parallel of latitude $90^\circ - K$ is represented by a circle of radius $2 \sin \frac{K}{2}$. Draw the "graticule" or series of circles for latitudes $70^\circ \ldots 90^\circ$.

To find the Sine or Tangent of a Small Angle.

32. The sine or tangent of an angle of 1 minute is

$$\frac{1}{3437} \text{ or say } \frac{1}{3440}.$$

The sine or tangent of an angle of 1 second is

$$\frac{1}{206265} \text{ or say } \frac{1}{206000}.$$

The sines and tangents of small angles are proportional to the angles themselves. Consequently, if m is the number of minutes in a small angle θ, we have

$$\sin \theta = \frac{m}{3440} = \tan \theta,$$

and if n is the number of seconds in the same angle, then

$$\sin \theta = \frac{n}{206000} = \tan \theta.$$

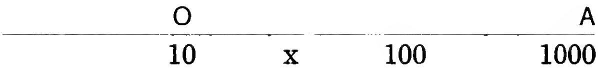
Hence problems relating to the sines and tangents of small angles can be worked on the front scales. A mark (′) is placed on the S scale corresponding when the slide is run home to 3440 on the top scale of the ruler, and in like manner a second mark (″) corresponding to 20600. This is supposed to save trouble if we are dealing with a series of small angles, some over 34′ and some under. For an alternative method see page 38.

CHAPTER VI.

THE LOGOLOGARITHMIC SCALES.

33. The logologarithmic scale was invented by Dr. Peter Roget, and fully described by him in the *Phil. Transactions* for 1815.

On the line OA, starting from O, measure off a distance representing $\log\log \mathbf{x}$ on a suitable scale, and mark the corresponding point $\mathbf{x}$.



O will be marked **10**, because

$$\log\log \mathbf{10} = \log \mathbf{1} = \mathbf{0}.$$

If **10** inches represents unity, the reading **100** will be **3·01** inches to the right of O, because

$$\log\log \mathbf{100} = \log \mathbf{2} = \mathbf{\cdot 301}.$$

Similarly the reading **1000** will be **4·77** inches to the right of O, because

$$\log\log \mathbf{1000} = \log 3 = \cdot 477.$$

A number $\mathbf{n}$ between **1** and **10** will lie to the left of O, because in this case $\log\log \mathbf{n}$ is negative.

A number lying between O and **1** has a negative logarithm; and as a negative quantity has no real logarithm, the $\log\log \mathbf{x}$ scale cannot go below **1**. In fact the reading **1** is infinitely distant from the reading **10**.

We may, however, construct a $\log(-\log \mathbf{x})$ scale for numbers lying between **0** and **1**.

The starting point O will be marked **·1**, because

$$\log(-\log \mathbf{\cdot 1}) = \log \mathbf{1} = \mathbf{0}.$$

This extension of the scale appears to have been first suggested by M. Blanc (Dyck's *Catalog Mathematischer Modelle*, p. 145).

If an ordinary logarithmic scale on a slide is applied to this log log ruler (the length representing unity being the same on both), so that **1** on the slide is opposite **2** on the log log scale or "ruler," then in Fig. 46, where **x** on the slide is opposite **b** on the rule, we have

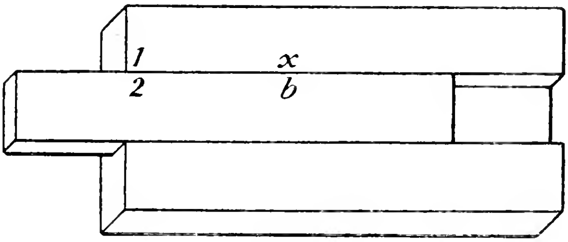


Fig. 46.

$$\log \log \mathbf{b} - \log \log 2 = \log \mathbf{x} - \log 1,$$

$$\frac{\log \mathbf{b}}{\log 2} = \mathbf{x},$$

$$\log \mathbf{b} = \mathbf{x} \log 2 = \log 2^{\mathbf{x}},$$

$$\mathbf{b} = 2^{\mathbf{x}}.$$

The general appearance of the two scales when set is shown in Fig. 47, in which the log log scale is uppermost.

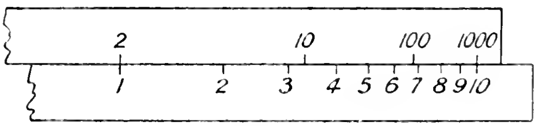


Fig. 47.

It will be noticed that, unlike the log scale, the log log scale does not repeat itself; there is one and only one position on the scale for each number. Hence, only a limited range of numbers can be dealt with in any actual instrument.

In the scales arranged by the authors for use with the **25** cm. ruler, the logologarithmic scales are placed on a separate slide.

Numbers greater than unity are on the front of the slide,

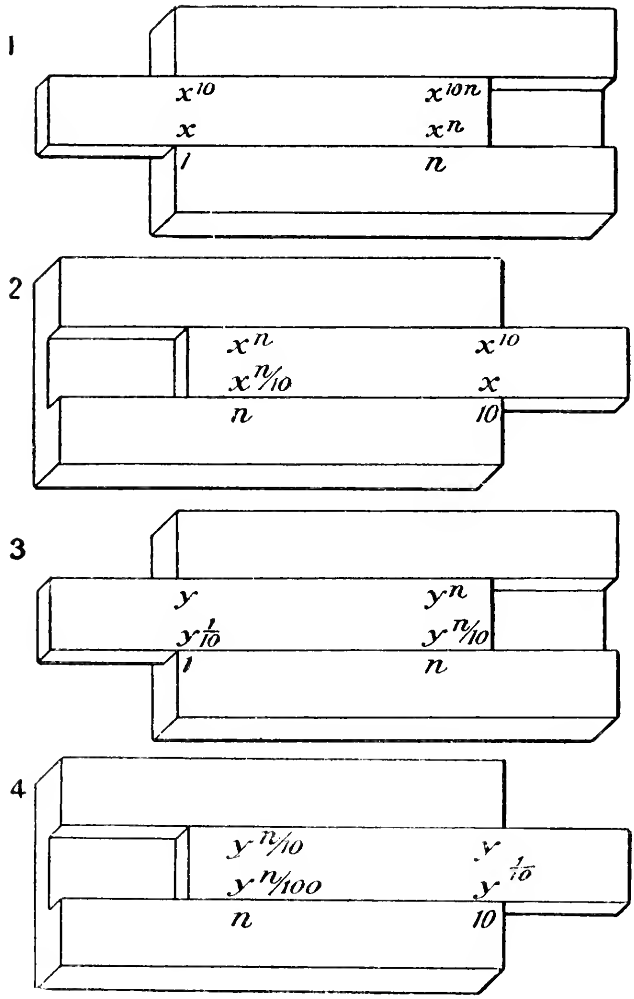


FIG. 48.

which is marked E. Numbers less than unity are on the back of the slide, which is marked – E.

On the front, E is a scale in two* portions, the lower portion running from 1·07 to 2, the upper portion continuing from 2 to 1000.

On the back, – E is a scale in two* portions, the upper portion running from ·001 to 5, the lower portion continuing from ·5 to ·93.

* In some cases three, beginning at 1·007, and ending at ·993 respectively.

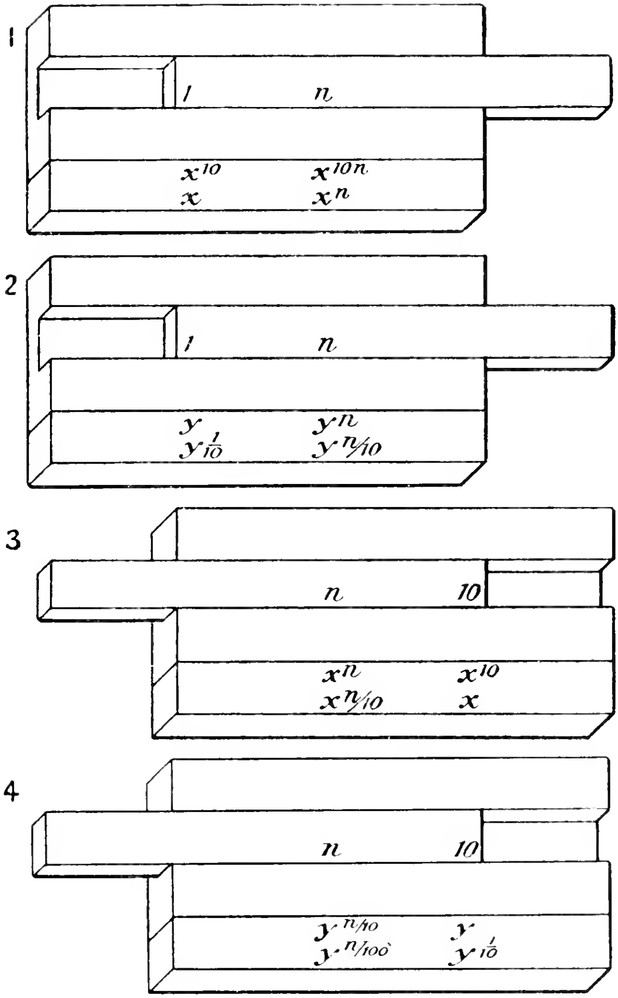


FIG. 49.

Mathematical Principle of the Scales.

34. The distance of any number **x** on **E** measured along the scale from the point marked **10** represents the logarithm of the logarithm of **x**. Similarly the distance of any number **x** on $-$**E** from the point marked **·1** represents log($-$ log **x**).

This slide can be used in two different ways:

(A) It may replace the ordinary slide and be used in conjunction with the bottom scale on the ruler, considered as running from **1** to **10**;

or (B) by means of a special clip it may be fixed to the ruler, and be used in conjunction with the bottom scale of the ordinary slide, considered as running from 1 to 10.

The mathematical principle of the instrument is exhibited in Figs. 48 and 49.

Notice first that the upper portion of the E scale is the continuation of the lower portion.

Save for reasons of portability, the two parts would be in one straight line, as in Fig. 50.

The foregoing statement, and what follows, applies to the – E as well as to the E scale.

The distance from any point P on the lower portion to the point Q directly above it on the upper portion is exactly equal to the length of the bottom scale of the rule.

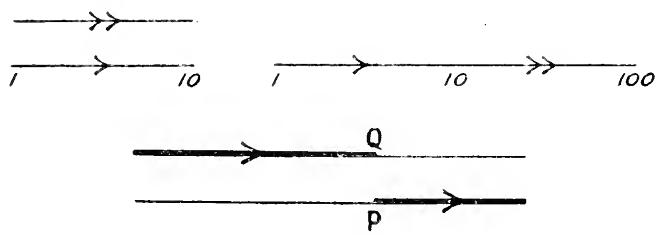


FIG. 50.

Consequently, if Q is marked x, the graduation z at P is determined by the equation

$$\log\log x - \log\log z = \log 10,$$

$$\log x = 10 \log z,$$

$$z = x^{\frac{1}{10}} \quad \text{or} \quad x = z^{10}.$$

EXAMPLES.

1. Read off powers of 2, 3, 1·1, ·5 on the E and – E scales respectively.

2. Draw a graph of $y = e^x$ for values of x from 1 to 4.

3. Find $(1{\cdot}5)^{1{\cdot}7}$.

4. Make a table of values of **p** and **v** from the formula

$$pv^{1\cdot065} = 479.$$

v = volume in cubic feet of 1 lb. of steam,

p = pressure in lb./in^2.

5. **Compound Interest.**—Find how long a sum of money will take to double itself at **r** per cent. compound interest. A being amount and P principal,

$$A = P(1 + r)^n,$$

and in this question $A = 2P$,

so that $(1 + r)^n = 2.$

Thus, at 5 %, $r = 1\cdot05$

and $(1\cdot05)^n = 2.$

$$\log\log 2 - \log\log(1\cdot05) = \log n.$$

The setting is shown in Fig. 51.

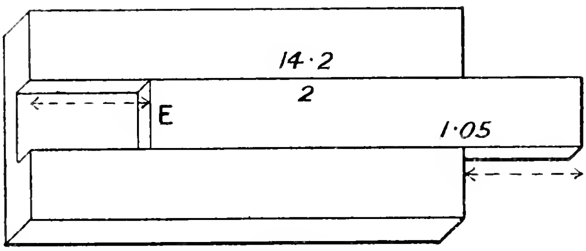


FIG. 51.

6. Find $\log_e 3\cdot5$.

If $\log_e 3\cdot5 = n,$

then $e^n = 3\cdot5,$

and the setting is shown in Fig. 52.

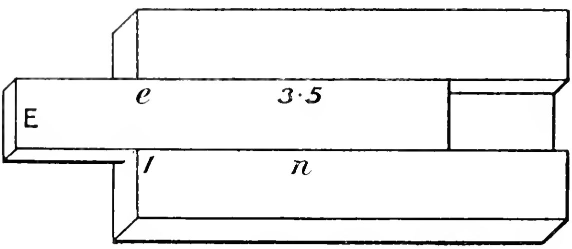


FIG. 52.

7. Plot a graph of

$$y = e^{-\frac{x^2}{100}} \quad (0 < x < 10).$$

8. **Young's Velocity Ratio Problem.**—A given velocity ratio is to be obtained by the use of several pairs of cog wheels, only two sizes of wheels being used, and the total number of teeth being a minimum.

If x is the ratio of the diameters of the wheels and n the number of pairs of wheels, x^n is equal to given velocity ratio and $(x+1)n$ is to be a minimum.

This leads to the equation

$$1 + \frac{1}{x} = \log_e x,$$

or, writing y for $\frac{1}{x}$,

$$1 + y = -\log_e y \quad \text{or} \quad \left(\frac{1}{e}\right)^{1+y} = y.$$

Thus, y is found by trial thus,

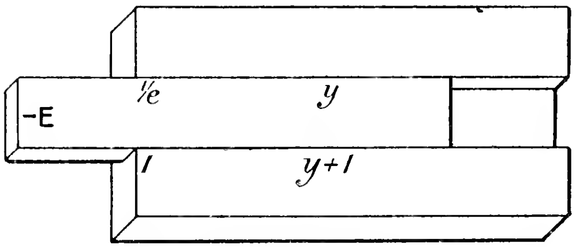


Fig. 53.

the point being noted where the reading on the bottom scale exceeds the reading on "$-E$" scale by unity.

9. Draw a graph of Vallier's equation for powder-pressure curve in a gun. $y = xe^{1-x}$.

The most General Form of Slide-Rule.

35. Slide-rules containing several slides have been proposed, and also slide-rules containing several scales side by side on the same part of the rule or slide. Both are somewhat troublesome to use.

If a pair of scales for $\log \mathbf{f}(\mathbf{x})$ are provided in conjunction with a pair of scales for $\log \mathbf{n}$, the general equation

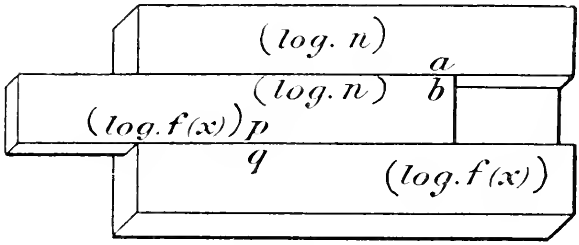


FIG. 54.

corresponding to the setting shewn will be

$$\log \mathbf{a} - \log \mathbf{f}(\mathbf{q}) = \log(\mathbf{b}) - \log \mathbf{f}(\mathbf{p}),$$

or

$$\frac{\mathbf{a}}{\mathbf{b}} = \frac{\mathbf{f}(\mathbf{p})}{\mathbf{f}(\mathbf{q})}.$$

A spare slide clipped to the ordinary rule furnishes a practical method of using such scales for the functions

$$\mathbf{f}(\mathbf{x}) = \sin \mathbf{x},$$

$$\mathbf{f}(\mathbf{x}) = \tan \mathbf{x},$$

$$\mathbf{f}(\mathbf{x}) = \log \mathbf{x}.$$

CHAPTER VII.

THE CENTRAL SCALE.

36. The central scale on the back of the slide is simply a uniformly divided scale, that length which on the bottom scale of the ruler represents unity or log 10 being divided into equal parts numbered from right to left, and reading from 0 to 1.

Consequently, when the slide is upset, inverted, and run home, if the cursor is set to read a given number on the bottom scale of the ruler, it will read on the central scale of the slide, the decimal part considered always as positive of the logarithm of that number.

The integral part or *characteristic* must be found by inspection as explained in Section 3. For example, we find that in most of the formulae which have been suggested for the flow of water in pipes the fifth root of a number is required, say of 170.

The characteristic of log 170 is 2.

The decimal part is found, as just described, to be ·23.

$$\text{Hence} \quad \log(170)^{\frac{1}{5}} \text{ or } \tfrac{1}{5}\log(170) = \frac{2{\cdot}23}{5} = {\cdot}446.$$

We now require the number whose logarithm is ·446, and reversing the process just described, we find that the significant figures of this number are 279, and by p. 6 we know that the number must be 2·79, whence

$$\sqrt[5]{170} = 2{\cdot}79.$$

The following contrivance enables the logarithm of a number to be read without upsetting or inverting the slide (see Fig. 55).

The slide is pulled out to the right until its end is opposite the given number **n** on the bottom scale of the ruler. The instrument being turned over, the decimal part of log **n** is read on the central scale opposite one of the right-hand marks on the back of the ruler.

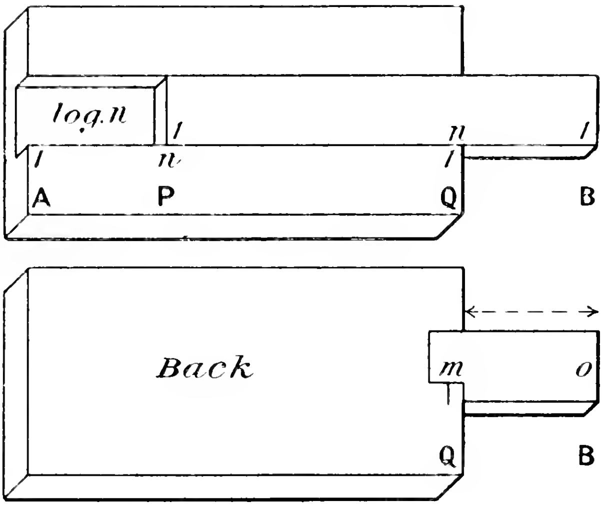


FIG. 55.

To find $\log \tan \theta$ we may upset the slide and run it home, and, using the cursor, find the reading **a** on the central scale opposite θ on the T scale. We then have, in Fig. 56,

FIG. 56.

$$\log \tan \mathbf{45} - \log \tan \theta = \mathbf{a},$$

or

$$\log \tan \theta = -\mathbf{a}.$$

Adding **10** to both sides of the equation we have the *tabular* logarithmic tangent

$$\mathrm{L} \tan \theta = \mathbf{10} - \mathbf{a}.$$

Similarly, to find $\log \sin \theta$, upset the slide and, using the cursor, find the reading **c** on the central scale corresponding to θ° on the S scale. Noting that the length which represents unity on the central scale represents **2** on the **S** scale, we have, in Fig. 57,

$$\log \sin 90^\circ - \log \sin \theta = 2c;$$

$$\therefore \log \sin \theta = -2c,$$

and, as above,

$$L \sin \theta = 10 - 2c.$$

FIG. 57.

CHAPTER VIII.

PLOTTING CURVES FROM EQUATIONS.

The Parabola.

37. $y^2 = ax$. If it is desired to plot the curve for values of x which increase uniformly, as for example when using squared paper, we may proceed thus:

Pull out the slide (right or left) until its end is below a on the top scale of the ruler.

Read the values of x on the top scale of the slide, and the corresponding values of y (using the cursor) on the bottom scale of the ruler.

The Circle $\qquad y = a\sqrt{1 - \frac{x^2}{a^2}},$

and

The Ellipse. $\qquad y = b\sqrt{1 - \frac{x^2}{a^2}}.$

Pull out the slide until a on its bottom scale is over an end of the ruler, and opposite x on the bottom scale of the slide is $\frac{x}{a}$ on the bottom scale of the ruler, and $\frac{x^2}{a^2}$ on the top scale of the ruler, so that reading the top scale of the ruler backwards $\left(\text{which is equivalent to subtracting } \frac{x^2}{a^2} \text{ from unity}\right)$ we write down the series of values of $1 - \frac{x^2}{a^2}$.

Now pull out the slide until b on its bottom scale is over an end of the ruler.

Setting the cursor to the successive values on the top scale of the ruler, of $1 - \frac{x^2}{a^2}$ as just written down, read the set of corresponding values of $b\sqrt{1-\frac{x^2}{a^2}}$ on the bottom scale of the slide.

The work may be conveniently arranged in a tabular form thus, taking

$$a=2, \qquad b=\sqrt{2},$$

we find

x	First operation. $1-\frac{x^2}{2^2}$.	Second operation. $y=b\sqrt{1-\frac{x^2}{2^2}}$.	Remarks.
·1	·9975	1·41	Slide to Left for 1st operation; to Right for 2nd.
·3	·9775	1·398	
·5	·9375	1·369	
·7	·877	1·324	
·9	·797	1·26	
1	·75	1·221	Slide shifted to Right for 1st operation.
1·1	·691	1·176	
1·2	·64	1·131	
1·3	·577	1·075	
1·4	·512	1·01	
1·5	·438	·936	Slide shifted to Left for 2nd operation.
1·6	·36	·848	
1·7	·277	·745	
1·8	·19	·617	
1·9	·097	·441	
1·95	·05	·314	
2	0	0	

It will be noticed that the axes of coordinates may be any two conjugate diameters of the ellipse, not necessarily the principal axes.

Alternatively we see that

$$x = a \cos \alpha,$$
$$y = b \sin \alpha$$

are the coordinates of a point on the ellipse, and by computing **x** and **y** for a series of values of α, using the **S** scale in the manner already explained, the curve might be plotted.

The Hyperbola.

$$(1) \quad xy = c^2,$$

$$(2) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The equation (1) represents a rectangular hyperbola with the asymptotes as coordinate axes.

Inverting the slide and setting an end of the slide opposite **c** on the bottom scale of the ruler, the corresponding values of **y** on the top scale of the slide may be read off by means of the cursor below the values of **x** on the top scale of the ruler (see p. 36).

Equation (2) represents an hyperbola referred to its axes.

The equation may be written

$$y = b\sqrt{\frac{x^2}{a^2} - 1}.$$

Pull out the slide (normal position) until **a** on its bottom scale is opposite an end of the ruler and opposite **x** on the bottom scale of the slide is $\frac{x}{a}$ on the bottom scale of the ruler and $\frac{x^2}{a^2}$ $\left(\text{read as } \frac{x^2}{a^2} - 1, \text{ by subtracting 1 mentally}\right)$ on the top scale of the ruler.

The series of values of $\frac{x^2}{a^2} - 1$ is written down.

Now pull out the slide until **b** on its bottom scale is opposite an end of the ruler. Setting the cursor to the successive values (on the top scale of the ruler) of $\frac{x^2}{a^2} - 1$, read the corresponding values of $b\sqrt{\frac{x^2}{a^2} - 1}$ on the bottom scale of the slide. Here also the work may conveniently be arranged in the tabular form already explained.

Barlow Curves.

$$xy^2 = \text{constant} = c, \text{ suppose.}$$

Invert the slide, and pull it out until an end of the slide is opposite **c** on the top scale of the ruler. Below **x** on the top scale of the slide read **y** on the bottom scale of the ruler.

We might equally well read **y** on the bottom scale of the slide immediately below **x** on the top scale of the ruler.

Lemniscate. $r = a\sqrt{\cos 2\theta}$.

Apart from a purely geometrical construction, this seems most easily plotted by polar coordinates.

Using the S scale, upset the slide and bring an end of it opposite **a**. Reading the scale backwards, so that

90° reads 0,

70 ,, 20°,

and so on, we write down a set of values of **r** corresponding to assumed values of 2θ. The angle θ can be set off in the manner shewn in Ex. 2, page 72, and the corresponding values of **r** can then be plotted.

Other examples of curves will be found in Chapter X.

CHAPTER IX.

EFFECT ON RESULTS OF ERRORS IN WORKING.

38. Let **z** be the true final answer of a problem worked on the instrument, and suppose that, owing to an aggregate error, due to the various settings and readings, the answer is found to be **z + h**, so that **h** is the error of the final answer, and suppose that the point read to give this answer is at a distance **d** inches from its true position. Let this distance **d** represent log(**1 + n**), so that **n** remains constant if **d** does so.

Then we have, in Fig. 58,

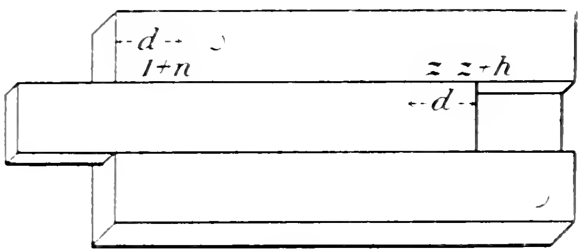


FIG. 58.

$$\log(\mathbf{z}+\mathbf{h}) - \log \mathbf{z} = \mathbf{d} = \log(\mathbf{1}+\mathbf{n});$$

$$\therefore \frac{\mathbf{z}+\mathbf{h}}{\mathbf{z}} = \mathbf{1}+\mathbf{n};$$

$$\therefore \frac{\mathbf{h}}{\mathbf{z}} = \mathbf{n} \quad \text{or} \quad \mathbf{h} = \mathbf{nz};$$

consequently the error **h** in the result is directly proportional to the result itself as long as **n** remains constant, that is, as long as **d** remains constant.

In other words, if we double the result but still have the same error **d** on the scale, we also double the *total* error of the result, but the *percentage* error of the result is unchanged, and this result is obvious, because, taking any setting $\frac{1}{a}=\frac{m}{ma}$, an error in **m** produces the same proportionate error in **ma**.

Hence, if two similar problems involve the same number of operations, we may expect the same *percentage error* in each, irrespective of the magnitude of the results. This principle applies to every case where the answer is read on the ruler, even when the trigonometric scales on the slide have been employed. But when the result is *read* on the S or T scale different considerations apply. The effect of a given error can, however, be worked out in a similar manner. Thus, if an angle is to be found from its sine, we have, θ being the angle read, $\theta + h$ the true angle, and **d** the error on the scale, as before.

$$\log \sin(\theta + h) - \log \sin \theta = d = \log(1 + n);$$

$$\therefore \frac{\sin(\theta + h)}{\sin \theta} = 1 + n;$$

$$\therefore 1 + h \cot \theta = 1 + n,$$

since **h** is small,

whence $$h = n \tan \theta,$$

shewing that the effect of the error **d** on the result is very large when θ is nearly 90°.

Again, in finding an angle from its tangent, we have, using the same notation and remembering that **h** is small,

$$\log \tan(\theta + h) - \log \tan \theta = \log(1 + n);$$

$$\therefore 1 + n = \frac{\tan(\theta + h)}{\tan \theta}$$

$$= \frac{h + \tan \theta}{1 - h \tan \theta} \frac{1}{\tan \theta}$$

$$= (1 + h \cot \theta)(1 + h \tan \theta)$$

$$= 1 + \frac{h}{\sin \theta \cos \theta}, \text{ whence } h = n \sin \theta \cos \theta = \frac{n}{2} \sin 2\theta.$$

This shews that the effect of the error is greatest when θ is nearly 45°.

It is difficult to lay down any precise rule as to the value of **d** or **n**, as this clearly varies to some extent with each different kind of problem. However, the following statement seems fairly well borne out by practice.

Let $\mathbf{e}$ = the probable error (estimated in length) of a setting of the slide or cursor or of a reading;

$\mathbf{m}$ = the number of settings or readings;

$\mathbf{p}$ = the probable error of the answer (estimated in length).

Then
$$\mathbf{p} = \mathbf{e}\sqrt{\mathbf{m}+1}.$$

$\mathbf{e}$ is independent of the length of the instrument and of the scales used.

For the average man, $\mathbf{e}$ is about $0{\cdot}1$ millimetre.

By means of the equations of the various scales given in Chap. X., $\mathbf{p}$, which is an error in length, may be evaluated in the same terms as the answer.

To keep $\mathbf{p}$ small, it is clear that $\mathbf{m}$ must be kept as small as possible, and hence great importance has been attached in the foregoing explanations to obtaining each answer with the fewest possible settings. It may be remarked that the utmost possible accuracy can only be got by sacrificing rapidity in use.

CHAPTER X.

MISCELLANEOUS NOTES.

Equations to the Various Scales.

39. The following is the mathematical description of the various scales on the instrument. The names previously given to the various scales are altered in this chapter. The object of the alteration is to assist in deciding which scales to employ in the solution of a given problem with the slide-rule.

Let **d** denote the distance between the extreme right-hand graduation of the particular scale in question (the slide not being inverted) and the graduation, the reading at which on the same scale is **s**. When the slide is run home all these extreme right-hand graduations coincide.

Let **c** be a constant common to all the scales and **n** any integer, not necessarily the same in each equation.

Taking the ruler first, we have:

(1) On its bevelled edge a scale of centimetres.

(2) **The "log" scale of the ruler**; previously termed the Top Scale (page 19).

Equation $d + cn = c \log s$ (s running from 10^n to 10^{n-2}).

(3) In the bed of the recess cut for the slide there is a scale of inches.

(4) **The "2 log" scale of the ruler**; previously termed the Bottom Scale.

Equation $d + 2cn = 2c \log s$ (s running from 10^n to 10^{n-1}).

(5) On the lower edge of the ruler a scale of inches.

Secondly, on the front of the slide we have:

(6) **The "log" scale of the slide**; previously termed the Top Scale.

Equation $d + cn = c \log s$ (s running from 10^n to 10^{n-1}).

(7) **The "2 log" scale of the slide**; previously termed the Bottom Scale.

Equation $d + 2cn = 2c \log s$ (s running from 10^n to 10^{n-1}).

Thirdly, on the back of the slide :

(8) **The "log sin" scale of the slide**; previously termed the S Scale.

Equation $d = c \log \sin s$ (s running from 90° to $34'$).

(9) **The "2 log tan" scale of the slide**; previously termed the T Scale.

Equation $d = 2c \log \tan s$ (s running from 45° to $5^\circ\, 43'$).

(10) **The "– 2" scale of the slide**; previously termed the Central Scale.

Equation $d = -2cs$ (s running from 0 to 1).

In constructing the scales we may give any arbitrary integral value to n.

In working with the instrument no notice is usually taken of the position of the decimal point, either in the data or in the result. Thus, when we bring 3 on the slide under 4 on the ruler, we do not pause to consider whether the 3 stands for 3 or 30, or what is the value of n in 3×10^n.

However, when dealing with exponentials or when using the scales on the ruler in conjunction with the trigonometrical scales, we must either attend to the value of n, that is, to the position of the decimal point, or rely on our knowledge of arithmetic or trigonometry to prevent obtaining an erroneous result.

The Mnemonic.

40. The mnemonic referred to on page 16 is useful in practice, and, when a series of results are required from a slide-rule, it is handy to work with equations of the form

$$\text{(i)} \quad \frac{y}{x} = \frac{a}{b},$$

$$\text{(ii)} \quad \frac{\;\frac{y}{1}\;}{\frac{1}{x}} = \frac{\;\frac{a}{1}\;}{\frac{1}{b}},$$

where the two variables are vertically below each other. The instrument can then be kept set to $\frac{a}{b}$ or $\frac{a}{\frac{1}{b}}$, and, for a series of given values of either **y** or **x**, the required values of **x** or **y** can be read off. When using equations of the forms above, readings are not in general taken from the indices at the back of the ruler.

The formal proof of the mnemonic is as follows: Taking the "log" scales, for example, suppose the slide pulled out at random and that two numbers **x** and **b** (lying between 10^n and 10^{n-2}) on the slide are opposite the numbers **y** and **a** (lying between 10^n and 10^{n-2}) on the ruler.

We shall refer to these positions as setting (i) if the slide is in its normal position and setting (ii) if the slide is inverted. Since the distance between any two points on the ruler is equal to the distance between the corresponding points on the slide, we have, from setting (i),

$$\log y - \log a = \log x - \log b \quad \text{or} \quad \frac{y}{x} = \frac{a}{b} \text{ (slide normal)};$$

and from setting (ii),

$$\log y - \log a = \log b - \log x \quad \text{or} \quad \frac{y}{1/x} = \frac{a}{1/b} \text{ (slide inverted)};$$

consequently, each of these settings determines the value of **y** from the corresponding equation when **a**, **b**, and **x** are given but restricted as just stated.

If now we have to determine **y** from the equations (i) or (ii), where **a**, **b**, and **x** are any numbers whatever without restriction, we notice that if we change the values of **a**, **b**, or **x**, preserving their significant figures unaltered, we may or may not change the value of **y**, but we always preserve its significant figures unaltered.

There are then certain definite numbers, lying between 10^n and 10^{n-2}, which possess the significant figures of **a**, **b**, **x**, and **y**, and for which the above settings are possible.

Hence, without troubling as to what these definite numbers are, we can find the significant figures of **y**, the required result, from the above settings, by using the significant figures of the data in setting the instrument.

In the diagrams of settings given in this chapter we have in general employed symbols such as **a** to represent "the significant figures of the definite number **a**."

By the mnemonic already referred to, equation (i) is "set" upon the slide-rule as shewn in this diagram, numbers being set and read by their significant figures only as just explained.

$$\text{(i)}\quad \frac{\mathbf{y}\quad \mathbf{a}\quad \text{on the "log" scale of the ruler}}{\mathbf{x}\quad \mathbf{b}\quad \text{on the "log" scale of the slide}},$$

while (ii) becomes

$$\frac{\mathbf{y}\quad \mathbf{a}\quad \text{on the "log" scale of the ruler}}{\mathbf{x}\quad \mathbf{b}\quad \text{on the "log" scale of the slide inverted}}.$$

We may look upon these two cases as typical. If we recollect them and also the equations to each of the scales on the ruler and slide, we should have little difficulty in deciding on a convenient manner for tackling a series of calculations. As already stated, the position of the decimal point in the result must in general be obtained apart from the instrument. Hence, in the case of a problem involving unfamiliar data, it may be advisable to have recourse at once to logarithms, if only a single answer is required. But, as will appear subsequently, if a series of results are wanted for varying data, the slide-rule may be useful.

The next dozen cases are merely variations of the two cases just given. The expressions on the right, being constant, might occur in various forms. If, in practice, they should assume a form differing from that given, as, for instance, $\frac{\mathbf{a}^2}{\mathbf{b}}$ instead of $\frac{\mathbf{a}}{\mathbf{b}}$, no difficulty will arise, for the manner of dealing with any such variations will be similar to that employed in treating the variations on the left of the equations.

Variations of Equation (i).

(i)a $\frac{\mathbf{y}^2}{\mathbf{x}} = \frac{\mathbf{a}}{\mathbf{b}}$, which, according to the mnemonic, gives

$$\frac{\mathbf{y}\quad \text{on the "2 log"}\quad \mathbf{a}\quad \text{on the "log" scale of the ruler}}{\mathbf{x}\quad \text{and}\quad \mathbf{b}\quad \text{on the "log" scale of the slide}}.$$

(i)*b* $\qquad \frac{y^2}{x^2} = \frac{a}{b}$, which gives

$$\frac{\mathbf{y} \text{ on the "2 log" } \mathbf{a} \text{ on the "log" scale of the ruler}}{\mathbf{x} \text{ on the "2 log" } \mathbf{b} \text{ on the "log" scale of the slide}}.$$

(i)*c* $\qquad \frac{\sin y}{x} = \frac{a}{b}$, which gives

$$\frac{\mathbf{y} \text{ and } 90^\circ \text{ on the "log sin" scale of the upset slide}}{\mathbf{x} \text{ and } \mathbf{k} \text{ on the "log" scale of the ruler}},$$

k being found from $k = \frac{b}{a}$.

(i)*d* $\qquad \frac{\sin y}{x} = \frac{\sin a}{b}$, which gives

$$\frac{\mathbf{y} \text{ and } \mathbf{a} \text{ on the "log sin" scale of the upset slide}}{\mathbf{x} \text{ and } \mathbf{b} \text{ on the "log" scale of the ruler}}.$$

(i)*e* $\qquad \frac{\tan y}{x}$ or $\frac{\cot(90 - y)}{x} = \frac{a}{b}$, which gives

$$\frac{\mathbf{y} \text{ and } 45^\circ \text{ on the "2 log tan" scale of the slide upset}}{\mathbf{x} \text{ and } \mathbf{k} \text{ on the "2 log" scale of the ruler}},$$

k being found from $k = \frac{b}{a}$.

(i)*f* $\qquad \frac{\tan y}{x} = \frac{\tan a}{b}$, which gives

$$\frac{\mathbf{y} \text{ and } \mathbf{a} \text{ on the "2 log tan" scale of the slide upset}}{\mathbf{x} \text{ and } \mathbf{b} \text{ on the "2 log" scale of the ruler}}.$$

Variations of Equation (ii).

(ii)*a* $\qquad \frac{y^2}{\frac{1}{x}} = \frac{a}{\frac{1}{b}}$, which gives

$$\frac{\mathbf{y} \text{ on the "2 log" } \mathbf{a} \text{ on the "log" scale of the ruler}}{\mathbf{x} \text{ and } \mathbf{b} \text{ on the "log" scale of the slide inverted}}.$$

(ii)*b* $$\frac{y^2}{\frac{1}{x^2}} = \frac{a}{\frac{1}{b}}, \text{ which gives}$$

$$\frac{\mathbf{y} \text{ on the "2 log" } \mathbf{a} \text{ on the "log" scale of the ruler}}{\mathbf{x} \text{ on the "2 log" } \mathbf{b} \text{ on the "log" scale of the slide inverted}}.$$

(ii)*c* $$\frac{\sin y}{\frac{1}{x}} = \frac{a}{\frac{1}{b}}, \text{ which gives}$$

$$\frac{\mathbf{y} \text{ and } \mathbf{90^\circ} \text{ on the "log sin" scale of the slide upset and inverted}}{\mathbf{x} \text{ and } \mathbf{k} \text{ on the "log" scale of the ruler}},$$

k being found from $k = ab$.

(ii)*d* $$\frac{\sin y}{\frac{1}{x}} = \frac{\sin a}{\frac{1}{b}}, \text{ which gives}$$

$$\frac{\mathbf{y} \text{ and } \mathbf{a} \text{ on the "log sin" scale of the slide upset and inverted}}{\mathbf{x} \text{ and } \mathbf{b} \text{ on the "log" scale of the ruler}}.$$

(ii)*e* $$\frac{\tan \overline{90 - y}}{x} \text{ or } \frac{\cot y}{x} = \frac{a}{b},$$

whence $$\frac{\tan y}{\frac{1}{x}} = \frac{b}{a},$$

which gives

$$\frac{\mathbf{y} \text{ and } \mathbf{45^\circ} \left\{\text{on the "2 log tan" scale of the slide upset and inverted}\right\}}{\mathbf{x} \text{ and } \mathbf{k} \text{ on the "2 log" scale of the ruler}},$$

k being found from $k = \frac{b}{a}$.

(ii)*f* $$\frac{\tan \overline{90 - y}}{x} \text{ or } \frac{\cot y}{x} = \frac{\cot a}{b},$$

whence $$\frac{\tan y}{\frac{1}{x}} = \frac{\tan a}{\frac{1}{b}},$$

which gives

$$\frac{\mathbf{y} \text{ and } \mathbf{a} \left\{\begin{array}{l}\text{on the "2 log tan" scale of the slide upset} \\ \text{and inverted}\end{array}\right\}}{\mathbf{x} \text{ and } \mathbf{b} \text{ on the "2 log" scale of the ruler}}.$$

Some examples, numbered to correspond with the above equations, will now be given.

Ex. (i)*a*. The uncorrected heights of several objects **y** miles off, the angles to which have been observed with a theodolite, have to be corrected for curvature and refraction. Set the slide-rule for determining **x**, the corrections in feet.

Here $\frac{y^2}{x} = \frac{13}{7}$, which gives

$$\frac{\mathbf{y} \text{ on the "2 log" } \mathbf{13} \text{ on the "log" scale of the slide}}{\mathbf{x} \text{ and } \mathbf{7} \text{ on the "log" scale of the ruler}}.$$

A few results are tabulated below:

2·13 miles.	2·44 feet.
7·62	31·25
8·37	37·5

Ex. (i)*b*. Two shells weigh respectively **45** and **50** lbs.; set the slide-rule so as to determine at what relative velocities **y** and **x**, their striking energies will be equal.

Here $\frac{y^2}{x^2} = \frac{50}{45}$, which gives

$$\frac{\mathbf{y} \text{ on the "2 log" } \mathbf{5} \text{ on the "log" scale of the ruler}}{\mathbf{x} \text{ on the "2 log" } \mathbf{45} \text{ on the "log" scale of the slide}}.$$

A few results are tabulated below:

x	y
2050	2160
1900	2003
1800	1897
1700	1792

In Siemens' Dynamometer we have $\frac{C^2}{C_2^2} = \frac{\phi_1}{\phi_2}$, which is of the same form.

Ex. (i)*e*. From the station S and the zero line SZ the angles 7° 27′, 12° 28′, 33° 16′, 43° 9′ have to be plotted. A point P on the line SZ is taken, such that an arc of 45° drawn with centre S will fall within the paper. From P a perpendicular PH is erected. Set the slide-rule so that lines drawn from S to points to be determined in PH shall give the required angles. Let the distance along the line PH = **x**, and let PS = 7·93 inches. Then

$$\tan \mathbf{y} = \frac{\mathbf{x}}{7{\cdot}93};$$

$$\therefore \frac{\tan \mathbf{y}}{\mathbf{x}} = \frac{\tan 45^\circ}{79{\cdot}3} \text{ or } \frac{10 \tan 5^\circ 43'}{7{\cdot}93},$$

which gives

$$\frac{\mathbf{y} \text{ and } 45^\circ \text{ on the "2 log tan" scale of the slide upset}}{\mathbf{x} \text{ and } 793 \text{ on the "2 log" scale of the ruler}}.$$

The numerical results are tabulated below:

y	x
7° 27′	1·04 inches.
19° 28′	2·803
33° 16′	5·202
43° 9′	7·43

Ex. (i)*f*. In a tangent galvanometer a known current C_1, say 1 milliampere, produces a deflection α, say 20°. Find the current **x** which will produce a deflection **y**.

The formula is $\frac{\mathbf{x}}{C_1} = \frac{\tan \mathbf{y}}{\tan \alpha}$ or $\frac{\mathbf{x}}{\tan \mathbf{y}} = \frac{C_1}{\tan \alpha}$,

so that we have, assuming **y** and α under 45°,

$$\frac{\mathbf{x} \text{ and } C_1 \text{ on the "2 log" scale of the ruler}}{\mathbf{y} \text{ and } \alpha \text{ on the "2 log tan" scale of the slide upset}}.$$

It may be more convenient, working as in (i)*c* and (ii)*e*, to put $\frac{C_1}{\tan \alpha} = \mathbf{k}$, when we have $\frac{\mathbf{x}}{\tan \mathbf{y}} = \frac{\mathbf{k}}{\tan 45^\circ}$, which gives

$$\frac{\mathbf{x} \text{ and } \mathbf{k} \text{ on the "2 log" scale of the ruler}}{\mathbf{y} \text{ and } 45^\circ \text{ on the "2 log tan" scale of the slide upset}}$$

for readings of **y** up to 45°.

For readings between 45° and 90°, we have

$$\frac{\text{x} \quad \text{and} \quad \text{k} \quad \text{on the "2 log" scale of the ruler}}{\text{y read backwards and } 45^\circ \left\{\begin{array}{l}\text{on the "2 log tan" scale of the} \\ \text{slide upset and inverted}\end{array}\right\}}.$$

"**y** read backwards" gives $90 - y$. This operation is done by eye on the scale, and not by subtraction on paper or by mental arithmetic For $\tan 45^\circ$ we may of course substitute $10 \tan 5^\circ 43'$ in the equations, and we may then use $5^\circ 43'$ instead of 45° on the instrument.

Ex. (ii)*a*. Given that in a hollow cylinder of 3″ inside and 8·3 outside radius, the inside radial pressure is 20 tons, and the inside circumferential tension is 26 tons, find the tension **t** and pressure **p** at radii $y = 4''$, $5''$, $6''$, and $7''$ from the formulae

$$(1)\ (26 + 20)3^2 = (p + t)y^2, \qquad (2)\ t - p = 6\,;$$

$$\therefore \frac{y^2}{\frac{1}{(p+t)}} = \frac{3^2}{\frac{1}{46}},$$

which gives

$$\frac{\text{y} \quad \text{and} \quad 3 \quad \text{on the "2 log" scale of the slide inverted}}{\text{p + t} \quad \text{and} \quad 46 \quad \text{on the "log" scale of the ruler}}.$$

The work is tabulated below:

y	p + t	$p = \frac{p + t - 6}{2}$	t = p + 6
3	46·0	20·0	26·0
4	25·9	9·95	15·95
5	16·6	5·30	11·3
6	11·5	2·75	8·75
7	8·45	1·22	7·22
8·3	6·0	0	6

Ex. (ii)*a*. Five road bearers of 6″ diameter will safely carry a load when the span is 15′. What should be the diameter **y** of five road bearers to give the same safety with the same load, if the span is 20′? The load varies as the cube of the diameter, and inversely as the length, so

$$\frac{y^3}{20}=\frac{6^3}{15}.$$

The original equation (ii)*a* is $\dfrac{y^2}{\frac{1}{x}}=\dfrac{a}{\frac{1}{b}}$.

Putting $\qquad x=y, \quad b=6^3, \quad a=\dfrac{20}{15},$

we have $$\frac{y^2}{\frac{1}{y}}=\frac{\frac{20}{15}}{\frac{1}{6^3}}=\frac{1{\cdot}333}{\frac{1}{216}},$$

which gives

$$\frac{y \text{ on the "2 log" } 1333 \text{ on the "log" scale of the ruler}}{y \text{ and } 216 \text{ on the "log" scale of the slide inverted}},$$

whence $\qquad y=6{\cdot}6''.$

Ex. (ii)*b*. The original equation (ii)*b* is $\dfrac{y^2}{\frac{1}{x^2}}=\dfrac{a}{\frac{1}{b}}$.

Put $\qquad y=\alpha, \quad \dfrac{a}{\frac{1}{b}}=q^2, \quad x=\lambda,$ and we have

α^2 on the "log,"

$$\frac{\alpha \text{ on the "2 log" } q \text{ on the "2 log" scale of the ruler}}{\lambda \text{ and } 1 \text{ on the "2 log" scale of the slide inverted}},$$

which is what is required for the solution of a cubic equation. (See Fig. 30.)

Ex. (ii)*c*. To support a weight **a** by means of a force **x** inclined to the horizontal at an angle **y**, the vertical component of the force must equal **a**, and consequently

$$x \sin y = a \quad \text{or} \quad a \operatorname{cosec} y = x,$$

whence $$\frac{x}{\frac{1}{\sin y}}=\frac{a}{\frac{1}{\sin 90^\circ}}=\frac{a/100}{\frac{1}{\sin 34'}},$$

which gives

$$\frac{\mathbf{x} \text{ and } \mathbf{a} \text{ on the ``log'' scale of the ruler}}{\mathbf{y} \text{ and } 90^\circ \text{ on the ``log sin'' scale of the slide upset and inverted}}.$$

Ex. (ii)*d*. With reference to Ex. (i), p. 38, this method might be used when **y** was over 34′.

In that example we had $\frac{121}{\mathbf{x}} = \sin \mathbf{y}$,

whence
$$\frac{\mathbf{x}}{\frac{1}{\sin \mathbf{y}}} = \frac{12100}{\frac{1}{\sin 34'}},$$

which gives

$$\frac{\mathbf{x} \text{ and } 121 \text{ on the ``log'' scale of the ruler}}{\mathbf{y} \text{ and } 34' \text{ on the ``log sin'' scale of the slide upset and inverted}}.$$

Ex. (ii)*e*. Referring to Ex. (ii)*c*, the horizontal force $\mathbf{x}_0$ necessary to balance the horizontal component of **x** is

$$\mathbf{x}_0 = \mathbf{a} \cot \mathbf{y};$$

$$\therefore \frac{\tan \mathbf{y}}{\frac{1}{\mathbf{x}_0}} = \frac{\tan 45^\circ}{\frac{1}{\mathbf{a}}} \quad \text{or} \quad \frac{\frac{1}{\tan \mathbf{y}}}{\mathbf{x}_0} = \frac{\frac{1}{\tan 45^\circ}}{\mathbf{a}},$$

which gives

$$\frac{\mathbf{y} \text{ and } 45 \left\{\begin{array}{l}\text{on the ``2 log tan'' scale of the slide inverted}\\ \text{and upset}\end{array}\right\}}{\mathbf{x}_0 \text{ and } \mathbf{a} \text{ on the ``2 log'' scale of the ruler}}.$$

Ex. (ii)*f*. Required in inches the distances between the contours on a map drawn to a scale of $\frac{1}{10560}$ for angles of slope of 6°, 8°, 10°, and 12°, the contours on the ground being spaced at 20-feet vertical intervals.

Calling the required distances **x** and the angles **y**, we have

$$\tan \mathbf{y} = \frac{6{\cdot}67 \times 36}{\mathbf{x} \times 10560},$$

whence
$$\frac{\tan \mathbf{y}}{\frac{1}{\mathbf{x}}} = \frac{6{\cdot}67 \times 36}{10560} = \frac{1}{\frac{1}{{\cdot}02273}} = \frac{10 \tan 5^\circ\, 43'}{\frac{1}{{\cdot}02273}},$$

which gives

$$\frac{\mathbf{y} \text{ and } 5^\circ 43' \left\{\text{on the “2 log tan” scale of the slide upset and inverted}\right\}}{\mathbf{x} \text{ and } 2273 \text{ on the “2 log” scale of the ruler}}.$$

The results are tabulated below:

y	x
6°	·2166 inch.
8°	·1638
10°	·129
12°	·107

Variation of the Mnemonic when the Indices at the Back of the Ruler are used.

41. If we wish to use the indices at the back of the ruler we must transpose our equations, so that unity may be under the sine or tangent required or given. Thus, in Fig. 36, p. 55, we have

$$\mathbf{x} = \text{cosec}\, \mathbf{a} = \frac{1}{\sin \mathbf{a}};$$

$$\therefore \frac{\mathbf{x}}{1} = \frac{1}{\sin \mathbf{a}},$$

which gives

$$\frac{\mathbf{x} \text{ and } 1 \text{ on the “log” scale of the ruler}}{1 \left\{\text{on the right-hand back index}\right\} \alpha \text{ on the “log sin” scale of the slide}}.$$

From the same figure, we have

$$\mathbf{x} = \sin^{-1} \frac{\mathbf{a}}{\mathbf{b}};$$

$$\therefore \sin \mathbf{x} = \frac{\mathbf{a}}{\mathbf{b}},$$

which gives

$$\frac{\mathbf{x} \text{ on the “log sin” scale}}{1 \text{ on the right-hand back index}} \quad \frac{\mathbf{a} \text{ on the “log” scale of the slide}}{\mathbf{b} \text{ on the “log” scale of the ruler}},$$

and so on.

The reader will find no difficulty in applying this method to examples involving tangents, cotangents, secants, and cosines, but the use of the indices at the back of the ruler is apt to lead to re-settings, and so to waste time.

Example illustrating Reading backwards.

42. If **y** is to be found from the equation

$$\mathbf{y} = 1 - \mathbf{a}\left(\frac{\mathbf{b}}{\mathbf{c}}\right)^2,$$

for slide-rule use it would not be convenient to write the above

$$\mathbf{y} = \frac{\mathbf{c}^2 - \mathbf{ab}^2}{\mathbf{c}^2} \quad \text{for a series of readings.}$$

We can obtain **y** directly from one setting by reading the top scale of the ruler backwards; that is, we write the equation

$$\frac{1 - \mathbf{y}}{\mathbf{a}} = \frac{\mathbf{b}^2}{\mathbf{c}^2},$$

which gives

$$\frac{\mathbf{y}\ \left\{\begin{matrix}\text{read backwards} \\ \text{on the “log”}\end{matrix}\right\} \qquad \mathbf{b}\ \text{ on the “2 log” scale of the ruler}}{\mathbf{a}\ \text{ on the “log”} \qquad \mathbf{c}\ \text{ on the “2 log” scale of the slide}}.$$

Another Example of Reading backwards.

If we have $\mathbf{x} = \mathbf{k} \sec^2\theta,$

we transform it into

$$\sqrt{\mathbf{x}} = \frac{\sqrt{\mathbf{k}}}{\cos\theta} = \frac{\mathbf{c}}{\cos\theta}, \text{ suppose,}$$

whence

$$\frac{\sqrt{\mathbf{x}}}{\dfrac{1}{\sin(90^\circ - \theta)}} = \frac{\mathbf{c}}{\dfrac{1}{\sin 90^\circ}},$$

which gives

$$\frac{\sqrt{\mathbf{x}}\ \text{ and }\ \mathbf{c}\ \text{ on the “log” scale of the ruler}}{\theta\ \text{ and }\ 90^\circ\ \left\{\begin{matrix}\text{read backwards on the “log sin” scale of the} \\ \text{slide upset and inverted}\end{matrix}\right\}}.$$

x can subsequently be read by means of a separate operation with the cursor.

Other Examples of Transposing an Equation.

43. As already remarked, whenever a long series of calculations of the same nature has to be performed, it is advisable to consider the most convenient form in which to express the equation for slide-rule use.

Some of the equations in this chapter present a singular appearance to a mathematician. But there is a reason for the forms adopted, namely, to compel the equations to disclose by their appearance a convenient mode of dealing with them on the instrument.

A. If we have $\theta = \sec^{-1}\sqrt{\frac{x}{k}}$, and require θ for a series of given values of x, we may proceed as follows:

$$x = k\sec^2\theta;$$

$$\therefore \frac{x-k}{k} = \tan^2\theta;$$

$$\therefore \frac{x-k}{\tan^2\theta} = \frac{k}{\tan^2 45^\circ},$$

which gives, if θ is less than 45°,

$$\frac{x-k \text{ and } k \text{ on the ``log'' scale of the ruler}}{\theta \text{ and } 45^\circ \text{ on the ``log tan'' scale of the slide}}.$$

If θ is greater than 45°, we have

$$\frac{x-k}{\frac{1}{\tan^2(90-\theta)}} = \frac{k}{\frac{1}{\tan^2 45^\circ}},$$

which gives

$$\frac{x-k \text{ and } k \text{ on the ``log'' scale of the ruler}}{\theta \text{ read backwards and } 45^\circ \left\{\begin{array}{l}\text{on the ``2 log tan'' scale of}\\ \text{the slide upset and inverted}\end{array}\right\}}.$$

If k is an integer, this is clearly a convenient form to use.

B. Again, if $x = k\sqrt{\sin y}$,

we have $$\frac{x^2}{\sin y} = \frac{k^2}{\sin 90^\circ},$$

which gives

$$\frac{x \text{ and } k \text{ on the ``2 log'' scale of the ruler}}{y \text{ and } 90^\circ \text{ on the ``log sin'' scale of the slide upset}}.$$

C. If an equation assumes the form

$$x = \sqrt{\frac{y^3}{k}},$$

we have

$$\frac{x^2}{\frac{1}{k}} = \frac{y^2}{\frac{1}{y}},$$

which gives

$$\frac{\mathbf{x} \text{ and } \mathbf{y} \text{ on the ``2 log'' scale of the ruler}}{\mathbf{k} \text{ and } \mathbf{y} \text{ on the ``log'' scale of the slide inverted}}.$$

D. To support a weight **w** by means of two ropes of diameter **x**, inclined to the horizontal at an angle **y**, we must have, if **t** is the tension on each rope per square inch of cross section,

$$\frac{2\pi d^2}{4} t \sin y = w$$

or

$$d^2 = a \operatorname{cosec} y \quad (a \text{ being a constant}),$$

$$\frac{d^2}{\frac{1}{\sin y}} = \frac{a}{\frac{1}{\sin 90^\circ}},$$

so that we have

$$\frac{\mathbf{d} \text{ on the ``2 log''} \quad \mathbf{a} \quad \text{on the ``log'' scale of the ruler}}{\mathbf{y} \quad \text{and} \quad 90^\circ \quad \left\{\begin{matrix}\text{on the ``log sin'' scale of the slide} \\ \text{upset and inverted}\end{matrix}\right\}}.$$

E. Given (i) $p = \frac{v}{608 \cdot 3}\sqrt{\frac{w}{d}} - \cdot 14d$

and (ii) $p = \frac{wv^3}{693000000d},$

find **p**, the penetration in inches with wrought iron, by formula (ii), for values of **v**, 2000 f.-s., 1900 f.-s., 1800 f.-s., 1700 f.-s., 1600 f.-s.; and, by formula (i), for values of **v** in 100's below 1600 f.-s., given $w = 380$ and $d = 9 \cdot 2$. Formula (ii) may be written, disregarding decimal points,

$$\frac{p^2}{v^2} = \frac{vw}{693d} = \frac{v}{1675},$$

whence

$$\frac{\text{p on the "2 log" \quad v \quad on the "log" scale of the ruler}}{\text{v on the "2 log" \quad 1675 \quad on the "log" scale of the slide}}.$$

v	p
2000 f.-s.	21·8 inches
1900	20·2
1800	18·6
1700	17·1
1600	15·6

Formula (i) may be written, disregarding the decimal point,

$$\frac{(p+\cdot 14d)^2}{v^2}=\frac{w}{37d}=\frac{1}{897},$$

whence

$$\frac{\text{(p + ·14d) on the "2 log" and \quad 1 \quad on "log" scale of the ruler}}{\text{v \quad on the "2 log" and \quad 897 \quad on "log" scale of the slide}}.$$

v	p + ·14d = p + 1·29	p
1500 f.-s.	15·80	14·51 inches
1400	14·80	13·51
1300	13·74	12·45
1200	12·67	11·38
1100	11·63	10·34
1000	10·57	9·28

Exponentials.

44. The examples which follow deal with exponentials, some of which might be more conveniently treated with the E scales (p. 74 *et seq*). It will be seen, however, that no great difficulties arise even with the ordinary instrument.

If we have to find **y** from

$$y = ad^{bx} = a\,10^{bx \log d},$$

then

$$\log y = \log a + bx \log d.$$

Suppose the decimal portion of **bx** log **d** to be ± **m**, **m** being essentially a positive quantity. Omitting the characteristics or integral portions of the logarithms, we have either

$$\log y = \log a + m$$

or

$$\log y = \log a - m,$$

and we employ the central or " − 2 " scale to effect the addition or subtraction of **m**.

If **bx** log **d** is positive, we therefore have

$$y = a\,10^{m}$$

or

$$\frac{y^2}{1/10^{(-2)m}} = \frac{a^2}{1/1},$$

whence we have, by the mnemonic,

$$\frac{\mathbf{y} \text{ and } \mathbf{a} \quad \text{on the "2 log" scale of the ruler}}{\mathbf{m} \text{ and } 1 \text{ or } 0 \quad \left\{\begin{array}{c}\text{on the "}-2\text{" scale of the slide upset and}\\ \text{inverted}\end{array}\right\}},$$

or, again, we may write the equation

$$\frac{y^2}{1} = \frac{a^2}{10^{(-2)m}},$$

whence, by the mnemonic,

$$\frac{\mathbf{y} \text{ and } \mathbf{a} \quad \text{on the "2 log" scale of the ruler}}{1 \text{ or } 0 \text{ and } \mathbf{m} \quad \text{on the "}-2\text{" scale of the slide upset}}.$$

Similarly, if **bx** log **d** is negative, we have

$$\frac{y^2}{10^{-2(m)}} = \frac{a^2}{1},$$

whence, by the mnemonic,

$$\frac{\mathbf{y} \text{ and } \mathbf{a} \quad \text{on the "2 log" scale of the ruler}}{\mathbf{m} \text{ and } 1 \text{ or } 0 \quad \text{on the "}-2\text{" scale of the slide upset}},$$

or we may write the equation

$$\frac{y^2}{1/1} = \frac{a^2}{1/10^{(-2)m}},$$

whence

y	and	**a**	on the "2 log" scale of the ruler
1 or **0**	and	**m**	on the "-2" scale of the slide upset and inverted

If the problem is reversed, and being given **y** we require to find **x**, we can determine the value of **m** by one of the above settings, and thus the *decimal* portion of **bx** log **d** is known.

The characteristic or integral portion, **i**, may then be obtained by inspection or by the roughest approximation. Then **x** may be found by the ordinary use of the slide-rule from

$$bx \log d = i + m.$$

1. A wet rope is coiled round a cylindrical post, the coefficient of friction being ·31. Neglecting the stiffness of the rope, find the weights which could be held by a force of 25 lbs., the rope making 1, 2, 3, 4, 5, or 6 complete turns.

The formula is $T_1 = T_0 e^{\mu x} = 25 \times 10^{\frac{\cdot 31x}{2\cdot 3}}$,

and, arranging the work in a tabular form, we have

x	bx log e $= \frac{\cdot 31x}{2\cdot 3}$	y
2π	·82	165 pounds
4π	1·64	1090
6π	2·46	7205
8π	3·28	47600
10π	4·1	314600
12π	4·92	2075000

Of course the results in the last three cases are of merely theoretical interest. We may note, as a check, that these results are in geometric progression, and set the instrument in the ordinary way, so as to read off the series thus :

$$\frac{165}{1090} = \frac{1090}{7210} = \frac{7210}{47600}, \text{ etc.}$$

2. The "logarithmic decrement" λ is found by the equation

$$\frac{c_1}{c_n} = e^{(n-1)\lambda},$$

where c_1 and c_n are the amplitudes of the first, and nth, semi-vibrations (see Glazebrook and Shaw, *Practical Physics*).

If we suppose $n = 10$; $c_1 = 210$; $c_n = 94$;

we have $$\frac{210}{94} = e^{9\lambda} = 10^{3 \cdot 91\lambda}$$

From the slide-rule we get

210 and 94	on the "2 log" scale of the ruler	
349 and 1	on the "– 2" scale of the slide upset and inverted.	

Hence the integral portion of $3{\cdot}91\lambda$ being evidently zero, we have

$$3{\cdot}91\lambda = {\cdot}349,$$

which gives $$\lambda = {\cdot}0893.$$

3. A uniform flexible chain is suspended from two points in the same horizontal line and at a distance $2a$ apart. The lowest point of the chain is at a distance d below the points of suspension.

The equation of the curve is

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right).$$

Let $$\frac{y}{c} = \frac{u}{2} \text{ and } \frac{x}{c} = v.$$

Then $$u = e^{v} + e^{-v}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

an equation common to all curves of this kind.

Now when $x = 0$, $y = c$,

and by the data when $x = a$, $y = c + d$,

and $$\frac{c + d}{c} = \frac{u_0}{2}; \quad \frac{a}{c} = v_0, \text{ suppose.} \ldots\ldots\ldots\ldots (2)$$

To plot the curve (1), we have

$$u = e^v + e^{-v} = 10^{\frac{v}{2\cdot3}} + 10^{-\frac{v}{2\cdot3}}.$$

Writing m for the decimal portion of $\frac{v}{2\cdot3}$ and putting

$$10^m = z_1 \quad \text{and} \quad 10^{-m} = z_2,$$

we have

$$\frac{1}{z_1^2} = \frac{10^{(-2)m}}{1} = \frac{z_2^2}{1},$$

whence z_1 may be found by the setting

$$\frac{z_1 \text{ on the "2 log" scale}}{1 \text{ on the "2 log" scale}} \quad \text{and} \quad \frac{1 \left\{\begin{matrix}\text{on the central back} \\ \text{index of the ruler}\end{matrix}\right\}}{m \left\{\begin{matrix}\text{on the "}-2\text{" scale} \\ \text{of the slide}\end{matrix}\right\}},$$

while the same setting gives, without shifting the slide,

$$\frac{1 \text{ on the "2 log" scale}}{z_2 \text{ on the "2 log" scale}} \quad \text{and} \quad \frac{1 \left\{\begin{matrix}\text{on the back index} \\ \text{of the ruler}\end{matrix}\right\}}{m \left\{\begin{matrix}\text{on the "}-2\text{" scale} \\ \text{of the slide}\end{matrix}\right\}}.$$

An index may be marked on the left-hand back of the ruler, so that the slide can be read when out to the left.

There are, of course, many other settings that will give the same results. Some of them will lead to the central scale being read backwards, some to the cursor being required near one end of the ruler. The position of the decimal point in z and z_2 is fixed as already explained.

A few readings are given below:

v	$\frac{v}{2\cdot3}$	$z_1 = e^v$	$z_2 = e^{-v}$	u
·1	·0434	1·105	·91	2·015
·2	·0868	1·224	·817	2·041
·3	·1302	1·351	·74	2·091
·4	·1736	1·491	·671	2·162
...	...	...	...	...

We are now in a position to plot the curve whose equation is

$$u = e^{v} + e^{-v}.$$

Next we find u_0, v_0 graphically using the plotted curve, and the equation obtained from (2) by eliminating c, namely:

$$\frac{\frac{u_0}{2} - 1}{d} = \frac{v_0}{a}.$$

From the value of v_0 we have c from the equation (2), and the catenary can be drawn.

In like manner we may draw the two catenaries which have the same directrix and points of suspension; a problem which arises in considering the stability of liquid films. (Minchin, *Statics*, vol. 2, p. 187.)

If the whole length of the chain is required, we have

$$S = c\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right) = c\left(10^{\frac{a}{2\cdot 3c}} - 10^{-\frac{a}{2\cdot 3c}}\right) = 2\sqrt{d^2 + 2cd},$$

and putting $2v$ for the decimal portion of $\frac{a}{2\cdot 3c}$, and taking

$$z_1 = c\,10^{2v} \text{ and } z_2 = c\,10^{-2v},$$

we have $$\frac{z_1}{c} = \frac{1}{10^{-2v}} = \frac{c}{z_2},$$

whence we have

$$\frac{z_1 \text{ on the "log" scale and } 1 \begin{Bmatrix}\text{on the central back index of} \\ \text{the ruler}\end{Bmatrix}}{c \text{ on the "log" scale and } v \text{ on the "}-2\text{" scale of the slide}},$$

and, without shifting the slide,

$$\frac{c \text{ on the "log" scale and } 1 \begin{Bmatrix}\text{on the central back index of} \\ \text{the ruler}\end{Bmatrix}}{z_2 \text{ on the "log" scale and } v \text{ on the "}-2\text{" scale of the slide}}.$$

4. We have approximately

$$q = \left\{1 - (\cdot 75)^{\frac{z^2}{z_0^2}}\right\}^{\frac{1}{2}},$$

where q is the chance of hitting a target, affording a zone z, z_0

the "50 % zone" when all causes of error are taken into account.

From the above

$$1 - q^2 = 10^{\left(\frac{z^2}{z_0^2}\log(\cdot 75)\right)} = 10^{-\cdot 1248\frac{z^2}{z_0^2}},$$

and by the mnemonic, putting the decimal portion of

$$\cdot 1248\frac{z^2}{z_0^2} = m,$$

we have $$\frac{(1-q^2)^2}{10^{-2m}} = \frac{1}{1},$$

whence

$$\frac{q^2 \text{ read backwards and } 1 \text{ on the "2 log" scale of the ruler}}{m \qquad \text{and} \qquad 1 \text{ on the "}-2\text{" scale of the slide upset}}.$$

Having found q^2, we obtain q by taking the square root in the ordinary manner.

Whenever the chance q is under ·3 in the formula, $q = \cdot 538\frac{z}{z^0}$ is to be preferred.

If $\cdot 1248\frac{z^2}{z_0^2}$ is greater than unity, the decimal value of q^2 will commence with as many "9's" as there are integers in the expression $\cdot 1248\frac{z^2}{z_0^2}$.

5. If $$y = e^{-bx}\sin ax$$
$$= 10^{-bx\log e}\sin ax,$$

and it is required to find y when x is given, we have

$$\log y = \log \sin ax - bx \log e.$$

Putting θ for that angle between 0° and 90° which has its sine numerically equal to sin ax and $2m$ for the decimal portion of $bx \log e$, m being essentially positive, we have (omitting the characteristics or integral portions of the logarithms)

$$\log y = \log \sin \theta - 2m.$$

We may write this equation

$$\frac{y}{\sin\theta} = \frac{1}{10^{-(-2m)}},$$

whence we have

$$\frac{\mathbf{1} \quad \text{and} \quad \mathbf{y} \quad \text{on the "log" scale of the ruler}}{\mathbf{m} \left\{\begin{matrix}\text{on the "}-2\text{" scale}\\ \text{read backwards}\end{matrix}\right\} \text{and } \theta \left\{\begin{matrix}\text{on the "log sin" scale of}\\ \text{the slide upset}\end{matrix}\right\}},$$

or, as an alternative, we may write the equation

$$\frac{\mathbf{y}}{1/10^{-(-2)\mathbf{m}}} = \frac{1}{1/\sin\theta},$$

which gives

$$\frac{\mathbf{y} \quad \text{and} \quad \mathbf{1} \quad \text{on the "log" scale of the ruler}}{\mathbf{m} \left\{\begin{matrix}\text{on the "}-2\text{" scale}\\ \text{read backwards}\end{matrix}\right\} \text{and } \theta \left\{\begin{matrix}\text{on the "log sin" scale of the}\\ \text{slide upset and inverted}\end{matrix}\right\}}.$$

The sign of $\mathbf{y}$ in the final result is the same as that of $\sin \mathbf{ax}$.

If

$$\mathbf{y} = e^{-\cdot 5\mathbf{x}} \sin(153\mathbf{x})^\circ$$

$$= 10^{-\frac{\cdot 5}{2\cdot 3}\mathbf{x}} \sin(153\mathbf{x})^\circ,$$

we have

$$\log \mathbf{y} = \log \sin(153\mathbf{x})^\circ - \cdot 2175\mathbf{x},$$

and we may arrange the work in a tabular form. A few results are given:

x	m = ·2175x	153x	θ	y
·1	·01087	15·3		·2517
·3	·0326	30·6		·619
·5	·9544	45·9		·757
·7	·0761	76·5		·7
...	...	...	...	...
1·5	·163	229·5	49·5	– ·3597
...	...	...	...	...

Owing to the difficulty of using the cursor at the end of the ruler, it will be found handy to use the second method of setting the slide-rule.

Some Trigonometrical Equations.

45. To solve the equation

$$\cot \mathbf{x} = \mathbf{a} - \mathbf{x}.$$

Taking as an example $\mathbf{a} = 3$, we can obtain a rough approximation by drawing the graphs of $\mathbf{y} = \cot \mathbf{x}$ and of $\mathbf{y} = 3 - \mathbf{x}$ and finding where they intersect.

Using two slide-rules, we write the equation

$$(3 - \mathbf{x}) \tan \left(\frac{180\, \mathbf{x}}{\pi}\right)^{\circ} = 1,$$

and set the instruments as follows:

$$(1) \quad \frac{180^\circ \quad \text{and} \quad \theta \quad \text{on the "log" scale of the ruler}}{\pi \quad \text{and} \quad \mathbf{x} \quad \text{on the "log" scale of the slide}}.$$

$$(2) \quad \frac{45^\circ \quad \text{and} \quad \theta \quad \left\{\begin{array}{c}\text{on the "2 log tan" scale of the slide} \\ \text{upset and inverted}\end{array}\right\}}{1 \quad \text{and} \quad \mathbf{x} \quad \left\{\begin{array}{c}\text{read backwards from 3 on the "2 log"} \\ \text{scale of the ruler}\end{array}\right\}}.$$

We thus find by trial where the coincidence, *i.e.* the simultaneous readings of $\mathbf{x}$ and θ occur, reading the value of $\mathbf{x}$ in circular measure and its equivalent θ in degrees. With two people, one for each slide-rule, the value of $\mathbf{x}$ is readily found. A further approximation may be found if required by the well-known method based on Taylor's theorem.

Alternative Method. The equation being $\cot \mathbf{x} = 3 - \mathbf{x}$ (where $\mathbf{x}$ is, of course, in circular measure), we write it

$$\frac{180^\circ}{\pi} \cot \theta = \frac{180^\circ}{\pi} \times 3 - \theta, \quad \text{or} \quad 57{\cdot}3^\circ \cot \theta = 171{\cdot}9^\circ - \theta,$$

so that $\mathbf{x}$ is the circular measure of θ, which is in degrees.

This gives

$$\frac{\theta \quad \text{and} \quad \left\{\begin{array}{c}5^\circ 43' \text{ on the "2 log tan" scale of the slide} \\ \text{upset and inverted}\end{array}\right\}}{171{\cdot}9 - \theta \quad \text{and} \quad 57{\cdot}3 \text{ on the "2 log" scale of the ruler}}.$$

$171{\cdot}9 - \theta$ is read as θ by counting θ backwards on the 2 log scale of the ruler from $171{\cdot}9$, and the coincidence of readings gives θ, say $20{\cdot}7^\circ = 20^\circ\, 42'$. Other solutions are $341^\circ\, 19'$; $531^\circ\, 56'$.

Similarly, to solve $x - \sin x = a = \frac{1}{4}$, suppose, we write the equation

$$\frac{180^\circ}{\pi} \sin\theta = \theta - \frac{180^\circ}{4\pi} \quad \text{or} \quad 57{\cdot}3^\circ \sin\theta = \theta - 14{\cdot}3,$$

x being the circular measure of θ, which is in degrees.

Whence

$$\frac{\theta - 14{\cdot}3 \text{ and } 57{\cdot}3^\circ \text{ on the "log" scale of the ruler}}{\theta \text{ and } 90^\circ \text{ on the "log sin" scale of the slide upset}}.$$

We can read $\theta - 14{\cdot}3$ as θ by counting the numbers on the scale as 14·3 more than they are marked, thus reading 24·3 at 10, 25·3 at 11, and so on.

The coincidence of readings (θ on the log sin scale and θ read mentally from the log scale) fixes θ, say $67{\cdot}33^\circ = 67^\circ\ 20'$. The problem occurs in constructing maps on Mollweides' projection.

Ex. 1. A long cylinder, diameter 12 inches, specific gravity ·3, floats in water with its axis horizontal. What is the length of the water line on its ends?

Ex. 2. A circular arc is 12 feet long and its chord 10 feet long. What is its area?

Ex. 3. Solve the equations

$$\cos x = x, \quad \tan x = x.$$

Hyperbolic Functions.

46. The equation $\cdot 4343u = \log_{10} \tan\left(45^\circ + \frac{v}{2}\right)$ enables the value of v to be found when u is given. We then have

$$\sinh u = \mathbf{sh}\,(u) = \tan v.$$
$$\cosh u = \mathbf{ch}\,(u) = \sec v.$$
$$\tanh u = \mathbf{th}\,(u) = \sin v.$$

Thus the values of the hyperbolic functions of u can be found on an emergency; but the employment of log sinh and log cosh scales would obviously be more convenient.

Ex. If $u = \cdot 336$,

then $v = 18^\circ\ 50'$, and $\sinh u = \cdot 3424$, $\cosh u = 1{\cdot}0570$.

MISCELLANEOUS EXAMPLES.

1. Find how much chlorine (in litres at 0° C. and 760 mm. pressure) can be obtained by treating ten pounds of manganese dioxide containing 7 per cent. of impurities with excess of hydrochloric acid.

2. How much potassium chlorate would be required to prepare 200 cubic feet of oxygen at 0° C. and 760 mm. pressure? Set the instrument to give the volume of this quantity of gas at t° C. and n millimetres pressure from its volume at 0° C. and 760 mm. pressure. Take $t = 10° 53'$, $n = 787$.

3. Find the percentage composition of calcium sulphate:

Ca	40
S	32
O_4	64
	136

4. The R.F. of a map is $\frac{1}{a}$ and the R.F. of a new map of the same country is $\frac{1}{b}$.

If a is 25000 and b 10600, what are the scales in inches to the mile?

If a certain distance represents 500 yards on the first map, how many does it represent on the second map?

The size of a map is $15'' \times 12''$.

What will be the sides of a new map of the same extent of country, but of $\frac{3}{5}$ths of the area of the old map?

Evidently the sides of the new map will be $15\sqrt{\frac{3}{5}}$ and $12\sqrt{\frac{3}{5}}$.

If 3 inches represent 1000 yards on the first map, what does 3 inches represent on the second map?

Given that a scale of 6″ to one mile has a representative fraction of $\frac{1}{10560}$, set the slide-rule to read the "inches to a mile," for a scale whose representative fraction is $\frac{1}{1970}$.

A map is drawn to a scale of $\frac{1}{25000}$; set the slide-rule so as to give miles on the ground in terms of centimetres on the map.

5. Calculate the distance of the horizon at sea from the formula

$$d^2 = 2ha,$$

where d = distance of horizon,
a = earth's radius = 3970 miles approximately,
h = observer's height above sea-level.

Verify the rules*

Distance in miles $= \frac{5}{4}\sqrt{\text{height in feet}}$.
Height in feet $= \frac{2}{3}(\text{distance in miles})^2$.

6. If the altitude of a star is observed with reference to the visible horizon by a person h feet above sea-level, the apparent altitude is increased by the "dip" which is given by the formula

Dip in circular measure $= \sqrt{\frac{2h}{a}}$, where a is the earth's radius in feet.

Shew that Dip in minutes $= 1{\cdot}06\sqrt{h}$ nearly.

7. If the effect of refraction is to decrease the dip and increase the distance of the visible horizon each by 8 per cent., amend the rules in Questions 5 and 6 to allow for this.

8. Owing to refraction, the apparent altitude α of a star exceeds its true altitude by $y = 57 \cot \alpha$ seconds nearly. Find y when

$$\alpha = 10^\circ,\quad 20^\circ,\quad 30^\circ,\quad 40^\circ,\quad 60^\circ,\quad 80^\circ.$$

If the apparent horizon is used, there will be a further correction for refraction at the horizon.

9. For angles of slope not exceeding 20°, if we regard the tangent and circular measure as equal, we have

$$\frac{\text{Vertical interval in yards}}{\text{Horizontal equivalent in yards}} = \frac{\theta^\circ \times \pi}{180^\circ}.$$

Calculate the H.E. for V.I. of 20 feet and angles of slope of 1°, 2°, 3°, 4°, 7°, 10°, and 20°.

Notice that $\frac{\text{V.I. in } \textit{feet}}{\text{H.E. in yards}} = \frac{\theta^\circ}{19{\cdot}1}$ nearly,

a formula sometimes employed.

* The employment in this and other examples of two different units for the measurement of the same physical quantity (here a length) cannot be defended. But in some of these cases the practice, though a vicious one, is so established that we have submitted to it.

10. Calculate the error committed by employing the formula of Qn. 9 for an angle of 20°.

[It will be noticed that the calculation in Qn. 10 is just as easy on the slide-rule as that of Qn. 9, and that consequently nothing is gained in practice by adopting the approximate method of Qn. 9.]

11. On the "normal system" the distance on the map between the contour lines is the same for the same gradient, whatever the scale of the map.

If on a scale of 6 inches to the mile the V.I. is 20 feet, what will be the V.I. on a scale of 4 inches to the mile and on a scale whose R.F. is $\frac{1}{5000}$?

12. If distances are measured on sloping ground a reduction is required to obtain the true horizontal distances before plotting on a map. Find the percentage reductions for angles of slope of 1°, 2°, 5°, 10°, and 20°.

[Notice that if $\mathbf{d}$ is the distance on the slope, and the inclination of the slope is θ, the true horizontal distance is $\mathbf{d}\cos\theta$, the reduction is $\mathbf{d}(1-\cos\theta)$ or $2\mathbf{d}\sin^2\frac{\theta}{2}$, and the percentage reduction $200\sin^2\frac{\theta}{2}$.]

13. An observer A at a height $\mathbf{h}$ feet above the sea notes the angle of depression δ minutes of a distant object and the azimuth angle θ of the object from a station B.

The horizontal distance R_A of the object from A is given in feet by

$$\frac{\mathbf{h}}{R_A} = \tan\delta = \frac{\delta}{3440} \text{ nearly},$$

and the horizontal distance of B from the object is

$$R_B' = R_A - AB\cos\theta \text{ nearly}.$$

Compute R_A in yards from

$$\mathbf{h} = 73 \text{ feet}.$$

$\delta = 10,\ 17,\ 25,\ 40,\ 60,\ 70$ minutes.

Work out the corrections for R_B if

$$AB = 174 \text{ yards},$$

$$\theta = 10^\circ,\ 20^\circ,\ 30^\circ,\ 50^\circ,\ 70^\circ.$$

14. Work out the true values of R_A and R_B when $\delta = 70'$, $\theta = 10°$, and compare with the results of Qn. 13.

15. If $\mathbf{h}$ = height of tide at high water above a datum point, $\mathbf{a}$ = height of mean sea-level above the datum point, then the correction to be added to, or subtracted from, $\mathbf{a}$ to find the height of the tide $\mathbf{t}$ hours after high water is

$$(\mathbf{h} - \mathbf{a}) \cos\left(\frac{\mathbf{t} \times 180°}{\mathbf{n}}\right) \text{ approximately,}$$

$\mathbf{n}$ being the interval between high and low water in hours.

Compute the height $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$ hours after high water, taking

$$\mathbf{n} = 6, \qquad \mathbf{h} = 20, \qquad \mathbf{a} = 8,$$

all heights being in feet.

From the *Admiralty Tide Tables* we find that on December 25th, 1900, it was high tide at Dover at 0 h. 58 m. a.m. and at 1 h. 20 m. p.m.,

that $\mathbf{h} = 18$ ft. 6 in.
and $\mathbf{a} = 9$ ft. $4\frac{1}{2}$ in.,

Find the height of the tide at 10 a.m.

16. The difference of height in feet H, between two places, is given by the formula

$$H = 60370\left(1 + \frac{\mathbf{t}_1 + \mathbf{t}_2}{500}\right) \log \frac{\mathbf{h}_2}{\mathbf{h}_1},$$

where $\mathbf{t}_1$ = Centigrade temperature at upper station,
$\mathbf{t}_2$ = " " lower "
$\mathbf{h}_1$ = height of mercury barometer at upper station,
$\mathbf{h}_2$ = " " " lower "

Find H when

$\mathbf{t}_1 = 15°$ C., $\mathbf{h}_2 = 30$ inches,
$\mathbf{t}_2 = 19°$ C., $\mathbf{h}_1 = 29{\cdot}5$, $29{\cdot}8$, $29{\cdot}9$, 29 inches.

Compare the results obtained from the approximate formulae:

$$\text{(i)} \quad H = 60000 \log \frac{\mathbf{h}_2}{\mathbf{h}_1},$$

$$\text{(ii)} \quad H = 55000 \frac{\mathbf{h}_2 - \mathbf{h}_1}{\mathbf{h}_2 + \mathbf{h}_1}.$$

17. The ordinary slide-rule is not, generally speaking, sufficiently accurate for dealing with astronomical or important surveying observations. It may, however, serve if rough approximations only are required, or if rapid checks are desired of the leading figures in results which have been calculated from tables.

The time of rising of the sun is given by

$$\cos(15t)^\circ = \tan\alpha \tan\lambda, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where t = number of hours from midnight to sunrise,

α = sun's declination,

λ = latitude of place of observation.

The sun's declination is given approximately by the equation

$$\sin\alpha = \sin\frac{360n}{365}\sin 23^\circ\ 30', \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where n = number of days which have elapsed since March 21st.

Find time of sunrise on May 1st in latitude 50° N.

This method is only approximate, as (i) the effect of *refraction* has been disregarded; (ii) the result gives the *apparent*, not *mean*, time of rising; (iii) the formula (2) is extremely rough.

18. The equation $x^4 - 54x^2 + 96x - 51 = 0$

has a root between 6 and 7. Find a closer approximation to this root.

19. When a star is due east or west, the latitude λ of the place of observation is given by the formula

$$\sin\lambda = \frac{\sin\delta}{\sin\alpha},$$

where δ = declination of star, from *Nautical Almanac*,

α = altitude of star, by observation, corrected for refraction and, if necessary, dip.

Take $\delta = 16^\circ\ 20'$ (Aldebaran),

$\alpha = 29^\circ\ 21'$,

and find λ.

The objection to this method is that, outside a fixed observatory, it is difficult to catch the star exactly due E. or W.

The reader will easily verify that every case of the solution of right-angled spherical triangles depends on two formulae:

$$\cos\alpha = \cos\beta\cos\gamma$$

or

$$\cos\alpha = \cot\beta\cot\gamma,$$

and consequently any right-angled spherical triangle can be solved with a slide-rule.

In a sundial in latitude λ, the graduation for **t** o'clock makes an angle θ with the N.-S. line, where

$\tan\theta = \sin\lambda\tan(15t)^\circ$ for a horizontal dial,

$\tan\theta = \cos\lambda\tan(15t)^\circ$ for a vertical dial facing S.

20. Find the stresses in the diagonal braces and in the flanges of a Warren girder, in which the braces slope at (1) **60**°, (2) **67½**°, supposing the lower flange to be in eight bays and each joint loaded with **16** cwts., the lower flange being in tension and the girder supported at its extremities.

21. If one wheel of a field gun is above the other, so that the line of trunnions is inclined ϕ° upwards to the left, then, when α° is the elevation of the gun, the shot will be deflected through an angle θ, where $\tan\theta = \sin\phi\tan\alpha$. Verify the approximate rule. "Deflection correction required in minutes (to the left) = elevation in degrees multiplied by inclination of trunnions in degrees," that is $\theta' = \alpha^\circ \times \phi^\circ$, taking $\phi = 4^\circ$ and $\alpha = 29{\cdot}7^\circ$.

The wheels of a field gun are **60** in. apart nearly.

22. If a shot is affected by three accidental causes of error which would produce probable errors p_1, p_2, p_3 respectively, the total probable error is

$$P = \sqrt{p_1^2 + p_2^2 + p_3^2}.$$

Compute P when

$p_1 = 45$ yards,

$p_2 = 12, 20, 30, 35$ yards,

$p_3 = 5, 7, 9, 11$ yards.

The above formula is based on the assumption that the errors each follow the normal law of error.

The very small effect of p_3 on the answer should be noticed.

23. If P is the *probable error* of a shot (*i.e.* that error which the actual error of one shot is equally likely to be above or below), the chance q that a shot has an error numerically less than nP is given *approximately* by the formula

$$q=\sqrt{1-(\tfrac{3}{4})^{n^2}}.$$

This result is due to Monsieur Gazot, Capitaine d'Artillerie.

Work out q for n = ·1, ·25, ·5, 2·, 3·, 4·. For the mode of working see page 113.

The results as calculated from the exact formula

$$q=\int_0^{n\rho}\frac{2}{\sqrt{\pi}}e^{-t^2}dt$$

are

n = ·1	q = ·0538
= ·25	·1339
= ·5	·2641
= 2	·8227
= 3	·957
= 4	·993

where ρ = ·4769.

24. The greatest and least tensions in the chain of a suspension bridge are given in pounds by

$$T_1=\frac{wa}{2}\sqrt{1+\left(\frac{a}{4d}\right)^2}\text{ and }T_0=\frac{wa^2}{8d},$$

where w = weight per foot run horizontal in pounds = 1400 pounds, say ;

a = total span in feet = 90, say ;

d = dip at centre in feet = 10, say.

Find T_1 and T_0 in pounds and tons.

25. Find the combined resistance of the three branches of a divided circuit whose separate resistances are 39·3, 11·7, and 2·1 ohms.

26. Given Matthiessen's standard for copper (resistance of copper wire 1 foot long, weighing 1 grain, is ·1516 ohm), find the resistance of a mile of copper wire, weighing 100 lb., whose percentage of purity is 98.

27. The calories of heat produced in a conductor of resistance R by a current of C amperes in t seconds is

$$H = C^2Rt \times \cdot 24.$$

Take $C = 4\cdot5$ amperes, $R = 5\cdot3$ ohms, $t = 5$ minutes, and find how much this will heat 3 pints of water originally at 10° C., assuming that all the heat developed goes towards raising the temperature of the water.

28. An incandescent lamp absorbing 60 watts is immersed in a given quantity of water, say a litre, at 10° C. How long will it take to raise the temperature to the boiling point, assuming that all the heat developed goes towards raising the temperature of the water?

A very large number of other practical examples can be obtained from Molesworth's *Pocket Book of Engineering Formulae*, or from the official text-books of Fortification, Gunnery, or Electricity.

ANSWERS.

p. 17. **3.** 3·9, say £3. 18s. 0*d*. per cent.

4. 49·6, 9·3, 78·2, 5·38.

6. Mn 62·79, K 31·9,
O 37·21. Cl 28·92,
O 39·18.

p. 24. **2.** ·782, 2·475, 3·91 seconds.

3. 19·23 yards. **5.** 11·619 by 9·295 inches.

p. 28. **1.** 550 lb., 500 lb., 474 lb.

2. 40·44 lb., ·212 inch. **3.** 3·517, 14·068, 31·65 lb.

4. ·225, 31288, 38213, 2246, 701·8 ft.-tons.

p. 29. **5.** ·6, 25·65, 28·35, 10·8, 7·05 inches.

6. 3·972, 2·583 tons/in². **7.** 14550, 39600, 133200 lb.-in.

p. 30. **8.**

$b \backslash d$	5	10·5	13
5	62500	131250	162500
10·5	275600	578800	716000
13	422500	817250	1098500

d	l	$b=5$	10·5	13	
5	10	4166	8750	10833	pounds.
	20	2083	4375	5416	
	24	1736	3645	4513	
10·5	10	18375	38600	47800	
	20	9190	19300	23900	
	24	7660	16080	19900	
13	10	28160	59150	73230	
	20	14080	29570	36600	
	24	11730	24640	30510	

p. 30. 11. ·616, 2·465, 3·853, 15·42 tons.

12.

$d \backslash b$	8	12	15
10	·381	·303	·2039
12·5	·193	·155	·1944
17	·0768	·0617	·0415

inches, with load at centre.

p. 31. 13. 53·6 lb. 14. 7717 lb.

15. 64, 217, 165 H.P.

p. 32. 16. 15·26 f./s.

17. *a* (i) 1·899, *b* (i) 1·266, *c* (i) 1·216,
a (ii) 1·323. *b* (ii) ·882. *c* (ii) ·847 inch.

18. 1178 lb.

p. 33. 19. 89·35 amp.

p. 41. 2. 18·6 lb. 3. 172·8, 337·5, 583·2 lb.

p. 42. 4. 1·0496, ·7281, ·6615, ·6141, ·5489 in.

p. 44. 150 or 5,
– 50 or – 15,

p. 45. 60 or 5,
20 or 15.
4·561 or ·438,
5·372 or – ·372,
– 499·82 or – ·18,
49·96 or ·04,
49·5967 or ·4033,
16·86 or – 11·86.

p. 48. 1. 6, 4, – 2.
8·021.
4·592.
14·15, – 14·13, – ·02.
5·408.
– 2·7785, ·2891, 2·4894.
– ·283.
– 1·891.
– 1·279. Write the equation $x^2(3 - x) = 7$.

p. 49. 4. 1·458 feet, 2 feet, 2·54 feet.

p. 64. 1. 2172, 2204, 2453, 2602 yards.

p. 78. 3. 1·992.

MISCELLANEOUS EXAMPLES.

p. 117. **1.** 546·5 litres. **2.** 20·6 kilos.

3. 29·41, 23·53, 47·06.

4. 2·534 in., 5·86 in., 212 yards, 1291 yards.

p. 118. **8.** 5′ 23″, 2′ 36″, 1′ 38″, 1′ 8″, 32·9″, 10″.

9. 382, 191, 127, 95·5, 54·5, 38·2, 19·1 yards.

p. 119. **10.** ·768 of a yard. About 4 per cent.

11. 30, 9·47. **12.** ·0152, ·0609, ·3805, 1·51, 6·03.

13. R = 8370, 4924, 3348, 2093, 1395, 1196 yards.
Corrections 171·3, 163·5, 150·6, 111·8, 59·52.

p. 120. **14.** 1195 and 1023. Errors 1·14 and 1·4 yards.

15. 19·89, 11·08, 9·52, 9·265.

16. 449·9, 178·3, 89, 903·8 feet.
(i) 437·9, 174·3, 87, 883·4 ,,
(ii) 437, 183·9, 86·8, 881·3 ,,

p. 121. **17.** 4 h. 45 m.; the true time being about 4 h. 33 m.

19. 35°.

p. 122. **20.** Stresses in diagonals:
(i) 64·7, 46·2, 27·7, 9·25 cwt.
(ii) 60·6, 43·3, 26, 8·67 ,,

22.

p_1 \ p_2	5	7	9	11
12	46·84	47·09	47·43	47·85
20	49·49	49·74	50·06	50·46
30	54·31	54·53	54·83	55·19
35	57·22	57·43	57·71	58·06

p. 123. **23.** ·0537, ·133, ·263, ·826, ·961, ·994.

24. 52·7 tons, 59·7 tons. **25.** 1·7 ohms.

26. 6·036 ohms. **27.** 4·5°.

p. 124. **28.** 104 minutes.

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